7

Linear Phase Perfect Reconstruction QMF Banks

7.0 INTRODUCTION

In some applications it is desirable to have a filter bank in which the analysis filters $H_k(z)$ are constrained to have linear phase. Such systems are called linear phase filter banks. These should not be confused with filter banks free from phase distortion, that is, filter banks for which the distortion function $T(z)$ has linear phase. For example the system in Design example 5.3.2 is not a linear phase filter bank (since the impulse response coefficients in Table 5.3.3 do not exhibit any symmetry), yet it is a perfect reconstruction system. On the other hand, the system in Design example 5.2.1 is a linear phase QMF bank (all filters have linear phase), but it is not a PR system since there is residual amplitude distortion (Fig. 5.2-4(b)).

In this chapter we show how to design linear phase filter banks which at the same time satisfy the perfect reconstruction (PR) property. The basic results were independently reported in Nguyen and Vaidyanathan [1989] and Vetterli and Le Gall [1988, 1989].

7.1 SOME NECESSARY CONDITIONS

In both the design examples mentioned above, the analysis filters $H_1(z)$ and $H_0(z)$ are constrained in some manner. In Design example 5.3.2 they are power complementary, whereas in Design example 5.2.1, $H_1(z) = H_0(-z)$. It turns out that, in order to design FIR linear phase QMF banks which at the same time enjoy the PR property, it is *necessary* to give up the power complementary property, as well as the relation $H_1(z) = H_0(-z)$. We begin the chapter by explaining why.

Power Complementary Constraint Must be Avoided

Suppose $H(z)$ and $G(z)$ are linear phase FIR filters, which at the same time satisfy the power complementary property. We will show that $H(z)$ is a sum of two delays, that is, $H(z) = az^{-K} + bz^{-L}$, where K and L are integers. $G(z)$ has similar form. As a result, the responses $|H(e^{j\omega})|$ and $|G(e^{j\omega})|$ are very restricted.

To prove this we assume that $H(z)$ and $G(z)$ are causal with the impulse response coefficients $h(0) \neq 0, g(0) \neq 0$. If this is not the case, we can redefine $H(z)$ and $G(z)$ by shifting the impulse responses, which does not affect either the linear phase property or the power complementary property. Let N denote the order of $H(z)$, so $h(N) \neq 0$. Let $N > 0$. (Otherwise there is nothing to prove.) The power complementary property implies

$$\widetilde{H}(z)H(z) + \widetilde{G}(z)G(z) = c^2 > 0. \tag{7.1.1}$$

Equating like powers on both sides we see that $G(z)$ also has order N. So $H(z) = \sum_{n=0}^{N} h(n)z^{-n}$ and $G(z) = \sum_{n=0}^{N} g(n)z^{-n}$, with $g(N) \neq 0$. From Sec. 2.4.2 we know that the linear phase property of $H(z)$ and $G(z)$ implies

$$H(z) = e^{-j\alpha}z^{-N}\widetilde{H}(z), \quad G(z) = e^{-j\beta}z^{-N}\widetilde{G}(z), \tag{7.1.2}$$

for real α, β. Substituting into (7.1.1) and simplifying we get

$$\Big(e^{j\alpha/2}H(z) + je^{j\beta/2}G(z)\Big)\Big(e^{j\alpha/2}H(z) - je^{j\beta/2}G(z)\Big) = c^2 z^{-N}. \tag{7.1.3}$$

Since all quantities on the left hand side are FIR, this implies

$$e^{j\alpha/2}H(z) + je^{j\beta/2}G(z) = pz^{-K}, \;\; e^{j\alpha/2}H(z) - je^{j\beta/2}G(z) = qz^{-L}, \tag{7.1.4}$$

where $pq = c^2$, and $K + L = N$. Adding and subtracting these two equations we get

$$H(z) = az^{-K} + bz^{-L}, \quad G(z) = \gamma\big(az^{-K} - bz^{-L}\big) \tag{7.1.5}$$

for appropriate a, b and γ, with $|\gamma| = 1$.

Consequences. As a consequence of this result, we have to remove the power complementary restriction on the analysis filters in order to obtain good responses. Since paraunitariness of the polyphase matrix $\mathbf{E}(z)$ (Sec. 6.2.2) implies that $H_0(z), H_1(z)$ are power complementary, it is necessary to *give up* paraunitariness of $\mathbf{E}(z)$ as well.

The Relation $\mathbf{H_1(z) = H_0(-z)}$ Must be Avoided

In Sec. 5.2 we studied alias-free FIR QMF banks with analysis filters related as $H_1(z) = H_0(-z)$. The overall distortion function is $T(z) = 0.5\big[H_0^2(z) - H_0^2(-z)\big]$. If $H_0(z)$ has linear phase, then $T(z)$ has linear phase, and phase distortion is eliminated. This system however suffers from amplitude distortion, that is, $|T(e^{j\omega})|$ is not perfectly flat. The residual amplitude distortion can be made very small using Johnston's procedure (Design example 5.2.1). This means that we have already seen examples of linear phase FIR QMF banks which "almost" satisfy the PR property. By increasing the order of $H_0(z)$ we can decrease the amplitude distortion to any desired degree [while maintaining the attenuation requirements of $H_0(z)$], so that the system gets as close to PR as we wish.

With this system, however, we can never achieve PR property *exactly*! We proved this in Sec. 5.2.1 by showing that the distortion function of the alias-free system has the form $T(z) = 2z^{-1}E_0(z^2)E_1(z^2)$, where $H_0(z) = E_0(z^2) + z^{-1}E_1(z^2)$. For perfect reconstruction $T(z)$ has to be a delay, that is, $H(z)$ has to be a sum of two delays, which is not useful. As a result, it is necessary to give up the condition $H_1(z) = H_0(-z)$ as well.

7.2 LATTICE STRUCTURES FOR LINEAR PHASE FIR PR QMF BANKS

Recall that neither the relation $H_1(z) = H_0(-z)$ nor the power complementary property is necessary for perfect reconstruction in FIR QMF banks. The condition (5.6.10) is really (necessary and) sufficient. It turns out that we can design very good linear phase analysis filters which at the same time satisfy this condition. As a first step, we generate an example with nontrivial analysis filters, such that neither the power complementary property nor the relation $H_1(z) = H_0(-z)$ is satisfied.

Example 7.2.1 An FIR Linear-Phase PR QMF Bank

Consider the analysis bank of Fig. 7.2-1(a). Here the polyphase matrix $\mathbf{E}(z) = \mathbf{T}_1\mathbf{\Lambda}(z)\mathbf{T}_0$, where

$$\mathbf{T}_0 = \begin{bmatrix} 1 & k \\ k & 1 \end{bmatrix}, \quad \mathbf{\Lambda}(z) = \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix}, \quad \mathbf{T}_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{7.2.1}$$

Assume k is real and $k \neq \pm 1$. This ensures that $\mathbf{T}_0$ is nonsingular. A corresponding synthesis bank which gives rise to perfect reconstruction is shown in Fig. 7.2-1(b). This is obtained by taking

$$\mathbf{R}(z) = cz^{-1}\mathbf{E}^{-1}(z) = c\mathbf{T}_0^{-1} \begin{bmatrix} z^{-1} & 0 \\ 0 & 1 \end{bmatrix} \mathbf{T}_1^{-1}$$

for appropriate c. The analysis and synthesis filters are verified to be

$$\begin{aligned} H_0(z) &= 1 + kz^{-1} + kz^{-2} + z^{-3}, \ H_1(z) = 1 + kz^{-1} - kz^{-2} - z^{-3}, \\ F_0(z) &= 1 - kz^{-1} - kz^{-2} + z^{-3}, \ F_1(z) = -1 + kz^{-1} - kz^{-2} + z^{-3}. \end{aligned} \tag{7.2.2}$$

Many points are worth noting here. The synthesis filters satisfy $F_0(z) = H_1(-z)$ and $F_1(z) = -H_0(-z)$, consistent with the alias cancelation condition (5.1.7). The analysis filters evidently have linear phase, and are nontrivial in the sense that they are not just sums of two delays. However, they are not power complementary, nor is the relation $H_1(z) = H_0(-z)$ satisfied. Finally, the synthesis filters are *not* given by $F_k(z) = z^{-3}H_k(z^{-1})$ as in a paraunitary perfect reconstruction system.

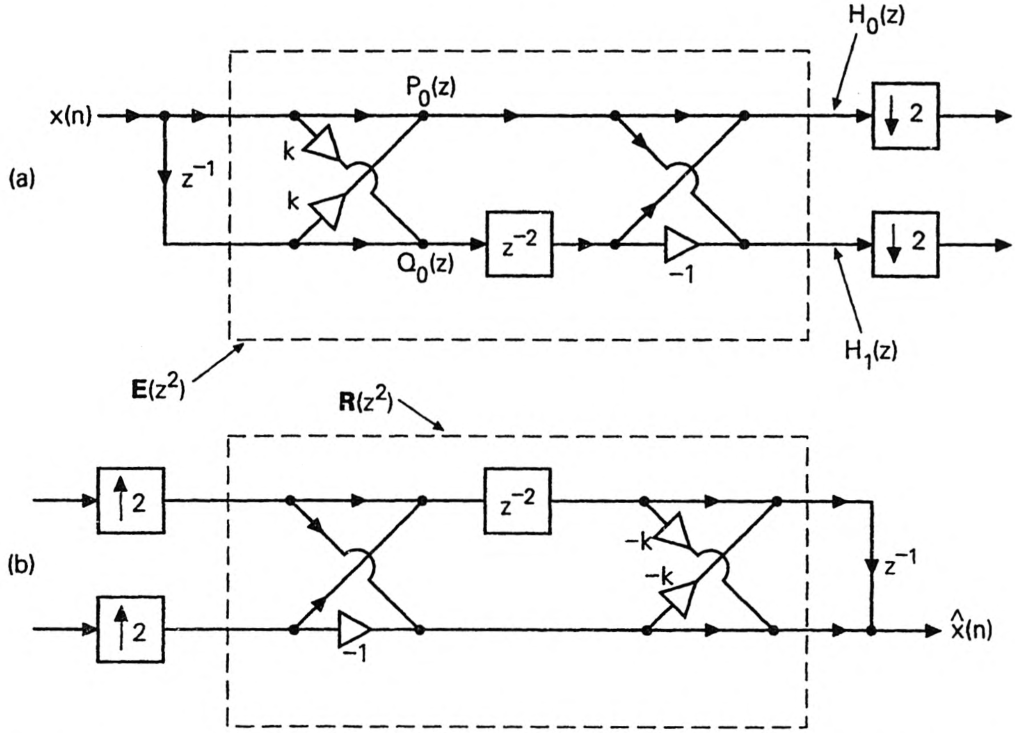

Figure 7.2-1 Example of linear phase PR QMF bank. (a) Analysis bank, and (b) synthesis bank.

What is the trick behind the success of this example? The matrices $\mathbf{T}_0$ and $\mathbf{T}_1$ have, no doubt, been 'carefully' chosen. The choice of $\mathbf{T}_0$ is such that $Q_0(z)$ is the Hermitian image of $P_0(z)$ (see Fig. 7.2-1(a). The choice of $\mathbf{T}_1$ is such that $H_0(z)$ and $H_1(z)$ are the sum and difference of the image pair $P_0(z), z^{-2}Q_0(z)$, so that $h_0(n)$ is symmetric and $h_1(n)$ is antisymmetric!

Example 7.2.2

We can in fact generate similar examples of arbitrary order. To demonstrate this, consider Fig. 7.2-2(a) in which

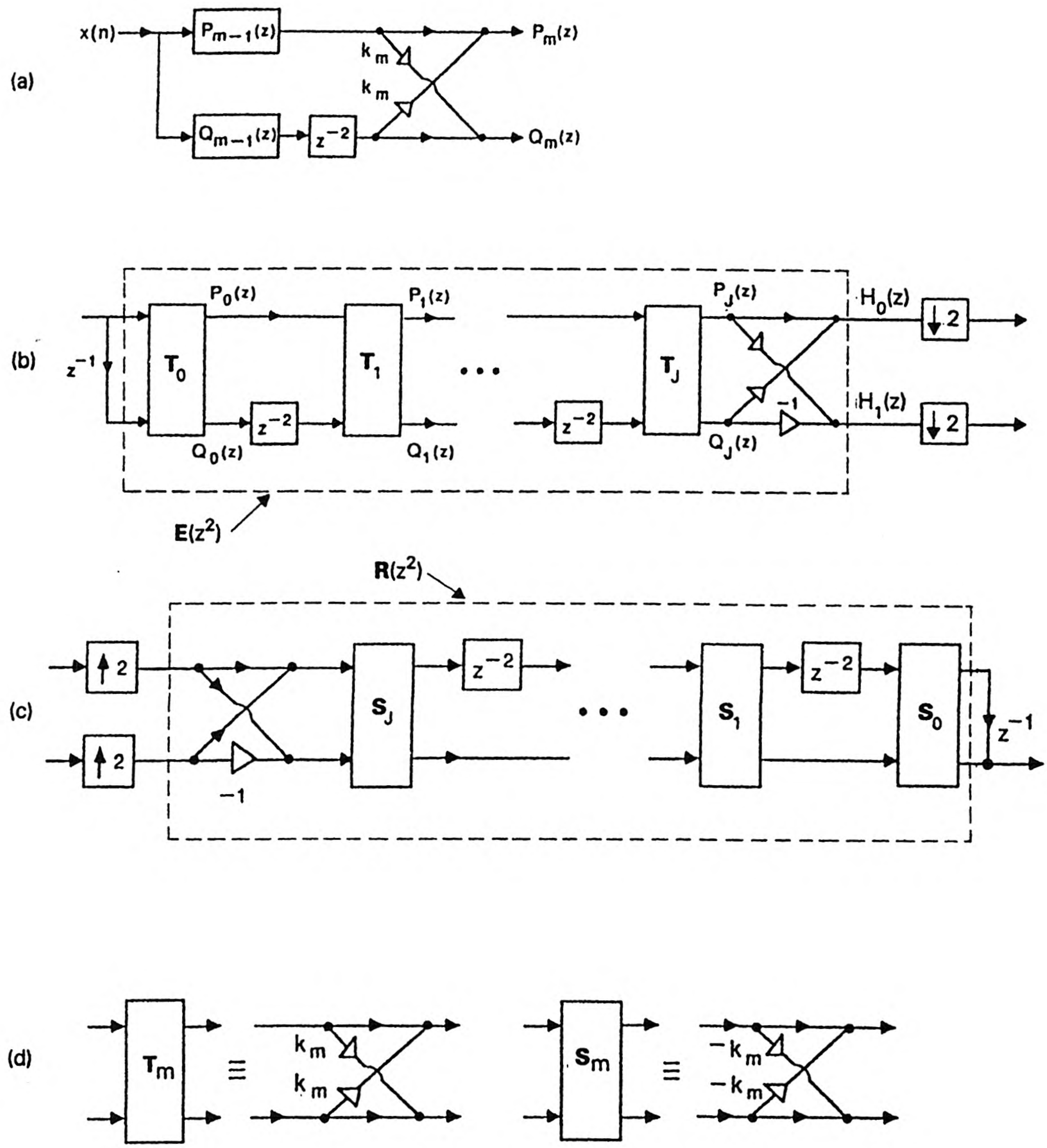

Figure 7.2-2 More general linear phase FIR PR QMF bank. (a) Basic generation technique for analysis bank. (b) Complete analysis bank. (c) Complete synthesis bank. (d) Details of the building blocks.

$$P_{m-1}(z) = \sum_{n=0}^{2m-1} p_{m-1}(n)z^{-n}, \quad Q_{m-1}(z) = \sum_{n=0}^{2m-1} q_{m-1}(n)z^{-n}, \qquad (7.2.3)$$

are real coefficient polynomials and k_m is real. Let $Q_{m-1}(z)$ be the Hermitian image of $P_{m-1}(z)$, i.e., $Q_{m-1}(z) = z^{-(2m-1)}P_{m-1}(z^{-1})$. Then the transfer functions

$$\begin{aligned} P_m(z) &= P_{m-1}(z) + k_m z^{-2} Q_{m-1}(z), \\ Q_m(z) &= k_m P_{m-1}(z) + z^{-2} Q_{m-1}(z), \end{aligned} \qquad (7.2.4)$$

are also Hermitian images, i.e., satisfy $Q_m(z) = z^{-(2m+1)}P_m(z^{-1})$. (This can be verified by substitution.) By repeated application of this, we see that the cascaded lattice structure shown in Fig. 7.2-2(b) has the property

$$Q_J(z) = z^{-N}P_J(z^{-1}), \quad N = 2J+1. \qquad (7.2.5)$$

The analysis filters in this figure are given by

$$\begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} P_J(z) \\ Q_J(z) \end{bmatrix}. \qquad (7.2.6)$$

From this it is easily verified that these filters satisfy

$$H_0(z) = z^{-N}H_0(z^{-1}), \quad H_1(z) = -z^{-N}H_1(z^{-1}), \qquad (7.2.7)$$

that is, in terms of impulse response coefficients,

$$h_0(n) = h_0(N-n), \quad h_1(n) = -h_1(N-n), \qquad (7.2.8)$$

so that they have linear phase. Figure 7.2-2(c) shows a synthesis bank which results in perfect reconstruction. This is obtained by choosing the polyphase matrix of the synthesis bank to be $\mathbf{R}(z) = cz^{-J}\mathbf{E}^{-1}(z)$. Here $\mathbf{S}_m = (1-k_m^2)\mathbf{T}_m^{-1}$. The synthesis filters satisfy (5.1.7) within a scale factor (Problem 7.1).

The main point of the example is that we can generate linear phase FIR PR QMF banks in which the analysis filters are nontrivial (i.e., not restricted to be a sum of two delays). The next design example shows that these analysis filters can provide excellent attenuation as well.

Design Example 7.2.1. Linear Phase FIR PR Lattice

Consider a lattice with $J = 31$ so that the filters have order $N = 63$. The lattice coefficients should now be optimized in order to minimize an appropriate objective function. The simple function (6.4.9) is not suitable any more because $H_0(z)$ is not power symmetric, and moreover there is no power complementary relation between $H_0(z)$ and $H_1(z)$. It is necessary to define an objective function which reflects the passbands *and* stopbands of *both* filters. For example we can take

$$\begin{aligned}\phi = &\int_0^{\omega_p}\Big[1-|H_0(e^{j\omega})|\Big]^2 d\omega + \int_{\omega_S}^{\pi} |H_0(e^{j\omega})|^2 d\omega \\ &+ \int_{\omega_S}^{\pi}\Big[1-|H_1(e^{j\omega})|\Big]^2 d\omega + \int_0^{\omega_p} |H_1(e^{j\omega})|^2 d\omega.\end{aligned} \tag{7.2.9}$$

Figure 7.2-3(a) shows the analysis filter responses of the optimized design. The filter coefficients are tabulated in Nguyen and Vaidyanathan [1989]. The transition bandwidth is about 0.172π. For comparison, we show in Fig. 7.2-3(b) the responses of Johnston's 64D filters (which also have order 63, and the same transition bandwidth). Johnston's filters offer a minimum stopband attenuation of 65 dB, in comparison to only 42.5 dB offered by the perfect reconstruction system. † The peak amplitude distortion of Johnston's 64D QMF bank is about 0.002 dB. Johnston's 32D filter, on the other hand, has nearly the same attenuation as the PR system.

Even though the PR system has higher order for a given attenuation, it can be implemented more efficiently than Johnston's filters, because of the lattice structure (see *computational complexity* below).

Recall that in order to obtain perfect reconstruction using linear phase filters, we had to give up the relation $H_1(z) = H_0(-z)$ as well as the power complementary property. Also the plot of $|H_0(e^{j\omega})|^2 + |H_1(e^{j\omega})|^2$ is very flat for Johnston's design but not for the linear phase PR pair (see Fig. 7.2-4). In spite of this the linear phase lattice structure enjoys perfect reconstruction because the quantity $|H_0(e^{j\omega})|^2 + |H_1(e^{j\omega})|^2$ is *not* proportional to the amplitude distortion unlike in Johnston's design!

Computational Complexity of Linear Phase QMF Lattice

The lattice structure of Fig. 7.2-2(b) has $J+1$ sections, with two multipliers per section. However, each section can be rearranged, permitting an implementation with only one multiplier (and three adders) per section (Problem 7.2)‡. From this we deduce that the analysis bank requires $0.25(N+1)+1$ MPUs and $0.75(N+1)+1$ APUs (where $N =$ filter order). In our example $N = 63$ so this reduces to 17 MPUs and 49 APUs. For comparison suppose we consider Johnston's 32D filter, which has nearly the same

† Improved optimization has recently been reported by Nguyen [1992a], whereby the perfect reconstruction system can provide almost as good attenuation as Johnston's filters of the same order.

‡ This is unlike in the paraunitary lattice (Fig. 6.4-2), which required a minumum of two multipliers per section.

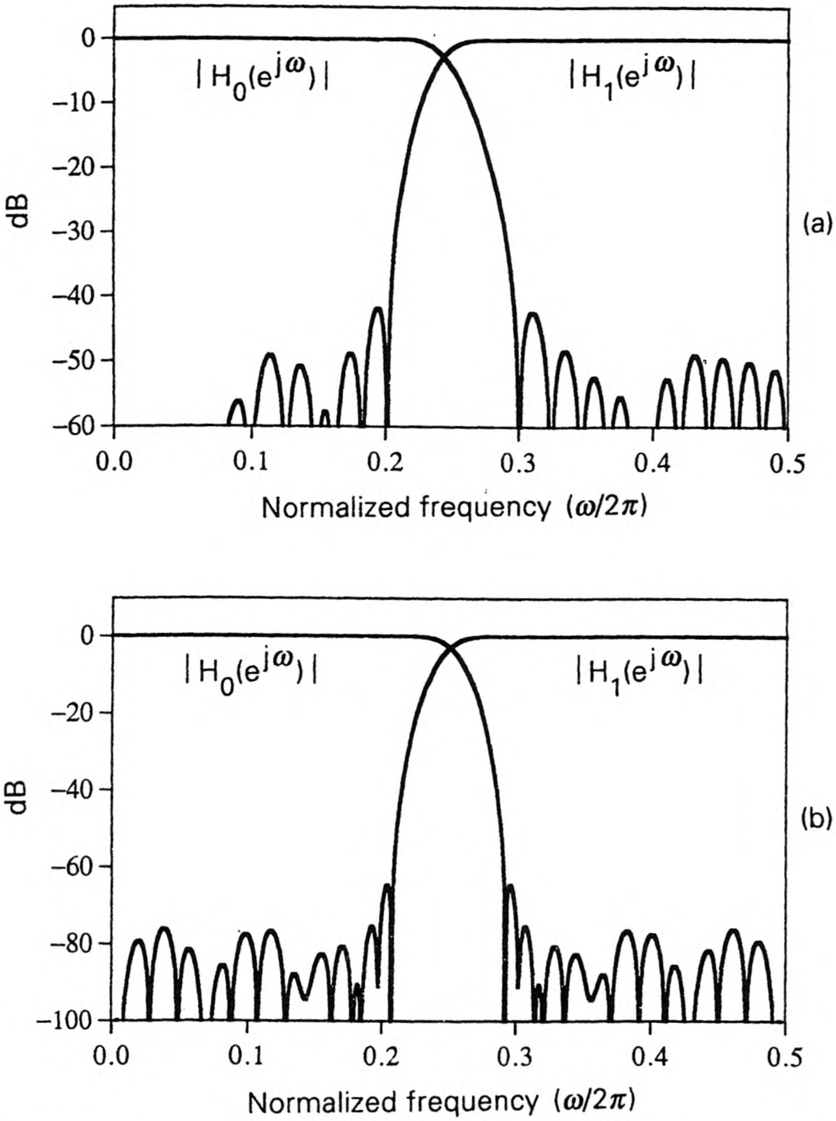

Figure 7.2-3 Design example 7.2.1 (Linear phase QMF banks). Magnitude responses of analysis filters for (a) perfect reconstruction system, and (b) Johnston's 64D system. Both systems have analysis filters of length 64. (© Adopted from 1989 IEEE.)

specifications (including minimum stopband attenuation) as the linear-phase lattice filters, and has a peak amplitude distortion of 0.025 dB. This can be implemented with a total of 16 MPUs and 16 APUs for the analysis bank. Summarizing, the linear phase PR QMF lattice has about the same number of MPUs as Johnston's filters with same specifications (and amplitude distortion of 0.025 dB). The number of APUs, however, is higher.

Table 7.2.1 summarizes the comparison between the linear phase PR QMF bank and Johnston's design.

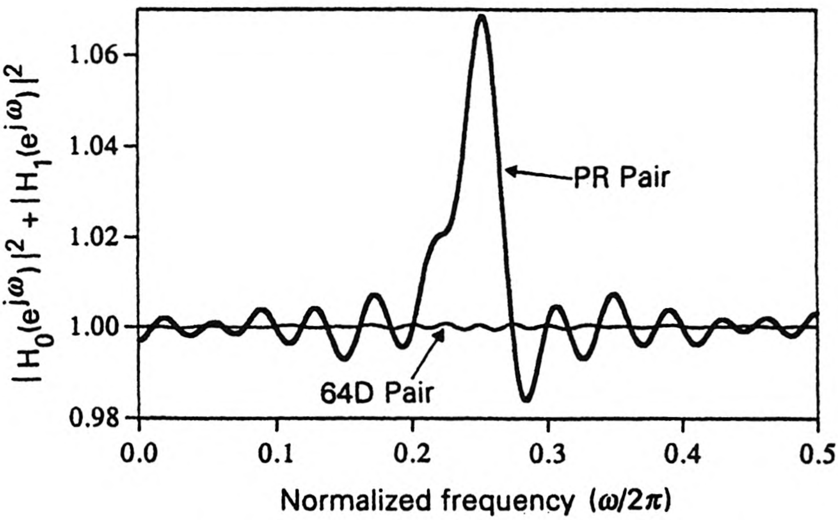

Figure 7.2-4 Pertaining to Design example 7.2.1. (© Adopted from 1989 IEEE.)

Initialization of the Lattice Parameters for Optimization

Since Johnston's filters have linear phase and "almost" satisfy the PR property, it is possible to obtain a lattice structure for these, which "almost" resembles Fig. 7.2-2(b). This can be used to initialize the parameters k_m.

Let us make the above statement more precise. Given the Nth order pair $H_0(z), H_1(z)$ from Johnston's design, suppose we define $P_J(z)$ and $Q_J(z)$ according to (7.2.6). Then (7.2.5) is satisfied because (7.2.7) holds. Define $P'_N(z) = P_J(z), Q'_N(z) = Q_J(z)$ for convenience. We can now construct a pair of lower order transfer functions as follows:

$$\begin{aligned} P'_{N-1}(z) &= P'_N(z) - \ell_N Q'_N(z), \\ z^{-1}Q'_{N-1}(z) &= -\ell_N P'_N(z) + Q'_N(z). \end{aligned} \tag{7.2.10}$$

We choose $\ell_N = p'_N(N)/q'_N(N)$ so that $P'_{N-1}(z)$ has degree $N-1$. Because of the relation (7.2.5), this same value of ℓ_N ensures that $Q'_{N-1}(z)$ defined in (7.2.10) is causal with degree $N-1$ or less. Assuming $\ell_N^2 \neq 1$ we can invert (7.2.10) to get

$$\begin{aligned} (1-\ell_N^2)P'_N(z) &= P'_{N-1}(z) + \ell_N z^{-1}Q'_{N-1}(z), \\ (1-\ell_N^2)Q'_N(z) &= \ell_N P'_{N-1}(z) + z^{-1}Q'_{N-1}(z), \end{aligned} \tag{7.2.11}$$

This gives rise to the lattice representation of Fig. 7.2-5(a). It can be verified from above that $Q'_{N-1}(z) = z^{-(N-1)}P'_{N-1}(z^{-1})$, so that we can repeat the process, resulting in Fig. 7.2-5(b). ‡ Here the scale factors $(1 - \ell_m^2)$ have all been lumped together into α.

TABLE 7.2.1 Comparison between the linear-phase perfect reconstruction design and Johnston's 32D design.

Feature	Johnston's 32D Pair of Filters	Linear phase QMF lattice
Phase response	Linear	Linear
Filter order	31	63
Stopband Attenuation	38dB	42.5dB
Peak amplitude distortion	0.025dB	No error
Number of MPU for analysis bank	16	17
Number of APU for analysis bank	16	49
Power Complementarity	Approximately holds	Does not hold
Relation between Analysis Filters	$H_1(z) = H_0(-z)$	Not explicit. Implicitly such that det E(z)=delay
Overall Group Delay of QMF bank	31	63
Aliasing	Canceled	Canceled
Phase distortion	Eliminated	Eliminated
Amplitude distortion	Minimized	Eliminated

‡ Readers familiar with linear predictive coding will notice the resemblance to the LPC lattice [Markel, and Gray, Jr., 1976]. However, there are two differences. In the LPC lattice the coefficients ℓ_m (called *reflection coefficients*) are typically bounded as $\ell_m^2 < 1$. Also the rightmost section, which generates $H_0(z), H_1(z)$, is absent.

So Johnston's analysis bank can be represented in this manner, provided $\ell_m^2 \neq 1$ for any m (and this is the case in all practical examples). This structure, however, is not in polyphase form because the system inside broken lines in Fig. 7.2-5(b) is not a function of z^{-2} (so that it is not equal to $\mathbf{E}(z^2)$). However, since the filters have linear phase and almost satisfy the PR property, the coefficients $\ell_2, \ell_4, \ldots$ and so on, turn out to be very close to zero. By setting these to zero, the remaining coefficients ℓ_{2m+1} can be used to initialize the coefficients k_n in the linear-phase lattice of Fig. 7.2-2(b). Such initialization leads to significantly faster convergence of optimization, as compared to random initialization. This method was used in the above design example.

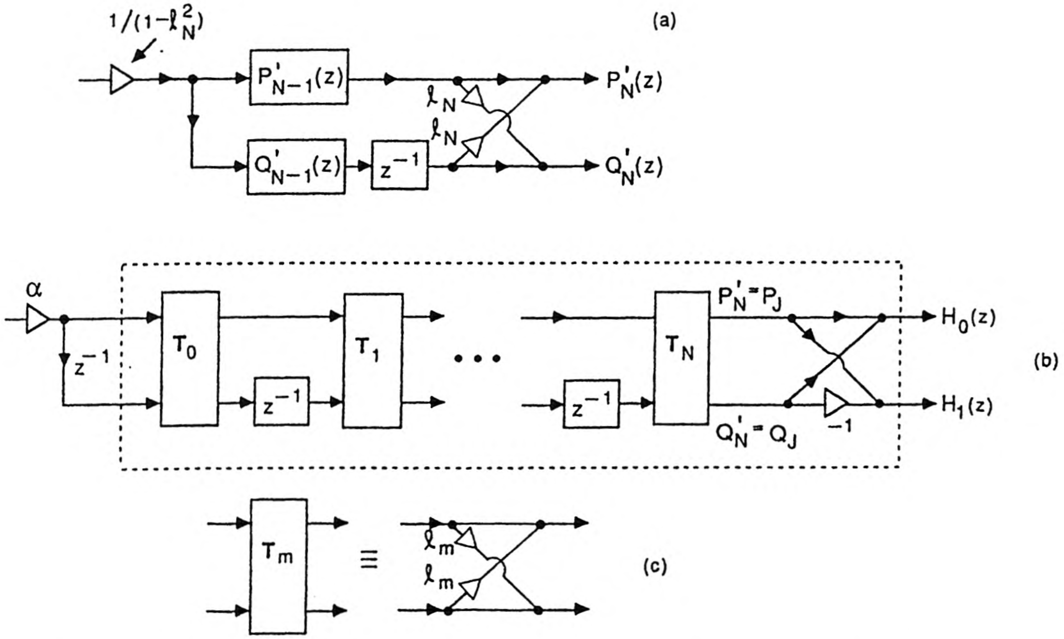

Figure 7.2-5 Lattice structure for an arbitrary (i.e. not necessarily PR) linear phase pair $[H_0(z), H_1(z)]$.

7.3 FORMAL SYNTHESIS OF LINEAR PHASE FIR PR QMF LATTICE

In Example 7.2.2 $H_0(z)$ and $H_1(z)$ are filters with odd order ($N = 2J + 1$) satisfying (7.2.7) and the PR condition [det $\mathbf{E}(z)$] = delay. The next question is, given such a pair of FIR filters, is it always possible to find a structure like Fig. 7.2-2(b)? In other words, does the lattice cover every analysis bank satisfying the said properties? The answer is *yes* with some minor exceptions which will be made clear soon.

The given set of filters $H_0(z), H_1(z)$ can always be expressed as in (7.2.6) by defining $P_J(z)$ and $Q_J(z)$ as

$$\begin{bmatrix} P_J(z) \\ Q_J(z) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix}. \tag{7.3.1}$$

With analysis filters expressed in polyphase form (5.6.11), we have

$$\begin{bmatrix} P_J(z) \\ Q_J(z) \end{bmatrix} = \underbrace{\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\mathbf{E}(z^2)}_{\mathbf{F}_J(z^2)}\begin{bmatrix} 1 \\ z^{-1} \end{bmatrix}. \tag{7.3.2}$$

In other words

$$\begin{bmatrix} P_J(z) \\ Q_J(z) \end{bmatrix} = \mathbf{F}_J(z^2)\begin{bmatrix} 1 \\ z^{-1} \end{bmatrix}, \tag{7.3.3}$$

and the PR condition (5.6.10) is equivalent to the condition

$$\det \mathbf{F}_J(z) = -c_0 z^{-m_0} \qquad \text{(PR condition)}. \tag{7.3.4}$$

The linear phase condition (7.2.7) is, of course, equivalent to (7.2.5).

So the problem of designing linear phase FIR PR QMF banks can be transformed to that of finding a lattice structure for $[P_J(z)\ Q_J(z)]^T$ which satisfies the properties (7.3.4) and (7.2.5). For convenience of discussion let us write

$$P_J(z) = \sum_{n=0}^{N} p_J(n) z^{-n}, \quad Q_J(z) = \sum_{n=0}^{N} q_J(n) z^{-n} \tag{7.3.5}$$

so that (7.2.5) is equivalent to

$$p_J(n) = q_J(N-n) \qquad \text{(linear-phase condition)}. \tag{7.3.6}$$

The following Lemma is crucial to our discussion.

♠**Lemma 7.3.1.** Let $P_J(z)$ and $Q_J(z)$ be as in (7.3.5), and satisfy (7.3.6) and (7.3.4), where $\mathbf{F}_J(z)$ is defined as in (7.3.3). Let N be odd, that is, $N = 2J+1$. Assume $p_J(n), q_J(n)$ are real, and $0 \neq p_J(0)$ and $p_J(0) \neq \pm p_J(N)$. Then we can find two FIR filters $P_{J-1}(z) = \sum_{n=0}^{N-2} p_{J-1}(n) z^{-n}$ and $Q_{J-1}(z) = \sum_{n=0}^{N-2} q_{J-1}(n) z^{-n}$ and a real $k_J \neq \pm 1$ such that

$$\begin{bmatrix} P_{J-1}(z) \\ z^{-2} Q_{J-1}(z) \end{bmatrix} = \frac{1}{1-k_J^2}\begin{bmatrix} 1 & -k_J \\ -k_J & 1 \end{bmatrix}\begin{bmatrix} P_J(z) \\ Q_J(z) \end{bmatrix}. \tag{7.3.7}$$

Moreover $Q_{J-1}(z) = z^{-(N-2)} P_{J-1}(z^{-1})$ and $p_{J-1}(0) \neq 0$. $\diamond$

Remark. The condition $k_J^2 \neq 1$ automatically ensures that the 2×2 matrix in (7.3.7) is nonsingular. By inverting it, we can obtain the lattice structure of Fig. 7.2-2(a) with J in place of m. The remainder $[\,P_{J-1}(z) \quad Q_{J-1}(z)\,]^T$ has all the properties of $[\,P_J(z) \quad Q_J(z)\,]^T$ so that we can repeat this process provided $p_{J-1}(0) \neq \pm p_{J-1}(N-2)$. So we can obtain the lattice structure shown in Fig. 7.2-2(b), for the system $[\,P_J(z) \quad Q_J(z)\,]^T$ *provided that* each one of the remainders satisfies

$$p_m(0) \neq \pm p_m(2m+1). \tag{7.3.8}$$

This means that we can implement the analysis bank $[\,H_0(z) \quad H_1(z)\,]^T$ as in Fig. 7.2-2(b).

Proof of Lemma 7.3.1. From (7.3.7) it is clear that k_J has to satisfy $p_J(N) - k_J q_J(N) = 0$ so that

$$k_J = \frac{p_J(N)}{q_J(N)} = \frac{p_J(N)}{p_J(0)} \neq \pm 1. \tag{7.3.9}$$

With this choice of k_J, the coefficient of z^{-N} in $P_J(z) - k_J Q_J(z)$ drops off.

We now show that the coefficient of $z^{-(N-1)}$ in $P_J(z) - k_J Q_J(z)$ is also zero, so that $P_{J-1}(z)$ has order $N-2$ as claimed. For this note that this coefficient is

$$p_J(N-1) - k_J q_J(N-1) = \left[p_J(N-1)q_J(N) - q_J(N-1)p_J(N)\right]/q_J(N), \tag{7.3.10}$$

by (7.3.9). With

$$\mathbf{F}_J(z) = \begin{bmatrix} A(z) & B(z) \\ C(z) & D(z) \end{bmatrix}, \tag{7.3.11a}$$

the condition (7.3.4) implies

$$A(z)D(z) - B(z)C(z) = -c_0 z^{-m_0}. \tag{7.3.11b}$$

But we have

$$\begin{aligned}
A(z) &= p_J(0) + p_J(2)z^{-1} + \ldots + p_J(N-1)z^{-M}, \\
B(z) &= p_J(1) + p_J(3)z^{-1} + \ldots + p_J(N)z^{-M}, \\
C(z) &= q_J(0) + q_J(2)z^{-1} + \ldots + q_J(N-1)z^{-M}, \\
D(z) &= q_J(1) + q_J(3)z^{-1} + \ldots + q_J(N)z^{-M},
\end{aligned} \tag{7.3.12}$$

where $M = (N-1)/2$. The coefficient of z^0 in the LHS of (7.3.11b) is

$$p_J(0)q_J(1) - q_J(0)p_J(1). \tag{7.3.13a}$$

The coefficient of z^{-2M} is, on the other hand,

$$p_J(N-1)q_J(N) - p_J(N)q_J(N-1). \qquad (7.3.13b)$$

Since $N \geq 3$, we have $2M > 0$. So (7.3.11b) implies that at least one of (7.3.13a),(7.3.13b) is zero. By using the image property (7.3.6) it is verified that (7.3.13a) and (7.3.13b) have the same value. Setting this to zero we see that (7.3.10) is indeed zero. So $P_{J-1}(z)$ has the stated form.

We can verify that $Q_{J-1}(z) = z^{-(N-2)}P_{J-1}(z^{-1})$ by substituting (7.3.6) into (7.3.7). So $Q_{J-1}(z)$ has the state form too. Inverting (7.3.7) one obtains

$$\begin{bmatrix} P_J(z) \\ Q_J(z) \end{bmatrix} = \begin{bmatrix} 1 & k_J \\ k_J & 1 \end{bmatrix} \begin{bmatrix} P_{J-1}(z) \\ z^{-2}Q_{J-1}(z) \end{bmatrix}, \qquad (7.3.14)$$

from which it follows that $p_J(0) = p_{J-1}(0)$, so $p_{J-1}(0) \neq 0$ indeed. $\bigtriangledown\bigtriangledown\bigtriangledown$

Condition on $\mathbf{H_0(z), H_1(z)}$ for Lattice Realization

The analysis filters have the form $H_0(z) = \sum_{n=0}^{N} h_0(n)z^{-n}$, $H_1(z) = \sum_{n=0}^{N} h_1(n)z^{-n}$, $N \geq 3$. The condition $p_J(0) \neq \pm p_J(N)$ in Lemma 7.3.1 can be satisfied by assuming that neither of $h_0(N), h_1(N)$ is zero. Now what does $p_J(0) = 0$ mean? Since (7.3.13a) equals zero, this means $q_J(0) = 0$ or $p_J(1) = 0$. This means that either $h_0(0) = h_1(0) = 0$ or that $P_J(z)$ and $Q_J(z)$ have the form $P_J(z) = z^{-2}P_{J-1}(z)$ and $Q_J(z) = Q_{J-1}(z)$. The former case is trivial and can be avoided by shifting. The latter case implies that we can interchange the roles of $P_J(z)$ and $Q_J(z)$ and then take $k_J = 0$. So the situations created by violation of the condition '$0 \neq p_J(0) \neq \pm p_J(N)$', can be handled easily. It is, however, still possible to find examples where (7.3.8) is violated for some value of $m < J$. In such cases the lattice cannot be synthesized.

It nevertheless remains a significant fact that the lattice of Fig. 7.2-2 can be used to generate a wide class of linear phase FIR PR systems. More general study of two-channel linear phase FIR PR systems can be found in Nguyen and Vaidyanathan [1989]. For instance, it can be shown that one can take both $h_0(n)$ and $h_1(n)$ to be symmetric (provided their orders are even and unequal). Such systems have a different type of lattice structure. Also see [Vetterli and Le Gall, 1989] and Nguyen and Vaidyanathan [1990] for M-channel linear phase FIR perfect reconstruction systems. For a general theory of M-channel linear phase FIR paraunitary filter banks see [Soman, Vaidyanathan, and Nguyen, 1992].

7.1. The lattice structure shown in Fig. 7.2-2 (b) and (c) represents a perfect reconstruction QMF bank. Express the synthesis filters $F_k(z)$ in terms of the analysis filters $H_k(z)$.

7.2. Consider the two multiplier lattice section shown in Fig. P7-2(a).

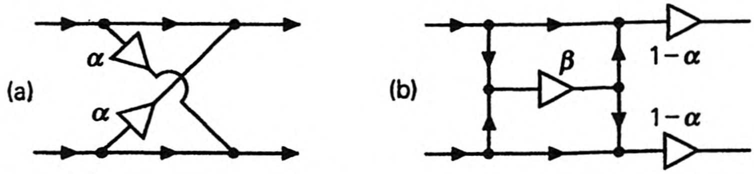

Figure P7-2

Show that this can be redrawn as shown in Fig. P7-2(b) where $\beta = \alpha/(1-\alpha)$, assuming $\alpha \neq 1$. Hence show that the linear phase perfect reconstruction system (Fig. 7.2-2) can be rearranged so that the analysis bank requires a total of $0.25(N+1)+1$ MPUs and $0.75(N+1)+1$ APUs.

7.3. Consider the analysis bank structure given below, where α_m, β_n are real.

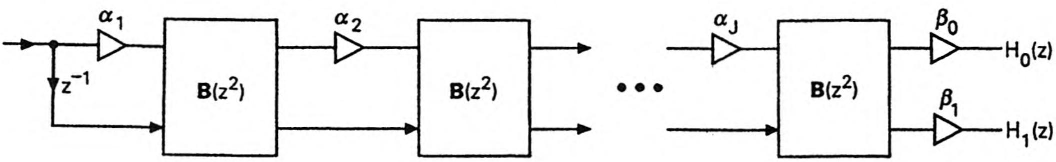

Figure P7-3

The building block $\mathbf{B}(z)$ has the form

$$\mathbf{B}(z) = \begin{bmatrix} 1+z^{-1} & 1 \\ 1+z^{-1}+z^{-2} & 1+z^{-1} \end{bmatrix}. \tag{P7.3a}$$

Evidently the analysis filters $H_0(z)$ and $H_1(z)$ are causal and FIR. Show that they have linear phase. More specifically show that the impulse responses satisfy

$$h_0(n) = h_0(N_0 - n), \quad h_1(n) = h_1(N_1 - n) \tag{P7.3b}$$

where $N_0 = 2J, N_1 = 2J+2$. Find a set of FIR synthesis filters [in terms of $H_0(z)$ and $H_1(z)$] which result in perfect reconstruction. (*Note.* this problem shows that we can obtain an FIR perfect reconstruction QMF bank in which both analysis filters are symmetric unlike in Sec. 7.2 where $H_1(z)$ was antisymmetric.)

7.4. Let $H(z)$ and $G(z)$ be two real coefficient linear phase FIR filters satisfying $|H(e^{j\omega})|^2 + |G(e^{j\omega})|^2 = 1$ for all ω. Prove that $|H(e^{j\omega})|^2 = \alpha \cos^2(a\omega + b)$ for some real α, a, b.

7.5. Let $H(z)$ be a linear phase FIR filter, and let $\widetilde{H}(z)H(z)$ satisfy the Mth band property.

a) For $M = 2$ (i.e., $\widetilde{H}(z)H(z)$ is half-band) show that $H(z)$ can have at most two nonzero coefficients.

b) For $M > 2$, a similar statement is not true. Show, by construction, that the number of nonzero coefficients of $H(z)$ can exceed L, for any arbitrary integer L.

Hints. Use the results of Sec. 4.6.3 and 7.1.