PART 2 Multirate Filter Banks

5

Maximally Decimated Filter Banks

5.0 INTRODUCTION

The basic philosophy of subband coding was explained in Chap. 4. The analysis/synthesis system used for this purpose is the maximally decimated filter bank. Figure 5.1-1(a) shows the two channel version, popularly called the Quadrature Mirror Filter (QMF) bank. This system was introduced in the mid seventies [Croisier, et al., 1976], and has since then been studied by many other researchers, as we cite at the appropriate sections. The input signal $x(n)$ is first filtered by two filters $H_0(z)$ and $H_1(z)$, typically lowpass and highpass as shown in part (b). Each signal $x_k(n)$ (subband signal) is therefore approximately bandlimited to a total width of π (in the frequency region $0 \leq \omega < 2\pi$). The subband signals are decimated by a factor of 2 to produce $v_k(n)$.

Each decimated signal $v_k(n)$ is then coded in such a way that the special properties of the subband (such as energy level, perceptive importance and so on) are exploited. At the receiver end, the received signals are decoded to produce (approximations of) the signals $v_0(n)$ and $v_1(n)$ which are then passed through two-fold expanders. The output signals $y_0(n)$ and $y_1(n)$ are then passed through the filters $F_0(z)$ and $F_1(z)$ (whose purpose we will explain) to produce the output signal $\widehat{x}(n)$.

$H_0(z)$ and $H_1(z)$ are called analysis filters, and the pair $[H_0(z), H_1(z)]$ the analysis bank. This pair followed by the two decimators is the *decimated analysis bank.* Similarly $F_0(z)$ and $F_1(z)$ are the synthesis (or reconstruction) filters, and the pair $[F_0(z), F_1(z)]$ the synthesis bank. In this chapter we will see that the reconstructed signal $\widehat{x}(n)$ differs from $x(n)$ due to *three* reasons: aliasing, amplitude distortion, phase distortion, It will be shown that the filters can be designed in such a way that some or all of these distortions are eliminated. These results will then be extended to the case of M channel filter banks.

There is a fourth reason why the reconstructed signal differs from $x(n)$. This is due to the coding or quantization of the subband signals. The effect of this cannot be corrected, but can only be analyzed. This is done in Appendix C.

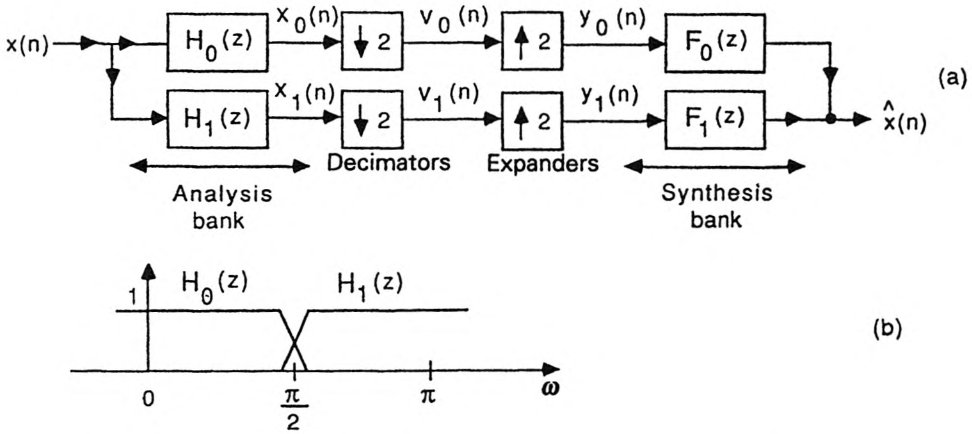


Figure 5.1-1 (a) The quadrature mirror filter bank and (b) typical magnitude responses.

5.0.1 A Brief History

For the two channel case, it was shown in Croisier, et al. [1976] that aliasing can be completely eliminated by a simple choice of the synthesis filters. Design techniques were later developed by other authors to minimize the remaining distortions [Johnston, 1980], [Jain and Crochiere, 1984], [Fettweis, et al., 1985] and efficient structures developed [Galand and Nussbaumer, 1984]. It was shown by Smith and Barnwell [1984] and Mintzer [1985] that all the three distortions mentioned above can be eliminated (i.e., perfect reconstruction achieved) in a two channel QMF bank with properly designed FIR filters, and further optimization techniques were developed [Grenez, 1988].

For the case of M channel filter banks, the conditions for alias cancelation and perfect reconstruction are much more complicated. The pseudo QMF technique was introduced [Nussbaumer, 1981], as a means of obtaining approximate alias cancelation in this case, and has since been developed by a number of authors [Rothweiler, 1983], [Chu, 1985], [Masson and Picel, 1985], and [Cox, 1986]. The general theory of perfect reconstruction in the M channel case was developed by a number of authors [Ramstad, 1984b], [Smith and Barnwell, 1985], [Vetterli, 1985], [Princen and Bradley, 1986], [Wackershruther, 1986b], [Vaidyanathan, 1987a,b], [Nguyen and Vaidyanathan, 1988], and [Viscito and Allebach, 1988a]. Vetterli and Vaidyanathan showed independently that the use of polyphase components leads to considerable simplification of the theory. A technique for the design of M channel perfect reconstruction systems was developed [Vaidyanathan, 1987a,b], based on polyphase matrices with the so-called paraunitary property. It has since been shown that the two channel perfect reconstruction system developed in Smith and Barnwell [1984] and Mintzer [1985] satisfy the paraunitary property. (This same property also finds application in the theory of orthonormal wavelet transforms, which we will study in Chap. 11.)

A particular class of M-channel perfect reconstruction systems was subsequently developed, with the property that all the analysis filters are derived starting from a prototype, by modulation. This has the advantage of economy during the design as well as implementation phases. The theory was developed by Malvar [1990b], Koilpillai and Vaidyanathan [1991 and 1992], and Ramstad [1991] independently. It turns out that these systems can be regarded as the generalization of the so-called lapped orthogonal transforms independently devloped by Cassreau [1985] and studied in Malvar and Staelin [1989], and Malvar [1990a].

Further advancement in the theory and design of filter banks have been made by several authors, but these will not be covered in our limited exposure here. This includes time domain design techniques, [Nayebi, et al., 1990], nonuniform filter banks, and filter banks with noninteger decimation ratios [Hoang and Vaidyanathan, 1989], [Kovačević and Vetterli, 1991a], [Nayebi, et al., 1991a], and filter banks with minimum reconstruction delay [Nayebi, et al., 1991b]. Also see Padmanabhan and Martin [1992], Horng, Samueli, and Willson [1991], and Horng and Willson [1992]. In Problems 5.25 and 5.32 we will consider some issues pertaining to nonuniform QMF banks.

5.0.2 Chapter Outline

In this chapter we present a detailed study of the QMF bank and its M channel extensions. Section 5.1 analyzes the various errors (aliasing, amplitude and phase distortions) created by the two channel QMF bank, and develops conditions for alias cancelation. Section 5.2 describes an alias-free system in greater detail. Section 5.3 considers a special class of alias free systems called the power symmetric QMF banks. These systems have very low complexity, and yet provide freedom from aliasing and amplitude distortion.

In Sec. 5.4 to 5.6 we extend these ideas for the case of M channel filter banks, and develop the theory of perfect reconstruction based on polyphase matrices. Section 5.7 develops the general theory of alias-free systems. Tree-structured filter banks are considered in Sec. 5.8, and Sec. 5.9 develops the theory of transmultiplexers.

The study of filter banks will be continued in the next few chapters. Paraunitary perfect reconstruction systems will be introduced in Chap. 6, along with several structures for implementing these systems. Some of the structures have the property that the perfect reconstruction property is retained in spite of coefficient quantization. Pseudo QMF banks and cosine modulated perfect reconstruction banks will be studied in Chap. 8.

5.1 ERRORS CREATED IN THE QMF BANK

The decimated signals $v_k(n)$ are encoded using one of many possible coding techniques [Jayant and Noll, 1984], and the resulting signals are actually transmitted. The receiver reconstructs an approximation $v'_k(n)$ of $v_k(n)$ from these encoded signals. The decoding error $v_k(n) - v'_k(n)$ represents a nonlinear distortion (like quantization error). This is called the *subband quantization error.* It cannot be corrected, that is, there is no way to exactly reconstruct $v_k(n)$ from $v'_k(n)$.

The subband quantization error will be treated in greater detail in Appendix C. In this chapter will ignore this error, that is, assume $v'_k(n) = v_k(n)$. The QMF bank still suffers from three fundamental errors, viz., aliasing, amplitude distortion, and phase distortion to be described next.

5.1.1 Aliasing and Imaging

In practice, the analysis filters have nonzero transition bandwidth and stopband gain. The signals $x_k(n)$ are, therefore, not bandlimited, and their decimation results in aliasing. To study this effect further, consider Fig. 5.1-2 where two situations are shown. In Fig. 5.1-2(a), the responses $|H_0(e^{j\omega})|$ and $|H_1(e^{j\omega})|$ do not overlap. Assuming that the stopband attenuations are sufficiently large, the effect of aliasing is not serious. In Fig. 5.1-2(b), however, the responses overlap, and each subband signal can in general have substantial energy for a bandwidth exceeding the ideal passband region. Decimation of these signals therefore results in aliasing regardless of how good the stopbands of the filters are.

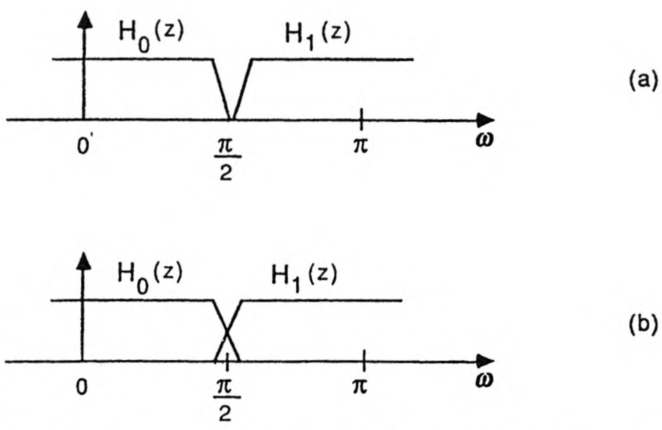


Figure 5.1-2 Two possible magnitude responses for the analysis filters. (a) Nonoverlapping, and (b) overlapping.

In principle it is true that the choice of filters as in Fig. 5.1-2(a) takes care of this problem. However, non overlapping responses imply severe attenuation of the input signal around $\omega = \pi/2$. Even though this can be compensenated, in principle, by appropriately boosting this frequency region [by proper design of $F_0(z)$ and $F_1(z)$], it will result in severe amplification of noise (such as coding noise, channel noise and filter roundoff noise). A second solution would be to make the transition widths of the responses very narrow but this requires very expensive filters. The overlapping response in Fig. 5.1-2(b) is therefore the more practical choice. Even though this results in aliasing, this effect can be canceled by careful choice of the *synthesis filters* as we will see.

Expression for the Reconstructed Signal

Using the results developed in Sec. 4.1.1 it is easy to find an expression for $\widehat{X}(z)$. From Fig. 5.1-1(a) we have

$$X_k(z) = H_k(z)X(z), \quad k = 0, 1. \tag{5.1.1}$$

The z-tranforms of the decimated signals $v_k(n)$ are [from (4.1.13) with $M = 2$]

$$V_k(z) = \frac{1}{2}[X_k(z^{1/2}) + X_k(-z^{1/2})], \quad k = 0, 1. \tag{5.1.2}$$

The second term above represents aliasing. The z-transform of $Y_k(z)$ is $V_k(z^2)$ so that

$$\begin{aligned} Y_k(z) = V_k(z^2) &= \frac{1}{2}[X_k(z) + X_k(-z)] \\ &= \frac{1}{2}[H_k(z)X(z) + H_k(-z)X(-z)], \quad k = 0, 1. \end{aligned} \tag{5.1.3}$$

The reconstructed signal is

$$\widehat{X}(z) = F_0(z)Y_0(z) + F_1(z)Y_1(z). \tag{5.1.4}$$

Substituting from (5.1.3) and rearranging, we finally obtain

$$\begin{aligned} \widehat{X}(z) =& \frac{1}{2}[H_0(z)F_0(z) + H_1(z)F_1(z)]X(z) \\ &+ \frac{1}{2}[H_0(-z)F_0(z) + H_1(-z)F_1(z)]X(-z). \end{aligned} \tag{5.1.5}$$

Or, in matrix-vector notation,

$$2\widehat{X}(z) = [\,X(z) \quad X(-z)\,] \underbrace{\begin{bmatrix} H_0(z) & H_1(z) \\ H_0(-z) & H_1(-z) \end{bmatrix}}_{\mathbf{H}(z)} \begin{bmatrix} F_0(z) \\ F_1(z) \end{bmatrix}. \tag{5.1.6}$$

The matrix $\mathbf{H}(z)$ is called the alias component (AC) matrix. The term which contains $X(-z)$ originates because of the decimation. On the unit circle, $X(-z) = X(e^{j(\omega-\pi)})$ which is a right-shifted version of $X(e^{j\omega})$ by an amount π. This term takes into account aliasing due to the decimators and imaging due to the expanders. We refer to this just as the alias term or alias component.

Alias Cancelation

From (5.1.5) it is clear that we can cancel aliasing by choosing the filters such that the quantity $H_0(-z)F_0(z) + H_1(-z)F_1(z)$ is zero. Thus the following choice cancels aliasing:

$$F_0(z) = H_1(-z), \quad F_1(z) = -H_0(-z). \tag{5.1.7}$$

Given $H_0(z)$ and $H_1(z)$, it is thus possible to *completely* cancel aliasing by this choice of synthesis filters. If the analysis filters have large transition bandwidths and low stopband attenuations, this implies large aliasing errors, but yet these errors are canceled by the choice (5.1.7).

So, the basic philosophy in the QMF bank is that we *permit* aliasing in the analysis bank instead of trying to avoid it. We then choose the synthesis filters so that the alias component in the upper branch is canceled by that in the lower branch.

Pictorial viewpoint. It helps to visualize the mechanism of alias cancelation in terms of frequency response plots. For this refer to Fig. 5.1-3 which shows an arbitrary input spectrum $X(e^{j\omega})$, the lowpass subband signal $X_0(e^{j\omega})$, and the decimated signal $V_0(e^{j\omega})$. The alias component $0.5X_0(-e^{j\omega/2})$ overlaps with $0.5X_0(e^{j\omega/2})$. The signal $Y_0(e^{j\omega})$ has contributions from $X(z)$ as well as $X(-z)$. The contribution which arises from $X(-z)$ (shaded region) is the alias component, and in general overlaps with the unshaded area.

In a similar way if we trace through the bottom channel, we can obtain qualitative plots of $X_1(e^{j\omega}), V_1(e^{j\omega})$ and $Y_1(e^{j\omega})$. The shaded areas in $Y_0(e^{j\omega})$ and $Y_1(e^{j\omega})$ represent aliasing (and imaging) effect(s), and dominantly occupy the *highpass* and *lowpass* regions, respectively. The filters $F_0(z)$ and $F_1(z)$, which are *lowpass* and *highpass* respectively, tend to eliminate these shaded portions. Because of the nonideal nature of these practical filters, the output of $F_0(z)$ still contains some residual shaded area (Fig. 5.1-3(h)), and so does the output of $F_1(z)$ (Fig. 5.1-3(i)). These two residual alias components can be made to cancel each other, and the choice (5.1.7) does precisely that.

The LPTV property. From Chap. 4 we know that the decimator and expander are linear and time varying (LTV) building blocks. So the QMF bank is a LTV system. Now (5.1.5) can be written as

$$\widehat{X}(z) = T(z)X(z) + A(z)X(-z). \tag{5.1.8}$$

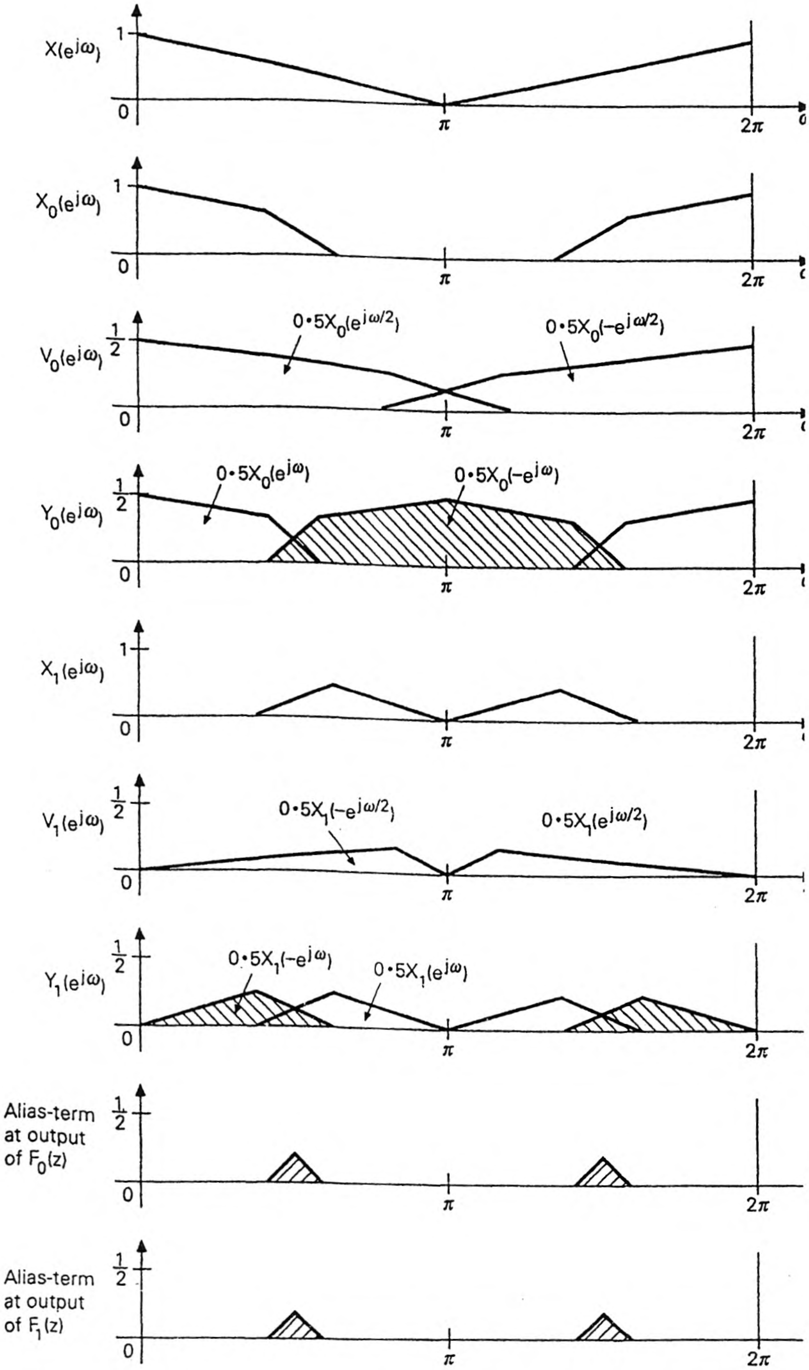


Figure 5.1-3 Various internal signals, and alias cancelation mechani: QMF bank. (© Adopted from 1990 IEEE.)

Denoting the impulse responses of $T(z)$ and $A(z)$ as $t(n)$ and $a(n)$, we can rewrite the above as

$$\widehat{x}(n) = \sum_k \Big(t(k) + (-1)^{k-n} a(k)\Big) x(n-k). \tag{5.1.9}$$

Defining $g_0(k) = t(k) + (-1)^k a(k)$ and $g_1(k) = t(k) - (-1)^k a(k)$, we then have

$$\widehat{x}(n) = \begin{cases} \sum_k g_0(k) x(n-k) & n \text{ even} \\ \sum_k g_1(k) x(n-k) & n \text{ odd}, \end{cases} \tag{5.1.10}$$

which proves that $\widehat{x}(n)$ is produced by passing $x(n)$ through the systems $G_0(z)$ and $G_1(z)$ in parallel, and taking the output of $G_0(z)$ for even n and that of $G_1(z)$ for odd n (Fig. 5.1-4). So the QMF bank is a linear periodically time varying (LPTV) system with period two. If aliasing is canceled (i.e., $A(z) = 0$), the system becomes LTI, and has transfer function $T(z)$.

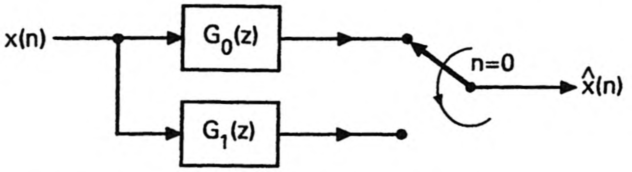


Figure 5.1-4 The QMF bank viewed as a LPTV system.

5.1.2 Amplitude and Phase Distortions

Suppose the choice (5.1.7) is made so that the QMF bank is free from aliasing. We then have

$$\widehat{X}(z) = T(z)X(z). \tag{5.1.11}$$

Thus even after aliasing is canceled, the signal $\widehat{x}(n)$ suffers from a linear shift-invariant distortion $T(z)$. Here

$$T(z) = \frac{1}{2}[H_0(z)F_0(z) + H_1(z)F_1(z)], \tag{5.1.12}$$

and is called the *distortion transfer function*, or "overall" transfer function of the alias-free system. Using (5.1.7) we get

$$T(z) = \frac{1}{2}[H_0(z)H_1(-z) - H_1(z)H_0(-z)]. \tag{5.1.13}$$

Letting $T(e^{j\omega}) = |T(e^{j\omega})|e^{j\phi(\omega)}$, we have

$$\widehat{X}(e^{j\omega}) = |T(e^{j\omega})|e^{j\phi(\omega)}X(e^{j\omega}). \tag{5.1.14}$$

Unless $T(z)$ is allpass (i.e., $|T(e^{j\omega})| = d \neq 0$ for all ω), we say that $\widehat{X}(e^{j\omega})$ suffers from "amplitude distortion." Similarly unless $T(z)$ has linear phase (that is, $\phi(\omega) = a + b\omega$ for constant a, b), $\widehat{X}(e^{j\omega})$ suffers from phase distortion.

We will use the following abbreviations for convenience: ALD (aliasing distortion), AMD (amplitude distortion), PHD (phase distortion).

Periodicity of $|T(e^{j\omega})|$. From (5.1.13) we see that $T(z)$ has the form $V(z) - V(-z)$. This means $T(z)$ has only odd powers of z, that is, $T(z) = z^{-1}S(z^2)$. So $|T(e^{j\omega})|$ has period π rather than 2π. For the real coefficient case this implies that $|T(e^{j\omega})|$ is symmetric with respect to $\pi/2$.

The Perfect Reconstruction (PR) QMF Bank

If a QMF bank is free from aliasing, amplitude distortion, and phase distortion, it is said to have the perfect reconstruction (abbreviated PR) property. This is equivalent to the condition $T(z) = cz^{-n_0}$. For a PR QMF bank we have

$$\widehat{X}(z) = cz^{-n_0}X(z), \quad \text{i.e.,} \quad \widehat{x}(n) = cx(n - n_0), \quad c \neq 0, \tag{5.1.15}$$

for all possible inputs $x(n)$. In other words, $\widehat{x}(n)$ is merely a scaled and delayed version of $x(n)$. This, of course, ignores the coding/decoding error and filter roundoff noise.

5.2 A SIMPLE ALIAS-FREE QMF SYSTEM

In the earliest known QMF banks the analysis filters were related as

$$H_1(z) = H_0(-z). \tag{5.2.1}$$

For the real coefficient case this means $|H_1(e^{j\omega})| = |H_0(e^{j(\pi-\omega)})|$. This ensures that $H_1(z)$ is a good highpass filter if $H_0(z)$ is a good lowpass filter. In fact $|H_1(e^{j\omega})|$ is a mirror image of $|H_0(e^{j\omega})|$ with respect to the quadrature frequency $2\pi/4$, justifying the name *quadrature mirror filters.*

With the choice (5.2.1) the alias cancelation constraint (5.1.7) becomes

$$F_0(z) = H_0(z), \quad F_1(z) = -H_1(z). \tag{5.2.2}$$

Thus all the four filters are completely determined by a single filter $H_0(z)$. The designer has to concentrate on the design of only this filter. According to the earliest nomenclatures, the system with the four filters related as above was known as the 'QMF' bank. But as a matter of convenience, the term 'QMF' has since been used to indicate generalized versions, for example, M-channel systems.

From (5.2.2) we see that $F_0(z)$ and $F_1(z)$ are lowpass and highpass respectively [consistent with the fact that $F_0(z)$ attenuates the 'highpass image' and $F_1(z)$ attenuates the 'lowpass image' created by the expanders]. With filters chosen as above, the distortion function is

$$T(z) = \frac{1}{2}\Big(H_0^2(z) - H_1^2(z)\Big) = \frac{1}{2}\Big(H_0^2(z) - H_0^2(-z)\Big). \tag{5.2.3}$$

5.2.1 Polyphase Representation

It is often beneficial, both conceptually and computationally, to represent the analysis and synthesis banks in terms of polyphase components (Section 4.3). Thus let

$$H_0(z) = E_0(z^2) + z^{-1}E_1(z^2) \qquad \text{(Type 1 polyphase).} \tag{5.2.4}$$

Since $H_1(z) = H_0(-z)$, we have $H_1(z) = E_0(z^2) - z^{-1}E_1(z^2)$, that is,

$$\begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} E_0(z^2) \\ z^{-1}E_1(z^2) \end{bmatrix}. \tag{5.2.5}$$

The synthesis filters $F_0(z)$ and $F_1(z)$, which satisfy (5.2.2), can also be represented in terms of $E_0(z)$ and $E_1(z)$ as follows:

$$[\,F_0(z) \quad F_1(z)\,] = [\,z^{-1}E_1(z^2) \quad E_0(z^2)\,] \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{5.2.6}$$

By using (5.2.5) and (5.2.6) we can draw the analysis and synthesis banks as in Fig. 5.2-1(a) and (b) respectively, and the complete QMF bank as in Fig. 5.2-2(a). By using the noble identities (Fig. 4.2-3) we can redraw this as in Fig. 5.2-2(b). The polyphase components are now operating at the lowest possible rate, so that the number of multiplications and additions per unit time (MPUs and APUs) is minimized†.

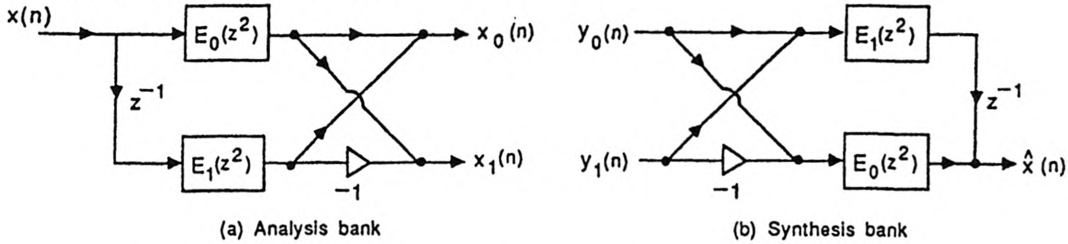


Figure 5.2-1 The analysis and synthesis banks in polyphase form.

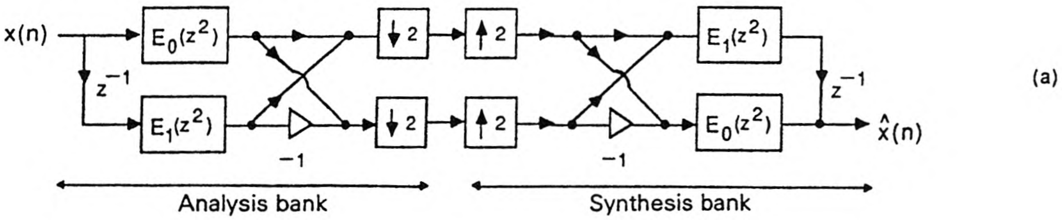
Limitations Imposed by the Constraint $H_1(z) = H_0(-z)$.

With the analysis filters related as $H_1(z) = H_0(-z)$ and synthesis filters chosen to cancel aliasing (eqn. (5.1.7)), the distortion function has the form (5.2.3). This can be written in terms of the polyphase components as

$$T(z) = 2z^{-1}E_0(z^2)E_1(z^2). \tag{5.2.7}$$

† As in Chap. 4, a unit of time is the separation between adjacent samples of the input $x(n)$.

This expression holds for any QMF bank (FIR or IIR; linear-phase or nonlinear phase) for which the filters are related by (5.2.1) and (5.2.2). From this expression we can draw a number of important conclusions.



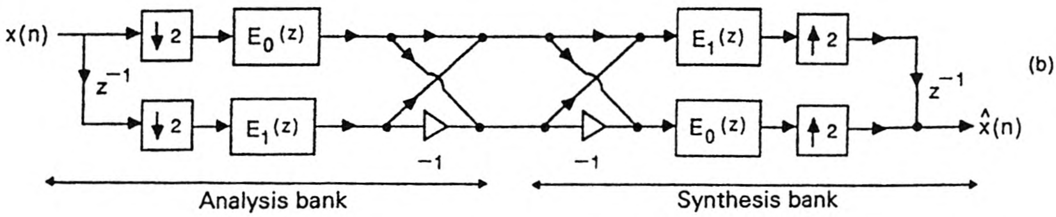


Figure 5.2-2 (a) The complete QMF bank in polyphase form. (b) Rearrangement using noble identities.

For example, let $H_0(z)$ be FIR so that $E_0(z)$, $E_1(z)$ and $T(z)$ are FIR as well. From (5.2.7) we note that amplitude distortion can be eliminated in this case if and only if each of the FIR functions $E_0(z)$ and $E_1(z)$ is a delay, that is, $E_0(z) = c_0 z^{-n_0}$ and $E_1(z) = c_1 z^{-n_1}$. This means

$$H_0(z) = c_0 z^{-2n_0} + c_1 z^{-(2n_1+1)}, \quad H_1(z) = c_0 z^{-2n_0} - c_1 z^{-(2n_1+1)}. \quad (5.2.8)$$

This conclusion holds whether or not $H_0(z)$ has linear phase.

Summarizing, if the analysis filters are related as $H_1(z) = H_0(-z)$ and $H_0(z)$ is FIR, we can eliminate amplitude distortion only if $H_0(z)$ and $H_1(z)$ have the above form! That is, the filters cannot have sharp cutoff and good stopband attenuations. We cannot, therefore, obtain useful FIR perfect reconstruction systems under the constraint $H_1(z) = H_0(-z)$.

If we choose $E_1(z) = 1/E_0(z)$, then (5.2.7) becomes a delay, thereby resulting in perfect reconstruction. But the filters become IIR.

5.2.2 Eliminating Phase Distortion with FIR Filters

A QMF bank in which the analysis *and* synthesis filters are FIR is said to be an FIR QMF bank. From Chap. 2 we know that FIR filters with exactly linear phase can be designed. If $H_0(z)$ has linear phase, then $T(z)$ given by (5.2.3) also has linear phase, thereby eliminating phase distortion.

The residual amplitude distortion $|T(e^{j\omega})|$ can now be analyzed with the help of (5.2.3). Let $H_0(z) = \sum_{n=0}^{N} h_0(n) z^{-n}$, with $h_0(n)$ real. The linear phase constraint requires $h_0(n) = \pm h_0(N-n)$. Since $H_0(z)$ has to be lowpass, the only possibility is $h_0(n) = h_0(N-n)$ (Section 2.4.2). With this choice

$$H_0(e^{j\omega}) = e^{-j\omega N/2} R(\omega), \tag{5.2.9}$$

where $R(\omega)$ is real for all ω. Substituting (5.2.9) into (5.2.3) and using the fact that $|H(e^{j\omega})|$ is an even function, we get

$$T(e^{j\omega}) = \frac{e^{-jN\omega}}{2}\Big(|H_0(e^{j\omega})|^2 - (-1)^N |H_0(e^{j(\pi-\omega)})|^2\Big). \tag{5.2.10}$$

Constraint on the order N. If N is even, then the above expression reduces to zero at $\omega = \pi/2$, resulting in severe amplitude distortion. So we have to chose N to be odd so that

$$\begin{aligned} T(e^{j\omega}) &= \frac{e^{-jN\omega}}{2}\Big(|H_0(e^{j\omega})|^2 + |H_0(e^{j(\pi-\omega)})|^2\Big) \\ &= \frac{e^{-jN\omega}}{2}\Big(|H_0(e^{j\omega})|^2 + |H_1(e^{j\omega})|^2\Big). \qquad \text{(from (5.2.1))} \end{aligned} \tag{5.2.11}$$

Minimization of Residual Amplitude Distortion

From the previous section we know that if $H_0(z)$ is FIR, then the constraint $H_1(z) = H_0(-z)$ rules out perfect reconstruction, unless the filters have the simple form (5.2.8). Having eliminated aliasing and phase distortion, we can therefore only *minimize* amplitude distortion, that is, we can make (5.2.11) only approximately constant.

If $H_0(z)$ has good passband and stopband responses, then $|T(e^{j\omega})|$ is almost constant in the passbands of $H_0(z)$ and $H_1(z)$. The main difficulty comes in the transition band region. The degree of overlap of $H_0(z)$ and $H_1(z)$ is very crucial in determining this distortion. To demonstrate this, Fig. 5.2-3 shows responses of three linear phase designs of $H_0(z)$. If the passband edge is too large as in curve 1 (i.e., $H_0(z)$ and $H_1(z)$ have too much overlap), $|T(e^{j\omega})|$ exhibits a peaking effect around $\pi/2$. If the passband edge is too small (curve 2), then $|T(e^{j\omega})|$ exhibits a dip around $\pi/2$. The third curve, where the passband edge is carefully chosen by trial and error, produces a much better response of $|T(e^{j\omega})|$.

The aim, therefore, is to adjust the coefficients of $H_0(z)$ so that the filters satisfy the condition

$$|H_0(e^{j\omega})|^2 + |H_1(e^{j\omega})|^2 = 1, \tag{5.2.12}$$

approximately. Systematic computer-aided optimization techniques for this have been developed [Johnston, 1980], [Jain and Crochiere, 1984]. In Johnston's technique, an objective function is formulated which reflects two things: (a) the stopband attenuation of the filter $H_0(z)$, and (b) the extent to which (5.2.12) is satisfied. For example the objective function could be

$$\phi = \alpha\phi_1 + (1-\alpha)\phi_2, \tag{5.2.13}$$

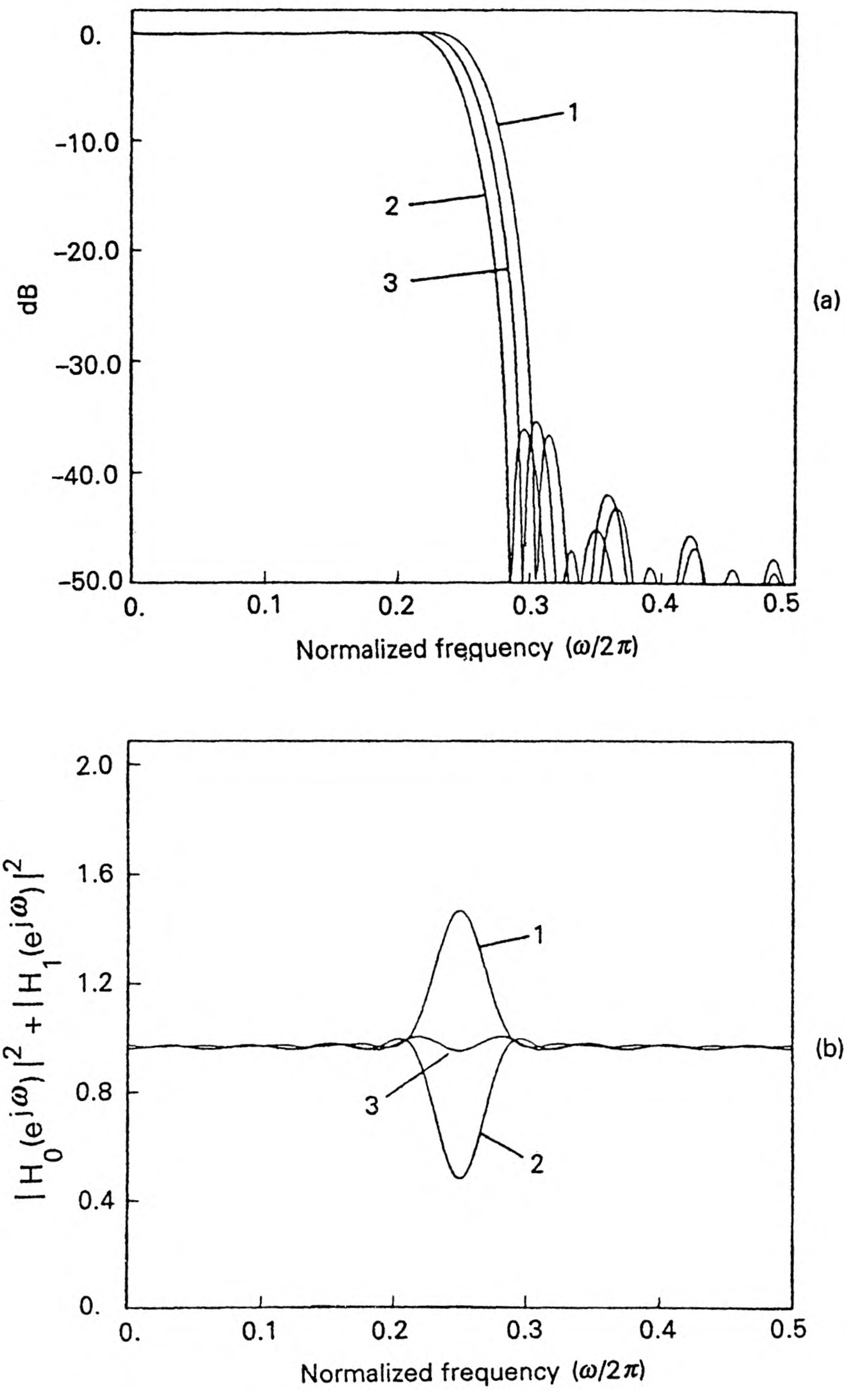


Figure 5.2-3 Amplitude distortion as a function of the degree of overlap between analysis filters. (© Adopted from 1990 IEEE.)

where

$$\phi_1 = \int_{\omega_S}^{\pi} |H_0(e^{j\omega})|^2 d\omega, \quad \phi_2 = \int_0^{\pi} \left(1 - |H_0(e^{j\omega})|^2 - |H_0(e^{j(\pi-\omega)})|^2\right)^2 d\omega, \tag{5.2.14}$$

and $0 < \alpha < 1$. The coefficients $h_0(n)$ of $H_0(z)$ are then optimized in order to minimize ϕ. Since $|T(e^{j\omega})|$ has symmetry with respect to $\pi/2$, we can replace $\int_0^{\pi}$ with $2\int_0^{\pi/2}$. The quantity ω_S is typically chosen as $\pi/2 + \epsilon$ for some small $\epsilon > 0$.

What controls the passband shape? If the optimized response is satisfactory, the quantities ϕ_1 and ϕ_2 will be very small, and (5.2.12) will hold approximately. This means that $|H_1(e^{j\omega})|$ (i.e., $|H_0(-e^{j\omega})|$) is close to unity in the stopband of $H_0(z)$. This is equivalent to saying that $|H_0(e^{j\omega})|$ is close to unity in its own passband. Summarizing, minimization of ϕ ensures that $H_0(z)$ has good stopband as well as passband responses.

Design Example 5.2.1: Johnston's Filters

Filters with a wide range of specifications have been designed, and impulse response coefficients tabulated in Johnston [1980]. These tables can also be found in Crochiere and Rabiner [1983]. Fig. 5.2-4(a) shows the magnitude response plots of the analysis filters for Johnston's 32D filter. For this design the filter order $N = 31$, $\omega_S = 0.586\pi$, and the minimum stopband attenuation is 38 dB. The quantity $|H_0(e^{j\omega})|^2 + |H_1(e^{j\omega})|^2$ (which is twice the amplitude distortion), is shown in Fig. 5.2-4(b). On a dB scale, this is close to 0 dB for all ω, with peak distortion equal to ± 0.025 dB.

Computational complexity. With N representing the order of $H_0(z)$, there are $N+1$ coefficients $h_0(n)$. There are $(N+1)/2$ coefficients in each of $E_0(z)$ and $E_1(z)$. So from Fig. 5.2-2(b) we see that the analysis bank requires a total of $N+1$ multiplications and additions, that is,

$$\frac{N+1}{2} \text{ MPUs} \quad \text{and} \quad \frac{N+1}{2} \text{ APUs}$$

(since these are performed after decimation). The synthesis bank has the same complexity.

For our design example, we have $N = 31$ so that the analysis bank can be implemented using 16 MPUs and 16 APUs. This is an efficient implementation exploiting two facts: (a) the presence of decimators and expanders, and (b) the relation $H_1(z) = H_0(-z)$. Once these are exploited the symmetry of $h_0(n)$ (due to linear phase) cannot, unfortunately, be exploited (Problem 5.3).

5.2.3 Eliminating Amplitude Distortion with IIR Filters

The question that arises now is this: is it possible to completely eliminate amplitude distortion, rather than just minimize it using a computer program? We address this now. In order to eliminate amplitude distortion,

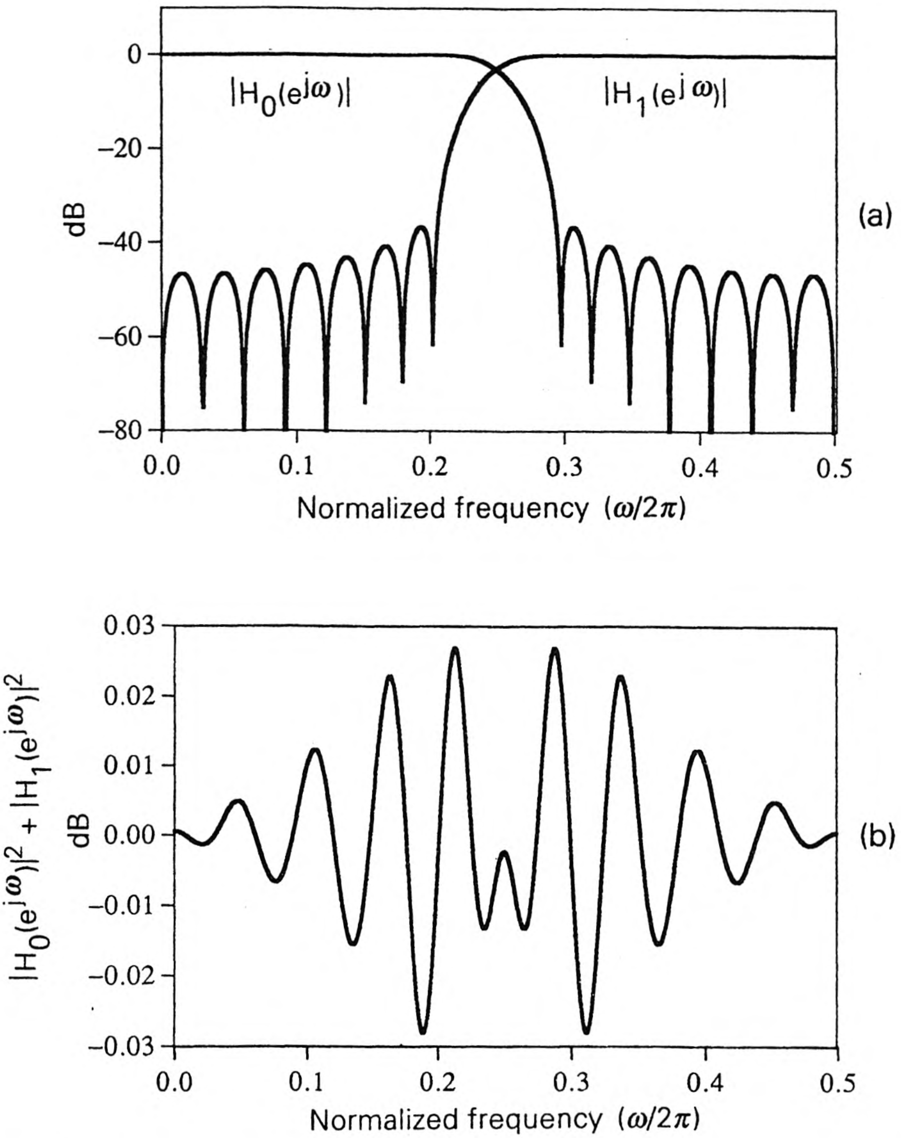


Figure 5.2-4 Design example 5.2.1 (Johnston's method). (a) Magnitude responses of the analysis filters, and (b) amplitude distortion measure.

we have to force $T(z)$ to be allpass. From (5.2.7) we see that this can be done by forcing $E_0(z)$ and $E_1(z)$ to be IIR and allpass [Vaidyanathan, et al, 1987], [Ramstad, 1988]. This also results in filters with a more general form than (5.2.8). Phase distortion still remains, and is governed by the phase responses of $E_0(z)$ and $E_1(z)$. †

To pursue this idea further, let us write the polyphase components as

$$E_0(z) = \frac{a_0(z)}{2}, \quad E_1(z) = \frac{a_1(z)}{2}, \tag{5.2.15}$$

where $a_0(z)$ and $a_1(z)$ are allpass, with $|a_0(e^{j\omega})| = |a_1(e^{j\omega})| = 1$. The analysis filter $H_0(z)$ now takes the form

$$H_0(z) = \frac{a_0(z^2) + z^{-1}a_1(z^2)}{2}. \tag{5.2.16}$$

Since $H_1(z) = H_0(-z)$, we have

$$\underbrace{\begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix}}_{\mathbf{h}(z)} = \underbrace{\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{\mathbf{R}} \underbrace{\begin{bmatrix} a_0(z^2) \\ z^{-1}a_1(z^2) \end{bmatrix}}_{\mathbf{a}(z)}. \tag{5.2.17}$$

The synthesis filters, which are given by (5.2.2), can be expressed as

$$[\,F_0(z) \quad F_1(z)\,] = \frac{1}{2}[\,z^{-1}a_1(z^2) \quad a_0(z^2)\,]\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{5.2.18}$$

The distortion function, which is allpass, is now given by

$$T(z) = \frac{z^{-1}}{2}a_0(z^2)a_1(z^2). \tag{5.2.19}$$

Figure 5.1-1(a) can now be redrawn as in Fig. 5.2-5, showing the complete QMF bank. This is free from aliasing and amplitude distortion, regardless of the details of the allpass functions $a_0(z)$ and $a_1(z)$!

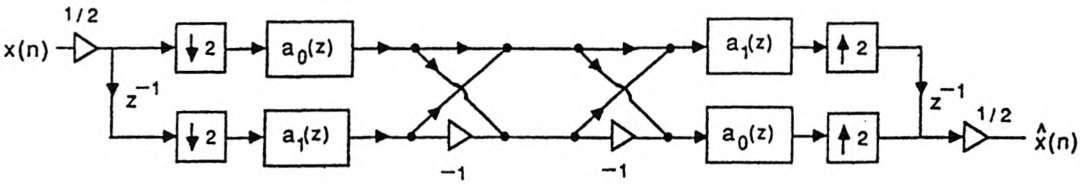


Figure 5.2-5 QMF bank with allpass polyphase components.

† The allpass constraint on $E_k(z)$ is, however, not *necessary*. For example, if $E_0(z) = 0.5 + z^{-1}$ and $E_1(z) = 1/(1 + 0.5z^{-1})$, then also $T(z)$ is allpass, that is, amplitude distortion is eliminated. However, since $H_0(z)$ has band edge around $\pi/2$ and since the coefficients $E_i(z)$ are decimated versions of $h_0(n+i)$, it is not counter-intuitive that $E_i(z)$ should be constrained to be allpass.

Can We Get Good Filter Responses with (5.2.16)?

The next question is, if we constrain the IIR analysis filter $H_0(z)$ to be of the form (5.2.16), is it still possible to have good attenuation characteristics? The answer is in the affirmative. For example, elliptic lowpass filters are of this form, if the bandedges and ripple sizes are chosen with appropriate symmetry (Fig. 5.2-6, to be explained next). With an elliptic filter so designed, we can easily identify the components $a_0(z)$ and $a_1(z)$ and then implement the structure of Fig. 5.2-5. It turns out that this technique is one of the most efficient ways (Sec. 5.3.5) to implement QMF banks free from aliasing and amplitude distortion. For example, we will see that if $H_0(z)$ is a fifth order elliptic filter, the entire analysis bank requires only one multiplication and three additions per input sample! In the next section we justify these claims, and also show how the filter $H_0(z)$ can be designed with the above constraint.

5.3 POWER SYMMETRIC QMF BANKS

We begin this section by summarizing the outcome of Sec. 5.2.3, concerning the IIR QMF bank. We assumed that the four filters are related as

$$H_1(z) = H_0(-z),\ F_0(z) = H_0(z),\ F_1(z) = -H_1(z).$$

This ensures that aliasing is canceled, and the distortion function is $T(z) = 2z^{-1}E_0(z^2)E_1(z^2)$, where $E_i(z)$ are polyphase components of $H_0(z)$, that is, $H_0(z) = E_0(z^2) + z^{-1}E_1(z^2)$. If $H_0(z)$ is IIR and the polyphase components $E_i(z)$ are allpass, then $T(z)$ becomes allpass. This, then, is a simple way to eliminate aliasing *and* amplitude distortion. The phase responses of $E_0(z)$ and $E_1(z)$ determine the remaining phase distortion.

♠**Main points of this section.** In this section we first study the properties of filters $H_0(z)$ for which $E_0(z)$ and $E_1(z)$ are allpass (i.e., filters which have the form (5.2.16) where $|a_0(e^{j\omega})| = |a_1(e^{j\omega})| = 1$) and then show how to design them.

1. We first show that if $H_0(z)$ is of the form (5.2.16), then it satisfies two symmetry-properties viz., numerator symmetry, and power symmetry (to be defined).
2. Conversely, we will show that if a transfer function satisfies these symmetry properties, it can be expressed as in (5.2.16). A more precise statement is given in Theorem 5.3.1.
3. As a consequence of the preceding result we will show the following: Let $H_0(z)$ be an odd order elliptic lowpass filter with ripple sizes δ_1, δ_2 and band edges ω_p, ω_S defined as usual [Figure 3.1-1(b)]. Suppose the response $|H_0(e^{j\omega})|^2$ exhibits symmetry with respect to $\pi/2$ as shown in Fig. 5.2-6. In other words, the ripple curve in the passband is a mirror image of the ripple curve in the stopband, with respect to the half-band frequency $\pi/2$. Mathematically this means $1 - (1 - 2\delta_1)^2 = \delta_2^2$, that is,

$$\delta_2^2 = 4\delta_1(1 - \delta_1), \tag{5.3.1a}$$

and also

$$\omega_p + \omega_S = \pi. \tag{5.3.1b}$$

(So if ω_S and δ_2 are specified then ω_p and δ_1 are determined, and the filter specifications are complete.) Under this symmetry condition, we can indeed express $H_0(z)$ as in (5.2.16), where $a_0(z)$ and $a_1(z)$ are unit-magnitude allpass filters. In other words the constraints (5.3.1) on the specifications ensure that the polyphase components of $H_0(z)$ are all-pass! We will present a modification of the standard elliptic filter design algorithm [Antoniou, 1979] to obtain the coefficients of $a_0(z)$ and $a_1(z)$, starting from the specifications ω_S and δ_2. $\diamond$

The reader interested only in the design procedure can proceed directly to Sec. 5.3.4.

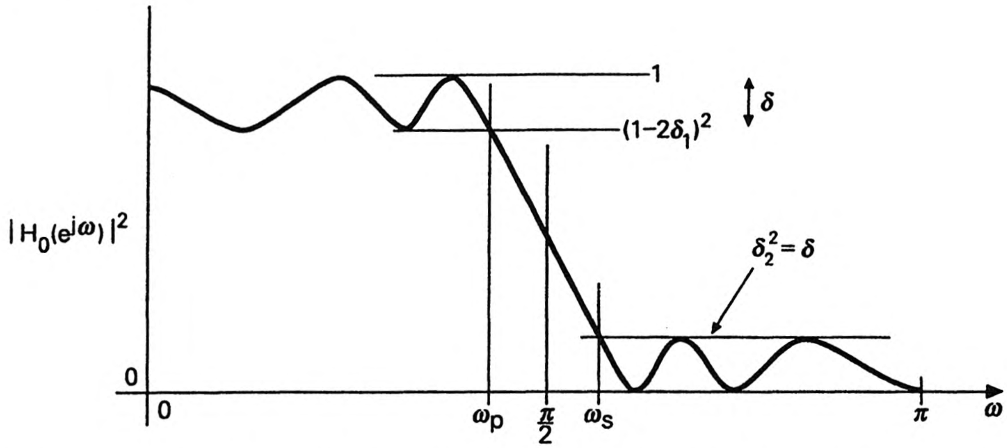


Figure 5.2-6 Square of the magnitude response function for a power symmetric filter.

5.3.1 Properties Induced by (5.2.16)

Power Symmetric Property

The quantities in (5.2.17) satisfy

$$\widetilde{\mathbf{a}}(z)\mathbf{a}(z) = 2, \quad \mathbf{R}^{\dagger}\mathbf{R} = 0.5\mathbf{I}, \tag{5.3.2}$$

so that $\widetilde{\mathbf{h}}(z)\mathbf{h}(z) = 1$. In terms of ω this means

$$|H_0(e^{j\omega})|^2 + |H_1(e^{j\omega})|^2 = 1. \tag{5.3.3}$$

So $H_1(z)$ is related to $H_0(z)$ in two ways: first by the constraint $H_1(z) = H_0(-z)$, and secondly by the power complementary property (5.3.3). Combining these we obtain the constraint

$$\widetilde{H}_0(z)H_0(z) + \widetilde{H}_0(-z)H_0(-z) = 1 \quad \text{(power symmetry condition)}. \tag{5.3.4}$$

Now, on the unit circle we have $|H_0(-e^{j\omega})| = |H_0(e^{j(\omega-\pi)})|$. For the real coefficient case this is the same as $|H_0(e^{j(\pi-\omega)})|$ so that (5.3.4) implies

$$|H_0(e^{j(\frac{\pi}{2}+\theta)})|^2 + |H_0(e^{j(\frac{\pi}{2}-\theta)})|^2 = 1, \tag{5.3.5}$$

for any real θ. This shows that the magnitude-squared function exhibits symmetry with respect to $\pi/2$, as demonstrated in Fig. 5.2-6. For this reason, (5.3.4) is called the *power symmetric property* and $H_0(z)$ is said to be power symmetric, even though (5.3.5) holds only for the real coefficient case. Also the right hand side of (5.3.4) is often permitted to be different from unity. We can restate (5.3.4) in any one of the following equivalent ways:

1. $\widetilde{H}_0(z)H_0(z)$ is a *half-band* filter [i.e., it satisfies (4.6.7c)].
2. $\widetilde{H}_0(z)H_0(z)\big|_{\downarrow 2} = 0.5$. Here the notation $A(z)\big|_{\downarrow 2}$ is as defined in Section 4.1.1. See, for example, (4.1.14).
3. $H_0(z)$ is power-symmetric.

Symmetry of Numerator of $H_0(z)$

From Sec. 3.4 we know that the allpass functions can be expressed as

$$a_0(z) = c_0 z^{-k_0}\frac{\widetilde{d}_0(z)}{d_0(z)}, \quad a_1(z) = c_1 z^{-k_1}\frac{\widetilde{d}_1(z)}{d_1(z)}, \tag{5.3.6}$$

where $|c_0| = |c_1| = 1$, and $k_i \geq$ order of $d_i(z)$. [By convention $d_i(z)$ is a polynomial in z^{-1}.] Substituting into (5.2.16) we obtain

$$H_0(z) = \frac{0.5\Big(c_0 z^{-2k_0}\widetilde{d}_0(z^2)d_1(z^2) + z^{-1}c_1 z^{-2k_1}\widetilde{d}_1(z^2)d_0(z^2)\Big)}{d_0(z^2)d_1(z^2)} \tag{5.3.7}$$

Thus, $H_0(z) = P_0(z)/d_0(z^2)d_1(z^2)$, that is, the denominator has only even powers of z^{-1}. It is easy to verify that the numerator $P_0(z)$ is generalized Hermitian (Sec. 2.3). For the most common case where $d_0(z)$ and $d_1(z)$ have real coefficients and $c_0 = c_1 = 1$, this means that $P_0(z)$ is *symmetric.* More specifically, $P_0(z^{-1}) = z^N P_0(z)$ where $N = 2(k_0 + k_1) + 1$. If $p_0(n)$ denotes the coefficients of $P_0(z)$, this property means $p_0(n) = p_0(N-n)$.

Irreducibility. It can be shown (Problem 5.7) that there are no common factors between $P_0(z)$ and the denominator $d_0(z^2)d_1(z^2)$, under the reasonable assumptions that (a) $d_0(z)$ and $d_1(z)$ have all zeros inside the unit circle, and (b) $d_0(z)$ and $d_1(z)$ do not have common factors. In practical examples such as Butterworth and elliptic filters, these two assumptions are true. The second assumption is reasonable because, if $(1-\alpha z^{-1})$ is a common factor between $d_0(z)$ and $d_1(z)$ then the allpass factor $(-\alpha^*+z^{-2})/(1-\alpha z^{-2})$ can be extracted from the right hand side of (5.2.16), and does not contribute to the magnitude response $|H_0(e^{j\omega})|$.

5.3.2 Power Symmetry and Numerator Symmetry Imply (5.2.16)

Assuming that (5.2.16) holds, we showed that $H_0(z)$ satisfies two symmetry properties. We now consider the converse, restricting our discussion to real coefficient filters. We show that if $H_0(z)$ is power symmetric and $P_0(z)$ symmetric, then $H_0(z)$ can be expressed as $(a_0(z^2) + z^{-1}a_1(z^2))/2$. The theorem below is a more precise statement of this. At this time, recall that $H_0(z)$ is said to be bounded real (BR) if it (a) has real coefficients, (b) is stable, and (c) satisfies $|H_0(e^{j\omega})| \leq 1$.

♠**Theorem 5.3.1.** Let $H_0(z) = P_0(z)/D(z)$ be the irreducible representation of a BR function with symmetric (or antisymmetric) numerator of odd order N. If $H_0(z)$ satisfies the power symmetric condition (5.3.4) then the following are true:

1. $H_0(z)$ can be expressed as in (5.2.16) where $a_0(z)$ and $a_1(z)$ are stable, real-coefficient, unit-magnitude allpass functions.
2. Moreover the order of $H_0(z)$ is $N = 2(k_0 + k_1) + 1$ where k_i is the order of $a_i(z)$. So there are no cancelations in (5.2.16). ◇

Some practical examples. As a special case, suppose $H_0(z)$ is an odd order elliptic lowpass filter satisfying (5.3.1a) and (5.3.1b). Then, the conditions of the theorem are satisfied. Notice, however, that power symmetric filters are not restricted to be elliptic. For example, odd order Butterworth filters can be designed to satisfy (5.3.4). Chebyshev filters, on the other hand, are not suitable because they are inherently nonsymmetric (the passband is equiripple and stopband monotone, or vice versa).

Proof of Theorem 5.3.1. Substituting $H_0(z) = P_0(z)/D(z)$ into the power symmetric condition (5.3.4) and rearranging, we obtain

$$\frac{\widetilde{P}_0(-z)P_0(-z)}{\widetilde{D}(-z)D(-z)} = \frac{\widetilde{D}(z)D(z) - \widetilde{P}_0(z)P_0(z)}{\widetilde{D}(z)D(z)}$$

Since $P_0(z)/D(z)$ is irreducible, there is no common factor of the form $(1 - \beta z^{-1})$, $\beta \neq 0$ between $P_0(z)$ and $D(z)$. Also since $\widetilde{P}_0(z) = z^N P_0(z)$ (by symmetry of $P_0(z)$), there are no such common factors between $\widetilde{P}_0(z)$ and $D(z)$ either. As a result, there are no common factors between the numerator and denominator of the left hand side of the above equation. The denominators on the two sides should therefore be equal except for a scale factor. Equating, in particular, the factors of these denominators which have zeros inside the unit circle, we obtain $D(-z) = cD(z)$. Assuming that $D(z)$ is normalized such that its constant coefficient is unity, we have $D(z) = D(-z)$. From this we also see that $D(z)$ has only even powers of z^{-1}, that is $D(z) = d(z^2)$.

Let $H_1(z) \stackrel{\Delta}{=} H_0(-z)$. Then

$$H_0(z) = \frac{P_0(z)}{d(z^2)}, \quad H_1(z) = \frac{P_0(-z)}{d(z^2)}$$

The power symmetric condition, (5.3.4) means that the function $H_1(z)$ is power complementary to $H_0(z)$. Since $P_0(z)$ is a odd order symmetric (antisymmetric) polynomial, $P_0(-z)$ is, therefore, antisymmetric (symmetric). Summarizing, $H_0(z)$ and $H_1(z)$ are a set of stable, real coefficient power complementary functions with symmetric and antisymmetric numerators. Moreover they have the same denominator. We can therefore apply Theorem 3.6.1 to conclude

$$\begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} A_0(z) \\ A_1(z) \end{bmatrix}, \tag{5.3.8}$$

where $A_0(z)$ and $A_1(z)$ are stable unit-magnitude allpass, with orders n_0 and n_1 such that $N = n_0 + n_1$. Notice now that since $H_1(z) = H_0(-z)$ we can always write the pair as in (5.2.5). Since the 2×2 matrix in (5.3.8) is nonsingular, we conclude by comparing (5.2.5) with (5.3.8) that $E_0(z^2) = A_0(z)/2$ and $z^{-1}E_1(z^2) = A_1(z)/2$. This proves that the polyphase components $E_0(z)$ and $E_1(z)$ are allpass with magnitude 0.5. Thus $H_0(z)$ can be expressed as in (5.2.16), where $a_0(z)$ and $a_1(z)$ are stable unit-magnitude allpass. Since $N = n_0 + n_1$ we have $N = 2(k_0 + k_1) + 1$. $\nabla\nabla\nabla$

Even-order filters. The above result is restricted to odd order filters. Recall from Sec. 3.6 that if $H_0(z)$ is an even order elliptic lowpass filter, then the allpass decomposition can still be done but the allpass filters now have complex coefficients [even though $H_0(z)$ has real coefficients]. Since the polyphase components evidently have real coefficients, these allpass filters cannot, therefore, be the polyphase components. In the even order case it is possible to use a modified IIR QMF bank which overcomes this difficulty (Problem 5.6).

5.3.3 Poles of Power Symmetric Elliptic Filters

Let $G(z)$ be a lowpass (or highpass) power symmetric elliptic filter. Then all its poles are located on the imaginary axis. Thus the poles have the form $j\beta_k$ (with $-1 < \beta_k < 1$ due to stability). The rest of this section is devoted to proving this, and can be skipped without loss of continuity.

Proof of the above claim. From Sec. 3.3.3 we know that for an Nth order elliptic filter $G(z)$ we can write

$$G(z^{-1})G(z) = \frac{1}{1 + \epsilon^2 R(z)R(z^{-1})}, \tag{5.3.9}$$

where $R(z)$ is a rational function of the form

$$R(z) = \left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)^{\ell} \prod_{k=1}^{m} \frac{(1 - z^{-1}e^{j\theta_k})(1 - z^{-1}e^{-j\theta_k})}{(1 - z^{-1}e^{j\omega_k})(1 - z^{-1}e^{-j\omega_k})}, \tag{5.3.10}$$

with $\ell = 1$ for odd N and $\ell = 0$ for even N. Here m is such that $N = 2m+\ell$. The frequencies θ_k are the reflection zeros (i.e., points where $|G(e^{j\omega})|$ attains the maximum of unity) and ω_k the transmission zeros [Fig. 5.3-1(a)]. From (5.3.10) we have the relation

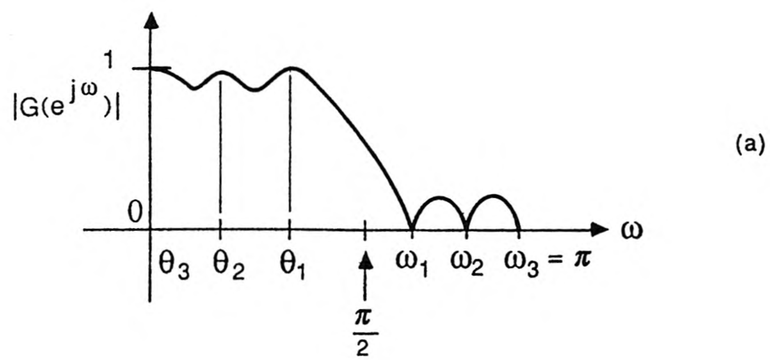
$$R(z) = (-1)^{\ell} R(z^{-1}). \tag{5.3.11}$$

The power-symmetric property means

$$G(z^{-1})G(z) = 1 - G(-z^{-1})G(-z). \tag{5.3.12}$$

Evidently the right hand side of (5.3.12) has reflection zeros at $\pi - \omega_k$ and transmission zeros at $\pi - \theta_k$. These, therefore, should agree with θ_k and ω_k respectively, that is, $\pi - \theta_k = \omega_k$ as demonstrated in Fig. 5.3-1. Substituting this into (5.3.10) we can show $R(-z) = 1/R(z)$, that is,

$$R(z)R(-z) = 1. \tag{5.3.13}$$



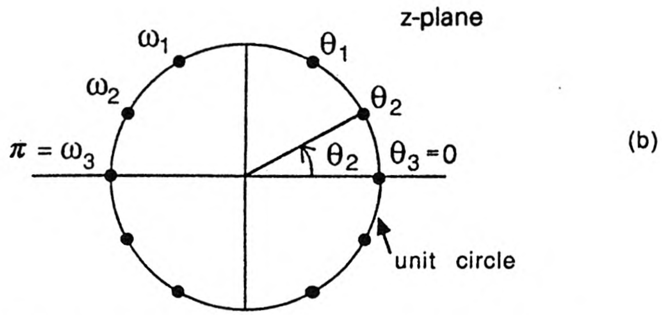


Figure 5.3-1 For a power symmetric elliptic lowpass filter $G(z)$, the relation $\omega_k + \theta_k = \pi$ holds.

Now by substituting (5.3.9) into the right hand side of (5.3.12) we get

$$G(z^{-1})G(z) = \frac{1}{1 + \frac{1}{\epsilon^2 R(-z)R(-z^{-1})}} = \frac{1}{1 + \frac{1}{\epsilon^2}R(z)R(z^{-1})} \tag{5.3.14}$$

using (5.3.13). By comparing (5.3.9) and (5.3.14) we conclude $\epsilon^2 = 1$. For a power symmetric elliptic filter $G(z)$ we thus have

$$G(z^{-1})G(z) = \frac{1}{1 + R(z)R(z^{-1})}, \tag{5.3.15}$$

where $R(z)$ is as in (5.3.10), with $\omega_k + \theta_k = \pi$. If p is a pole then $1 + R(p)R(p^{-1}) = 0$. In view of (5.3.11) and (5.3.13) this implies

$$\left|\frac{R(p)}{R(-p)}\right| = 1, \quad \text{if } p \text{ is a pole of } G(z). \tag{5.3.16}$$

From Fig. 5.3-1 we see that $\omega_k > \pi/2$ and $\theta_k < \pi/2$ for all k. So the poles of the rational function $R(z)/R(-z)$ are restricted to the open left-half of the z-plane. Moreover, $R(z)/R(-z)$ has unit magnitude on the imaginary axis so that by maximum modulus theorem (Sec. 3.4.1) we have $|R(z)/R(-z)| < 1$ for Re $z > 0$. By replacing z with $-z$ and repeating this argument we find that $|R(-z)/R(z)| < 1$ for Re $z < 0$. Summarizing, we have

$$\left|\frac{R(z)}{R(-z)}\right| \begin{cases} < 1, & \text{Re } z > 0, \\ = 1, & \text{Re } z = 0, \\ > 1, & \text{Re } z < 0. \end{cases} \tag{5.3.17}$$

which proves that all the poles of $G(z)$ are indeed on the imaginary axis of the z plane. $\nabla\nabla\nabla$

5.3.4 Design of Power Symmetric Filters

By Theorem 5.3.1, elliptic lowpass filters whose specifications satisfy (5.3.1) have the form (5.2.16). For example, with $N = 5$, we have $n_0 = 2, n_1 = 3$, that is, $k_0 = k_1 = 1$ so that a fifth order power symmetric elliptic filter can be expressed as

$$H_0(z) = 0.5\frac{\alpha_0 + z^{-2}}{1 + \alpha_0 z^{-2}} + 0.5z^{-1}\frac{\alpha_1 + z^{-2}}{1 + \alpha_1 z^{-2}} \tag{5.3.18}$$

where $0 < \alpha_0, \alpha_1 < 1$. Similarly a third order power symmetric elliptic filter can be expressed as

$$H_0(z) = 0.5\left(\frac{\alpha + z^{-2}}{1 + \alpha z^{-2}} + z^{-1}\right), \tag{5.3.19}$$

where $0 < \alpha < 1$. [Since the zeros of an elliptic filter are on the unit circle, we have $(1/3) < \alpha$ as well.] The constants c_0 and c_1 in (5.3.6) which are obviously real in this case, are taken to be unity so that $H_0(1) = 1$ as required.

Design Procedure (Butterworth and Elliptic Cases)

Our discussions are made easier in terms of analog filters, reviewed in Sec. 3.3. If we design a Butterworth filter with 3 dB point $\Omega_c = 1$ and obtain $H_0(z)$ using the bilinear transformation (3.3.1), then $H_0(z)$ automatically satisfies the power symmetric property (Problem 5.8).

We now consider the elliptic case. The design specifications are δ_1, δ_2, Ω_p and Ω_S. The parameter

$$r \triangleq \Omega_p/\Omega_S \tag{5.3.20}$$

governs the filter sharpness. The analog domain equivalent of the condition $\omega_p + \omega_S = \pi$ is, in view of the bilinear transform,

$$\Omega_p \Omega_S = 1. \tag{5.3.21}$$

Since the ripples are constrained as in (5.3.1a), we have only two degrees of freedom, viz., δ_2 and r. Given these specifications, if we compute δ_1 using (5.3.1a) and then use standard techniques to design the analog elliptic filter, the estimated filter order may not turn out to be an integer. If it is rounded to the nearest (or next higher) odd integer N, the resulting filter does not in general satisfy the desired ripple constraint.

It is, however, possible to modify the standard elliptic filter algorithm such that the condition (5.3.1a) is exactly satisfied after the order has been rounded up. In this process the value of δ_2 is readjusted (reduced), such that N is exactly an integer under the constraint (5.3.1a). Table 5.3.1 shows the design-algorithm for this, obtained by modifying the procedure given by Antoniou [1979]. The resulting analog elliptic filter has the desired r, and usually has smaller ripple size δ_2 than specified. In any case it satisfies the power symmetric condition (5.3.1a) exactly. If this is transformed into a digital filter by use of the bilinear transform, the resulting $H_0(z)$ satisfies (5.2.16). In particular the denominator has only even powers of z^{-1}, and the poles are all on the imaginary axis of the z-plane.

Identifying the two allpass filters. Since the above algorithm gives $H_0(z)$ in factored form, the poles are already known, so the method described in Sec. 3.6 can be used to identify $a_0(z)$ and $a_1(z)$. In the elliptic filter case, the pole interlace property can be used to simplify this identification (recall Fig. 3.6-5 and associated comments). Fig. 5.3-2 demonstrates this for $N = 7$. Once the poles of $a_0(z^2)$ and $z^{-1}a_1(z^2)$ are identified, the polynomials $d_0(z^2)$ and $d_1(z^2)$ in (5.3.6) are known. By setting $c_0 = c_1 = 1$ and taking $k_i =$ order of $d_i(z)$, we can identify $a_i(z)$. These are summarized in Table 5.3.1.

A second way to identify the allpass filters is as follows. We have $H_0(z) = P_0(z)/d(z^2)$, with the coefficients of $P_0(z)$ and $d(z)$ known from

TABLE 5.3.1 Design of power symmetric elliptic filters

We summarize the procedure to design an odd order, lowpass, power-symmetric elliptic filter $H_0(z)$. Let the filter order be $N = 2m + 1$.

Specifications.

The given specifications are ω_S and δ_2, i.e., the stopband edge and peak stopband ripple, as in Fig. 3.1-1(b). The minimum stopband attenuation is then $A_S = -20\log_{10}\delta_2$. The passband edge ω_p and peak passband ripple δ_1 are determined according to the halfband symmetry conditions

$$\omega_p = \pi - \omega_S, \qquad \text{and } 4\delta_1(1-\delta_1) = \delta_2^2.$$

Also recall $A_{max} = -20\log_{10}(1 - 2\delta_1)$.

Order estimation.

Define the quantities $r = \tan(0.5\omega_p)/\tan(0.5\omega_S)$, $\widehat{r} = \sqrt{1-r^2}$, $q_0 = 0.5(1-\sqrt{\widehat{r}})/(1+\sqrt{\widehat{r}})$,

$$q = q_0 + 2q_0^5 + 15q_0^9 + 150q_0^{13}, \quad \text{and} \quad D = \left(\frac{1-\delta_2^2}{\delta_2^2}\right)^2.$$

Estimate the order N to be the smallest odd integer such that

$$N \geq \frac{\log_{10} 16D}{\log_{10}(1/q)}$$

Readjusting ripple size.

Since N is obtained by rounding-up the right hand side above, the resulting peak ripple δ_2 is smaller than specified. To recompute this ripple first recompute D from

$$D = \frac{10^{N\log_{10}(1/q)}}{16}.$$

Then find readjusted δ_2 from $D = (1-\delta_2^2)^2/\delta_2^4$, and δ_1 from $4\delta_1(1-\delta_1) = \delta_2^2$. Values of A_S and A_{max} are also readjusted accordingly.

Computing the filter coefficients.

Let

$$\Lambda = \frac{1}{2N}\log_e\left(\frac{10^{0.05A_{max}}+1}{10^{0.05A_{max}}-1}\right) \quad \text{(use readjusted } A_{max}\text{)},$$

and $w = (1+r)/\sqrt{r}$. For $1 \leq k \leq m$ (where $m = (N-1)/2$) compute

$$\Omega_k = \frac{2q^{0.25}\sum_{i=0}^{\infty}(-1)^i q^{i(i+1)}\sin\big((2i+1)k\pi/N\big)}{1+2\sum_{i=1}^{\infty}(-1)^i q^{i^2}\cos\big(2\pi ki/N\big)},$$

$$v_k = \sqrt{\left(1 - r\Omega_k^2\right)\left(1 - \frac{\Omega_k^2}{r}\right)},$$

$$b_k = \frac{2v_k}{1+\Omega_k^2}, \quad \text{and} \quad \alpha_{k-1} = \frac{2-b_k}{2+b_k},$$

in the order mentioned. Usually, it is sufficient to retain five or six terms of the infinite summations above. The quantities α_k computed above are distinct and satisfy $0 < \alpha_k < 1$. Renumber them so that

$$0 < \alpha_0 < \alpha_1 < \ldots < \alpha_{m-1} < 1.$$

Define the polynomials

$$d_0(z) = \prod_{k \text{ even}} (1 + \alpha_k z^{-1}), \quad d_1(z) = \prod_{k \text{ odd}} (1 + \alpha_k z^{-1}).$$

Let k_0 and k_1 denote the orders of $d_0(z)$ and $d_1(z)$. Define the allpass functions

$$a_0(z) = \frac{z^{-k_0}\widetilde{d}_0(z)}{d_0(z)}, \quad a_1(z) = \frac{z^{-k_1}\widetilde{d}_1(z)}{d_1(z)}.$$

Then the lowpass power symmetric elliptic filter is $H_0(z) = 0.5\left[a_0(z^2) + z^{-1}a_1(z^2)\right]$. Its order is $N = 2(k_0 + k_1) + 1$.

---

the above design. From (5.2.16) we have $H_0(z) + H_0(-z) = a_0(z^2)$, i.e., $a_0(z^2) = [P_0(z) + P_0(-z)]/d(z^2)$. After reducing this rational function to its irreducible form, we can identify $d_0(z)$. Thus, $a_0(z)$ given by $z^{-k_0}\widetilde{d}_0(z)/d_0(z)$ is found. Similarly, $a_1(z)$ can be identified from $[P_0(z) - P_0(-z)]/d(z^2)$.

Design Example 5.3.1: Power Symmetric Elliptic Filter

Suppose we wish to design a power symmetric elliptic filter $H_0(z)$ with stop band edge $\omega_S = 0.608\pi$ and stopband attenuation $A_S = 35$dB. This A_S corresponds to $\delta_2 \approx 0.0178$. From ω_S we determine $\omega_p = \pi - \omega_S$. The quantities Ω_p and Ω_S can now be identified using (3.3.15). From these we obtain $r = \Omega_p/\Omega_S \approx 0.5$. If we compute δ_1 using (5.3.1a), then the required filter order N for this combination of δ_1, δ_2 and r is $N = 4.7$, which is not an integer. If this is readjusted to $N = 5$, the ripples will not satisfy (5.3.1a) any more.

By using the values of δ_2 and r in the algorithm of Table 5.3.1, we can obtain the readjusted ripple size $\delta_2 = 0.0132$ (i.e., $A_S = 37.58$dB). If this is used in (5.3.1a), we get $\delta_1 = 4.36 \times 10^{-5}$. These values of δ_1 and δ_2, together with the specified r (i.e., $r = 0.5$) imply a filter order $N = 5$, which is exactly an integer. This filter, therefore is power symmetric.

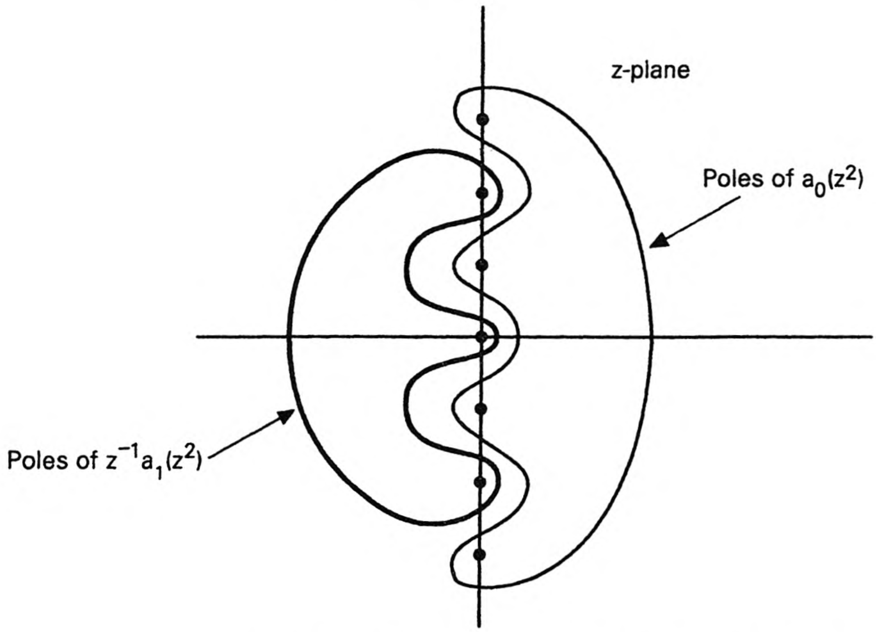


Figure 5.3-2 Grouping of poles into those of $a_0(z^2)$ and $z^{-1}a_1(z^2)$. Here $N = 7$.

The poles of this power symmetric elliptic filter are at the locations

$$z = 0, \quad z = \pm j\sqrt{\alpha_0}, \quad z = \pm j\sqrt{\alpha_1}, \tag{5.3.22}$$

where $\alpha_0 = 0.226634$ and $\alpha_1 = 0.703653$. See Fig. 5.3-3. So we associate the poles $z = \pm j\sqrt{\alpha_0}$ with $a_0(z^2)$, and the poles $z = 0$ and $z = \pm j\sqrt{\alpha_1}$ with $z^{-1}a_1(z^2)$. Thus the elliptic filter $H_0(z)$ has the form (5.3.18), with α_0 and α_1 as above. Fig. 5.3-4(a) shows the magnitude response of $H_0(z)$.

Phase distortion. The distortion function $T(z)$ is given by

$$T(z) = \frac{z^{-1}a_0(z^2)a_1(z^2)}{2} = 0.5z^{-1}\left(\frac{\alpha_0 + z^{-2}}{1 + \alpha_0 z^{-2}}\right)\left(\frac{\alpha_1 + z^{-2}}{1 + \alpha_1 z^{-2}}\right)$$

This is allpass with nonlinear phase response (i.e., nonconstant group delay). [The phase response is linear only if $a_0(z)$ and $a_1(z)$ are pure delays, which is uninteresting]. Figure 5.3-4(b) shows a plot of the group delay of $T(z)$ for the above example. This exhibits a variation from 3 samples to about 16 samples. Whether this is acceptable or not depends on the application in hand, and several subjective considerations come into play. For example, some amount of phase distortion is acceptable in speech processing, but not in image processing [Lim, 1990].

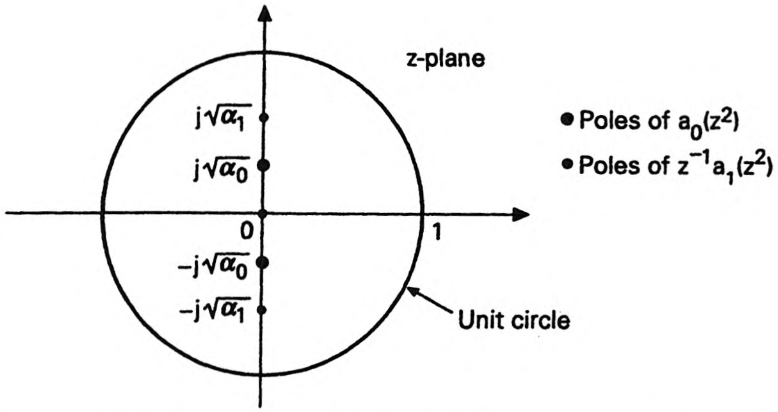


Figure 5.3-3 Identifying poles of $a_0(z^2)$ and $z^{-1}a_1(z^2)$ in Design example 5.3.1.

A Direct Optimization Approach (Non-Elliptic Design)

The fact that the poles of elliptic power symmetric filters are located on the imaginary axis implies that the denominators $d_0(z)$ and $d_1(z)$ of the allpass functions in (5.2.16) are of the form

$$d_0(z) = \prod_i (1 + \alpha_{0,i} z^{-1}), \quad d_1(z) = \prod_i (1 + \alpha_{1,i} z^{-1}), \tag{5.3.23}$$

with $0 < \alpha_{j,i} < 1$. This gives us the hint that if we wish to optimize the coefficients of $a_0(z)$ and $a_1(z)$ directly (rather than by designing an elliptic filter), then we can restrict $d_0(z)$ and $d_1(z)$ to be of this form. For example, we can optimize the parameters $\alpha_{j,i}$ in (5.3.23) in order to minimize the stopband energy

$$\phi = \frac{1}{\pi} \int_{\omega_S}^{\pi} |H_0(e^{j\omega})|^2 d\omega. \tag{5.3.24}$$

Such an optimization is generally fast because in practice we have very few parameters. A fifth order filter of the form (5.3.18) has only two parameters to optimize! Note that even though the passband error is not included in the objective function, it automatically turns out to be small because of the power symmetric condition ensured by (5.2.16).

Design example 5.3.1. Power symmetric elliptic filter *(continuation).*

Fig. 5.3-4(c) shows the plot of $|H_0(e^{j\omega})|$ designed by optimizing the function ϕ. Again the filter order is taken to be $N = 5$, i.e., the power symmetric filter is as in (5.3.18). In this example $\omega_S = 0.6\pi$ (the lower

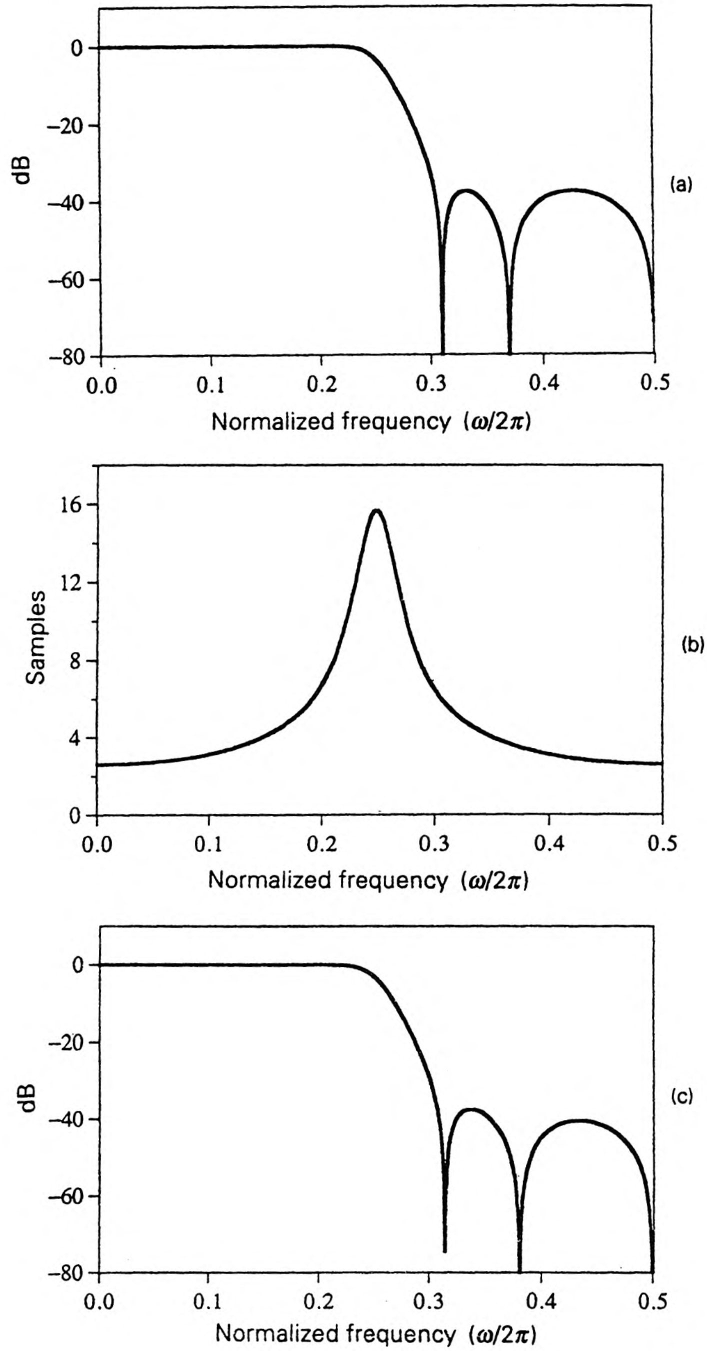


Figure 5.3-4 Design example 5.3.1. (a) Magnitude of elliptic power symmetric filter (b) its group delay response, and (c) magnitude of minimum energy power symmetric filter.

limit of the integral in (5.3.24)). The first peak ripple in the stopband is $A_S \approx 38$ dB. With (5.3.24) used as the objective function, the attenuation at ω_S is not equal to A_S, but typically less. In our example, the lowest frequency with attenuation equal to A_S is 0.614π. The optimized system has $\alpha_0 = 0.2121846$, and $\alpha_1 = 0.689796$. The optimized filter response is not equiripple, since this is not a minimax design. The peak ripple decreases as ω increases. This is desirable in some applications.

Table 5.3.2 gives the values of α_0 and α_1 in (5.3.18) which minimize (5.3.24), for various choices of ω_S appearing in (5.3.24). These are fifth order filters ($N = 5$), and cover a wide range of requirements. The table also shows the attenuation A_S at the location of the first peak-ripple in the stopband. The table serves as a quick design aid for IIR power symmetric filters which can be used to design alias-free QMF banks with freedom from amplitude distortion. For other combinations of N, ω_S and A_S, the reader can obtain designs by direct optimization of (5.3.24), or by using the algorithm of Table 5.3.1.

TABLE 5.3.2 Optimal IIR power symmetric filters with N=5

ω_S	α_0	α_1	A_S
0.550π	0.2790	0.7652	28.6
0.575π	0.2401	0.7231	33.2
0.600π	0.2122	0.6898	37.6
0.625π	0.1910	0.6626	41.7
0.650π	0.1744	0.6399	46.5
0.675π	0.1611	0.6206	50.0
0.700π	0.1502	0.6042	54.5

5.3.5 Low Complexity of the IIR Power Symmetric QMF Bank

We know that the allpass filters $a_j(z)$ have denominators of the form (5.3.23). So $a_j(z)$ is a product of k_j first order sections of the form

$$\frac{\alpha_{j,i} + z^{-1}}{1 + \alpha_{j,i} z^{-1}}, \tag{5.3.25}$$

with $0 < \alpha_{j,i} < 1$. Each of these sections can be implemented with one multiplier, two adders and two delays as shown in Fig. 3.4-4. So $a_j(z)$ can be implemented by cascading k_j such sections, requiring a total of k_j

multipliers, $2k_j$ adders, and $2k_j$ delays. In fact, it is possible to share a delay between adjacent sections, as demonstrated in Fig. 5.3-5.

The total complexity to implement $a_0(z)$ and $a_1(z)$ is equal to $k_0 + k_1 = 0.5(N-1)$ multiplications and $(N-1)$ additions. The outputs of $a_i(z)$ are added and subtracted, which costs two more adders. These multipliers and adders operate at the lower rate (see Fig. 5.2-5) so that the analysis bank requires

$$0.25(N-1) \text{ MPUs} \quad \text{and} \quad 0.5(N+1) \text{ APUs.} \tag{5.3.26}$$

The complexity of the synthesis bank is the same.

In our design example $N = 5$, so that the analysis bank requires one MPU and three APUs. For this cost, the analysis filters provide 37.6 dB stopband attenuation, and the QMF bank is entirely free from aliasing and amplitude distortion. This system, therefore, is very efficient indeed!

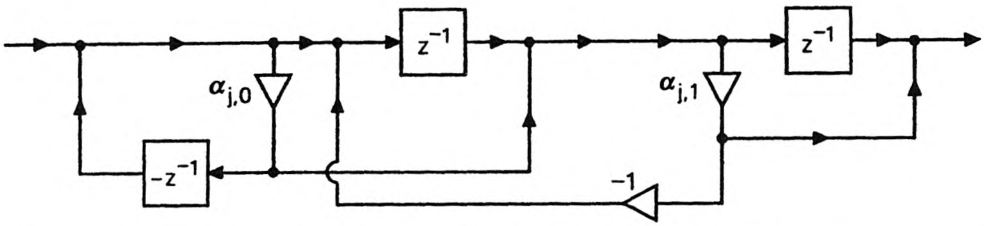


Figure 5.3-5 A cascade of two sections of the form (5.3.25). Each section is implemented as in Fig. 3.4-4, but a delay has been shared so that only three delays are required.

Robustness to Quantization

In any practical implementation, the multiplier coefficients are quantized (Chap. 9). In general this can result in the loss of some or all of the desirable properties (e.g., alias-cancelation, freedom from amplitude distortion, etc.). It is easy to verify that the allpass based structure of Fig. 5.2-5 is free from aliasing, as long as $a_i(z)$ in the analysis bank is quantized the same way as $a_i(z)$ in the synthesis bank. [This is because the alias cancelation condition (5.1.7) continues to hold.]

Furthermore, suppose the allpass filters are implemented such that they remain allpass inspite of multiplier quantization. This is easily ensured since $a_i(z)$ is a product of first order allpass functions which can be implemented as in Fig. 5.3-5 with real multiplier coefficients $\alpha_{j,i}$. Under this condition, the distortion function (5.2.19) continues to be allpass. Summarizing, the structure can be made free from aliasing as well as amplitude distortion, in spite of multiplier quantization.

5.3.6 FIR PR System with Power Symmetric Filters

We will now present an FIR perfect reconstruction system by modification of the above ideas. This system was introduced independently by Smith and

Barnwell [1984] and Mintzer [1985]. Let the synthesis filters be chosen in the usual way to cancel aliasing [i.e., as in (5.1.7)]. We have

$$\widehat{X}(z) = \frac{1}{2}[H_0(z)H_1(-z) - H_1(z)H_0(-z)]X(z), \tag{5.3.27}$$

as shown in Section 5.1.2. For perfect reconstruction, we require this to be a delay. Note that we have not made the assumption $H_1(z) = H_0(-z)$ here. In particular, therefore, the alias-free system need not satisfy (5.2.2).

Assume now that $H_0(z)$ is power symmetric, that is, (5.3.4) holds. By comparing this with (5.3.27), we see that if the filter $H_1(z)$ is chosen as

$$H_1(z) = -z^{-N}\widetilde{H}_0(-z), \tag{5.3.28}$$

for some odd N, then (5.3.27) reduces to $\widehat{X}(z) = 0.5z^{-N}X(z)$, that is, we have a perfect reconstruction system! In order for this system to be practical, $H_0(z)$ has to be FIR. (Otherwise $H_1(z)$ would be unstable for stable $H_0(z)$). By using (5.3.28) in (5.1.7) we see that the synthesis filters are given by

$$F_0(z) = z^{-N}\widetilde{H}_0(z), \quad F_1(z) = z^{-N}\widetilde{H}_1(z). \tag{5.3.29}$$

The above choices of filters can be rewritten in the time domain as

$$h_1(n) = (-1)^n h_0^*(N-n), \; f_0(n) = h_0^*(N-n), \text{ and } f_1(n) = h_1^*(N-n). \tag{5.3.30}$$

Assuming that $H_0(z)$ is causal, we see that the remaining filters are causal as long as $N \geq$ order of $H_0(z)$.

We can summarize these results as follows. Let

$$H_0(z) = \sum_{n=0}^{N} h_0(n)z^{-n} \tag{5.3.31}$$

be power symmetric [i.e., satisfies (5.3.4)]. Then the choice of the remaining three filters according to (5.3.30) results in a perfect reconstruction system satisfying $\widehat{x}(n) = 0.5x(n-N)$.

Other properties. It is easily verified that the filters chosen as above satisfy these properties: (a) $|F_k(e^{j\omega})| = |H_k(e^{j\omega})|$, that is, the synthesis filters have the same magnitude responses as the analysis filters, and (b) $|H_1(e^{j\omega})| = |H_0(-e^{j\omega})|$. In the real coefficient case, the second property means that if $H_0(z)$ is lowpass then $H_1(z)$ is highpass, with same ripple sizes. Note that the relation $H_1(z) = H_0(-z)$ is in general *not* satisfied by this system.

Design Procedure

Only the filter $H_0(z)$ remains to be designed. The power symmetric property means that the zero-phase filter $H(z) = \widetilde{H}_0(z)H_0(z)$ is a half-band filter. Note that $H(e^{j\omega})$ has to be nonnegative. The design steps for the real coefficient case ($h_0(n)$ real) are as follows:

1. First design a zero-phase FIR half band filter $G(z) = \sum_{n=-N}^{N} g(n)z^{-n}$ of order $2N$ (e.g., by using the McClellan-Parks algorithm). The half-band property can be achieved by constraining the bandedges to be such that $\omega_p + \omega_S = \pi$, and the peak ripples in the passband and stopband to be identical as shown in Fig. 5.3-6(a).
2. Then define $H(z) = G(z) + \delta$, where δ is the peak stopband ripple of $G(e^{j\omega})$. This ensures that $H(e^{j\omega}) \geq 0$, as seen from Fig. 5.3-6(a).
3. Finally compute a spectral factor $H_0(z)$ of the filter $H(z)$. In principle, this can be done by computing the zeros of $H(z)$ and assigning an appropriate subset to $H_0(z)$ (Sec. 3.2.5). However there exist more efficient techniques which do not require the computation of zeros. One of these, due to Mian and Nainer [1982], is described in Appendix D. Once $H_0(z)$ has been computed, the remaining three filters are obtained using (5.3.30).

Comments.

1. *Order is odd.* As shown in Sec. 4.6.1, the order of $G(z)$ in the above design is of the form $4J + 2$ so that the order of $H_0(z)$ is $2J + 1$, that is, odd. Since the integer N in (5.3.28) is also required to be odd, we can take N to be same as the order of $H_0(z)$. This also ensures that the filters defined as in (5.3.30) are causal.

2. *Choosing the specifications.* Let ω_S and A_S be the stopband edge and minimum stopband attenuation specified for $H_0(z)$. Then the filter $G(z)$ has the same stopband edge ω_S, and stopband attenuation $\approx 2A_S + 6.02$ dB (why?). The passband specifications of $G(z)$ are automatically determined by the half-band constraint as follows: (a) peak passband ripple is identical to peak stopband ripple, and (b) $\omega_p + \omega_S = \pi$.

3. *Efficient design of* $G(z)$. The half-band filter $G(z)$ can also be designed using a more efficient trick, which was outlined in Problem 4.30 (using slightly different notations for the filters).

4. *Phase of* $H_0(z)$. As explained in Sec. 3.2.5, the spectral factor $H_0(z)$ is not unique because of the many ways in which the zeros of $H(z)$ can be grouped into those of $H_0(z)$ and $\widetilde{H}_0(z)$. The efficient technique described in Appendix D gives a minimum-phase spectral factor (i.e., the zeros are on and inside the unit circle). If one desires to have a spectral factor with nearly *linear* phase response, it can be done by other groupings of the zeros [Smith and Barnwell, 1984]. However, $H_0(z)$ *cannot* have exactly linear phase, unless it has the form $az^{-K} + bz^{-L}$. This is because, if $H_0(z)$ has linear phase, then so does $H_1(z)$ defined according to (5.3.28). But $H_0(z)$ and $H_1(z)$ are *also* power complementary, and cannot therefore have more than two nonzero coefficients (as proved later in Sec. 7.1).

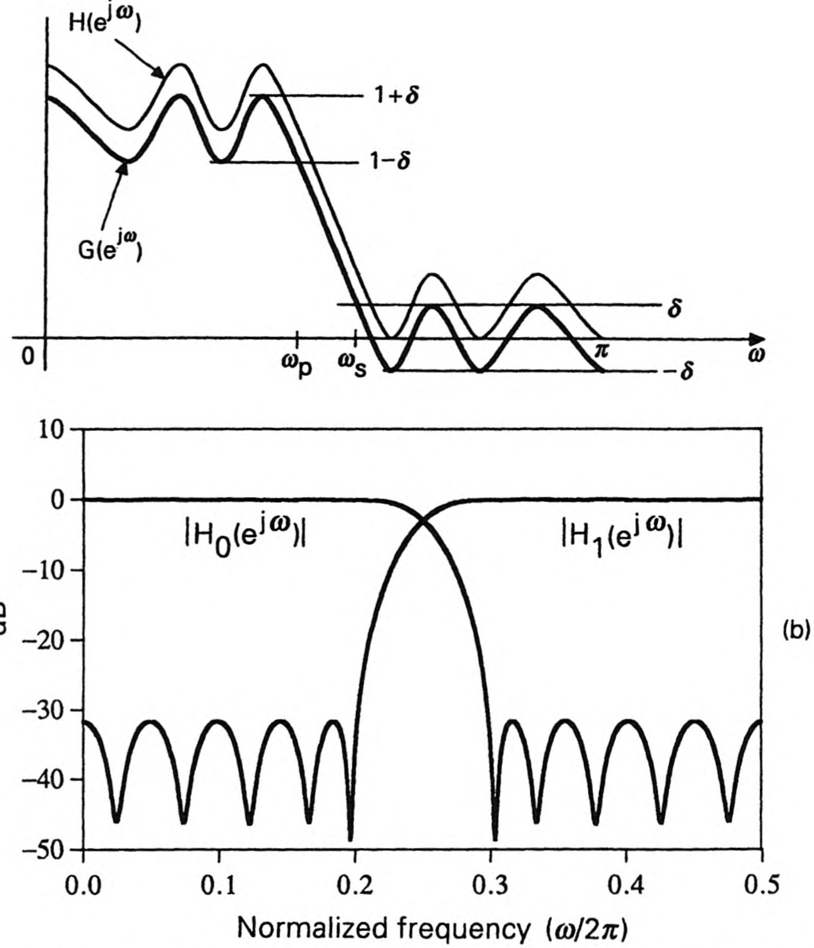


Figure 5.3-6 (a) Construction of a half-band filter $H(z)$ with $H(e^{j\omega}) \geq 0$. (b) Design example 5.3.2. Magnitude responses of the analysis filters of the perfect reconstruction system.

Design Example 5.3.2. FIR Power Symmetric Filter Bank

Suppose $H_0(z)$ is required to be a real-coefficient, equiripple, power symmetric FIR lowpass filter with specifications: $\omega_S = 0.6\pi$ and $A_S = 32$ dB. This means that the half-band filter $G(z)$ has stopband attenuation 70 dB (and stopband edge 0.6π). The required order of $G(z)$ (hence $H(z)$) turns out to be 38. So the power symmetric analysis filter $H_0(z)$ has order $N = 19$. The coefficients of the spectral factor $H_0(z)$ are found using the technique due to Mian and Nainer [1982], described in Appendix D. Table 5.3.3 shows the coefficients $h_0(n)$. The magnitude responses of the analysis filters are shown in Fig. 5.3-6(b).

Computational complexity. If implemented independently, each analysis filter would require $(N+1)$ multiplications and N additions. However, since the impulse responses are related as in (5.3.30), we can implement the analysis bank as shown in Fig. 5.3-7, requiring a total of $(N+1)$ multiplications and $2N$ additions. The total complexity of the direct form implementation is therefore $(N+1)$ MPUs and $2N$ APUs for the analysis bank (and the same for the synthesis bank).

TABLE 5.3.3 Filter coefficients in Design example 5.3.2

n	$h_0(n)$
0	0.1605476 e+00
1	0.4156381 e+00
2	0.4591917 e+00
3	0.1487153 e+00
4	−0.1642893 e+00
5	−0.1245206 e+00
6	0.8252419 e−01
7	0.8875733 e−01
8	−0.5080163 e−01
9	−0.6084593 e−01
10	0.3518087 e−01
11	0.3989182 e−01
12	−0.2561513 e−01
13	−0.2440664 e−01
14	0.1860065 e−01
15	0.1354778 e−01
16	−0.1308061 e−01
17	−0.7449561 e−02
18	0.1293440 e−01
19	−0.4995356 e−02

Instead of using the structure of Fig. 5.3-7 which exploits the relation between $H_1(z)$ and $H_0(z)$, we can also implement $H_0(z)$ and $H_1(z)$ individually in polyphase form. We then require only $(N+1)$ MPUs and N APUs for the entire analysis bank.

The above MPU and APU counts are higher than the cost for Johnston's designs ($0.5(N+1)$ MPUs and $0.5(N+1)$ APUs for the analysis bank). The increased complexity above is partly due to the fact that we have not *simultaneously* exploited the relation (5.3.30) and the decimation operations.

In Sec. 6.4 we will present a lattice structure for the QMF bank which overcomes this, and has the smallest possible complexity (same number of MPUs and APUs as Johnston's filters). This lattice has the additional advantage that the perfect reconstruction property is preserved in spite of multiplier quantization. Such a feature is not offered by the direct form structure (Fig. 5.3-7); for example, quantization of $h_0(n)$ results in the loss of power symmetric property (hence loss of perfect reconstruction).

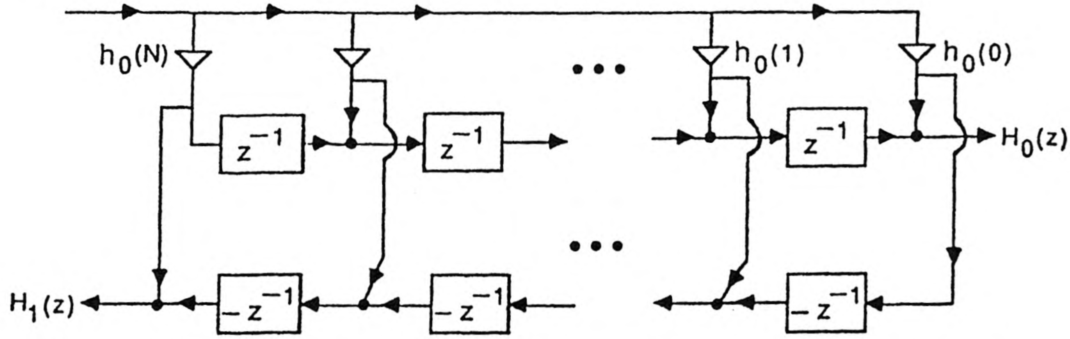


Figure 5.3-7 An $(N+1)$-multiplier implementation of the real-coefficient analysis bank satisfying $h_1(n) = (-1)^n h_0(N-n)$.

5.4 M-CHANNEL FILTER BANKS

For the two channel QMF bank, we considered a specific case where the analysis filters are related as $H_1(z) = H_0(-z)$, and studied it in detail. It is important to analyze the more general case [where restrictions such as $H_1(z) = H_0(-z)$ are not imposed a priori], so that we can understand the general conditions for alias cancelation and perfect reconstruction.

However, in attempting to study the general theory of alias cancelation and perfect reconstruction, it turns out to be more efficient to deal directly with the M-channel maximally decimated filter bank shown in Fig. 5.4-1. We, therefore, study this system in the next few sections. The special properties which arise for the two channel case ($M = 2$) will be pointed out at appropriate places, along with several examples.

In Fig. 5.4-1 the signal $x(n)$ is split into M subband signals $x_k(n)$ by the *M analysis filters $H_k(z)$*. Fig. 4.1-15(c) in Chap. 4 shows typical frequency responses of the analysis filters. Each signal $x_k(n)$ is then decimated by M to obtain $v_k(n)$. The decimated signals are eventually passed through M-fold expanders, and recombined via the synthesis filters $F_k(z)$ to produce $\widehat{x}(n)$. For convenience, and to be consistent with the literature, we sometimes refer to this system as the (M-channel) QMF bank, even though the name "QMF" is not justified any more. Many applications of this system were outlined in Chap. 4. More can be found in Chap. 10 and 11, where this system is used as a unifying tool for a number of diverse topics such as block filtering, nonuniform sampling, periodically time varying systems, and wavelet transform theory.

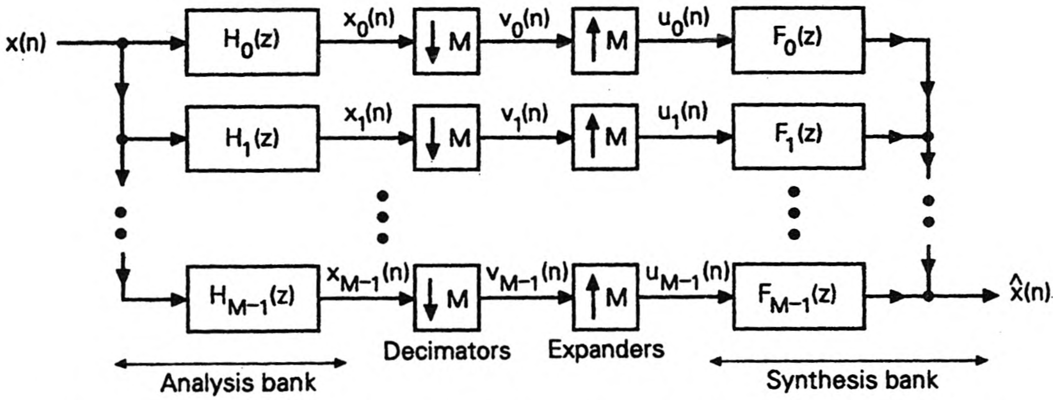


Figure 5.4-1 The M-channel (or M-band) maximally decimated filter bank. Also called M-channel QMF bank.

In this section we introduce the fundamentals of alias cancelation and perfect reconstruction. These results will be used in other chapters frequently. For notational convenience we define the vectors

$$\underbrace{\mathbf{h}(z) = \begin{bmatrix} H_0(z) \\ H_1(z) \\ \vdots \\ H_{M-1}(z) \end{bmatrix}}_{\text{Analysis bank}}, \quad \underbrace{\mathbf{f}(z) = \begin{bmatrix} F_0(z) \\ F_1(z) \\ \vdots \\ F_{M-1}(z) \end{bmatrix}}_{\text{Transposed synthesis bank}}, \quad \underbrace{\mathbf{e}(z) = \begin{bmatrix} 1 \\ z^{-1} \\ \vdots \\ z^{-(M-1)} \end{bmatrix}}_{\text{Delay chain}} \tag{5.4.1}$$

Notice that the analysis bank is a one-input M-output system with transfer matrix $\mathbf{h}(z)$; the synthesis bank is an M-input one-output system with transfer matrix $\mathbf{f}^T(z)$. The delay chain vector will be used in polyphase representations; this was already encountered in Chap. 4 [e.g., see Fig. 4.1-16(a)].

5.4.1 Expression for the Reconstructed Signal

We first obtain an expression for $\widehat{X}(z)$ in terms of $X(z)$, by ignoring the presence of coding and quantization errors. Each subband signal is given by

$$X_k(z) = H_k(z)X(z) \tag{5.4.2}$$

so that the decimated signals $v_k(n)$ have z-transform (Sec. 4.1.1)

$$V_k(z) = \frac{1}{M} \sum_{\ell=0}^{M-1} H_k(z^{1/M} W^\ell) X(z^{1/M} W^\ell), \tag{5.4.3}$$

where $W = W_M = e^{-j2\pi/M}$. The outputs of the expanders are therefore given by

$$U_k(z) = V_k(z^M) = \frac{1}{M} \sum_{\ell=0}^{M-1} H_k(zW^\ell) X(zW^\ell), \tag{5.4.4}$$

so that the reconstructed signal is

$$\widehat{X}(z) = \sum_{k=0}^{M-1} F_k(z) U_k(z) = \frac{1}{M} \sum_{\ell=0}^{M-1} X(zW^\ell) \sum_{k=0}^{M-1} H_k(zW^\ell) F_k(z). \tag{5.4.5}$$

We can rewrite this in the more convenient form

$$\widehat{X}(z) = \sum_{\ell=0}^{M-1} A_\ell(z) X(zW^\ell), \tag{5.4.6}$$

where

$$A_\ell(z) = \frac{1}{M} \sum_{k=0}^{M-1} H_k(zW^\ell) F_k(z), \quad 0 \le \ell \le M-1. \tag{5.4.7}$$

The quantity $X(zW^\ell)$ can be written for $z = e^{j\omega}$ as

$$X(e^{j\omega} W^\ell) = X(e^{j(\omega - \frac{2\pi\ell}{M})}). \tag{5.4.8}$$

For $\ell \neq 0$, this represents a shifted version of the spectrum $X(e^{j\omega})$. So the reconstructed spectrum $\widehat{X}(e^{j\omega})$ is a linear combination of $X(e^{j\omega})$ and its $M-1$ uniformly shifted versions.

5.4.2 Errors Created by the Filter Bank System

In a manner analogous to the two-channel QMF bank, the reconstructed signal $\widehat{x}(n)$ differs from $x(n)$ due to several reasons such as aliasing, imaging, amplitude distortion, and phase distortion as explained next.

Aliasing and Imaging

The presence of shifted versions $X(zW^\ell), \ell > 0$ is due to the decimation and interpolation operations. We say that $X(zW^\ell)$ is the ℓth aliasing term, and $A_\ell(z)$ is the gain for this aliasing term. It is clear that aliasing can be eliminated for *every possible* input $x(n)$, if, and only if,

$$A_\ell(z) = 0, \quad 1 \leq \ell \leq M-1. \tag{5.4.9}$$

We now demonstrate alias cancelation ideas graphically for $M = 3$. We have two alias cancelation conditions to satisfy, namely

$$H_0(zW)F_0(z) + H_1(zW)F_1(z) + H_2(zW)F_2(z) = 0, \tag{5.4.10}$$

$$H_0(zW^2)F_0(z) + H_1(zW^2)F_1(z) + H_2(zW^2)F_2(z) = 0. \tag{5.4.11}$$

Figure 5.4-2(a)–(c) show the magnitude responses of the three analysis filters, along with their shifted versions. It is assumed that $|H_k(e^{j\omega})|$ is symmetric with respect to zero-frequency, which is consistent with the common situation where the filter coefficients are real.

The signal which enters the filter $F_0(z)$ contains the terms

$$H_0(z)X(z), \quad H_0(zW)X(zW), \quad \text{and } H_0(zW^2)X(zW^2). \tag{5.4.12}$$

The purpose of the filter $F_0(z)$, broadly speaking, is to eliminate the terms involving $X(zW)$ and $X(zW^2)$. This is done if $F_0(z)$ attenuates the replicas $H_0(zW)$ and $H_0(zW^2)$, and retains only $H_0(z)$. For this reason, the response $|F_0(e^{j\omega})|$ resembles $|H_0(e^{j\omega})|$, as shown in Fig. 5.4-2(d). The responses of $F_1(z)$ and $F_2(z)$, based on same reasoning, are also indicated in the same figure.

Thus, the output of $F_0(z)$ is a lowpass filtered version of $x(n)$, plus some alias terms. Similarly, the output of $F_1(z)$ is a bandpass filtered version of $x(n)$ plus alias terms. The relation between these outputs and the so-called multiresolution components will be discussed in Section 5.8.

Note that if the filters were ideal, with responses given by

$$H_k(e^{j\omega}) = F_k(e^{j\omega}) = \begin{cases} \sqrt{M} & \text{(passband)} \\ 0 & \text{(stopband)} \end{cases}$$

then there is perfect reconstruction, that is, $\widehat{x}(n) = x(n)$. Since the filters $F_k(z)$ are not ideal in practice, they do not completely eliminate the shifted replicas $H_k(zW)$ and $H_k(zW^2)$. For instance, the three terms in (5.4.10) are not individually equal to zero. The *residual alias terms* are demonstrated in Fig. 5.4-2(e)-(g). The responses of $H_0(zW)F_0(z)$ and $H_1(zW)F_1(z)$ have an overlap, and so do the responses of $H_1(zW)F_1(z)$ and $H_2(zW)F_2(z)$. The basic idea behind alias cancelation is to choose the synthesis filters such that these overlapping terms cancel out.

Amplitude and Phase Distortions

Unless aliasing is canceled, the M-channel QMF bank is a periodically time varying system (LPTV) with period M. (This was shown in Section

5.1.1, by taking $M = 2$; also see Sec. 10.1.2.) If the aliasing terms are somehow eliminated by forcing $A_\ell(z) = 0$ for $\ell > 0$, we have

$$\widehat{X}(z) = T(z)X(z). \tag{5.4.13}$$

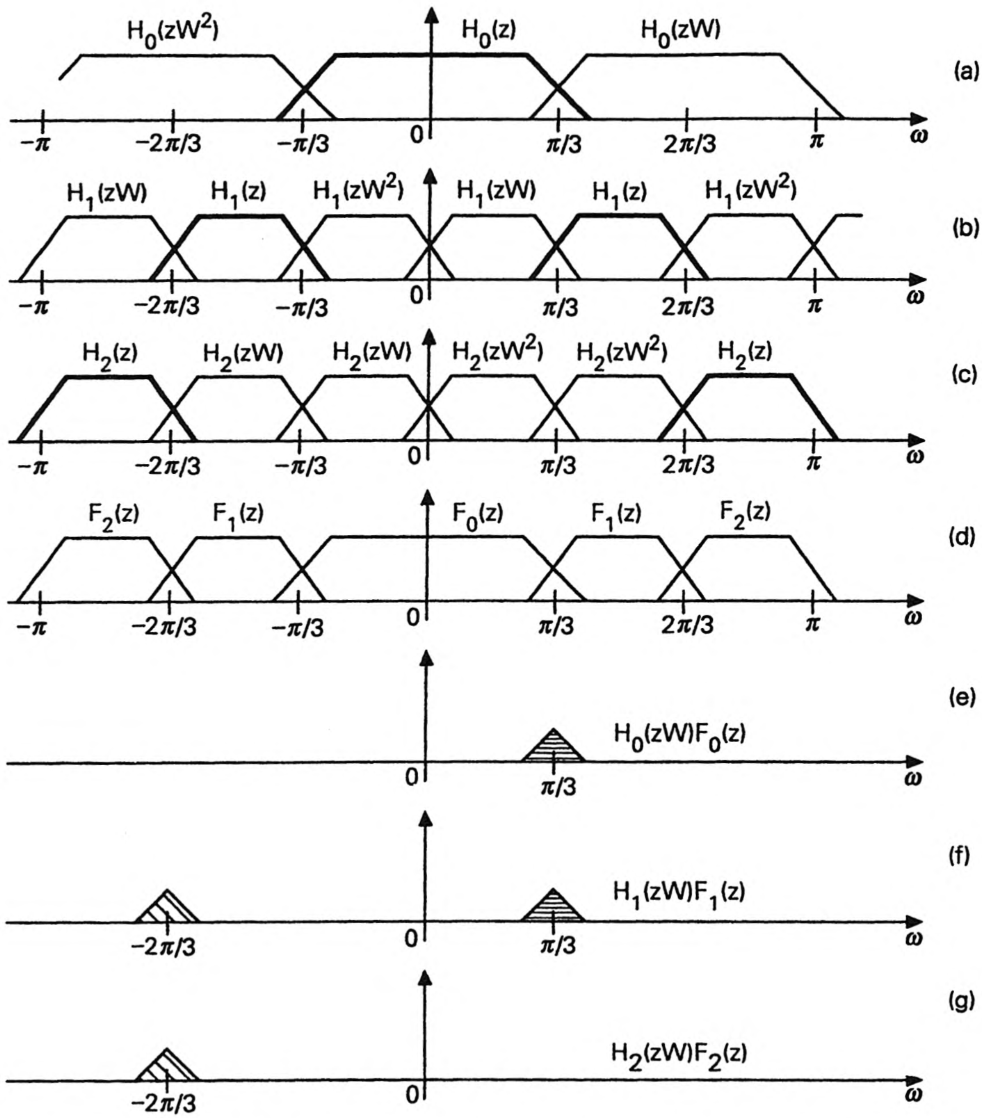


Figure 5.4-2 (a), (b), (c) Magnitude respones of analysis filters and various shifted versions. (d) Magnitude responses of synthesis filters. (e), (f), (g) Residual alias terms with $\ell = 1$, indicating overlap between adjacent-channel alias terms which can be canceled with each other.

Here $T(z)$ is the distortion function (or overall transfer function)

$$T(z) \triangleq A_0(z) = \frac{1}{M} \sum_{k=0}^{M-1} H_k(z) F_k(z). \tag{5.4.14}$$

Thus, when aliasing is canceled, the QMF bank is an LTI system with transfer function $T(z)$. If $|T(e^{j\omega})|$ is not a constant (i.e., $T(z)$ not allpass) we say that there is amplitude distortion, and if $T(z)$ has nonlinear phase we say that there is phase distortion.

Perfect reconstruction (PR) systems. If $H_k(z)$ and $F_k(z)$ are such that (a) aliasing is completely canceled and (b) $T(z)$ is a pure delay (i.e., $T(z) = cz^{-n_0}, c \neq 0$), then the system is free from aliasing, amplitude distortion and phase distortion. Such a system satisfies $\widehat{x}(n) = cx(n - n_0)$, and is called a perfect reconstruction system.

5.4.3 The Alias Component (AC) Matrix

We can rewrite (5.4.7) in matrix-vector form as

$$\underbrace{M \begin{bmatrix} A_0(z) \\ A_1(z) \\ \vdots \\ A_{M-1}(z) \end{bmatrix}}_{\mathbf{A}(z)}$$
$$= \underbrace{\begin{bmatrix} H_0(z) & H_1(z) & \dots & H_{M-1}(z) \\ H_0(zW) & H_1(zW) & \dots & H_{M-1}(zW) \\ \vdots & \vdots & \ddots & \vdots \\ H_0(zW^{M-1}) & H_1(zW^{M-1}) & \dots & H_{M-1}(zW^{M-1}) \end{bmatrix}}_{\mathbf{H}(z)} \underbrace{\begin{bmatrix} F_0(z) \\ F_1(z) \\ \vdots \\ F_{M-1}(z) \end{bmatrix}}_{\mathbf{f}(z)}, \tag{5.4.15}$$

To cancel aliasing, we have to force all elements on the left side to zero (except the top element). So, the conditions for alias cancelation can be written as

$$\mathbf{H}(z)\mathbf{f}(z) = \mathbf{t}(z) \tag{5.4.16}$$

where

$$\mathbf{t}(z) = \begin{bmatrix} MA_0(z) \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} MT(z) \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{5.4.17}$$

The $M \times M$ matrix $\mathbf{H}(z)$ is called the *Alias Component* (AC) matrix.

By combining (5.4.6) with (5.4.15) we can express

$$\widehat{X}(z) = \mathbf{A}^T(z)\mathbf{x}(z) = \frac{1}{M}\mathbf{f}^T(z)\mathbf{H}^T(z)\mathbf{x}(z), \tag{5.4.18}$$

where

$$\mathbf{x}(z) = \begin{bmatrix} X(z) \\ X(zW) \\ \vdots \\ X(zW^{M-1}) \end{bmatrix}. \tag{5.4.19}$$

It is clear that, given a set of analysis filters $H_k(z)$, we can in principle cancel aliasing by solving for the synthesis filters from (5.4.16) as

$$\mathbf{f}(z) = \mathbf{H}^{-1}(z)\mathbf{t}(z). \tag{5.4.20}$$

This works as long as [det $\mathbf{H}(z)$] is not identically zero. We can go a step further and obtain perfect reconstruction, simply by requiring that $\mathbf{t}(z)$ be of the form

$$\mathbf{t}(z) = \begin{bmatrix} z^{-n_0} \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{5.4.21}$$

Practical Difficulties with the AC Matrix Inversion

If we attempt to solve the alias cancelation or perfect reconstution problem by use of (5.4.20), then we would have to invert $\mathbf{H}(z)$. This is in principle possible, unless the determinant of $\mathbf{H}(z)$ is identically zero for all z. However the resulting filters $F_k(z)$ may not be practical. To elaborate this point, let us write (5.4.20) explicitly as (Appendix A)

$$\mathbf{f}(z) = \frac{\text{Adj } \mathbf{H}(z)}{\det \mathbf{H}(z)}\mathbf{t}(z). \tag{5.4.22}$$

Notice from here that $F_k(z)$ could be IIR even if each analysis filter $H_k(z)$ is FIR. The zeros of the quantity [det $\mathbf{H}(z)$] are related to the analysis filters $H_k(z)$ in a very complicated manner, and it is difficult to ensure that they are inside the unit circle [which is necessary for stability of $F_k(z)$].

If we are willing to give up perfect reconstruction, and be satisfied with alias cancelation, then we can replace (5.4.22) with

$$\mathbf{f}(z) = [\text{Adj } \mathbf{H}(z)]\mathbf{t}(z), \tag{5.4.23}$$

so that the distortion function after alias cancelation is

$$T(z) = cz^{-n_0}[\det \mathbf{H}(z)], \tag{5.4.24}$$

for some $c \neq 0$. The synthesis filters $F_k(z)$ are now FIR (whenever the analysis filters are FIR). But the entries of the matrix [Adj $\mathbf{H}(z)$] are determinants of $(M-1)\times(M-1)$ submatrices of $\mathbf{H}(z)$ and can represent FIR filters of very large order even if $H_k(z)$ have moderate order. Another difficulty with this approach is that if [det $\mathbf{H}(z)$] has zeros on the unit circle, say at $z = e^{j\omega_0}$, then $|T(e^{j\omega_0})| = 0$, that is, there is severe amplitude distortion around ω_0.

In the next section we will outline a different technique for perfect reconstruction, in which all the above difficulties 'go away'. This is based on the polyphase representation.

Singularity of $\mathbf{H}(e^{j\omega})$ versus Amplitude Distortion

Consider a QMF bank in which the filters have been chosen to cancel aliasing completely. This means that (5.4.16) holds with $\mathbf{t}(z)$ as in (5.4.17). If $T(z)$ has a zero at $z = e^{j\omega_0}$, then $\mathbf{t}(e^{j\omega_0}) = \mathbf{0}$ so that

$$\mathbf{H}(e^{j\omega_0})\mathbf{f}(e^{j\omega_0}) = \mathbf{0}. \tag{5.4.25}$$

Unless all synthesis filters $F_k(z)$ have a zero at ω_0, this implies that $\mathbf{H}(e^{j\omega_0})$ is singular. Summarizing, the situation $T(e^{j\omega_0}) = 0$ in a alias-free system implies singularity of the AC matrix at the frequency ω_0. We can restate this as follows: if the AC matrix is nonsingular for all ω, then the alias-free system cannot satisfy $T(e^{j\omega_0}) = 0$ for any ω_0 (unless $F_k(e^{j\omega_0}) = 0$ for all k, which does not happen in a good design).

In Section 5.2 we designed a class of two channel alias-free systems satisfying the constraint $H_1(z) = H_0(-z)$. In these systems the analysis filters had linear phase. The filter order was required to be odd, in order to avoid the situation $T(e^{j\pi/2}) = 0$. In Problem 5.19 we request the reader to verify the connection between that issue and the singularity of $\mathbf{H}(e^{j\pi/2})$.

5.5 POLYPHASE REPRESENTATION

In Sec. 4.3 we studied the polyphase representation, and found it to be very useful, both theoretically and in engineering practice. This representation finds application in filter bank theory as well [Vetterli, 1986], [Swaminathan and Vaidyanathan, 1986], [Vaidyanathan, 1987a,b].

We know from Sec. 4.3 that any transfer function $H_k(z)$ can be expressed in the form

$$H_k(z) = \sum_{\ell=0}^{M-1} z^{-\ell} E_{k\ell}(z^M) \quad \text{(Type 1 polyphase)}. \tag{5.5.1}$$

We can rewrite this as

$$\begin{bmatrix} H_0(z) \\ \vdots \\ H_{M-1}(z) \end{bmatrix} =$$

$$\begin{bmatrix} E_{00}(z^M) & E_{01}(z^M) & \dots & E_{0,M-1}(z^M) \\ \vdots & \vdots & \ddots & \vdots \\ E_{M-1,0}(z^M) & E_{M-1,1}(z^M) & \dots & E_{M-1,M-1}(z^M) \end{bmatrix} \begin{bmatrix} 1 \\ z^{-1} \\ \vdots \\ z^{-(M-1)} \end{bmatrix},$$

$$(5.5.2a)$$

that is, as

$$\mathbf{h}(z) = \mathbf{E}(z^M)\mathbf{e}(z), \tag{5.5.2b}$$

where

$$\mathbf{E}(z) = \begin{bmatrix} E_{00}(z) & E_{01}(z) & \dots & E_{0,M-1}(z) \\ E_{10}(z) & E_{11}(z) & \dots & E_{1,M-1}(z) \\ \vdots & \vdots & \ddots & \vdots \\ E_{M-1,0}(z) & E_{M-1,1}(z) & \dots & E_{M-1,M-1}(z) \end{bmatrix}, \tag{5.5.3}$$

and $\mathbf{h}(z)$ and $\mathbf{e}(z)$ are as in (5.4.1). Fig. 5.5-1 shows this idea pictorially. The matrix $\mathbf{E}(z)$ is the $M \times M$ Type 1 polyphase component matrix (or polyphase matrix) for the analysis bank.

We can express the set of synthesis filters also in an identical manner. Thus

$$F_k(z) = \sum_{\ell=0}^{M-1} z^{-(M-1-\ell)} R_{\ell k}(z^M) \quad \text{(Type 2 polyphase)}. \tag{5.5.4}$$

Using matrix notations we have

$$[\,F_0(z) \quad \dots \quad F_{M-1}(z)\,] =$$

$$[\,z^{-(M-1)} \quad z^{-(M-2)} \quad \dots \quad 1\,] \begin{bmatrix} R_{00}(z^M) & \dots & R_{0,M-1}(z^M) \\ R_{10}(z^M) & \dots & R_{1,M-1}(z^M) \\ \vdots & \ddots & \vdots \\ R_{M-1,0}(z^M) & \dots & R_{M-1,M-1}(z^M) \end{bmatrix}$$

$$(5.5.5a)$$

In terms of $\mathbf{e}(z)$ and the synthesis-bank vector $\mathbf{f}^T(z)$, this becomes

$$\mathbf{f}^T(z) = z^{-(M-1)}\widetilde{\mathbf{e}}(z)\mathbf{R}(z^M), \tag{5.5.5b}$$

where

$$\mathbf{R}(z) = \begin{bmatrix} R_{00}(z) & R_{01}(z) & \dots & R_{0,M-1}(z) \\ R_{10}(z) & R_{11}(z) & \dots & R_{1,M-1}(z) \\ \vdots & \vdots & \ddots & \vdots \\ R_{M-1,0}(z) & R_{M-1,1}(z) & \dots & R_{M-1,M-1}(z) \end{bmatrix} \tag{5.5.6}$$

The matrix $\mathbf{R}(z)$ is the Type 2 polyphase matrix for the synthesis bank. Fig. 5.5-2 shows this representation. In Sec. 5.6.3 we provide many examples.

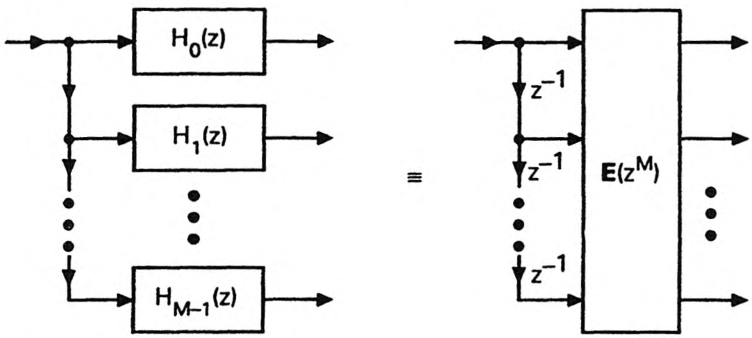


Figure 5.5-1 Type 1 polyphase representation of an analysis bank. $\mathbf{E}(z)$ is called the polyphase component matrix for the analysis bank.

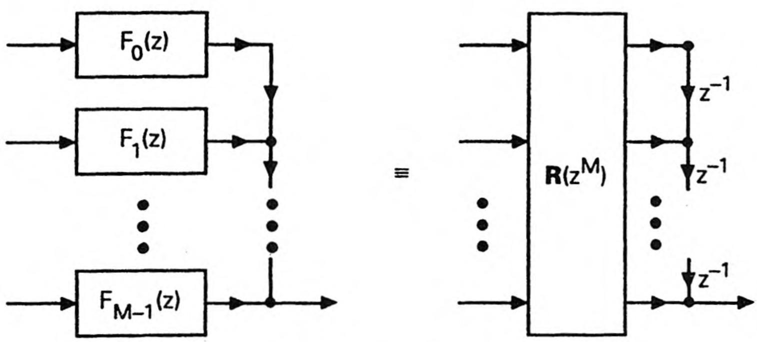


Figure 5.5-2 Type 2 polyphase representation of a synthesis bank. $\mathbf{R}(z)$ is the polyphase component matrix for the synthesis bank.

Using these two representations in the filter bank of Fig. 5.4-1, we obtain the equivalent representation shown in Fig. 5.5-3(a), which we refer to as the polyphase representation of the M-channel QMF bank.

By using noble identities (Fig. 4.2.3), we can redraw this in the equivalent form shown in Fig. 5.5-3(b). This simplified structure can even be used in practical implementations, and has the advantage that the filter coefficients (coefficients of $\mathbf{E}(z)$ and $\mathbf{R}(z)$) are operating at the *lower* rate.

Finally, we can combine the matrices and redraw the system as in Fig. 5.5-3(c), where the $M \times M$ matrix $\mathbf{P}(z)$ is defined as

$$\mathbf{P}(z) = \mathbf{R}(z)\mathbf{E}(z). \tag{5.5.7}$$

As we will see, these equivalent circuits are extremely useful for analytical study as well as in the design and efficient implementation of QMF banks.

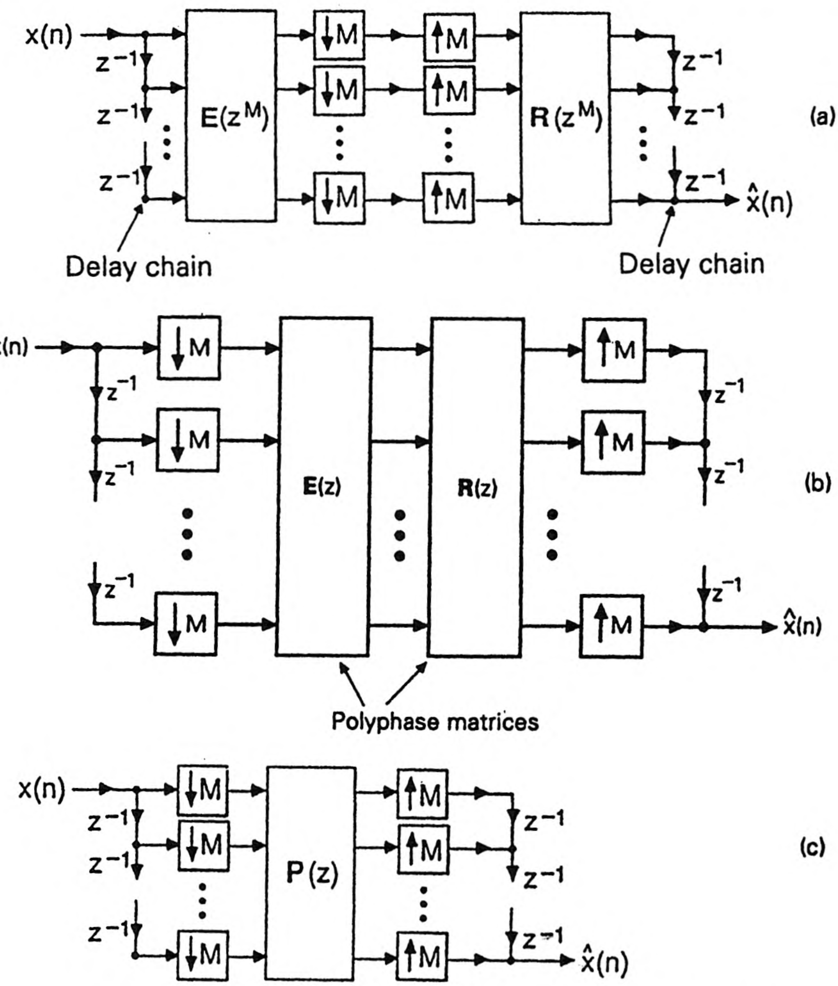


Figure 5.5-3 (a) Polyphase representation of an M-channel maximally decimated filter bank. (b) Rearragement using noble identitites. (c) Further simplification, where $\mathbf{P}(z) = \mathbf{R}(z)\mathbf{E}(z)$.

Causality

Unless mentioned otherwise, the analysis filters $H_k(z)$ will be assumed to be causal so that $\mathbf{E}(z)$ is causal. The synthesis filters $F_k(z)$, which are normally chosen to satisfy certain conditions (such as alias cancelation, perfect reconstruction and so on) can be made causal by insertion of appropriate delays.

Relation Between Polyphase Matrix and AC Matrix

The study of filter banks can be done using either the alias component matrix $\mathbf{H}(z)$ [Eq. (5.4.15)] or the polyphase matrix $\mathbf{E}(z)$. The later approach has the advantage that $\mathbf{E}(z)$ is a physical matrix which makes appearance in the polyphase implementation [Figs. 5.5-3(a),(b)]. However, all theoretical conclusions obtained from use of one of these matrices can also be obtained from the other.

We shall prove that the AC matrix $\mathbf{H}(z)$ and the polyphase component matrix $\mathbf{E}(z)$ of *any* M-channel analysis bank are related as

$$\mathbf{H}(z) = \mathbf{W}^{\dagger}\mathcal{D}(z)\mathbf{E}^{T}(z^{M}), \tag{5.5.8}$$

where

$$\mathcal{D}(z) = \operatorname{diag}[\,1 \quad z^{-1} \quad \ldots \quad z^{-(M-1)}\,]. \tag{5.5.9}$$

and $\mathbf{W}$ is the $M \times M$ DFT matrix.

To see this, note that the definitions of $\mathbf{H}(z)$ and $\mathbf{h}(z)$ give us

$$\begin{aligned}\mathbf{H}^{T}(z) &= [\,\mathbf{h}(z) \quad \mathbf{h}(zW) \quad \ldots \quad \mathbf{h}(zW^{M-1})\,] \\ &= \mathbf{E}(z^{M})[\,\mathbf{e}(z) \quad \mathbf{e}(zW) \quad \ldots \quad \mathbf{e}(zW^{M-1})\,],\end{aligned} \tag{5.5.10}$$

using $\mathbf{h}(z) = \mathbf{E}(z^{M})\mathbf{e}(z)$. From the definition of $\mathbf{e}(z)$ we find

$$\mathbf{e}(zW^{k}) = \mathcal{D}(z)\begin{bmatrix} 1 \\ W^{-k} \\ \vdots \\ W^{-(M-1)k} \end{bmatrix} \tag{5.5.11}$$

By using this in (5.5.10) (and remembering $\mathbf{W} = \mathbf{W}^{T}$), we obtain (5.5.8).

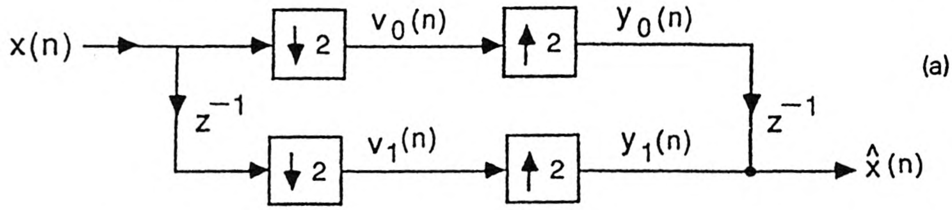
5.6 PERFECT RECONSTRUCTION (PR) SYSTEMS

Recall that a perfect reconstruction (PR) system satisfies $\widehat{x}(n) = cx(n-n_0)$. This means that aliasing has been canceled, and that $T(z)$ has been forced to be a delay. Such systems can indeed be designed. We will show that FIR PR systems can be built for arbitrary M. Moreover, these can be designed such that $H_k(z)$ provides as much attenuation as the user specifies. If designed properly, the implementation cost of such a system is quite competitive with the cost of well-known *approximate* reconstruction systems (Chap. 8).

5.6.1 The Delay Chain Perfect Reconstruction System

We begin with a very simple FIR perfect reconstuction system, and use it to build more useful systems. Consider Fig. 5.6-1(a) which is a two-channel system ($M = 2$) with analysis and synthesis filters

$$H_0(z) = 1,\ H_1(z) = z^{-1},\ F_0(z) = z^{-1},\ F_1(z) = 1. \tag{5.6.1}$$



$x(n)$:	$x(0)$	$x(1)$	$x(2)$	$x(3)$	$x(4)$	$x(5)$	$x(6)$	..
$v_0(n)$:	$x(0)$		$x(2)$		$x(4)$		$x(6)$	..
$v_1(n)$:	$x(-1)$		$x(1)$		$x(3)$		$x(5)$	..
$y_0(n)$:	$x(0)$	0	$x(2)$	0	$x(4)$	0	$x(6)$	...
$y_1(n)$:	$x(-1)$	0	$x(1)$	0	$x(3)$	0	$x(5)$	...
$\hat{x}(n)$:	$x(-1)$	$x(0)$	$x(1)$	$x(2)$	$x(3)$	$x(4)$	$x(5)$	$x(6)$..

(b)

Figure 5.6-1 (a) The delay chain perfect reconstruction QMF bank, and (b) its operation explained in the time domain.

By substituting in (5.4.5) we obtain $\widehat{X}(z) = z^{-1}X(z)$. The distortion function simplifies to $T(z) = z^{-1}$ so that this is a PR system indeed. It is instructive to see how the system works in the time domain. This is demonstrated in Fig. 5.6-1(b). The output of the upper decimator permits the even numbered samples $x(0), x(2), x(4)\ldots$, whereas the lower decimator permits odd numbered samples $x(-1), x(1), x(3), \ldots$ The expanders insert zero-valued samples as shown. The signals in these two branches are beautifully interlaced by the synthesis bank as indicated by the oblique arrows. So the reconstructed signal is precisely $x(n)$ except for one unit of delay.

Figure 5.6-2 shows the M-channel generalization of this. This is a filter-bank with analysis and synthesis filters

$$H_k(z) = z^{-k}, \quad F_k(z) = z^{-(M-1-k)}, \quad 0 \le k \le M-1. \tag{5.6.2}$$

By substituting into (5.4.5), one can verify that this is a perfect reconstruction system, with

$$\widehat{X}(z) = z^{-(M-1)}X(z), \quad \text{i.e.,} \quad \widehat{x}(n) = x(n-M+1). \tag{5.6.3}$$

So the overall system is an LTI system [with transfer function $T(z) = z^{-(M-1)}$] even though there are multirate building blocks in it.

Viewed in the time domain, we see that the kth channel passes the subset of input samples $x(nM-k)$. In other words, the analysis bank merely splits the input $x(n)$ into M subsequences

$$x(nM-k), \quad 0 \leq k \leq M-1. \tag{5.6.4}$$

These subsequences are then interlaced by the synthesis bank, in order to resynthesize $x(n)$.

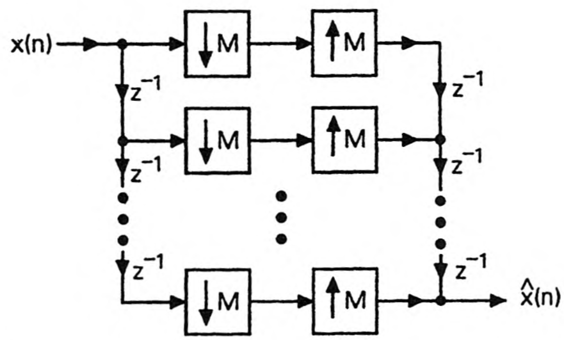


Figure 5.6-2 The delay chain perfect reconstruction system. Here $\widehat{x}(n) = x(n-M+1)$. Number of channels $= M$.

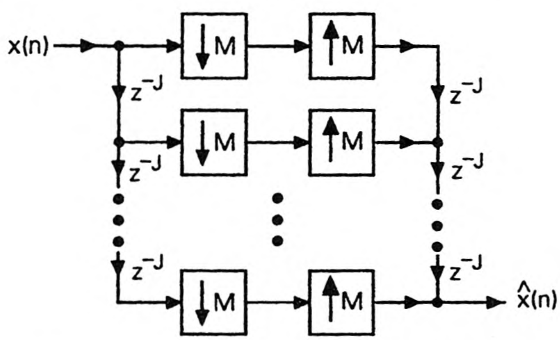


Figure 5.6-3 A generalization of Fig. 5.6-2. This is a perfect recronstruction system if and only if M and J are relatively prime. Again, M is the number of channels.

A further generalization is shown in Fig. 5.6-3. This is obtained by replacing each delay in Fig. 5.6-2 with z^{-J} where J is some integer. This is a perfect reconstruction system if and only if the integers M and J are relatively prime (Problem 5.15).

5.6.2 More General Perfect Reconstruction Systems

The above PR system has allpass analysis filters, which are not useful in practice. Our aim is to use this simple system to develop more useful and practical PR systems. For this imagine that we insert two matrices $\mathbf{E}(z)$ and $\mathbf{R}(z)$ in this system to obtain Fig. 5.5-3(b). It is clear that if

$$\mathbf{R}(z)\mathbf{E}(z) = \mathbf{I}, \tag{5.6.5}$$

then the output $\widehat{x}(n)$ is unchanged. Next, suppose we move $\mathbf{E}(z)$ and $\mathbf{R}(z)$ (using noble identities) to obtain Fig. 5.5-3(a). This system continues to be equivalent to Fig. 5.6-2, so that an observer who measures $\widehat{x}(n)$ [in response to $x(n)$] does not even notice our manipulations! In particular, Fig. 5.5-3(a) continues to have the perfect reconstruction property, except that the analysis filters can now be nontrivial.

We can now do our thinking backwards: suppose we are given a set of analysis filters $H_k(z), 0 \leq k \leq M-1$. This completely determines $\mathbf{E}(z)$ (Sec. 5.5). Assuming that $\mathbf{E}(z)$ can be inverted, we can then obtain a PR system by choosing $\mathbf{R}(z)$ to be $\mathbf{E}^{-1}(z)$ and then computing the synthesis filter coefficients from (5.5.5).

Matrix inversion again? The first thought that crosses the mind now is that this will bring home the same difficulties (including instability) we encountered in the inversion of the AC matrix $\mathbf{H}(z)$ (Sec. 5.4.3). As we will substantiate in Chap. 6, this alarm is unwarranted. We can avoid direct inversion of $\mathbf{E}(z)$ in many ways; one of these is to constrain it to be paraunitary (Sec. 6.1). Notice also that, unlike the AC matrix, $\mathbf{E}(z)$ is a *physical* matrix which will be used in implementation as well as in filter design.

Necessary and Sufficient Conditions for Perfect Reconstruction

The condition (5.6.5) is sufficient for perfect reconstruction, whether the system is FIR or IIR. It is clear that if we replace this with

$$\mathbf{R}(z)\mathbf{E}(z) = cz^{-m_0}\mathbf{I}, \tag{5.6.6}$$

we still have perfect reconstruction but now $T(z) = cz^{-(Mm_0+M-1)}$. More generally it can be shown that, the system has perfect reconstruction if and only if the product $\mathbf{R}(z)\mathbf{E}(z)$ has the form

$$\underbrace{\mathbf{R}(z)\mathbf{E}(z)}_{\mathbf{P}(z)} = cz^{-m_0}\begin{bmatrix} \mathbf{0} & \mathbf{I}_{M-r} \\ z^{-1}\mathbf{I}_r & \mathbf{0} \end{bmatrix} \quad \text{(most general PR condition),} \tag{5.6.7}$$

for some integer r with $0 \leq r \leq M-1$, some integer m_0, and some constant $c \neq 0$. Under this condition the reconstructed signal is $\widehat{x}(n) = cx(n-n_0)$, where $n_0 = Mm_0 + r + M - 1$. This result is a consequence of a general result which we will prove later in Sec. 5.7.2. It holds whether the system is FIR or IIR.

As a special case consider the two channel QMF bank. The matrices $\mathbf{E}(z), \mathbf{R}(z)$ and $\mathbf{P}(z)$ are now 2×2. This system has perfect reconstruction if and only if $\mathbf{P}(z)$ has the form

$$cz^{-m_0}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{or} \quad cz^{-m_0}\begin{bmatrix} 0 & 1 \\ z^{-1} & 0 \end{bmatrix}. \tag{5.6.8}$$

Every QMF bank satisfying (5.6.7) for some r can be obtained by starting from a QMF bank satisfying (5.6.5) and inserting a delay z^{-r} in front of each synthesis filter. (This will be shown in Sec. 5.7.2.) As a result $c, r,$ and m_0 are not fundamental quantities. We sometimes use the term 'perfect reconstruction' to imply the simpler condition (5.6.5).

Condition on determinant. The reader can verify (Problem 5.16) that under the condition (5.6.7) we have

$$\det \mathbf{R}(z) \det \mathbf{E}(z) = c_0 z^{-k_0}. \tag{5.6.9}$$

for some $c_0 \neq 0$ and some integer k_0. So any perfect reconstruction system (FIR or IIR) *has* to satisfy this determinant condition.

FIR Perfect Reconstruction Systems

Perfect reconstruction QMF banks with FIR filters $H_k(z)$ and $F_k(z)$ are of great interest in practice. For these systems the elements of $\mathbf{E}(z)$ and $\mathbf{R}(z)$ are FIR. The FIR nature of $\mathbf{E}(z)$ and $\mathbf{R}(z)$ implies that their determinants are FIR. If the product of these FIR functions has to be a delay [see (5.6.9)], then we must have

$$\det \mathbf{E}(z) = \alpha z^{-K}, \quad \alpha \neq 0, \quad K = \text{integer}. \tag{5.6.10}$$

Thus every FIR perfect reconstruction system must satisfy the above condition; and [det $\mathbf{R}(z)$] must have similar form.

Characterization using paraunitary and unimodular systems. In Chap. 6 we will study a particular family of causal FIR matrices called paraunitary matrices, which satisfy the condition (5.6.10) with $K =$ McMillan degree of $\mathbf{E}(z)$. In Chap. 13 we will encounter *another* family of causal FIR matrices called unimodular matrices, which, by definition, satisfy (5.6.10) with $K = 0$. It is shown in Vaidyanathan [1990b] that any causal FIR matrix satisfying (5.6.10) is a product of a paraunitary matrix and a unimodular matrix, motivating us to study these two classes of matrices in the chapters mentioned above.

5.6.3 Examples of Perfect Reconstruction Systems

Using the above principles we now generate a number of examples which demonstrate the idea of perfect reconstruction.

Example 5.6.1

Consider the two channel system in Fig. 5.6-4(a). By comparing with Fig. 5.6-1 we see that $\mathbf{E}(z) = \mathbf{T}$ and $\mathbf{R}(z) = c\mathbf{T}^{-1}$ so that the perfect reconstruction condition is satisfied, and $\widehat{x}(n) = cx(n-1)$. We can find the analysis and synthesis filters using (5.5.2) and (5.5.5), that is,

$$\begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix} = \mathbf{E}(z^2) \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix}, \quad [F_0(z) \quad F_1(z)] = [z^{-1} \quad 1]\mathbf{R}(z^2). \qquad (5.6.11)$$

Take an example with $c = 2$ and

$$\mathbf{T} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \text{so that} \quad c\mathbf{T}^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \mathbf{T}. \qquad (5.6.12)$$

This is shown in Fig. 5.6-4(b), and can be redrawn in the form of the usual QMF bank as in Fig. 5.6-4(c). So the filters are

$$\begin{aligned} H_0(z) &= 1 + z^{-1}, \quad H_1(z) = 1 - z^{-1}, \\ F_0(z) &= 1 + z^{-1}, \quad F_1(z) = -1 + z^{-1}. \end{aligned} \qquad (5.6.13)$$

This PR system is less trivial than Fig. 5.6-1(a) because the filters $H_0(z)$ and $H_1(z)$ are lowpass and highpass (rather than just allpass). We can generate endless examples like this. For example let

$$\mathbf{T} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}, \quad \text{so that} \quad \mathbf{T}^{-1} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}. \qquad (5.6.14)$$

We then have with $c = 1$,

$$\begin{aligned} H_0(z) &= 2 + z^{-1}, \quad H_1(z) = 3 + 2z^{-1}, \\ F_0(z) &= -3 + 2z^{-1}, \quad F_1(z) = 2 - z^{-1}. \end{aligned} \qquad (5.6.15)$$

In this case $\widehat{x}(n) = x(n-1)$. Notice that the condition $H_1(z) = H_0(-z)$ is not satisfied by this perfect reconstruction example.

Example 5.6.2.

Let

$$\mathbf{E}(z) = \begin{bmatrix} 1 + z^{-1} & 1 - z^{-1} \\ 1 - z^{-1} & 1 + z^{-1} \end{bmatrix}, \qquad (5.6.16)$$

which is FIR. Notice that the determinant of this matrix is a delay, as required by (5.6.10). We choose $\mathbf{R}(z)$ to satisfy (5.6.6), that is,

$$\mathbf{R}(z) = cz^{-m_0}\mathbf{E}^{-1}(z) = \frac{cz^{-m_0}}{4} \begin{bmatrix} 1 + z & 1 - z \\ 1 - z & 1 + z \end{bmatrix}, \qquad (5.6.17)$$

so that the perfect reconstruction condition holds. Choosing $c = 4$ and $m_0 = 1$, this becomes

$$\mathbf{R}(z) = \begin{bmatrix} 1+z^{-1} & -1+z^{-1} \\ -1+z^{-1} & 1+z^{-1} \end{bmatrix}. \tag{5.6.18}$$

The only purpose of m_0 has been to avoid the positive powers of z (non causal terms). The analysis and synthesis filters corresponding to the above $\mathbf{E}(z)$ and $\mathbf{R}(z)$ are

$$\begin{aligned} H_0(z) &= 1+z^{-1}+z^{-2}-z^{-3}, \quad H_1(z) = 1+z^{-1}-z^{-2}+z^{-3}, \\ F_0(z) &= -1+z^{-1}+z^{-2}+z^{-3}, \quad F_1(z) = 1-z^{-1}+z^{-2}+z^{-3}. \end{aligned} \tag{5.6.19}$$

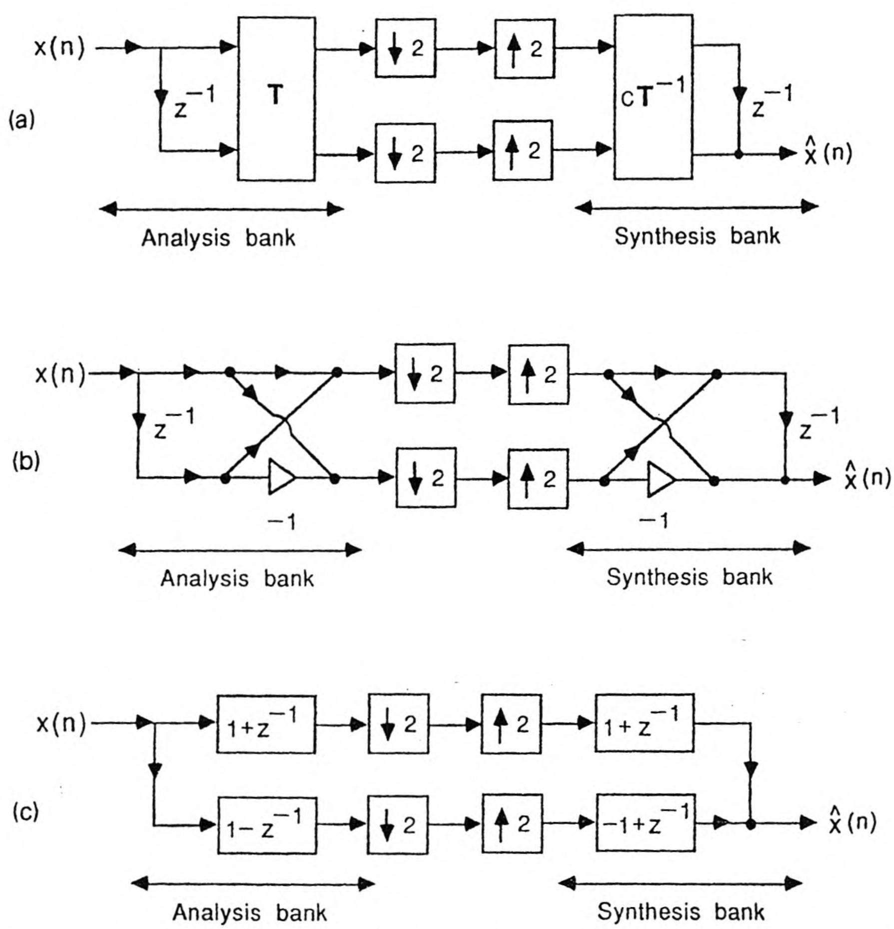


Figure 5.6-4 (a) Example of a perfect reconstruction system, (b) a specific choice of $\mathbf{T}$ and (c) redrawing in conventional form.

Example 5.6.3: The Uniform-DFT Filter Bank

A simple FIR perfect reconstruction system can be constructed by referring to Example 4.1.1 (the DFT filter bank) in Chap. 4. In that example, the analysis bank is as in Fig 4.1-16(a), so that the filters are related as

$$H_k(z) = H_0(zW^k), \tag{5.6.20}$$

where

$$H_0(z) = 1 + z^{-1} + \ldots + z^{-(M-1)}. \tag{5.6.21}$$

Notice that the filters have length M, which is equal to the number of channels. The frequency responses $H_k(e^{j\omega})$ are shifted versions of the lowpass response $H_0(e^{j\omega})$, as shown in Fig. 4.1-16. In this example, we clearly have $\mathbf{E}(z) = \mathbf{W}^*$, so that we can obtain a perfect reconstruction system by taking $\mathbf{R}(z) = \mathbf{W}$. Under this condition $\mathbf{R}(z)\mathbf{E}(z) = \mathbf{W}\mathbf{W}^* = M\mathbf{I}$ so that the reconstructed signal satisfies the perfect reconstruction property $\widehat{x}(n) = Mx(n-M+1)$. It can be shown that the synthesis filters are related as

$$F_k(z) = W^{-k}F_0(zW^k), \tag{5.6.22}$$

and that $F_0(z) = H_0(z)$. So each synthesis filter has precisely the same magnitude response as the corresponding analysis filter. Fig. 5.6-5 shows the complete analysis/synthesis system.

Recall from Fig. 4.1-16 that each analysis filter has about 13 dB attenuation, and adjacent responses have substantial overlap. This shows that there is substantial amount of aliasing error at the output of each decimator. However, the filters $F_k(z)$ and $H_k(z)$ are related in such a delicate manner that the aliasing has canceled off.

Higher Order FIR Perfect Reconstruction Systems

Even though (5.6.10) can be trivially satisfied by taking $\mathbf{E}(z)$ to be a constant nonsingular matrix (as we did in the above example), it is of greater practical interest to employ $\mathbf{E}(z)$ having higher degree, so that the filters $H_k(z)$ have higher order. In this way $|H_k(e^{j\omega})|$ can have higher stopband attenuation and sharper cutoff rate.

One way to obtain FIR $\mathbf{E}(z)$ of higher degree while at the same time satisfying (5.6.10) is shown in Fig. 5.6-6. Here $\mathbf{R}_m$ are constant $M \times M$ nonsingular matrices. Clearly

$$\mathbf{E}(z) = \mathbf{R}_J\mathbf{\Lambda}(z)\mathbf{R}_{J-1}\ldots\mathbf{\Lambda}(z)\mathbf{R}_0, \tag{5.6.23}$$

where

$$\mathbf{\Lambda}(z) = \begin{bmatrix} \mathbf{I}_{M-1} & \mathbf{0} \\ \mathbf{0} & z^{-1} \end{bmatrix}. \tag{5.6.24}$$

Evidently [det $\mathbf{E}(z)$] $= \alpha z^{-J}, \alpha \neq 0$. We can choose $\mathbf{R}(z) = z^{-J}\mathbf{E}^{-1}(z)$ so that it is causal. We then have

$$\mathbf{R}(z) = \mathbf{R}_0^{-1}\boldsymbol{\Gamma}(z)\mathbf{R}_1^{-1}\ldots\boldsymbol{\Gamma}(z)\mathbf{R}_J^{-1}, \tag{5.6.25}$$

where

$$\boldsymbol{\Gamma}(z) = \begin{bmatrix} z^{-1}\mathbf{I}_{M-1} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}. \tag{5.6.26}$$

Figure 5.6-7 shows the synthesis bank obtained in this manner. The filters $H_k(z)$ and $F_k(z)$ can be found using (5.5.2) and (5.5.5).

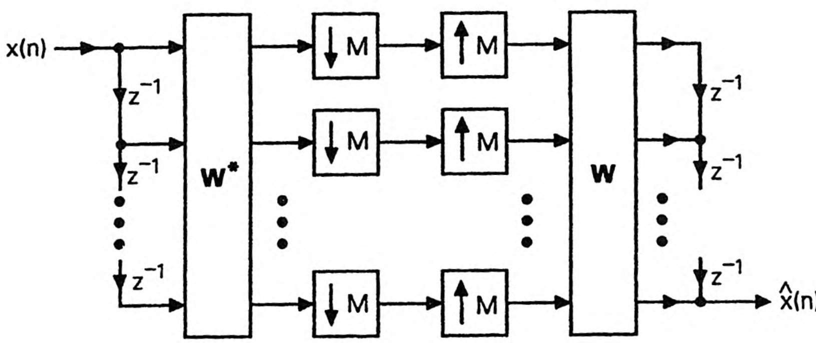


Figure 5.6-5 An FIR perfect reconstruction system with $\mathbf{E}(z) = \mathbf{W}^*$ and $\mathbf{R}(z) = \mathbf{W}$, where $\mathbf{W}$ = DFT matrix. Here $\widehat{x}(n) = Mx(n - M + 1)$.

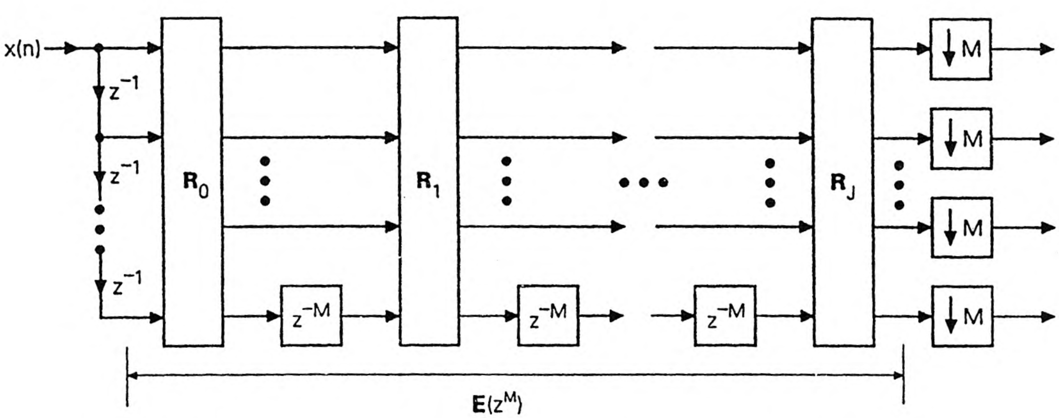


Figure 5.6-6 Analysis bank in which $\mathbf{E}(z)$ is a cascade of nonsingular matrices $\mathbf{R}_m$ separated by delays. Clearly [det $\mathbf{E}(z)$] = delay.

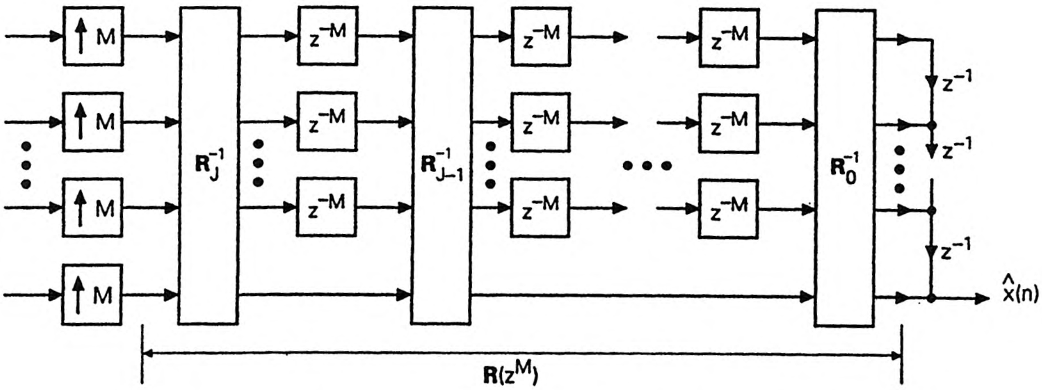


Figure 5.6-7 The synthesis-bank corresponding to Fig. 5.6-6, which would result in a perfect reconstruction system.

Example 5.6.4

Consider a special case with $J = 1$, and the matrices $\mathbf{R}_0$ and $\mathbf{R}_1$ chosen (rather arbitrarily) as

$$\mathbf{R}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}, \quad \mathbf{R}_1 = \mathbf{R}_0^T. \tag{5.6.27}$$

Since $\mathbf{R}_0$ is triangular, its determinant is the product of its diagonal elements, and is nonzero. So $\mathbf{R}_0$ is nonsingular, and

$$\mathbf{R}_0^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix}. \tag{5.6.28}$$

Also $\mathbf{R}_1^{-1}$ is the transpose of $\mathbf{R}_0^{-1}$. The matrix $\mathbf{E}(z)$ is

$$\mathbf{E}(z) = \mathbf{R}_1\mathbf{\Lambda}(z)\mathbf{R}_0 = \begin{bmatrix} 5+z^{-1} & 2+2z^{-1} & z^{-1} \\ 2+2z^{-1} & 1+4z^{-1} & 2z^{-1} \\ z^{-1} & 2z^{-1} & z^{-1} \end{bmatrix}. \tag{5.6.29}$$

The analysis filters obtained using (5.5.2) are given by

$$\begin{aligned} H_0(z) &= 5 + 2z^{-1} + z^{-3} + 2z^{-4} + z^{-5} \\ H_1(z) &= 2 + z^{-1} + 2z^{-3} + 4z^{-4} + 2z^{-5} \\ H_2(z) &= z^{-3} + 2z^{-4} + z^{-5} \end{aligned} \tag{5.6.30}$$

The synthesis filters for perfect reconstruction are obtained by taking

$$\mathbf{R}(z) = \mathbf{R}_0^{-1}\mathbf{\Gamma}(z)\mathbf{R}_1^{-1} = \begin{bmatrix} z^{-1} & -2z^{-1} & 3z^{-1} \\ -2z^{-1} & 5z^{-1} & -8z^{-1} \\ 3z^{-1} & -8z^{-1} & 1+13z^{-1} \end{bmatrix}. \tag{5.6.31}$$

By using (5.5.5) we obtain

$$\begin{aligned} F_0(z) &= 3z^{-3} - 2z^{-4} + z^{-5} \\ F_1(z) &= -8z^{-3} + 5z^{-4} - 2z^{-5} \\ F_2(z) &= 1 + 13z^{-3} - 8z^{-4} + 3z^{-5}. \end{aligned} \tag{5.6.32}$$

This example demonstrates that we can construct FIR perfect reconstruction systems of arbitrarily high order, by structuring $\mathbf{E}(z)$ and $\mathbf{R}(z)$ as in Figs. 5.6-6 and 5.6-7. Since the matrices $\mathbf{R}_m$ in Fig. 5.6-6 can be chosen arbitrarily (subject only to nonsingularity requirement), we can optimize the elements of $\mathbf{R}_m$ to obtain good filter responses $|H_k(e^{j\omega})|$. The resulting system is guaranteed to have perfect reconstruction. Practical design examples of this nature can be found in Chap. 6 to 8.

Example 5.6.5

Let $H_0(z)$ and $H_1(z)$ be related as $H_1(z) = H_0(-z)$ so that the analysis bank has the form (5.2.5). We then have

$$\mathbf{E}(z) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} E_0(z) & 0 \\ 0 & E_1(z) \end{bmatrix}. \tag{5.6.33}$$

Using (5.6.6) with $c = 2$ and $m_0 = 0$ results in

$$\mathbf{R}(z) = \begin{bmatrix} \frac{1}{E_0(z)} & 0 \\ 0 & \frac{1}{E_1(z)} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{5.6.34}$$

The analysis and synthesis banks can now be drawn as in Fig. 5.6-8. So in this case the PR system is obtained merely by using, on the synthesis bank side, the *reciprocals* of the polyphase components of $H_0(z)$. The synthesis filters are

$$F_0(z) = \frac{1}{E_1(z^2)} + z^{-1}\frac{1}{E_0(z^2)}, \quad F_1(z) = \frac{-1}{E_1(z^2)} + z^{-1}\frac{1}{E_0(z^2)}, \tag{5.6.35}$$

and are stable as long as the zeros of $E_i(z)$ are strictly inside the unit circle. In this case [i.e., with $H_1(z) = H_0(-z)$] there is no way to obtain perfect reconstruction if all the filters are required to be FIR (unless the filters have trivial responses). This is consistent with the observation made in Sec. 5.2, where we studied this case in detail. As a numerical example, let $E_0(z) = E_1(z) = 2 + z^{-1}$. Then

$$H_0(z) = 2 + 2z^{-1} + z^{-2} + z^{-3}, \quad H_1(z) = H_0(-z),$$
$$F_0(z) = \frac{0.5(1 + z^{-1})}{1 + 0.5z^{-2}}, \quad F_1(z) = \frac{0.5(-1 + z^{-1})}{1 + 0.5z^{-2}}. \tag{5.6.36}$$

In the above example, the requirement that the zeros of $E_i(z)$ be inside the unit circle, is severe. It puts severe constraints on the frequency response of $H_0(z)$. So this is not a very practical system.

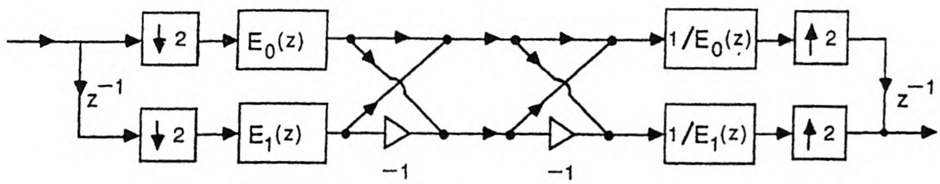


Figure 5.6-8 Another example of a PR QMF bank. The synthesis bank is IIR if analysis bank is FIR.

5.7 ALIAS-FREE FILTER BANKS

Alias cancelation is evidently a less stringent requirement than perfect reconstruction. Even though it is possible to achieve perfect reconstruction as explained in the previous section, it is important to study the most general conditions under which aliasing is canceled. We first demonstrate some useful M-channel alias-free QMF banks. We then study the general theory for alias cancelation.

5.7.1 Examples of Alias-Free Systems

Starting from the conceptually simple perfect reconstruction system of Fig. 5.6-2, we now obtain some examples of alias-free systems.

Example 5.7.1

Consider Fig. 5.7-1(a) in which we have M transfer functions $S_k(z)$ 'sandwiched' between the decimators and expanders. Evidently $\widehat{X}(z)$ is a linear combination of $X(z)$ and the alias components $X(zW^\ell)$. What is the set of necessary and sufficient conditions on $S_k(z)$ so that aliasing terms are canceled?

First, we claim that aliasing is absent if

$$S_k(z) = S(z), \quad \text{for all } k. \tag{5.7.1}$$

To see this, simply move $S(z)$ all the way to the right using the appropriate noble identity (Fig. 4.2-3). The result [Fig. 5.7-1(b)] is identical to the perfect reconstruction structure of Fig. 5.6-2, in cascade with $S(z^M)$. Under this condition we have

$$\widehat{X}(z) = z^{-(M-1)} S(z^M) X(z). \tag{5.7.2}$$

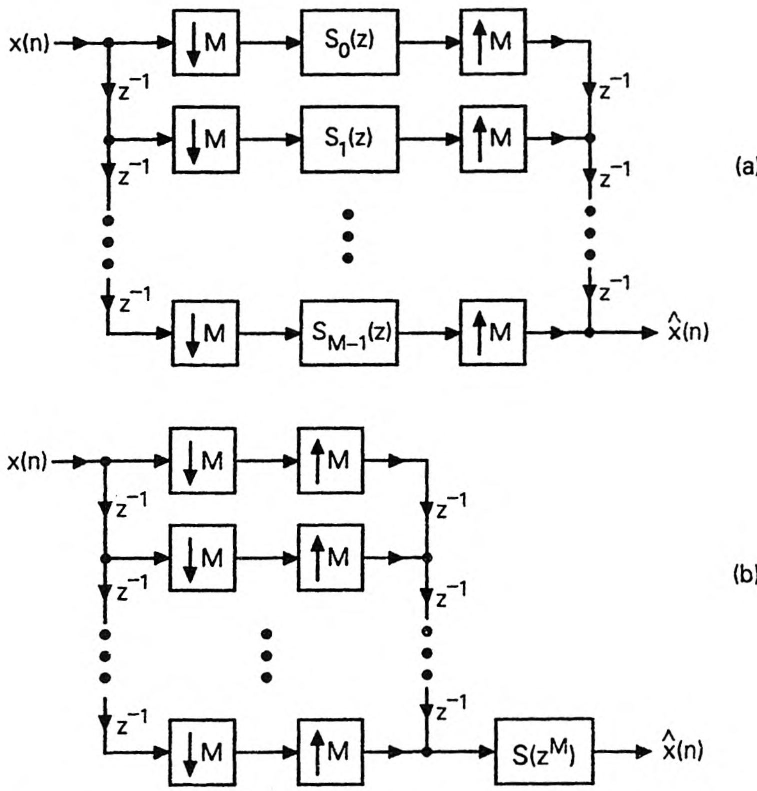


Figure 5.7-1 (a) Pertaining to Example 5.7.1 and (b) simplification when $S_k(z) = S(z)$ for all k.

So the system is alias free, and has distortion $T(z) = z^{-(M-1)} S(z^M)$. It turns out that (5.7.1) is also a *necessary* condition for alias cancelation. To see this, we first express $\widehat{X}(z)$ in terms of $X(z)$:

$$\widehat{X}(z) = \frac{z^{-(M-1)}}{M} \sum_{\ell=0}^{M-1} X(zW^{\ell}) \sum_{k=0}^{M-1} S_k(z^M) W^{-k\ell}. \tag{5.7.3}$$

This is free from the alias components $X(zW^{\ell}), \ell > 0$ [for all possible inputs $x(n)$] if, and only if,

$$\sum_{k=0}^{M-1} S_k(z)W^{-k\ell} = 0, \quad 1 \leq \ell \leq M-1. \tag{5.7.4}$$

$$\text{i.e.,} \qquad \mathbf{W}^{\dagger} \begin{bmatrix} S_0(z) \\ S_1(z) \\ \vdots \\ S_{M-1}(z) \end{bmatrix} = \begin{bmatrix} \times \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{5.7.5}$$

where $\times$ denotes a possibly nonzero entry. Since $\mathbf{W}\mathbf{W}^{\dagger} = M\mathbf{I}$, this implies

$$\begin{bmatrix} S_0(z) \\ S_1(z) \\ \vdots \\ S_{M-1}(z) \end{bmatrix} = \mathbf{W} \begin{bmatrix} S(z) \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{5.7.6}$$

for some $S(z)$, from which (5.7.1) follows. This result can be used to generate some useful alias-free systems, as demonstrated next.

Example 5.7.2.

Suppose each transfer function $S_k(z)$ in Fig. 5.7-1(a) is factorized into $S_k(z) = E_k(z)R_k(z)$ (Fig. 5.7-2(a)). By use of the noble identities we can move $E_k(z)$ all the way to the left and $R_k(z)$ all the way to the right (Fig. 5.7-2(b)). If we now insert a nonsingular matrix $\mathbf{T}$ and its inverse as shown, the input-output behavior of the system is still unchanged. In particular if the product

$$S_k(z) = E_k(z)R_k(z), \quad 0 \leq k \leq M-1, \tag{5.7.7}$$

is the same ($= S(z)$) for all k, then the system is free from aliasing, and $\widehat{X}(z)$ is given by (5.7.2), *regardless of* the choice of $\mathbf{T}$!

For example, imagine that $\mathbf{T} = \mathbf{W}^*$. Then the analysis bank is the familiar uniform-DFT bank. In this case, $E_k(z)$ and $R_k(z)$ are, respectively, the Type 1 and Type 2 polyphase components of the prototype filters $H_0(z)$ and $F_0(z)$. The filters are related by uniform shifts (precisely as in (5.6.20) and (5.6.22)). This little exercise shows that we can eliminate aliasing in a uniform-DFT filter bank by enforcing the condition that $R_k(z)E_k(z)$ be the same for all k, that is,

$$R_k(z)E_k(z) = S(z), \quad \text{for all } k.$$

One way to do so would be to take $R_k(z) = 1/E_k(z)$, which also yields perfect reconstruction. This choice, however, makes $R_k(z)$ (and hence

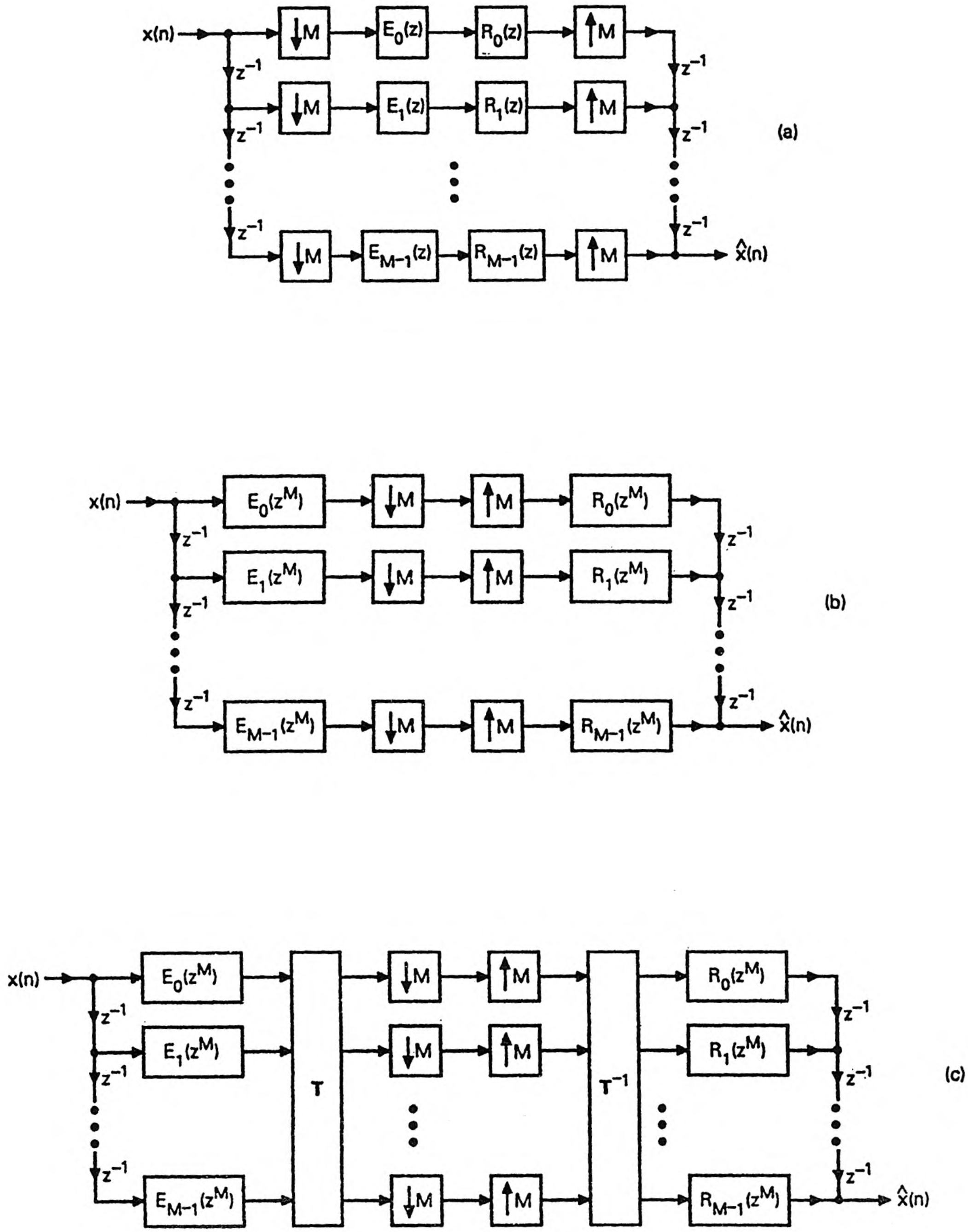


Figure 5.7-2 Step by step development of a fairly general alias-free system. All three systems have the same input/output behavior. Here $S_k(z) = E_k(z)R_k(z)$.

the synthesis filters) unstable unless $E_k(z)$ has all zeros inside the unit circle. A second way to enforce the above condition would be to take

$$R_k(z) = \prod_{\ell \neq k} E_\ell(z). \tag{5.7.8}$$

Then

$$R_k(z)E_k(z) = S(z) = \prod_{\ell=0}^{M-1} E_\ell(z), \quad 0 \leq k \leq M-1. \tag{5.7.9}$$

For large M, (5.7.8) implies that the synthesis filters have much higher order than the analysis filters. For $M = 2$, (5.7.8) means

$$R_0(z) = E_1(z), \quad R_1(z) = E_0(z).$$

This is consistent with the special cases we saw in Section 5.2. For example see Fig. 5.2-2 where the synthesis bank has $E_1(z)$ in the top branch and $E_0(z)$ in the bottom branch. Also in Fig. 5.2-5, the synthesis bank has $a_1(z)$ in the top branch and $a_0(z)$ in the bottom branch.

The ideas introduced above can also be used to compensate for channel distortion in QMF systems, as well as to design M-channel IIR systems free from amplitude distortion. We will skip these details (many of which are covered in Problems 5.21–5.23), and return to the general problem.

5.7.2 The Most General Alias-Free System

What is the most general set of *necessary and sufficient* conditions so that aliasing is canceled? One way to answer this question is to refer to (5.4.16), where $\mathbf{H}(z)$ is the alias component matrix (determined completely by the analysis bank) and $\mathbf{f}(z)$ is the synthesis filter bank. The filter bank is alias free if and only if the product $\mathbf{H}(z)\mathbf{f}(z)$ has the form (5.4.17).

We now obtain an equivalent set of necessary and sufficient conditions based on the polyphase matrices $\mathbf{E}(z)$ and $\mathbf{R}(z)$ [Vaidyanathan and Mitra, 1988]. We will show that the filter bank is alias free if and only if $\mathbf{P}(z)$, defined as the product $\mathbf{R}(z)\mathbf{E}(z)$, is a *pseudocirculant* matrix (defined below). Under alias free condition, additional properties of the distortion function $T(z)$ can be expressed entirely in terms of this matrix very conveniently.

Pseudocirculant Matrices

First, a matrix is said to be *circulant* if every row is obtained using a right-shift (by one position) of the previous row with the added requirement that the righmost element which 'spills over' in the process be 'circulated back' to become the leftmost element. Here is an example:

$$\begin{bmatrix} P_0(z) & P_1(z) & P_2(z) \\ P_2(z) & P_0(z) & P_1(z) \\ P_1(z) & P_2(z) & P_0(z) \end{bmatrix} \quad \text{(circulant matrix).} \tag{5.7.10}$$

Actually it is more approprite to call this a *right*-circulant, because the definition involves *right* shifts. In a similar way one can define left circulants. In this book, 'circulant' stands for 'right circulant', unless mentioned otherwise.

A pseudocirculant matrix is essentially a circulant matrix with the additional feature that the elements below the main diagonal are multiplied with z^{-1}. An example is:

$$\begin{bmatrix} P_0(z) & P_1(z) & P_2(z) \\ z^{-1}P_2(z) & P_0(z) & P_1(z) \\ z^{-1}P_1(z) & z^{-1}P_2(z) & P_0(z) \end{bmatrix} \quad \text{(pseudocirculant matrix).} \quad (5.7.11a)$$

In other words, the element that spills over during the right shift is circulated after multiplying with z^{-1}. In the time domain, the above matrix has the form

$$\begin{bmatrix} p_0(n) & p_1(n) & p_2(n) \\ p_2(n-1) & p_0(n) & p_1(n) \\ p_1(n-1) & p_2(n-1) & p_0(n) \end{bmatrix} \quad \text{(pseudocirculant matrix).} \quad (5.7.11b)$$

Evidently, all the rows of a $M \times M$ pesudocirculant matrix $\mathbf{P}(z)$ are determined by the 0th row which is

$$[\,P_0(z) \quad P_1(z) \quad \ldots \quad P_{M-1}(z)\,]. \qquad (5.7.12)$$

For a pseudocirculant, the kth column is obtainable from the $(k+1)$st column as follows: (a) shift the $(k+1)$st column upwards by one element, (b) circulate the element that spills over so that it becomes the bottom most element, and (c) multiply the circulated element with z^{-1}. The result is equal to the kth column. [The reader can verify this for (5.7.11a).] This can in fact be taken as an equivalent definition for pseudocirculants.

The occurence of pseudocirculant matrices in the context of multirate filter banks was noticed by Marshall [1982]. It was studied later in Vaidyanathan and Mitra [1988]. These matrices have also been found to arise in the context of block digital filtering [Barnes and Shinnaka, 1980]; Sec. 10.1 provides a more complete discussion. The following result was proved in Vaidyanathan and Mitra [1988].

♠**Theorem 5.7.1. Necessary and sufficient condition for alias cancelation.** The M-channel maximally decimated filter bank (Fig. 5.4-1) is free from aliasing if and only if the $M \times M$ matrix $\mathbf{P}(z)$ (defined as the product $\mathbf{R}(z)\mathbf{E}(z)$) is pseudocirculant. Under this condition $\widehat{X}(z) = T(z)X(z)$, and the distortion function $T(z)$ can be expressed as

$$T(z) = z^{-(M-1)}\Big(P_0(z^M)+z^{-1}P_1(z^M)+\ldots+z^{-(M-1)}P_{M-1}(z^M)\Big), \quad (5.7.13)$$

where $P_m(z)$ are the elements of the 0th row of $\mathbf{P}(z)$.

Proof. Consider Fig. 5.7-3 which is the familiar equivalent circuit for the QMF bank in terms of $\mathbf{P}(z)$. We will express $\widehat{X}(z)$ in terms of $X(z)$ and the elements $P_{s,\ell}(z)$ of $\mathbf{P}(z)$. First, using standard decimation formulas (Sec. 4.1.1) we have

$$C_\ell(z) = \frac{1}{M}\sum_{k=0}^{M-1}\left(z^{1/M}W^k\right)^{-\ell} X(z^{1/M}W^k), \quad 0 \le \ell \le M-1, \tag{5.7.14}$$

with $W = e^{-j2\pi/M}$. The outputs of $\mathbf{P}(z)$ are given by

$$B_s(z) = \sum_{\ell=0}^{M-1} P_{s,\ell}(z)C_\ell(z). \tag{5.7.15}$$

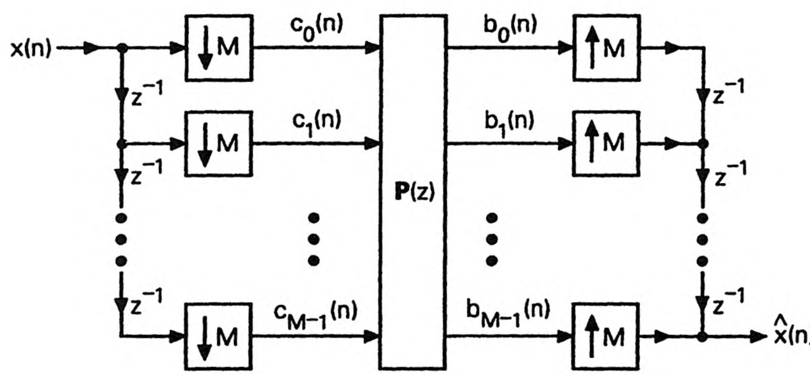


Figure 5.7-3 The equivalent circuit for the maximally decimated filter bank.

The reconstructed signal is

$$\begin{aligned}\widehat{X}(z) &= \sum_{s=0}^{M-1} z^{-(M-1-s)}B_s(z^M)\\ &= \sum_{s=0}^{M-1} z^{-(M-1-s)}\sum_{\ell=0}^{M-1} P_{s,\ell}(z^M)C_\ell(z^M)\\ &= \frac{1}{M}\sum_{s=0}^{M-1} z^{-(M-1-s)}\sum_{\ell=0}^{M-1} P_{s,\ell}(z^M)\sum_{k=0}^{M-1}\left(zW^k\right)^{-\ell} X(zW^k).\end{aligned} \tag{5.7.16}$$

This can be rearranged as

$$\widehat{X}(z) = \frac{1}{M}\sum_{k=0}^{M-1} X(zW^k)\sum_{\ell=0}^{M-1} W^{-k\ell}\sum_{s=0}^{M-1} z^{-\ell}z^{-(M-1-s)}P_{s,\ell}(z^M). \tag{5.7.17}$$

The terms of the form $X(zW^k), k \neq 0$ represent aliasing. The above expression is free from these aliasing terms [for all input signals $x(n)$] if and only if

$$\sum_{\ell=0}^{M-1} W^{-k\ell} \underbrace{\sum_{s=0}^{M-1} z^{-\ell} z^{-(M-1-s)} P_{s,\ell}(z^M)}_{V_\ell(z)} = 0, \quad k \neq 0. \tag{5.7.18}$$

This can be written using matrix notation as

$$\mathbf{W}^\dagger \begin{bmatrix} V_0(z) \\ V_1(z) \\ \vdots \\ V_{M-1}(z) \end{bmatrix} = \begin{bmatrix} \times \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{5.7.19}$$

where $\mathbf{W}$ is the $M \times M$ DFT matrix, and $\times$ indicates a possibly nonzero entry. Using the fact that $\mathbf{W}\mathbf{W}^\dagger = M\mathbf{I}$, we can rewrite this as

$$\begin{bmatrix} V_0(z) \\ V_1(z) \\ \vdots \\ V_{M-1}(z) \end{bmatrix} = \mathbf{W} \begin{bmatrix} \times \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{5.7.20}$$

This implies

$$V_\ell(z) = V(z), \quad 0 \leq \ell \leq M-1, \tag{5.7.21}$$

since the 0th column of $\mathbf{W}$ has all entries equal to unity. Thus the QMF bank is alias-free if and only $V_\ell(z)$ defined in (5.7.18) is the same for all ℓ.

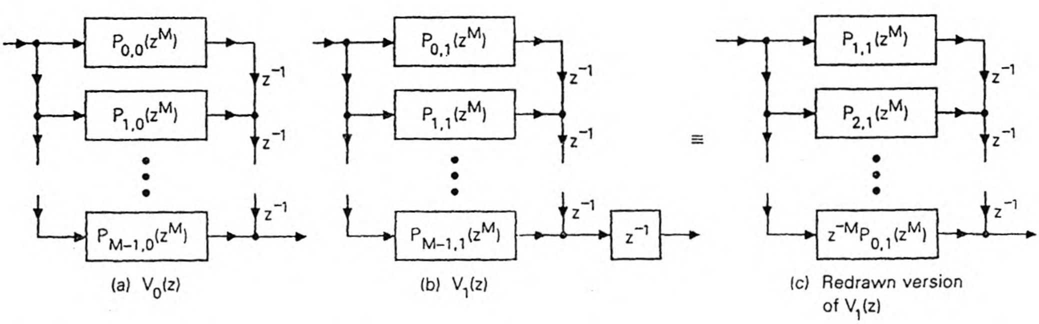


Figure 5.7-4 Comparing the Type 2 polyphase implementations of $V_0(z)$ and $V_1(z)$.

In Figs. 5.7-4(a) and (b) we demonstrate polyphase structures for $V_0(z)$ and $V_1(z)$. The structure for $V_1(z)$ can be rearranged as shown in Fig. 5.7-4(c). Because of the requirement $V_0(z) = V_1(z)$, the polyphase components in Figs. 5.7-4(a) and (c) should be the same. This shows that the 0th column of $\mathbf{P}(z)$ is an upwards-shifted version of the 1st column, with the top most element recirculated with a z^{-1} attached to it. Similarly we can verify that the ℓth column is obtained from the $(\ell+1)$st column in this manner. This proves that $\mathbf{P}(z)$ is pseudocirculant.

Having canceled aliasing, (5.7.18) holds so that $\widehat{X}(z) = T(z)X(z)$ with $T(z)$ obtained from (5.7.17) as

$$T(z) = \frac{1}{M}\sum_{s=0}^{M-1} z^{-(M-1-s)} \sum_{\ell=0}^{M-1} z^{-\ell} P_{s,\ell}(z^M). \tag{5.7.22}$$

Since the elements $P_{s,\ell}(z)$ are completely determined by the 0th row elements $P_{0,\ell}(z)$, we can rearrange this (Problem 5.29) into the form (5.7.13). This completes the proof. $\bigtriangledown\bigtriangledown\bigtriangledown$

The Special Case of Perfect Reconstruction (PR) Systems

A PR system is an alias free system with $T(z) =$ delay. The alias-free nature implies that $\mathbf{P}(z)$ is pseudocirculant. With the 0th row of $\mathbf{P}(z)$ as in (5.7.12), $T(z)$ has the form (5.7.13). This is a delay only if $P_m(z) = 0$ for all but one value of m in the range $0 \leq m \leq M-1$. And this nonzero $P_m(z)$ must have the form cz^{-m_0}. Summarizing, an alias free system has perfect reconstruction if and only if the pseudocirculant $\mathbf{P}(z)$ has 0th row equal to

$$[0 \ \ \ldots \ \ 0 \ \ cz^{-m_0} \ \ 0 \ \ \ldots 0]. \tag{5.7.23a}$$

In other words $\mathbf{P}(z)$ (i.e., $\mathbf{R}(z)\mathbf{E}(z)$) has the form

$$\mathbf{R}(z)\mathbf{E}(z) = cz^{-m_0}\begin{bmatrix} \mathbf{0} & \mathbf{I}_{M-r} \\ z^{-1}\mathbf{I}_r & \mathbf{0} \end{bmatrix}, \tag{5.7.23b}$$

for some r in $0 \leq r \leq M-1$. This was stated earlier in (5.6.7) without proof. Under this condition (5.7.13) reduces to

$$T(z) = cz^{-r}z^{-(M-1)}z^{-m_0M}. \tag{5.7.23c}$$

Some Practical Special Cases of Alias Free Systems

1. Consider the special case when $\mathbf{P}(z)$ is diagonal. This means that the structure is as in Fig. 5.7-1(a). The pseudocirculant condition on $\mathbf{P}(z)$ now means that all diagonal elements are identical, so that

$$\mathbf{P}(z) = S(z)\mathbf{I}. \tag{5.7.24}$$

This result agrees with the alias-cancelation condition obtained earlier in Example 5.7.1. In this case $T(z)$ reduces to

$$T(z) = z^{-(M-1)} S(z^M). \tag{5.7.25}$$

2. A generalization of the above is the case where $\mathbf{P}(z)$ has one nonzero entry per row. In this case, the pseudocirculant property means

$$\mathbf{P}(z) = S(z) \begin{bmatrix} \mathbf{0} & \mathbf{I}_{M-r} \\ z^{-1}\mathbf{I}_r & \mathbf{0} \end{bmatrix}, \tag{5.7.26}$$

where $0 \leq r \leq M-1$. The 0th row of $\mathbf{P}(z)$ has all zeros except $P_r(z) = S(z)$, so that (5.7.13) yields

$$T(z) = z^{-(M-1)} z^{-r} S(z^M). \tag{5.7.27}$$

The presence of r merely introduces additional delay. We can obtain this from the first special case simply by replacing each synthesis filter $F_k(z)$ with $z^{-r}F_k(z)$. Thus every PR system satisfying (5.7.26) can be obtained from a PR system satisfying (5.7.24) simply by replacing each synthesis filter $F_k(z)$ with $z^{-r}F_k(z)$. In this sense, the form (5.7.26) is only "trivially" more general than (5.7.24).

If $S(z)$ is a delay, then (5.7.26) reduces to the form (5.7.23b) implying perfect reconstruction. Practically all the alias-free systems we consider belong to simple special cases of the form (5.7.26), that is, $\mathbf{P}(z)$ is a pseudocirculant with one nonzero entry per row.

Further results on amplitude and phase distortion in alias-free systems can be found in Sec. 10.1. In particular, it will be shown that $T(z)$ is allpass (i.e., there is no amplitude distortion) if and only if the pseudocirculant $\mathbf{P}(z)$ satisfies a property called paraunitariness.

5.8 TREE STRUCTURED FILTER BANKS

Consider the structure shown in Fig. 5.8-1(a). Here a signal is split into two subbands, and after decimation, each subband is again split into two and decimated. The subbands are then recombined, two at a time, by use of two-channel synthesis banks. This system is said to be a maximally decimated (binary) tree structured filter bank. The complete system can be redrawn in the equivalent nontree form of Fig. 5.4-1, with $M = 4$. The resulting filters $H_m(z)$ and $F_m(z)$ $(0 \leq m \leq M-1)$ can be expressed in terms of the filters $H_i^{(k)}(z)$ and $F_i^{(k)}(z)$ (Problem 5.24).

Fig. 5.8-1(b) shows an example of the magnitude responses of the four analysis filters $H_m(z)$ for the two level tree. In this example, the tree filters $H_0^{(k)}(z)$ have the power symmetric response shown earlier in Fig. 5.3-4(a), and $H_1^{(k)}(z) = H_0^{(k)}(-z)$. Note that the four analysis filters are not equiripple, even though $H_0^{(k)}(z)$ and $H_1^{(k)}(z)$ are.

Suppose the filters $H_0^{(k)}(z), H_1^{(k)}(z), F_0^{(k)}(z)$ and $F_1^{(k)}(z)$ are such that the two-channel QMF bank with these filters is alias-free. Then the complete system is also alias-free. Similarly, if the two channel system has perfect reconstruction, then so does the complete system. (Problems 5.24 and 5.25). These results can also be extended to more than two levels of splitting. The results also extend to trees other than binary (e.g., split a signal into two subbands, then split one subband into three and the other into four, etc.).

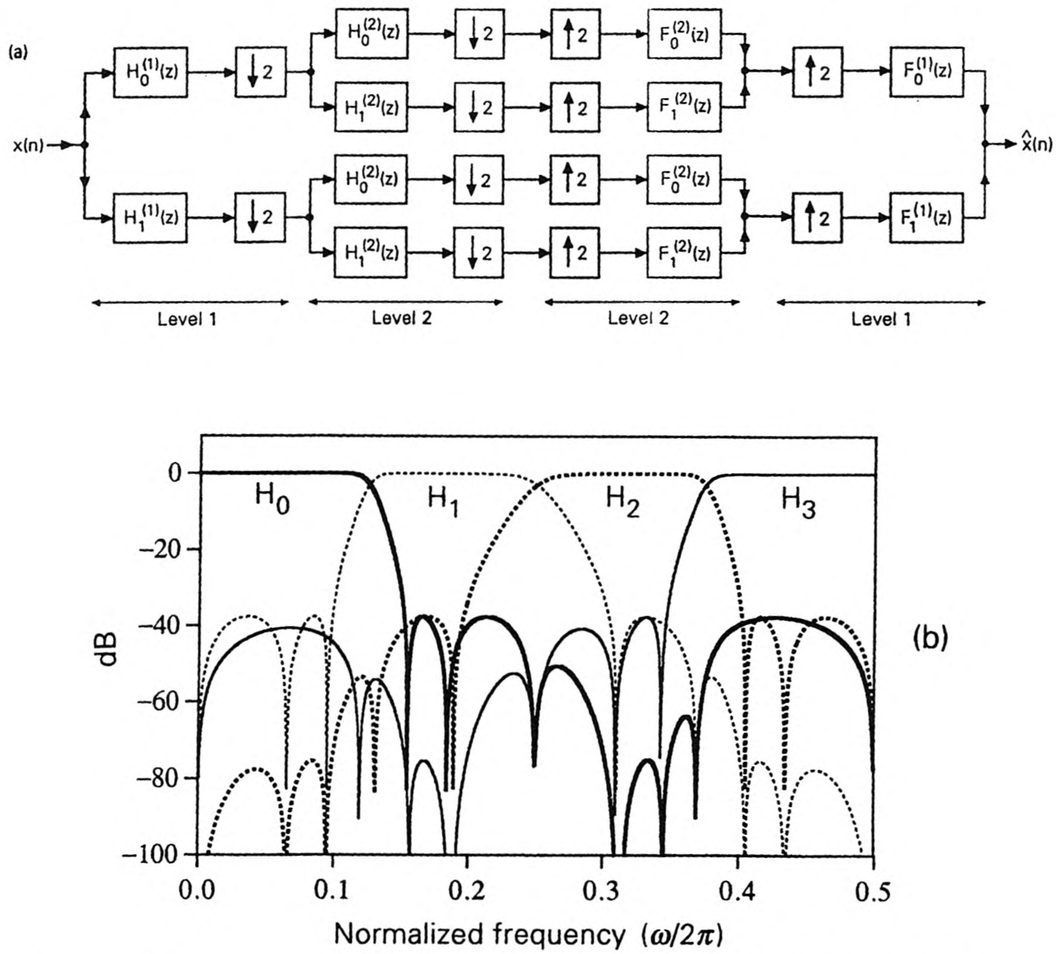


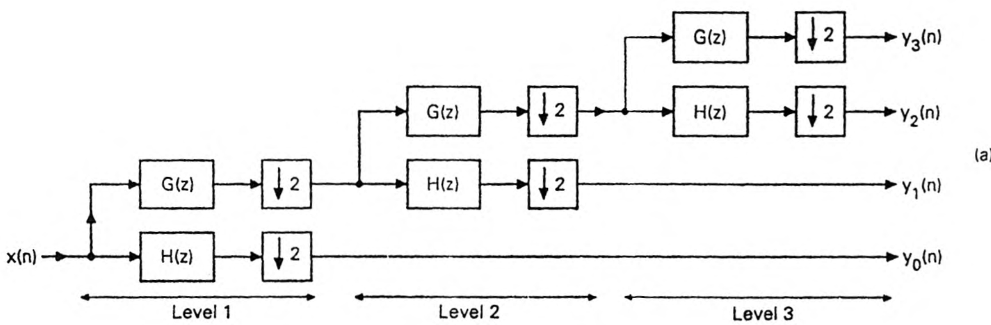
Figure 5.8-1 (a) A two-level maximally decimated tree structured filter bank, and (b) example of magnitude responses.

Assume that all the two-channel systems in Fig. 5.8-1(a) have perfect reconstruction. Suppose, however, that the upper two-channel QMF bank and the lower two-channel QMF bank at the second level do *not* have the same set of analysis and synthesis filters. Then it may be necessary to introduce appropriate scale factors and delays at proper places so that the complete system still has perfect reconstruction (why?).

Tree structured filter banks are used in a number of applications both in one and two-dimensional signal processing. We now mention two of these, which were originally intended for image processing. The presentation here is brief.

Multiresolution Analysis Algorithm

Consider the variation of the analysis bank shown in Fig. 5.8-2(a). This is equivalent to the system shown in Fig. 5.8-2(b). This is a four-channel system with unequal decimation ratios. (It is still a maximally decimated system.) Each $[G(z), H(z)]$ is typically a lowpass/highpass pair, as in a two channel QMF bank.



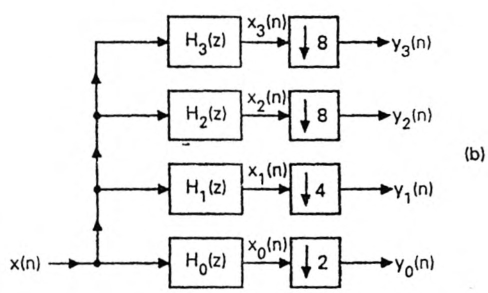


Figure 5.8-2 (a) A 3-level binary tree structured QMF bank, and (b) the equivalent four-channel system.

Figure 5.8-3(a) shows the synthesis bank that goes with this system, and Fig. 5.8-3(b) shows the non-tree equivalent structure. Assume that $G_s(z), H_s(z)$ are chosen so that the two channel QMF bank with filters

$G(z), H(z), G_s(z)$ and $H_s(z)$ has perfect reconstruction, with unit-gain and no delay. We then have $\widehat{x}(n) = x(n)$.

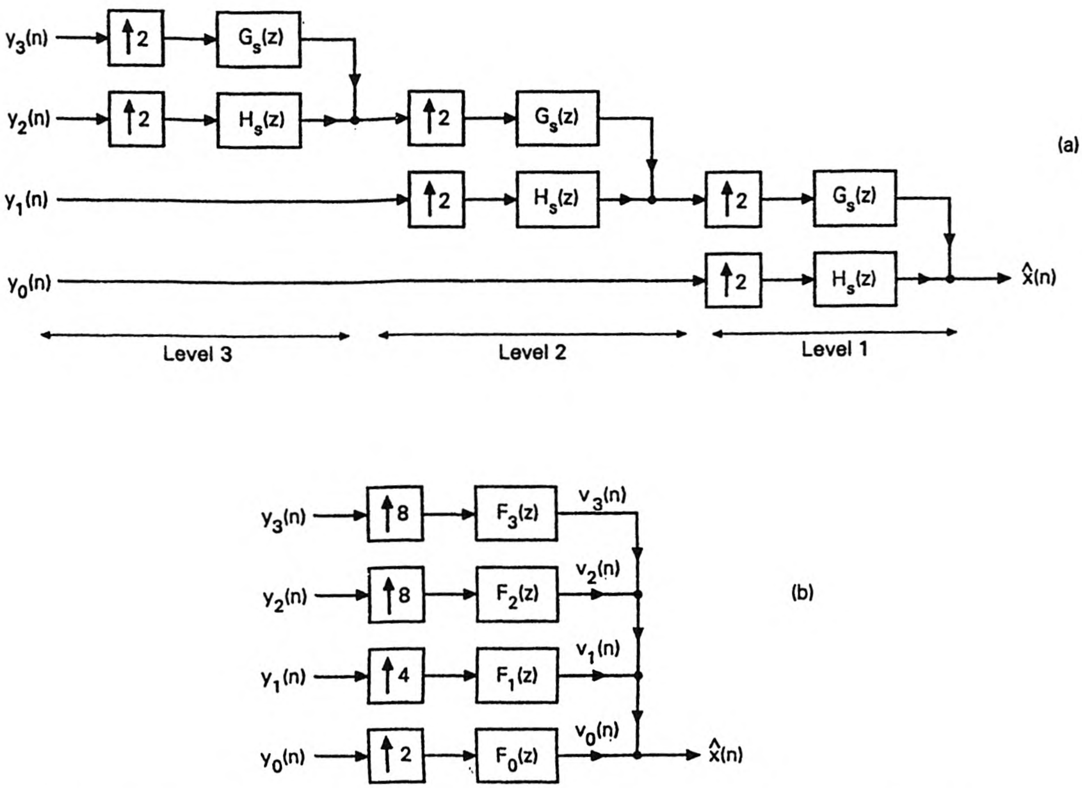


Figure 5.8-3 (a) The synthesis bank corresponding to Fig. 5.8-2, and (b) equivalent four-channel system.

Figure 5.8-4 shows typical frequency responses of the analysis and synthesis filters (assuming that $G(z)$ and $H(z)$ form a lowpass and highpass pair, with cutoff around $\pi/2$.) The signals $v_k(n)$ which are outputs of $F_k(z)$, are called *multiresolution* components. For example the signal $v_3(n)$ represents a lowpass version (or a 'coarse' approximation) of $x(n)$, subject to aliasing and other errors. (Note that $v_k(n)$ has the same 'sampling rate' as $x(n)$.) The signal $v_2(n)$ adds some high frequency (bandpass) details, so that $v_3(n) + v_2(n)$ is a finer approximation of $x(n)$. The signal $v_0(n)$ adds the finest ultimate (high-frequency) detail, so that $\widehat{x}(n) = x(n)$ (by perfect reconstruction property). An obvious generalization of the tree structure uses different filter pairs at different levels of the tree.

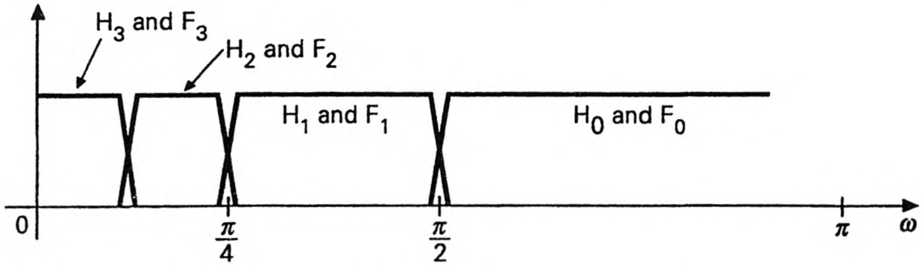


Figure 5.8-4 Typical appearances of magnitude responses of filters in the 3-level tree.

There are several ways in which this structure can be used to obtain image compression. For example, one can choose to retain only $v_3(n)$; or one can add a quantized version of $v_2(n)$ to $v_3(n)$. More generally we can attach decreasing weights (bits) to the finer and finer detail signals $v_k(n)$. This technique is the ingredient of Mallat's multiresolution algorithm for image compression [Mallat, 1989a,b]. The above observation can also be used to transmit finer and finer versions of video data (e.g., in teleconferencing).

The above algorithm is extremely appealing even from an intuitive and philosophical view point: any kind of 'learning' or 'understanding' in life always occurs at various levels of resolutions, which get finer and finer as we improve our skills. Think of the way we mature in any of these: baseball, music, scientific skills, writing skills ...

The Laplacian Pyramid

This is a well-known scheme for image coding [Burt and Adelson, 1983], and is demonstrated in Fig. 5.8-5. Here $G(z)$ is an FIR lowpass filter. The notation $\widetilde{G}(z)$ is defined as usual, so that $\widetilde{G}(e^{j\omega}) = G^*(e^{j\omega})$. Thus $\widehat{x}_0(n)$ is a coarse lowpass approximation of the input $x(n)$. This approximation introduces no phase distortion [since $\widetilde{G}(z)G(z)$ has zero phase]. We can subtract $\widehat{x}_0(n)$ from $x(n)$ to recover the high frequency details, denoted $d_0(n)$.

This process is now repeated on the decimated signal $x_1(n)$. The analysis bank, therefore, produces the highpass signals $d_0(n), \ldots d_{L-1}(n)$, and the lowpass signal $\widehat{x}_{L-1}(n)$. (In the figure $L = 2$.) These signals can then be recombined using a synthesis bank as demonstrated in the figure, to recover $x(n)$.

Notice that the perfect reconstruction property is trivially satisfied, regardless of the design of $G(z)$. This is not surprising because the difference signals (highpass signals) $d_k(n)$ are not maximally decimated. For example, $d_0(n)$ is not decimated at all. This results in increased data rate (nearly by a factor of two). In order for the scheme to be beneficial, this must be compensated by the compression obtainable by the quantization of the signals $d_k(n)$ and $\widehat{x}_{L-1}(n)$. Traditional QMF banks (such as Fig. 5.8-2) are, on the other hand, maximally decimated, and do not have this problem (but require special design procedures).

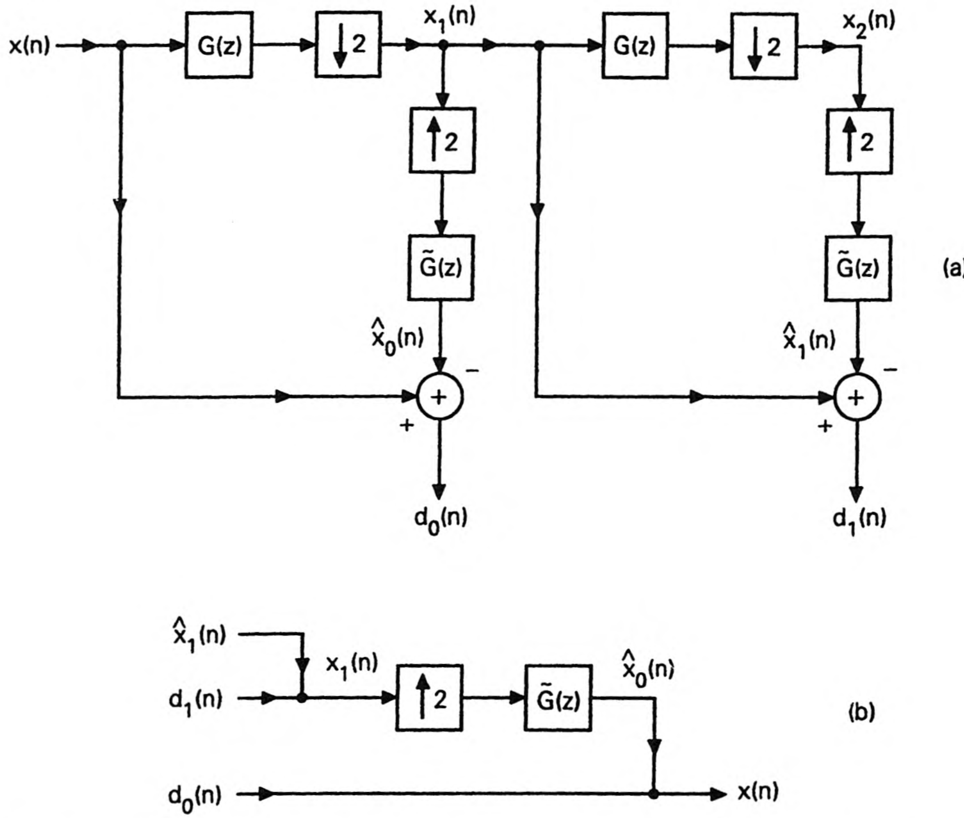


Figure 5.8-5 Burt and Adelson's algorithm. (a) analysis, and (b) synthesis.

5.9 TRANSMULTIPLEXERS

An introduction to transmultiplexers was given in Section 4.5.4, which the reader should review at this time. Figs. 4.5-4 and 4.5-5 demonstrate the time domain and frequency domain multiplexing operations, and Fig. 4.5-6 shows the complete TDM → FDM → TDM converter (transmultiplexer), also reproduced in Fig. 5.9-1.

Fig. 5.9-2 demonstrates how the signal $V_1(e^{j\omega})$ is generated starting from $X_1(e^{j\omega})$. If all the signals $x_k(n)$ are bandlimited to $|\omega| < \sigma_k$ with $\sigma_k < \pi$, there is no overlap between adjacent signals in the FDM format, that is, there exists a guard band between ajdacent frequency bins, as demonstrated in Fig. 5.9-3. In this case the FDM signals can be separated by filtering operations (followed by M-fold decimation to stretch the signal back to the full band $-\pi \leq \omega \leq \pi$). The presence of guard bands ensures that there is no cross talk between adjacent signals, even though the filters have nonzero

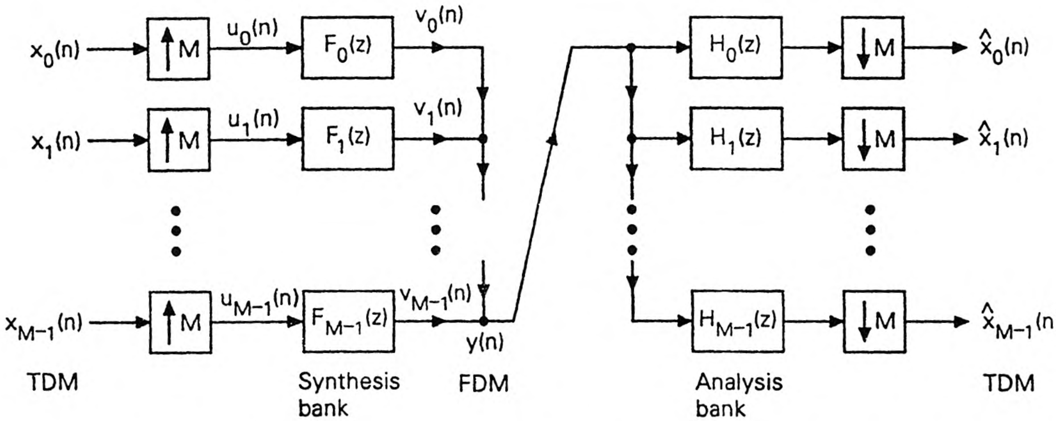


Figure 5.9-1 The transmultiplexer circuit, drawn in terms of filter bank notations.

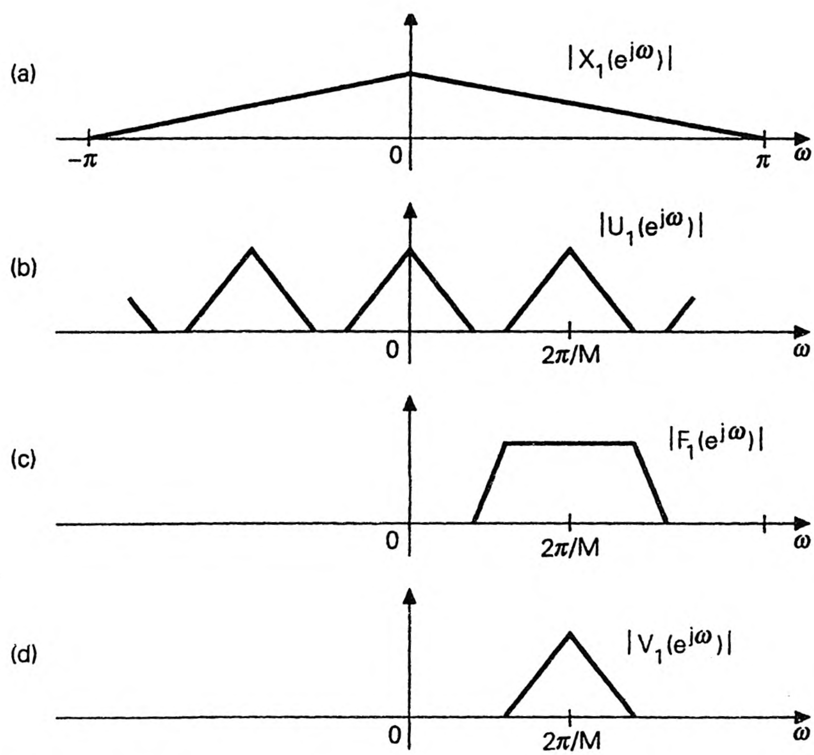


Figure 5.9-2 Generation of the signal $v_1(n)$ by use of interpolation and filtering.

transition band. A larger guard band implies larger permissible transition band (hence lower cost) for the filters $H_k(z)$, which attempt to recover the signals $x_k(n)$ from the FDM version. However, the existence of guard bands also means that the full channel bandwidth is *not* utilized in the transmission process.

The following observation was made in Vetterli [1986]: even if there are no guard bands (thereby permitting cross talk), we can subsequenty eliminate the cross talk in a manner analogous to alias cancelation in QMF banks. This idea makes judicious use of the relation between the mathematics of QMF banks and transmultiplexers as we will elaborate. We remind the reader that the term 'QMF', which is used for convenience, really stands for 'maximally decimated analysis synthesis systems'.

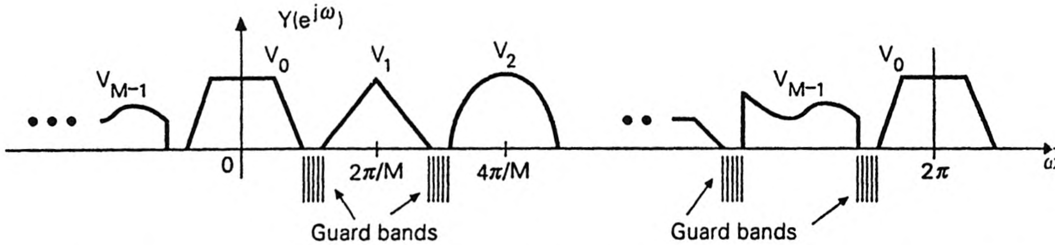


Figure 5.9-3 Stacking up the M signals $V_k(e^{j\omega})$ in the frequency domain, to obtain the FDM version $y(n)$.

5.9.1 Input-Output Relations for Transmultiplexers

We show that it is possible to achieve perfect cross talk elimination as well as perfect recovery of each TDM component $x_k(n)$ with finite-cost (in fact FIR) filters $H_k(z)$ and $F_k(z)$. In analogy with the QMF bank, we continue to use terms such as "analysis" and "synthesis" filters, and "filter banks" as indicated in Fig. 5.9-1. Notice the conceptual duality between the QMF bank and the transmultiplexer. In the former, we first "analyze" and then "synthesize"; this is in reverse order as compared to the transmultiplexer. (The QMF bank can also be conceptually looked upon as a FDM $\rightarrow$ TDM $\rightarrow$ FDM convertor.) We will see that the problem of designing filters for 'perfect reconstruction transmultiplexers' is same as the design of perfect reconstruction (PR) QMF banks.

The relation between $\widehat{x}_k(n)$ and $x_m(n)$ can be schematically represented as in Fig. 5.9-4. By using the polyphase identity (Fig. 4.3-13) we see that each branch in this figure is in reality an LTI system. We can therefore express

$$\widehat{X}_k(z) = \sum_{m=0}^{M-1} S_{km}(z) X_m(z), \quad 0 \le k \le M-1, \tag{5.9.1}$$

where $S_{km}(z)$ is the 0th polyphase component of $H_k(z)F_m(z)$. By defining

$$\mathbf{x}(n) = \begin{bmatrix} x_0(n) \\ \vdots \\ x_{M-1}(n) \end{bmatrix}, \quad \widehat{\mathbf{x}}(n) = \begin{bmatrix} \widehat{x}_0(n) \\ \vdots \\ \widehat{x}_{M-1}(n) \end{bmatrix}, \tag{5.9.2}$$

we can express (5.9.1) more compactly as

$$\widehat{\mathbf{X}}(z) = \mathbf{S}(z)\mathbf{X}(z). \tag{5.9.3}$$

So the transmultiplexer is an LTI system with transfer matrix $\mathbf{S}(z)$. The system is free from cross talk if and only if $\mathbf{S}(z)$ is diagonal. (This is the same as the requirement that the 0th polyphase component of $H_k(z)F_m(z)$ be zero unless $k = m$.) Under this condition, each reconstructed TDM signal $\widehat{x}_k(n)$ is related to the original signal $x_k(n)$ according to

$$\widehat{X}_k(z) = S_{kk}(z)X_k(z). \tag{5.9.4}$$

The transfer functions $S_{kk}(z)$ represent the distortions that remain after cross talk elimination. If $S_{kk}(z)$ is allpass for all k, there is no amplitude distortion; if $S_{kk}(z)$ has linear phase, there is no phase distortion. Finally, a perfect reconstruction (PR) transmultiplexer is one for which

$$S_{kk}(z) = c_k z^{-n_k}, \quad \text{for all } k, \tag{5.9.5}$$

for some nonzero c_k and integer n_k. The TDM signals are then recovered without error, that is, $\widehat{x}_k(n) = c_k x_k(n - n_k)$.

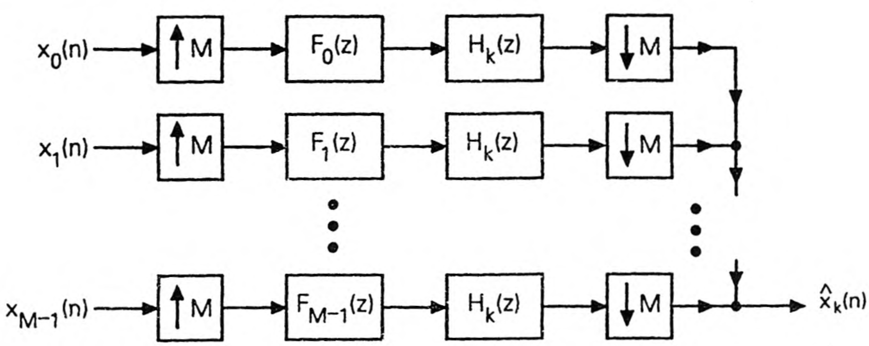


Figure 5.9-4 Equivalent circuit for generation of $\widehat{x}_k(n)$.

5.9.2 Study Based on Polyphase Matrices

The use of polyphase decomposition adds further insight into the operation of the transmultiplexer [Koilpillai et al., 1991]. As in Sec. 5.5, we can redraw the analysis and synthesis banks in terms of the polyphase matrices $\mathbf{E}(z)$ and

$\mathbf{R}(z)$. The resulting equivalent transmultiplexer circuit is shown in Fig. 5.9-5(a), which simplifies to Fig. 5.9-5(b) after invoking the noble identities. † This structure can be further simplified into the equivalent form shown in Fig. 5.9-5(c), by using the equivalence of Fig. 4.3-14. It is, therefore, clear that the transfer matrix $\mathbf{S}(z)$ can be expressed as

$$\mathbf{S}(z) = \mathbf{E}(z)\mathbf{\Gamma}(z)\mathbf{R}(z), \tag{5.9.6}$$

where

$$\mathbf{\Gamma}(z) = \begin{bmatrix} \mathbf{0} & 1 \\ z^{-1}\mathbf{I}_{M-1} & \mathbf{0} \end{bmatrix}. \tag{5.9.7}$$

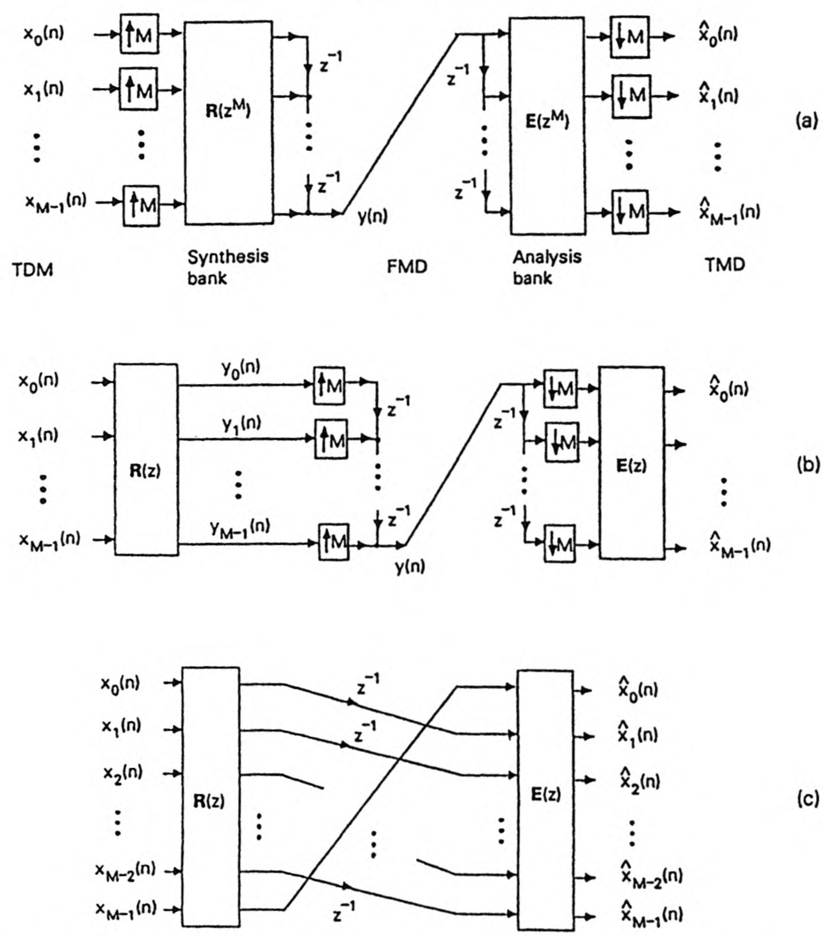


Figure 5.9-5 (a) Equivalent structures for the transmultiplexer in terms of polyphase matrices, (b) rearrangement using noble identitites, and (c) simplification using the equivalence of Fig. 4.3-14.

† Note that if we set $\mathbf{E}(z) = \mathbf{I}$ as a special case, then $y(n)$ becomes the TDM (rather than FDM) signal!

So the set of reconstructed signals $\widehat{\mathbf{x}}(n)$ is related to $\mathbf{x}(n)$ by the transfer matrix (5.9.6). From this expression we can explore the conditions for cross talk elimination and perfect reconstruction.

Perfect Reconstruction

A *sufficient condition* for perfect reconstruction is obtained by setting $\mathbf{S}(z) = cz^{-n_0}\mathbf{I}$. Now

$$\begin{aligned} &\mathbf{S}(z) = cz^{-n_0}\mathbf{I} \\ \Longleftrightarrow\ &\mathbf{E}(z)\mathbf{\Gamma}(z)\mathbf{R}(z) = cz^{-n_0}\mathbf{I} \\ \Longleftrightarrow\ &c\mathbf{E}^{-1}(z)\mathbf{R}^{-1}(z) = z^{n_0}\mathbf{\Gamma}(z) \\ \Longleftrightarrow\ &\mathbf{R}(z)\mathbf{E}(z) = cz^{-n_0}\mathbf{\Gamma}^{-1}(z). \end{aligned} \tag{5.9.8}$$

Substituting for $\mathbf{\Gamma}(z)$, this becomes

$$\mathbf{R}(z)\mathbf{E}(z) = cz^{-m_0}\begin{bmatrix} \mathbf{0} & \mathbf{I}_{M-1} \\ z^{-1} & \mathbf{0} \end{bmatrix}, \tag{5.9.9}$$

for appropriate integer m_0.

Relation to perfect reconstruction (PR) QMF banks. From the previous section we know that the product $\mathbf{R}(z)\mathbf{E}(z)$ of a PR QMF bank satisfies (5.7.23b) for some integer r in $0 \le r \le M-1$. If the QMF bank is such that $r = 1$, then this condition is same as (5.9.9). On the other hand, if $r \neq 1$, then we can insert appropriate amount of delay in front of the filters $F_k(z)$ to force $r = 1$.

The amount of delay to be introduced can be judged as follows: for arbitrary r the PR QMF bank has overall transfer function (5.7.23c). This has the form $cz^{-\ell M}$ for integer ℓ if and only if $r = 1$. So the amount of delay to be inserted is such that $T(z)$ takes this form. For example, suppose the PR QMF bank has $T(z) = cz^{-2}z^{-iM}$. If we insert the delay $z^{-(M-2)}$ in front of each $F_k(z)$, then $T(z)$ becomes $cz^{-(i+1)M}$. So insertion of this delay results in a PR QMF bank with $r = 1$. Its analysis and synthesis filters can then be used in the transmultiplexer to obtain perfect reconstruction!

Summary of perfect-reconstruction condition. This important conclusion can be summarized as follows: Let $H_k(z)$ and $F_k(z)$ be the analysis and synthesis filters of a perfect reconstruction QMF bank, with overall transfer function $T(z) = cz^{-L}$ for some $c \neq 0$ and integer L. Then the transmultiplexer with analysis filters $H_k(z)$ and synthesis filters $z^{-J}F_k(z)$ has perfect reconstruction property for some integer J in the range $0 \le J \le M-1$. The appropriate value of J is such that $L + J$ is a multiple of M. (That is, J is such that a QMF bank with filters $H_k(z)$ and $z^{-J}F_k(z)$ would have $T(z) = cz^{-\ell M}$ for some integer ℓ.)

Cross Talk Free Transmultiplexers

The next natural question is this: suppose we are not interested in perfect reconstruction, but only in perfect cross talk elimination, and minimization of other distortions. (This will cut the cost of filters to some extent.) Can we obtain such a system starting from a QMF bank? Since $\widehat{\mathbf{X}}(z) = \mathbf{S}(z)\mathbf{X}(z)$, the transmultiplexer is cross talk free if $\mathbf{S}(z)$ is diagonal.

We now show that this can be accomplished by starting from a suitable alias-free QMF bank. The most common alias-free QMF bank satisfies (5.7.26), where $\mathbf{P}(z) = \mathbf{R}(z)\mathbf{E}(z)$. We will assume $r = 1$, as this can be ensured by inserting the right amount of delay z^{-J} in front of $F_k(z)$. So we have

$$\mathbf{R}(z)\mathbf{E}(z) = S(z)\underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{I}_{M-1} \\ z^{-1} & \mathbf{0} \end{bmatrix}}_{z^{-1}\boldsymbol{\Gamma}^{-1}(z)}, \tag{5.9.10}$$

where $\boldsymbol{\Gamma}(z)$ is as in (5.9.7). The QMF bank satisfying (5.9.10) has distortion function (5.7.27), with $r = 1$. That is,

$$T(z) = z^{-M}S(z^M). \tag{5.9.11}$$

In other words, $T(z)$ is a function of z^M, i.e., z appears only in the form z^M. Now the condition (5.9.10) implies

$$z^{-1}S(z)\mathbf{R}^{-1}(z)\boldsymbol{\Gamma}^{-1}(z)\mathbf{E}^{-1}(z) = \mathbf{I}, \tag{5.9.12}$$

which in turn implies

$$\mathbf{E}(z)\boldsymbol{\Gamma}(z)\mathbf{R}(z) = z^{-1}S(z)\mathbf{I}. \tag{5.9.13}$$

The quantity on the left is the transfer matrix $\mathbf{S}(z)$ of the transmultiplexer with same analysis and synthesis filters as the QMF bank. So (5.9.13) is equivalent to

$$\mathbf{S}(z) = z^{-1}S(z)\mathbf{I}. \tag{5.9.14}$$

Since this is a diagonal matrix, cross talk has been eliminated, and the reconstructed signals satisfy

$$\frac{\widehat{X}_k(z)}{X_k(z)} = z^{-1}S(z). \tag{5.9.15}$$

Summary of cross talk cancelation condition. This result can be summarized as follows: Let $H_k(z)$ and $F_k(z)$ be the analysis and synthesis filters in a QMF bank satisfying (5.9.10). This QMF bank is therefore alias-free with distortion function $T(z) = z^{-M}S(z^M)$. If we now design a transmultiplexer with analysis filters $H_k(z)$ and synthesis filters $F_k(z)$, then it is free from cross talk. Moreover, the reconstructed signals satisfy $\widehat{X}_k(z) = z^{-1}S(z)X_k(z)$.

The cross talk free transmultiplexer in general suffers from amplitude and phase distortions since $S(z)$ in (5.9.15) is an arbitrary transfer function. This same $S(z)$ appears in (5.9.11) which represents the distortions in the alias-free QMF bank. If $S(z)$ is allpass, then both systems are free from amplitude distortion. If $S(z)$ has linear phase, then both systems are free from phase distortion.

Notice that the distortion functions $S_{kk}(z)$ in the transmultiplexer are not required to be same for all k. This freedom is not exploited above, because $S_{kk}(z) = z^{-1}S(z)$ for all k.

5.10 SUMMARY AND TABLES

In this chapter we studied the quadrature mirror filter bank. Both two-channel and M channel cases were considered.

For the two channel case we also presented design techniques for alias-free QMF banks; in the FIR case we showed how to eliminate phase distortion and minimize amplitude distortion. For the IIR case we showed that if the analysis filters are constrained to be power symmetric, we can design alias-free QMF banks free from amplitude distortion. The very low computational complexity of this IIR system was also demonstrated. With FIR filters, the same power symmetric condition was then used to obtain perfect reconstruction.

For the M channel case we developed the theory of alias cancelation and perfect reconstruction, and demonstrated the ideas with several examples. These results were extended to the study of transmultiplexers. We also considered tree structured filter banks.

Tables 5.10.1–5.10.4 summarize the main results of this chapter. Table 5.10.5 presents a summary of important matrix quantities, and the relations between them. In the next few chapters, we will present design techniques for M channel QMF banks.

TABLE 5.10.1 Two-channel QMF bank at a glance

1. Basic facts (Section 5.1)

Reconstructed signal: $\widehat{X}(z) = T(z)X(z) + A(z)X(-z)$.

$T(z) =$ Distortion function $= \frac{1}{2}[H_0(z)F_0(z) + H_1(z)F_1(z)]$.

$A(z) =$ Aliasing gain $= \frac{1}{2}[H_0(-z)F_0(z) + H_1(-z)F_1(z)]$.

Suff. cond. for alias cancelation: $F_0(z) = H_1(-z)$, $F_1(z) = -H_0(-z)$.

After aliasing is canceled $\widehat{X}(z) = T(z)X(z)$.

$T(z)$ not allpass $\Rightarrow$ amplitude distortion (AMD)
$T(z)$ not linear-phase $\Rightarrow$ phase distortion (PHD).

FIR QMF bank: $H_0(z), H_1(z), F_0(z), F_1(z)$ are FIR.
Linear-phase QMF bank: $H_0(z), H_1(z)$ have linear phase.

2. A simple choice of filters for alias cancelation (Section 5.2)

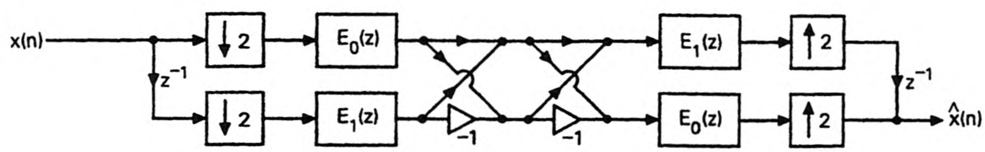
Choose $H_1(z) = H_0(-z), F_0(z) = H_0(z), F_1(z) = -H_1(z)$. Then

a) This is alias-free with $T(z) = \frac{1}{2}[H_0^2(z) - H_0^2(-z)]$.

b) Let $H_0(z) = E_0(z^2) + z^{-1}E_1(z^2)$, then $T(z) = 2z^{-1}E_0(z^2)E_1(z^2)$.

c) This expression for $T(z)$ (a consequence of the constraint $H_1(z) = H_0(-z)$) shows that perfect reconstruction is obtained if and only if $E_1(z) = az^{-b}/E_0(z)$, imposing severe restrictions on analysis filters. For example, in the FIR case $H_0(z)$ has to be a sum of two delays.

A polyphase implementation:



FIR case (see Table 5.10.2 for IIR case).

a) If $H_0(z)$ is linear-phase FIR with order N, then $T(z)$ has linear phase, and the system has only AMD. N must be odd, or else $T(e^{j\pi/2}) = 0$. Perfect reconstruction is not possible unless $E_0(z)$ and $E_1(z)$ are delays, which would make $H_0(z)$ trivial.

b) If N denotes the order (odd) of $H_0(z)$, the analysis bank requires $0.5(N+1)$ MPUs and $0.5(N+1)$ APUs (using polyphase form). This is true whether $H_0(z)$ has linear phase or not.

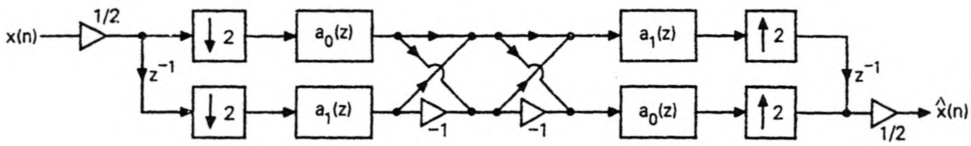
TABLE 5.10.2 IIR allpass based QMF banks

In what follows, the analysis and synthesis filters are related as $H_1(z) = H_0(-z), F_0(z) = H_0(z), F_1(z) = -H_1(z)$, so that aliasing is canceled.

1. **Power symmetric filters.**

   a) $H_0(z)$ is said to be power symmetric if $\widetilde{H}_0(z)H_0(z)$ is a half-band filter, i.e., $\widetilde{H}_0(z)H_0(z) + \widetilde{H}_0(-z)H_0(-z) = \beta$ for some nonzero constant β.

   b) Under some mild conditions (Theorem 5.3.1), an IIR power symmetric filter can be written as $H_0(z) = 0.5[a_0(z^2) + z^{-1}a_1(z^2)]$, where $a_0(z), a_1(z)$ are real-coefficient allpass. Then the QMF bank can be implemented as shown below.



   The distortion function is $T(z) = \frac{1}{2}z^{-1}a_0(z^2)a_1(z^2) =$ allpass, so that the QMF bank is free from AMD. Only PHD is still present, since aliasing has already been canceled.

2. **Power symmetric elliptic filters (Table 5.3.1 has design algorithm).**

   a) *Major fact.* If $H_0(z)$ is elliptic lowpass with ripples related as $\delta_2^2 = 4\delta_1(1-\delta_1)$ and band edges related as $\omega_p + \omega_S = \pi$, then it is power symmetric.

   b) *Low complexity.* If in addition the order N is odd, it can be expressed as $H_0(z) = [a_0(z^2) + z^{-1}a_1(z^2)]/2$, and the QMF bank implemented as above. Here $a_0(z), a_1(z)$ are real-coefficient allpass. The analysis bank requires only $0.25(N-1)$ MPUs and $0.5(N+1)$ APUs.

   c) *Pole locations.* Power symmetric elliptic filters have all poles on the imaginary axis. So the denominator has the form $D(z) = d(z^2)$.

TABLE 5.10.3 FIR power symmetric QMF banks

Basic result. (Section 5.3.6). Let $H_0(z) = \sum_{n=0}^{N} h_0(n)z^{-n}$ be power symmetric, that is, $\widetilde{H}_0(z)H_0(z)$ is a half-band filter, that is,

$$\widetilde{H}_0(z)H_0(z) + \widetilde{H}_0(-z)H_0(-z) = \beta$$

for some nonzero constant β. Then N is automatically odd (assuming that $h_0(0) \neq 0$ and $h_0(N) \neq 0$). Let the filters $H_1(z), F_0(z)$ and $F_1(z)$ be chosen as

$$H_1(z) = -z^{-N}\widetilde{H}_0(-z), \ F_0(z) = z^{-N}\widetilde{H}_0(z), \ F_1(z) = z^{-N}\widetilde{H}_1(z).$$

Then the two channel QMF bank has *perfect reconstruction.* All the filters are FIR and have same order N. Efficient lattice structures for this system will be presented in Section 6.4.

Design procedure. It only remains to design $H_0(z)$. This can be done by first designing a zero-phase FIR half-band filter $H(z)$ with $H(e^{j\omega}) \geq 0$ and taking $H_0(z)$ to be a spectral factor. See Section 5.3.6 for more details.

TABLE 5.10.4 Facts about M-channel QMF banks

Fig. 5.4-1 represents an M channel QMF bank. The reconstructed signal $\widehat{X}(z)$ is given by $\widehat{X}(z) = T(z)X(z) + \sum_{\ell=1}^{M-1} A_\ell(z)X(zW^\ell)$. This is a linear and time varying system. The terms $X(zW^\ell), \ell > 0$ are the alias terms. The system is free from aliasing if $A_\ell(z) = 0$ for $\ell > 0$. Under such condition, the QMF bank becomes a linear time invariant (LTI) system with transfer function $T(z) = \sum_{k=0}^{M-1} H_k(z)F_k(z)/M$, called the *distortion* function.

Any M-channel QMF bank can be redrawn in terms of the polyphase component matrices $\mathbf{E}(z)$ and $\mathbf{R}(z)$ (Fig. 5.5-3(a),(b)). This in turn can be redrawn in terms of a $M \times M$ matrix $\mathbf{P}(z) = \mathbf{R}(z)\mathbf{E}(z)$ (Fig. 5.5-3(c)).

1. The QMF bank is alias-free if and only if $\mathbf{P}(z)$ is a pseudocirculant (demonstrated in (5.7.11) for $M = 3$.)
2. Under this alias-free condition, the QMF bank is an LTI system with transfer function $T(z) = z^{-(M-1)} \sum_{k=0}^{M-1} z^{-k} P_k(z^M)$.
3. An alias-free system is free from amplitude distortion (i.e., $T(z)$ is stable allpass) if and only if $\mathbf{P}(z)$ is a lossless matrix. (To be proved later in Section 10.1.)
4. An alias-free system has perfect reconstruction if $T(z)$ is a delay, i.e., $T(z) = cz^{-n_0}$. This happens if and only if the pseudocirculant $\mathbf{P}(z)$ has the special form (5.7.23b). The most common special case has $r = 0$ so that $\mathbf{R}(z)\mathbf{E}(z) = cz^{-m_0}\mathbf{I}$, i.e.,

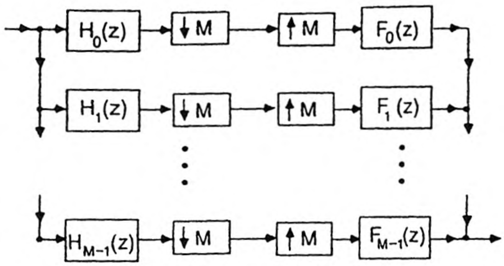
$$\mathbf{R}(z) = cz^{-m_0}\mathbf{E}^{-1}(z).$$

   Given a perfect reconstrucion (PR) QMF bank satisfying (5.7.23b) for some r in the range $0 \le r \le M-1$, we can obtain a PR QMF bank with a different value of r just by replacing the synthesis filters $F_k(z)$ with $z^{-m}F_k(z)$ for appropriate integer m.
5. An FIR QMF bank is one for which $H_k(z)$ as well as $F_k(z)$ are FIR. If such a system has PR property then

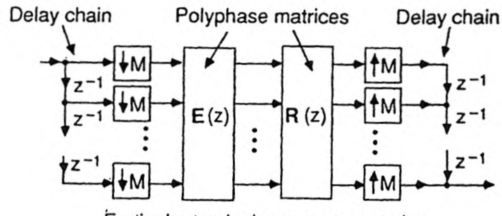
$$\det \mathbf{E}(z) = \alpha_0 z^{-K_0}, \quad \text{and} \quad \det \mathbf{R}(z) = \alpha_1 z^{-K_1}$$

6. *Special case where* $\mathbf{P}(z)$ *is diagonal.* A multirate system of the form shown in Fig. 5.7-1(a) is alias-free if and only if $S_k(z)$ is same for all k. Letting $S_k(z) = S(z)$ we then have $T(z) = z^{-(M-1)}S(z^M)$.

TABLE 5.10.5 Matrix notations in filter bank theory



Maximally decimated filter bank



Equivalent polyphase representation

$$\underbrace{\begin{bmatrix} H_0(z) \\ \vdots \\ H_{M-1}(z) \end{bmatrix}}_{\mathbf{h}(z)} = \underbrace{\begin{bmatrix} E_{00}(z^M) & E_{01}(z^M) & \cdots & E_{0,M-1}(z^M) \\ \vdots & \vdots & \ddots & \vdots \\ E_{M-1,0}(z^M) & E_{M-1,1}(z^M) & \cdots & E_{M-1,M-1}(z^M) \end{bmatrix}}_{\mathbf{E}(z^M)} \underbrace{\begin{bmatrix} 1 \\ z^{-1} \\ \vdots \\ z^{-(M-1)} \end{bmatrix}}_{\mathbf{e}(z)}$$

$$\underbrace{[F_0(z) \quad \ldots \quad F_{M-1}(z)]}_{\mathbf{f}^T(z)} =$$

$$\underbrace{[z^{-(M-1)} \quad z^{-(M-2)} \quad \ldots \quad 1]}_{z^{-(M-1)}\widetilde{\mathbf{e}}(z)} \underbrace{\begin{bmatrix} R_{00}(z^M) & \cdots & R_{0,M-1}(z^M) \\ R_{10}(z^M) & \cdots & R_{1,M-1}(z^M) \\ \vdots & \ddots & \vdots \\ R_{M-1,0}(z^M) & \cdots & R_{M-1,M-1}(z^M) \end{bmatrix}}_{\mathbf{R}(z^M)}$$

$$\text{AC matrix } \mathbf{H}(z) = \begin{bmatrix} H_0(z) & H_1(z) & \cdots & H_{M-1}(z) \\ H_0(zW) & H_1(zW) & \cdots & H_{M-1}(zW) \\ \vdots & \vdots & \ddots & \vdots \\ H_0(zW^{M-1}) & H_1(zW^{M-1}) & \cdots & H_{M-1}(zW^{M-1}) \end{bmatrix}$$

$$\text{The matrix } \mathbf{F}(z) = \begin{bmatrix} F_0(z) & F_1(z) & \cdots & F_{M-1}(z) \\ F_0(zW) & F_1(zW) & \cdots & F_{M-1}(zW) \\ \vdots & \vdots & \ddots & \vdots \\ F_0(zW^{M-1}) & F_1(zW^{M-1}) & \cdots & F_{M-1}(zW^{M-1}) \end{bmatrix}$$

$$\mathbf{H}(z) = \mathbf{W}^\dagger \mathcal{D}(z)\mathbf{E}^T(z^M) \quad \text{(Sec. 5.5)}$$

$$\mathbf{F}(z) = \mathbf{\Gamma W \Lambda}(z)\mathbf{R}(z^M) \quad \text{(see Sec. 11.4.3 later)}$$

PROBLEMS

5.1. Suppose the analysis filters in a two-channel QMF bank (Fig. 5.1-1(a)) are given by

$$H_0(z) = 2 + 6z^{-1} + z^{-2} + 5z^{-3} + z^{-5}, \quad H_1(z) = H_0(-z).$$

Find a set of stable synthesis filters that result in perfect reconstruction.

5.2. In Sec. 5.2 we considered QMF banks in which the filters are related as in (5.2.1) and (5.2.2). We saw that with $H_0(z)$ chosen to have real coefficients and linear phase, the distortion function is given by (5.2.10). If N is even this implies $T(e^{j\pi/2}) = 0$ so that the filter order N has to be odd. Now consider the modified QMF bank shown below where the filters could be FIR or IIR [Galand and Nussbaumer, 1984].

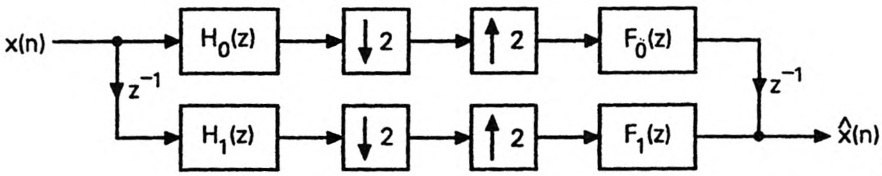


Figure P5-2

Express $\widehat{X}(z)$ in terms of $X(z)$. With $H_1(z) = H_0(-z)$, show that the choice $F_0(z) = H_0(z)$ and $F_1(z) = H_1(z)$ cancels aliasing. With this choice write down the distortion $T(z)$ in terms of $H_0(z)$.

a) Now let $H_0(z)$ be a real coefficient linear phase FIR lowpass filter of order N. Simplify $T(z)$ and show that there is no phase distortion. Also show that N has to be *even,* in order to avoid the condition $T(e^{j\pi/2}) = 0$.

b) For the system in part (a) with N even, what is the number of MPUs required to implement the analysis bank? (Try to exploit as many of the following facts as you can: (i) the relation $H_1(z) = H_0(-z)$, (ii) the linear phase property, and (iii) the presence of decimators). How does this compare with the numbers we obtained for the case of Fig. 5.1-1(a) with odd N?

5.3. Consider Fig. 5.2-2(b). Here the analysis filters are related as $H_1(z) = H_0(-z)$. Assuming that $H_0(z)$ is a real coefficient Nth order filter (N odd), we know that the analysis bank requires $0.5(N+1)$ MPUs. This implementation uses two facts, namely that the coefficients of $H_1(z)$ are related to those of $H_0(z)$, and that we decimate the filter outputs. Curiously enough, we have not used the fact that $H_0(z)$ has linear phase, i.e., $h_0(n) = h_0(N-n)$. At first sight it appears that there should be some way to reduce complexity further by exploiting this relation. This, however, is not true.

a) Prove that linear-phase of $H_0(z)$ implies that $E_0(z)$ is the Hermitian image of $E_1(z)$.

b) In view of Problem 4.17, it appears therefore that we can share the multipliers between $E_0(z)$ and $E_1(z)$. This, however, is not true. To see this, consider the decimated outputs $v_0(n)$ and $v_1(n)$ (Fig. 5.1-1). Show that if the product $x(i)h_0(j)$ is computed in the process of evaluating $v_0(n)$, this same product is never computed in the evaluation of $v_1(m)$ for any choice of n, m (unless the input sequence $x(n)$ is restricted to have special values). (*Note.* Another way to look at this is as follows. The analysis bank requires the implementation of the two systems $E_0(z) + E_1(z)$ and $E_0(z) - E_1(z)$. Each of these resembles Fig. P4-17(b), and can therefore be implemented with $(N+1)/2$ multipliers each. If each of these systems is implemented this way, we cannot share the multipliers in $E_0(z)+E_1(z)$ with those in $E_0(z) - E_1(z)$. This is because the multiplier α cannot be shared between $(x+y)\alpha$ and $(x-y)\alpha$.)

c) What is the story if N is even? Can we exploit linear phase property of $H_0(z)$ to implement the analysis bank with only about $0.25(N+1)$ MPUs? Explain.

5.4. For the case of an odd order power symmetric BR transfer function we presented a result which showed that it can be expressed as in (5.2.16) so that the polyphase components are allpass (Theorem 5.3.1). For the even order case, the situation is somewhat different. Suppose for example that $H_0(z)$ is an IIR elliptic lowpass filter with even order $N > 0$. It can be shown [Vaidyanathan, et al., 1987] that this can be expressed as

$$H_0(z) = \frac{A(z) + A_*(z)}{2} \qquad (P5.4a)$$

where $A(z)$ is a unit-magnitude allpass function of order $N/2$ with complex coefficients, and $A_*(z)$ is obtained by conjugating the coefficients of $A(z)$. (You can accept this as a fact for this Problem). Suppose now that $H_0(z)$ is, in addition, power symmetric. Define the new real coefficient transfer function

$$H_1(z) = \frac{A(z) - A_*(z)}{2j} \qquad (P5.4b)$$

a) Show that $\widetilde{H}_0(z)H_0(z) + \widetilde{H}_1(z)H_1(z) = 1$.

b) Show that $H_1(z) = \widehat{c}H_0(-z)$ where $\widehat{c} = \pm 1$.

c) Let $E_0(z)$ and $E_1(z)$ be the polyphase components of $H_0(z)$, i.e., $H_0(z) = E_0(z^2) + z^{-1}E_1(z^2)$. Show that $z^{-i}E_i(z^2) = c_i A(z) + c_i^* A_*(z)$ for some c_i.

d) Hence show that $E_i(z)$ cannot be allpass (*Hint.* Use Problem 3.21.)

The application of this problem in QMF banks is considered in Problem 5.6.

5.5. Let $H_0(z)$ be as in Problem 5.4.

a) Show that $A_*(z) = \pm jA(-z)$.

b) Show that $A(z)$ has the form

$$A(z) = c\prod_{k=1}^{M}\left(\frac{-j\beta_k + z^{-1}}{1 + j\beta_k z^{-1}}\right) \qquad (P5.5)$$

where $c = e^{\pm j\pi/4}$ and $-1 < \beta_k < 1$.

5.6. Consider the system shown in Fig. P5-2 again. Suppose that the analysis filters are related as $H_1(z) = H_0(-z)$ and let the synthesis filters be chosen as in Problem 5.2 to eliminate aliasing. Assume, however, that $H_0(z)$ is an elliptic power symmetric filter with even order $N > 0$. We know from Problem 5.4 that the polyphase components are not allpass.

a) Find the distortion function $T(z)$ and show that amplitude distortion has actually been eliminated.

b) Draw the complete QMF bank in terms of $A(z)$. In your scheme, what is the number of MPUs required to implement the analysis bank? (*Note.* When the multipliers are complex, you must carefully count the number of real MPUs).

5.7. Let $H_0(z)$ be as in (5.2.16) where $a_0(z)$ and $a_1(z)$ are allpass filters as in (5.3.6). We can then write $H_0(z) = P_0(z)/d_0(z^2)d_1(z^2)$ where $P_0(z)$ is a polynomial in z^{-1}. Assume that (i) $d_0(z)$ and $d_1(z)$ have all poles inside the unit circle, and (ii) $d_0(z)$ and $d_1(z)$ have no common factors of order ≥ 1. Show that there are no common factors (of order ≥ 1) between $P_0(z)$ and the denominator $d_0(z^2)d_1(z^2)$.

5.8. Let $H_0(z)$ be a digital filter obtained from an analog Butterworth filter $H_a(s)$ (Sec. 3.3.2), using the bilinear transform (3.3.1). Suppose the Butterworth filter has 3dB point $\Omega_c = 1$. Show then that $H_0(z)$ is power symmetric.

5.9. Consider a QMF bank with analysis filters related by $H_1(z) = H_0(-z)$ so that (5.2.5) holds. $H_0(z)$ and $H_1(z)$ could be FIR or IIR, but assume that they are stable.

a) Assume that the polyphase components $E_0(z)$ and $E_1(z)$ have all zeros *outside* the unit circle (the poles, of course, are inside). Find a set of stable synthesis filters so that aliasing as well as amplitude distortion are eliminated.

b) Repeat (a) under the condition that $E_0(z)$ and $E_1(z)$ have some zeros inside and some outside (but none *on*) the unit circle.

5.10. Let $H_0(z) = P_0(z)/D(z)$ and $H_1(z) = P_1(z)/D(z)$ with

$$P_k(z) = \sum_{n=0}^{N} p_k(n)z^{-n}, \quad D(z) = 1 + \sum_{n=1}^{N} d(n)z^{-n}. \qquad (P5.10)$$

Assume $D(z)$ has all zeros inside the unit circle. Suppose the following conditions are true: (i) $|H_0(e^{j\omega})|^2 + |H_1(e^{j\omega})|^2 = 1$ for all ω. (ii) $P_0(z)$ is Hermitian and $P_1(z)$ is skew Hermitian. Show that we can express these transfer functions as $H_0(z) = [A_0(z) + A_1(z)]/2$ and $H_1(z) = [A_0(z) - A_1(z)]/2$ where $A_0(z)$ and $A_1(z)$ are stable unit-magnitude allpass.

5.11. Let $H_0(z)$ be a causal stable rational transfer function with $|H_0(e^{j\omega})| \leq 1$, with irreducible representation $H_0(z) = P_0(z)/D_0(z)$. Assume that if α is a zero of $D_0(z)$ then $1/\alpha^*$ cannot be a zero of $P_0(z)$. This means that there are no nontrivial allpass factors in $H_0(z)$.

a) Let $H_1(z)$ be a causal stable system such that $|H_1(e^{j\omega})|^2 + |H_0(e^{j\omega})|^2 = 1$, and let $H_1(z) = P_1(z)/D_1(z)$ be an irreducible representation. Assume that $H_1(z)$ has no nontrivial allpass factors. Show that $D_1(z) = cD_0(z)$ for some constant c.

b) Assume now that $H_0(z)$ above is power symmetric. Show that its denominator can be written in the form $D_0(z) = 1 + d(2)z^{-2} + \ldots d(2K)z^{-2K}$. That is, $D_0(z) = G(z^2)$ for some FIR $G(z)$.

5.12. Let $H_0(z)$ be a stable IIR power symmetric transfer function with possibly complex coefficients. Let $H_0(z) = P_0(z)/D(z)$ be an irreducible representation, where

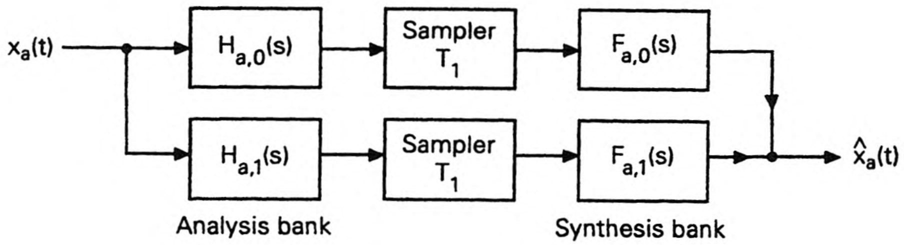
$$P_0(z) = \sum_{n=0}^{N} p_0(n) z^{-n}, \quad D(z) = 1 + \sum_{n=1}^{N} d(n) z^{-n}, \qquad (P5.12a)$$

with none of $d(N), p_0(0), p_0(N)$ equal to zero. Assume that there are no allpass factors (of order > 0) in $H_0(z)$. Suppose N is odd and $P_0(z)$ Hermitian. Prove that $H_0(z)$ can be expressed as

$$H_0(z) = \frac{a_0(z^2) + z^{-1} a_1(z^2)}{2} \qquad (P5.12b)$$

where $a_0(z)$ and $a_1(z)$ are stable (rational) unit-magnitude allpass functions. Thus, the polyphase components are allpass. (*Hint.* use Problem 5.10.)

5.13. *Analog QMF bank.* In this problem we consider an extension of the maximally decimated filter bank system for the case where the input is a continuous-time signal. Consider the following 'two-channel filter bank' system, where $x_a(t)$ is a continuous-time signal with Laplace transform $X_a(s)$.



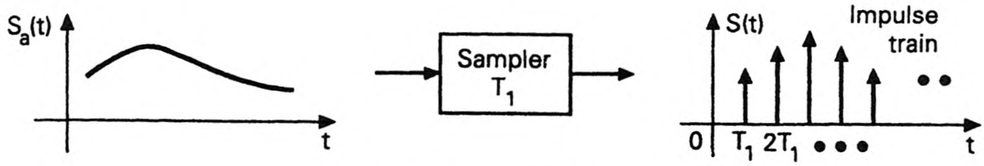


Figure P5-13(a),(b)

The device labeled "sampler" operates as follows: in response to a continuous time input $s_a(t)$, it produces the sampled version

$$s(t) = \sum_{n=-\infty}^{\infty} s_a(nT_1) \delta_a(t - nT_1),$$

where $\delta_a(.)$ is the Dirac delta function (Sec. 2.3). This is illustrated in Fig. P5-13(b).

a) Express $\widehat{X}_a(j\Omega)$ in terms of $X_a(j\Omega)$, and the various filter transfer functions. You will see that this expression is a linear combination of an infinite number of terms of the form

$$X_a(j\Omega - j\frac{2\pi m}{T_1}), \quad -\infty \leq m \leq \infty.$$

b) Suppose $x_a(t)$ is a σ-bandlimited signal, that is, $X_a(j\Omega) = 0$ for $|\Omega| \geq \sigma$. (Assume $0 < \sigma < \pi$.) We know that the Nyquist sampling rate is $\Theta = 2\sigma$. (If we sample $x_a(t)$ at this rate, we can recover $x_a(t)$ from the samples with the help of a ideal lowpass filter with passband $-\sigma < \Omega < \sigma$.) Define the corresponding Nyquist sampling period $T = 2\pi/\Theta$. Suppose $T_1 = 2T$, that is, each channel in the figure performs sampling at *half* the Nyquist rate (so that the total number of samples per unit time, counting both channels, corresponds to Nyquist sampling). Show that, for any fixed frequency Ω, only *two* out of the infinite number of terms in the expression for $\widehat{X}_a(j\Omega)$ can be nonzero.

c) The aim is to choose the synthesis filters such that aliasing and other distortions are eliminated. Continuing with part (b), assume that the synthesis filters satisfy $F_{a,k}(j\Omega) = 0$ for $|\Omega| \geq \sigma$. Show that we can obtain perfect reconstruction (that is, $\widehat{x}_a(t) = x_a(t)$) by solving for $F_{a,k}(j\Omega)$ from the equations

$$\begin{bmatrix} H_{a,0}(j\Omega) & H_{a,1}(j\Omega) \\ H_{a,0}(j\Omega + j\sigma) & H_{a,1}(j\Omega + j\sigma) \end{bmatrix} \begin{bmatrix} F_{a,0}(j\Omega) \\ F_{a,1}(j\Omega) \end{bmatrix} = \begin{bmatrix} T_1 \\ 0 \end{bmatrix}, \qquad (P5.13a)$$

for $-\sigma < \Omega \leq 0$, and

$$\begin{bmatrix} H_{a,0}(j\Omega) & H_{a,1}(j\Omega) \\ H_{a,0}(j\Omega - j\sigma) & H_{a,1}(j\Omega - j\sigma) \end{bmatrix} \begin{bmatrix} F_{a,0}(j\Omega) \\ F_{a,1}(j\Omega) \end{bmatrix} = \begin{bmatrix} T_1 \\ 0 \end{bmatrix}, \qquad (P5.13b)$$

for $0 \leq \Omega < \sigma$. In other words, given the analysis filters $H_{a,0}(s)$ and $H_{a,1}(s)$, we can solve for the frequency responses of the synthesis filters from the above equations. The sets of equations to be used depends on the frequency region as indicated. (Outside this frequency region we just take $F_{a,k}(j\Omega) = 0$.) (*Note.* This idea works as long as the 2×2 matrices in the equations above are nonsingular, but the resulting synthesis filters, in general, are not guaranteed to be stable or realizable!)

d) Continuing with part (c), let $H_{a,0}(s) = 1$, and let $H_{a,1}(s) = s$ (i.e., a differentiator). Verify that the matrices above are nonsingular. So we can indeed find synthesis filters for perfect reconstruction. Find expressions for $F_{a,0}(j\Omega)$ and $F_{a,1}(j\Omega)$ in the above frequency regions. Show that these synthesis filters have impulse responses

$$f_{a,0}(t) = 4\sin^2(\sigma t/2)/\sigma^2 t^2, \quad f_{a,1}(t) = 4\sin^2(\sigma t/2)/\sigma^2 t. \qquad (P5.13c)$$

Evidently these are noncausal (unrealizable) continuous-time filters.

Note: The above scheme gives rise to a number of generalizations to Nyquist sampling theorem. If $x_a(t)$ is σ-bandlimited, Nyquist theorem says that we can reconstruct it (by lowpass filtering) from its samples uniformly spaced apart by T seconds. According to above scheme, we can split $x_a(t)$ into two signals and sample each at *half the rate*, and still reconstruct the original version. In part (d) we are essentially sampling $x_a(t)$ and its derivative (output of $H_{a,1}(s)$) at half the Nyquist rate. We can recover $x_a(t)$ from these two undersampled signals by using the filters in (P5.13c). This gives a proof of the *derivative sampling theorem,* originally proposed in [Shannon, 1949] four decades ago. More generally, if we consider the M-channel version of this problem we will find that we can recover a bandlimited signal by sampling it and its $M-1$ derivatives M times slower than the Nyquist rate. As part (d) shows, the filters required to do this reconstruction are, as such, unrealizable. (In fact the ideal lowpass filter which is used to reconstruct a bandlimited signal from its 'traditional Nyquist-rate samples' is also unstable and noncausal.) These filters should therefore be replaced with practical approximations.

5.14. Given below are three sets of FIR analysis banks for a 3 channel maximally decimated QMF bank (Fig. 5.4-1, with $M = 3$). In each case, answer the following: (i) Is it possible to obtain a set of FIR synthesis filters for perfect reconstruction? If so find them. (ii) If not, find a set of IIR synthesis filters for perfect reconstruction. (iii) In the latter event, are the synthesis filters stable?

a) $H_0(z) = 1,\ H_1(z) = 2 + z^{-1},\ H_2(z) = 3 + 2z^{-1} + z^{-2}$.
b) $H_0(z) = 1,\ H_1(z) = 2 + z^{-1} + z^{-5},\ H_2(z) = 3 + 2z^{-1} + z^{-2}$.
c) $H_0(z) = 1,\ H_1(z) = 2 + z^{-1} + z^{-5},\ H_2(z) = 3 + z^{-1} + 2z^{-2}$.

5.15. Prove that the structure of Fig. 5.6-3 has perfect reconstruction property if and only if the integers M and J are relatively prime. Under this condition, find $\widehat{x}(n)$ in terms of $x(n), M$, and J.

5.16. Consider the $M \times M$ matrix

$$\begin{bmatrix} \mathbf{0} & \mathbf{I}_{M-r} \\ z^{-1}\mathbf{I}_r & \mathbf{0} \end{bmatrix}. \qquad (P5.16)$$

Show that its determinant is of the form $\pm z^{-r}$. This shows that if the product $\mathbf{P}(z) = \mathbf{R}(z)\mathbf{E}(z)$ takes the form (5.6.7), then its determinant is a delay, that is, has the form (5.6.9).

5.17. Suppose the filter bank of Fig. 5.4-1 is alias-free, and let $T(z)$ be the distortion function. Suppose we define a new filter bank in which the analysis and synthesis filters are interchanged, that is, $F_k(z)$ are the analysis filters and $H_k(z)$ the synthesis filters. Show that the resulting system is free from aliasing and has the same distortion function $T(z)$. So we can swap each $F_k(z)$ with corresponding $H_k(z)$, without changing these input/output properties! (*Hint.* Use AC matrix formulation cleverly.)

5.18. Consider the M channel maximally decimated system of Fig. 5.4-1. Let the choice of filters be such that this is a perfect reconstruction system. Suppose we replace each synthesis filters $F_k(z)$ with $F_k(zW^{\ell})$, where $W = e^{-j2\pi/M}$, and ℓ is an integer (independent of k) with $0 \le \ell \le M-1$. Let $\widehat{x}_1(n)$ be the new output of the QMF bank. How is it related to the input $x(n)$? Given $\widehat{x}_1(n)$, would you be able to recover $x(n)$? If so, how?

5.19. Consider the two channel QMF bank with analysis filters related as $H_1(z) = H_0(-z)$. Suppose the synthesis filters are chosen as $F_0(z) = H_0(z)$ and $F_1(z) = -H_1(z)$, so that aliasing is canceled.

a) Write down the AC matrix $\mathbf{H}(z)$, and express its determinant in terms of $H_0(z)$.

b) Show that the distortion function $T(z)$ is zero for some z if and only if $\mathbf{H}(z)$ is singular (i.e., the determinant equals zero) for this value of z.

c) Suppose $H_0(z) = \sum_{n=0}^{N} h(n)z^{-n}$. Let N be even and let $h(n)$ be real with $h(n) = h(N-n)$ (Type 1 linear phase). Show that $\mathbf{H}(z)$ is singular for $z = e^{j\pi/2}$. In view of part (b) this proves that $T(e^{j\pi/2}) = 0$, a conclusion we already know from Sec. 5.2.2.

5.20. Consider the following uniform DFT analysis bank,

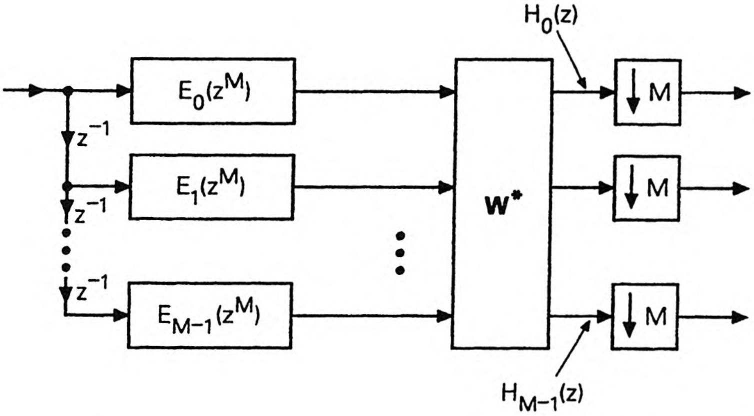


Figure P5-20

where $E_k(z)$ are stable allpass functions with $|E_k(e^{j\omega})| = 1$ for all ω. (Evidently, these are the polyphase components of $H_0(z)$.)

a) Show that the anlaysis filters are power complementary.

b) Show that each analysis filter is a spectral factor of an Mth band filter, that is, show that $\widetilde{H}_k(z)H_k(z)$ satisfies the Mth band property.

c) Draw a (stable) synthesis bank structure so that (i) aliasing is canceled, and (ii) $T(z)$ becomes allpass thereby eliminating amplitude distortion.

5.21. Consider Fig. 5.7-2(c) with $\mathbf{T} = \mathbf{W}^*$ (uniform DFT analysis bank). Suppose $R_k(z)$ are chosen as in (5.7.8), so that the product $R_k(z)E_k(z)$ is independent of k. This ensures that aliasing has been canceled.

a) Just as a review, verify that the uniform-shift relations $H_k(z) = H_0(zW^k)$ and $F_k(z) = W^{-k}F_0(zW^k)$ hold, where $W = e^{-j2\pi/M}$.

b) Express the distortion function $T(z)$ in terms of $E_k(z), 0 \le k \le M-1$.

c) Show that the AC matrix $\mathbf{H}(z)$ is a left circulant.

d) Find the determinant of $\mathbf{H}(z)$ in terms of $E_k(z)$. (Review of Sec. 5.5 helps here.) Show that this determinant is equal to $cz^{-K}T(z)$ where $c \neq 0$, and K is some integer. Thus $\mathbf{H}(e^{j\theta})$ is singular if and only if $T(e^{j\theta}) = 0$.

e) Suppose $H_0(z) = \sum_{n=0}^{N} h(n)z^{-n}$. Assume $h(n)$ is real with $h(n) = h(N-n)$ (linear phase FIR). This property imposes certain constraints on the polyphase components $E_k(z)$ (Problem 4.22). In particular, for some combinations of N and M, it is possible that some polyphase component $E_L(z)$ is an *odd* order filter with *symmetric* impulse response. This implies $E_L(e^{j\pi}) = 0$. Using part (b) prove that this implies $T(e^{j\omega}) = 0$ for $\omega = \pi/M, 3\pi/M, \ldots$ etc. To avoid this situation, the relative values of M and N must be carefully chosen. Explain how. (You can do it using the notation 'm_0' from Problem 4.22) (*Hint.* For $M = 2$, this should reduce to the requirement that N be odd, as seen in Sec. 5.2.2.)

5.22. In the above problem we took $R_k(z)$ according to (5.7.8). This has a disadvantage: each filter $R_k(z)$, which is a product of $M-1$ of the $E_\ell(z)$'s, can have very high order (for large M), so that the synthesis filters have high order. We can partially rectify this situation if we take a closer look at the form of $E_k(z)$. Thus, let

$$E_k(z) = \frac{N_{k,1}(z)N_{k,2}(z)}{D_k(z)}. \qquad (P5.22a)$$

where $N_{k,1}(z), N_{k,2}(z)$ and $D_k(z)$ are polynomials in z^{-1}. Here $N_{k,2}(z)$ is the part with all zeros inside the unit circle, (and $N_{k,1}(z)$ has zeros on and/or outside).

a) Show that the choice

$$R_k(z) = \frac{D_k(z)\prod_{\ell\neq k} N_{\ell,1}(z)}{N_{k,2}(z)}. \qquad (P5.22b)$$

cancels aliasing. (Note that this choice gives stable synthesis filters.)

b) With such choice of $R_k(z)$, what is the distortion function $T(z)$?

c) Making the further assumption that $N_{k,1}(z)$ has no zeros *on* the unit circle for any k, how would you modify $R_k(z)$ [without destroying stability of $R_k(z)$, and the alias-free property] so that $T(z)$ now becomes allpass (thereby eliminating amplitude distortion)?

5.23. Consider the following M channel multirate system, which is essentially a QMF bank with the additional transfer functions $C_k(z)$ inserted.

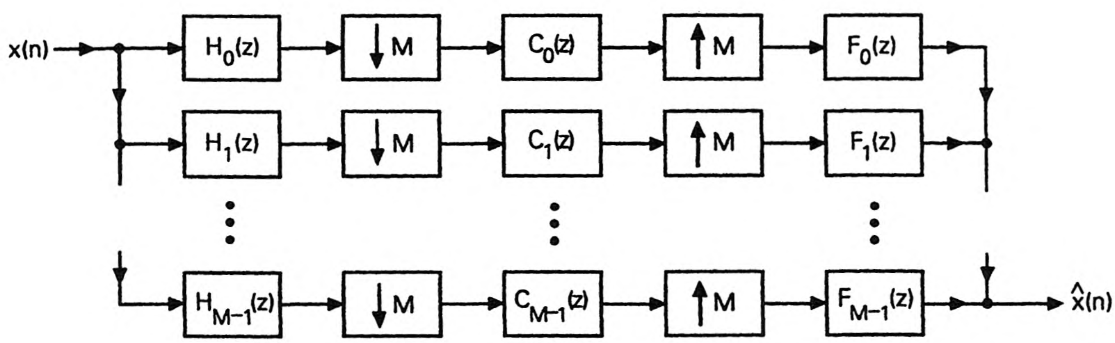


Figure P5-23

We can imagine that $C_k(z)$ represents the amplitude and phase distortions introduced by the kth channel. Assume throughout that the functions $F_k(z)$,

$H_k(z)$ and $C_k(z)$ are rational and stable; unless stated otherwise, do not make specific assumptions about *zeros* of these transfer functions.

a) Suppose $H_k(z)$ and $F_k(z)$ are such that the system is alias-free in absence of channel distortion $C_k(z)$ (i.e., with $C_k(z)$ replaced with unity for all k). Now with $C_k(z)$ present, find a modified set of (stable) synthesis filters $G_k(z)$ to retain alias-free property.

b) Repeat part (a) by replacing "alias-free" with "alias-free and free from amplitude distortion" everywhere. Assume, however, that $C_k(z)$ has no zeros on the unit circle. (*Hint.* First write the numerator of $C_k(z)$ as $A_k(z)B_k(z)$ where $A_k(z)$ has all zeros inside the unit circle and $B_k(z)$ has them outside.)

c) Repeat part (a) by replacing "alias-free" with "perfect reconstruction" everywhere. Assume now that $C_k(z)$ has no zeros on or outside the unit circle.

Hint. This is not tedious, but you have to think straight!

5.24. Consider the tree structure shown in Fig. 5.8-1(a).

a) Show that the complete system is equivalent to Fig. 5.4-1 with $M = 4$ (four-band QMF bank) and identify $H_k(z)$ and $F_k(z)$ for $0 \leq k \leq 3$, in terms of the filters in Figure 5.8-1(a).

b) Assume that the two-channel QMF bank with filters $H_0^{(1)}(z)$, $H_1^{(1)}(z)$, $F_0^{(1)}(z)$, and $F_1^{(1)}(z)$ is alias-free with distortion function $T^{(1)}(z)$. Let the same be true of $H_0^{(2)}(z), H_1^{(2)}(z), F_0^{(2)}(z)$, and $F_1^{(2)}(z)$, with distortion function $T^{(2)}(z)$. Prove that the equivalent four-band QMF bank is alias-free, and find its distortion function $T(z)$ in terms of $T^{(1)}(z)$ and $T^{(2)}(z)$.

c) Continuing with part (b), prove that $T(z)$ is allpass if $T^{(1)}(z)$ and $T^{(2)}(z)$ are allpass. Thus, if each two-channel QMF bank is free from amplitude distortion, then so is the overall four-channel system. Similarly verify that $T(z)$ has linear phase if $T^{(1)}(z)$ and $T^{(2)}(z)$ have linear phase.

d) If each two-channel QMF bank

$$H_0^{(k)}(z),\ H_1^{(k)}(z),\ F_0^{(k)}(z),\ F_1^{(k)}(z), \quad k = 1, 2,$$

is a perfect reconstruction system, verify that the same is true for the equivalent four-channel system.

Note. These results can be extended to tree structures with more than two (say m) levels. We can thus build QMF banks with $M = 2^m$, with any set of desired properties (such as freedom from selected set of distortions etc.), including perfect reconstruction. For composite M which is not a power of two, the idea can be extended. Thus if $M = 3 \times 2$, we can build the QMF bank in terms of two-channel systems and three channel systems. So tree structures cover a wide class of useful filter banks.

5.25. Tree structures can be used to obtain QMF banks in which the decimation ratio is not the same for all channels (called nonuniform filter banks). Consider the system shown in Fig. P5-25(a).

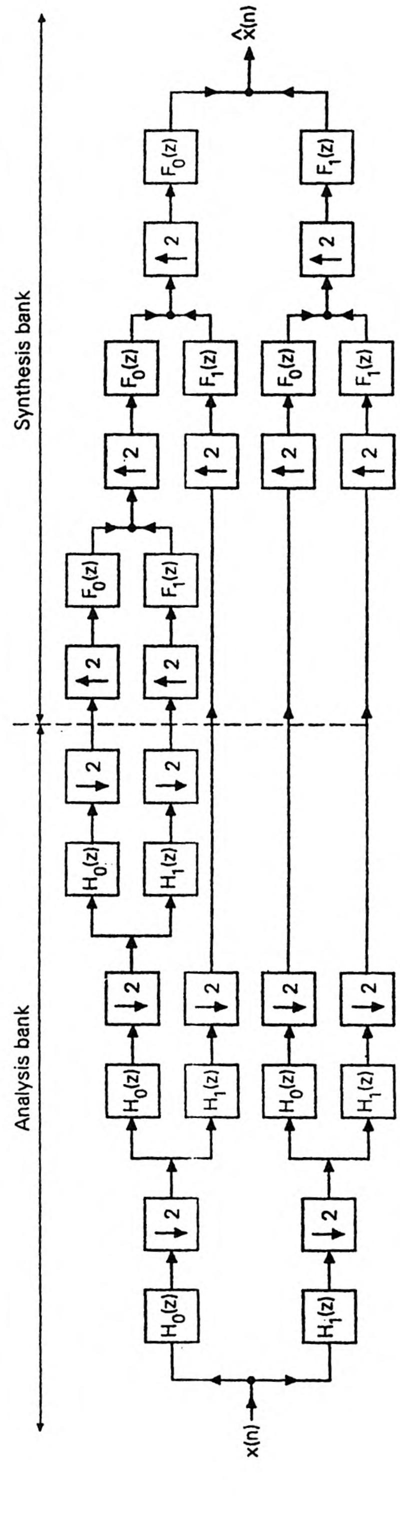


Figure P5-25(a)

This is equivalent to the following five channel system.

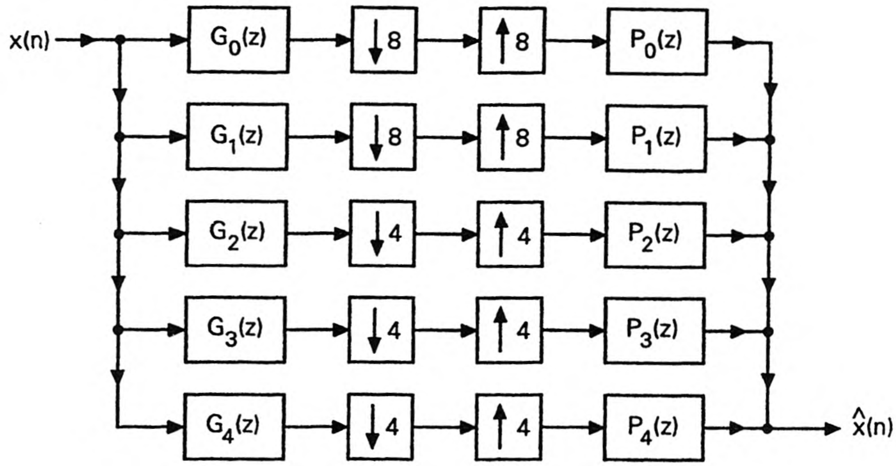


Figure P5-25(b)

a) Identify the filters $G_k(z)$ and $P_k(z)$ in terms of the filters $H_0(z)$, $H_1(z)$, $F_0(z)$ and $F_1(z)$. Suppose $H_0(z)$ and $H_1(z)$ are real coefficient filters with magnitude responses as shown below.

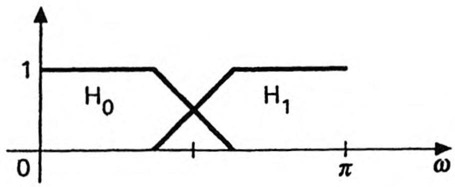


Figure P5-25(c)

Sketch the magnitude responses $|G_k(e^{j\omega})|$ for $0 \leq k \leq 4$. Thus, the filters have unequal bandwidths, and the decimation ratios are inversely proportional to these bandwidths. This is called a nonuniform (maximally decimated) QMF bank.

b) Suppose the filters $H_0(z), H_1(z), F_0(z)$ and $F_1(z)$ are such that the traditional two channel QMF bank (Fig. 5.1-1(a)), has perfect reconstruction property, with distortion function $T(z) = 1$. Show that the five-channel nonuniform system also has perfect reconstruction property.

c) Suppose the filters $H_0(z), H_1(z), F_0(z)$ and $F_1(z)$ are such that the traditional two channel QMF bank is alias-free, with distortion function $T(z)$. Does the above five channel nonuniform system remain alias-free? If not, how would you modify the structure of Fig. P5-25(a) to obtain this property, and what is the resulting distortion function?

5.26. Consider a transmultiplexer with $M = 2$.

a) Let the analysis filters be $H_0(z) = 1 + z^{-1}$ and $H_1(z) = 1 - z^{-1}$. Find a set of FIR synthesis filters $F_0(z)$ and $F_1(z)$ such that the system has perfect reconstruction property.

b) Let the analysis filters be $H_0(z) = 1 + z^{-1}$ and $H_1(z) = 1 + z^{-1} + z^{-2}$. Can we find FIR synthesis filters such that there is perfect reconstruction? If so find them.

5.27. Consider a three-channel transmultiplexer with *synthesis* filters

$$F_0(z) = 1, \ F_1(z) = 2 + z^{-1}, \ F_2(z) = 3 + 2z^{-1} + z^{-2}.$$

Find a set of FIR analysis filters such that perfect reconstruction property is satisfied.

5.28. Suppose we have a three channel alias-free QMF bank with distortion function $T(z) = z^{-2}/(1 - az^{-1})$. Find closed form expressions for the elements of the 3×3 matrix $\mathbf{P}(z)$ [Fig. 5.5-3(c)] in terms of a, z.

5.29. Assuming that the matrix $\mathbf{P}(z)$ is pseudocirculant, verify that (5.7.22) indeed reduces to (5.7.13) (with $P_m(z)$ denoting $P_{0,m}(z)$).

5.30. Consider the following multirate system.

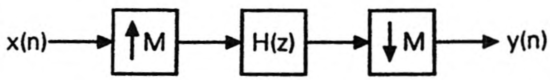


Figure P5-30

In each of the following cases, what can you say about the input output relation of the system? Give as much information as possible, based on given data.

a) $M = 2$ and $H(z)$ is an IIR power symmetric elliptic filter of odd order.

b) M is arbitrary, and $H(z)$ is a zero-phase Mth band lowpass filter.

5.31. Consider the following M-channel analysis/synthesis system.

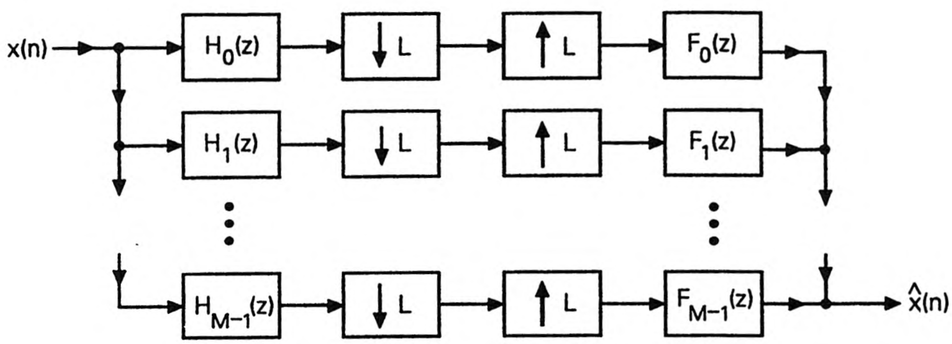


Figure P5-31(a)

This reduces to the QMF bank of Fig. 5.4-1 if $L = M$. If $L < M$, this is called a *nonmaximally* decimated QMF bank. With such a system, elimination of aliasing turns out to be relatively easy (as this exercise will demonstrate).

a) Find an expression for $\widehat{X}(z)$.

b) Suppose $M = 4$ and $L = 3$. Suppose the analysis bank is the uniform DFT bank, i.e., $H_k(z) = H_0(zW_4^k), 0 \leq k \leq 3$. Assume that $H_0(z)$ has response shown below.

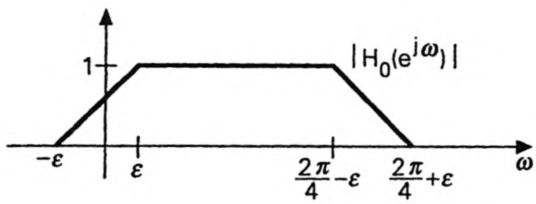


Figure P5-31(b)

How small would ϵ have to be so that $H_0(z)$ does not overlap with the aliased versions $H_0(zW_3^n), n = 1, 2$? With such ϵ, show typical responses of $F_k(z), 0 \leq k \leq 3$ such that aliasing terms are eliminated. (The trivial choice $F_k(z) = 0$ is forbidden, of course!). What is the distortion transfer function after such elimimation of aliasing?

Note. In nonmaximally decimated systems, the total number of samples per unit time at the output of the decimated analysis bank is evidently more than for $x(n)$. This is the price paid to obtain the simplicity of alias elimination.

5.32. Consider the following system which is a general M channel nonuniform filter bank.

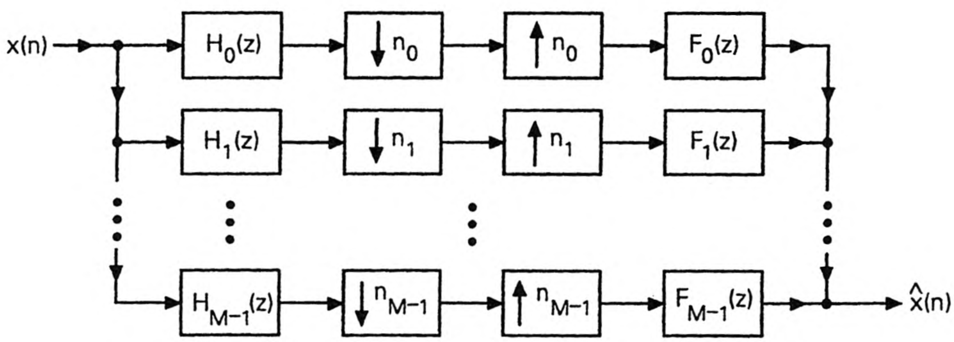


Figure P5-32

If the integers n_k are such that

$$\sum_{k=0}^{M-1} \frac{1}{n_k} = 1, \qquad (P5.32a)$$

the system is said to be maximally decimated. (Note that this condition holds for Fig. 5.4-1, where $n_k = M$ for all k. Also the tree structured system in Problem 5.25 is a special case of this nonuniform filter bank, with decimation ratios 8, 8, 4, 4, 4.) The kth analysis filter $H_k(z)$ has total passband width $\approx 2\pi/n_k$, so that it makes sense to decimate its output by n_k. This system suffers from the usual set of errors (aliasing, amplitude distortion, and phase distortion) as does

Fig. 5.4-1. Given an arbitrary set $\{n_i\}$ of M integers n_k satisfying (P5.32a), it is not in general possible to find nonideal filters $H_k(z)$ and $F_k(z)$ to eliminate aliasing completely. The alias terms at the output of the kth expander are

$$X(zW_{n_k}^n), \quad 1 \leq n \leq n_k - 1. \tag{$P5.32b$}$$

(W_m stands for $e^{-j2\pi/m}$.) So, unlike in Fig. 5.4-1, the "shifted versions" created by different channels have different amounts of shift. For example, the 0th channel generates

$$X(zW_{n_0}), X(zW_{n_0}^2)\dots \tag{$P5.32c$}$$

Unless each of these is also generated by at least one other channel, we cannot "cancel" all alias terms with nonideal filters.

If a set of integers $\{n_i\}$ is such that every shifted copy of $X(e^{j\omega})$ appears at the output of at least two expanders, we say that $\{n_i\}$ is a *compatible set.* Compatibility of $\{n_i\}$ is thus a *necessary* condition for complete alias cancelation in Fig. P5-32. If the filter bank is derived from a tree structure, (i.e., by starting from a system similar to Fig. P5-25(a) where uniform filter banks, not necessarily two-channel each, are used repeatedly), then the compatibility property is satisfied automatically because we know the system can be designed to be alias free (Problems 5.24, 5.25).

Which of the following sets are compatible?

a) $(2, 3, 6)$

b) $(2, 6, 6, 6)$

c) For large sets of integers [e.g., part e) below], it is tedious to directly check compatibility. Devise an efficient test for compatibility of a given set $\{n_i\}$.

d) Using the test developed above, show that the set $(2, 6, 10, 12, 12, 30, 30)$ is compatible.

e) Show that the set in part d) cannot be derived from a tree structure. Thus, there exist compatible sets which are not derived from tree structures.

5.33. Consider Fig. 5.4-1. Suppose the filters are chosen such that the system has the perfect reconstruction (PR) property. Now suppose that we replace each of the analysis and synthesis filters with $H_k(z^2)$ and $F_k(z^2)$, for $0 \leq k \leq M-1$. Does the resulting system still have the PR property?

5.34. In Problem 5.33, suppose we replace $H_k(z)$ and $F_k(z)$ with $H_k(z^L)$ and $F_k(z^L)$ for some integer $L > 0$. (So with $L = 2$, we obtain Problem 5.33). Find a necessary and sufficient condition on L such that the resulting system continues to have the PR property.