10

Multirate filter bank theory and related topics

10.0 INTRODUCTION

In this chapter we study the interrelation between multirate filter bank theory, and several "neighbouring" topics in signal processing. In Sec. 10.1 we consider the connection between alias-free (maximally decimated) filter banks, block digital filtering, and linear periodically time varying (LPTV) systems. We will see that the pseudocirculant matrix defined in Sec. 5.7.2 unifies these three topics in a nice way. In Sec. 10.2 we study a number of unconventional sampling theorems (such as "difference-sampling," and nonuniform sampling) using the framework of multirate filter banks. Readers who have looked at Problem 5.13 will recall that Shannon's well-known derivative sampling theorem can be derived based on an analog filter bank [Papoulis, 1977b], and [Brown, 1981]. Such a viewpoint not only simplifies the understanding of these sampling techniques, but also opens up new digital ways to reconstruct signals from uncoventionally sampled data.

Further applications can be found in Vetterli [1988], where a multirate filter bank framework is used for the efficient implementation of FIR and IIR filters. Also see Sathe and Vaidyanathan [1993] where the role of pseudocirculat matrices in random process theory is discussed.

10.1 BLOCK FILTERS, LPTV SYSTEMS, AND MULTIRATE FILTER BANKS

10.1.1 Block Filtering

The processing of a scalar signal in blocks is a common approach in many applications. Block processing has been studied by a number of authors [Burrus, 1971], [Mitra and Gnanasekaran, 1978], [Barnes and Shinnaka, 1980], and [Clark, et al, 1981]. One example of block processing was indicated in Sec. 6.6 (transform-coding and LOT). Block digital filtering, in particular, is a technique to implement a scalar filter $H(z)$ in such a way as to increase the parallelism in the computations. This finds application in high speed digital filtering, that is, where the sampling rate is very high.

Definition of Block Digital Filters

Let $x(n)$ and $y(n)$ denote, respectively, the input and output of the scalar filter $H(z)$. Consider two vector sequences $\mathbf{x}_B(n)$ and $\mathbf{y}_B(n)$ defined by

$$\mathbf{x}_B(n) = \begin{bmatrix} x(nM+M-1) \\ x(nM+M-2) \\ \vdots \\ x(nM) \end{bmatrix}, \quad \mathbf{y}_B(n) = \begin{bmatrix} y(nM+M-1) \\ y(nM+M-2) \\ \vdots \\ y(nM) \end{bmatrix}. \tag{10.1.1}$$

We say that the vector sequences $\mathbf{x}_B(n)$ and $\mathbf{y}_B(n)$ are *blocked versions* of (or blocked sequences corresponding to) the scalar sequences $x(n)$ and $y(n)$. The block-length (or size) is M.

Now imagine that we have a system which generates the sequence $\mathbf{y}_B(n)$ in response to $\mathbf{x}_B(n)$. Evidently this is an M-input M-output system. Not surprisingly this is an LTI system (Problem 13.24), and can be characterized by a $M \times M$ transfer matrix $\mathbf{H}(z)$. In other words we have $\mathbf{Y}_B(z) = \mathbf{H}(z)\mathbf{X}_B(z)$, where

$$\mathbf{X}_B(z) = \sum_n \mathbf{x}_B(n)z^{-n}, \quad \mathbf{Y}_B(z) = \sum_n \mathbf{y}_B(n)z^{-n}. \tag{10.1.2}$$

The matrix $\mathbf{H}(z)$ is called the *blocked version* of $H(z)$. From its definition it is clear that $\mathbf{H}(z)$ is completely determined by the scalar system $H(z)$. Figure 10.1-1 is a summary of the situation. The "blocking mechanism" can be considered to be a serial to parallel converter of data, and the "unblocking mechanism" a parallel to serial converter.

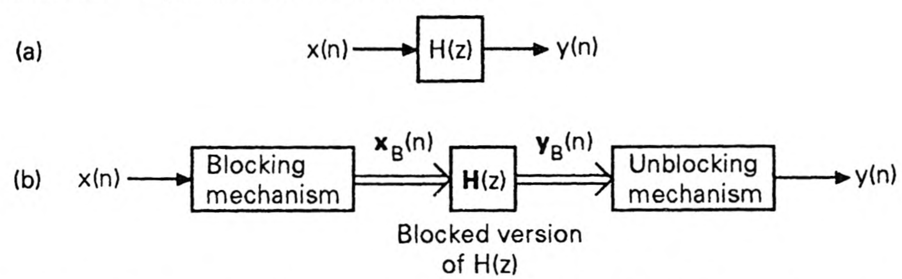


Figure 10.1-1 (a) A scalar transfer function, and (b) its blocked implementation.

Multirate Filter-Bank Notation

Figure 10.1-2 shows a schematic diagram of block digital filtering in terms of multirate notation. Here the signals $x_k(n)$ are given by $x_k(n) =$

$x(nM+k)$. Similarly $y_k(n) = y(nM+k)$, so that the blocked versions (10.1.1) can also be represented as

$$\mathbf{x}_B(n) = \begin{bmatrix} x_{M-1}(n) \\ \vdots \\ x_1(n) \\ x_0(n) \end{bmatrix}, \quad \mathbf{y}_B(n) = \begin{bmatrix} y_{M-1}(n) \\ \vdots \\ y_1(n) \\ y_0(n) \end{bmatrix}. \tag{10.1.3}$$

The transfer matrix $\mathbf{H}(z)$ in this figure produces $\mathbf{y}_B(n)$ in response to $\mathbf{x}_B(n)$, and is therefore the blocked version of $H(z)$.

Let $X_\ell(z)$ and $Y_\ell(z)$ denote the z-transforms of $x_\ell(n)$ and $y_\ell(n)$. Then the z-transforms of $x(n)$ and $y(n)$ can be expressed as

$$X(z) = \sum_{\ell=0}^{M-1} z^{-\ell} X_\ell(z^M), \quad Y(z) = \sum_{\ell=0}^{M-1} z^{-\ell} Y_\ell(z^M). \tag{10.1.4}$$

In other words, the components of $\mathbf{x}_B(n)$ and $\mathbf{y}_B(n)$ are the polyphase componenents of $x(n)$ and $y(n)$, respectively.

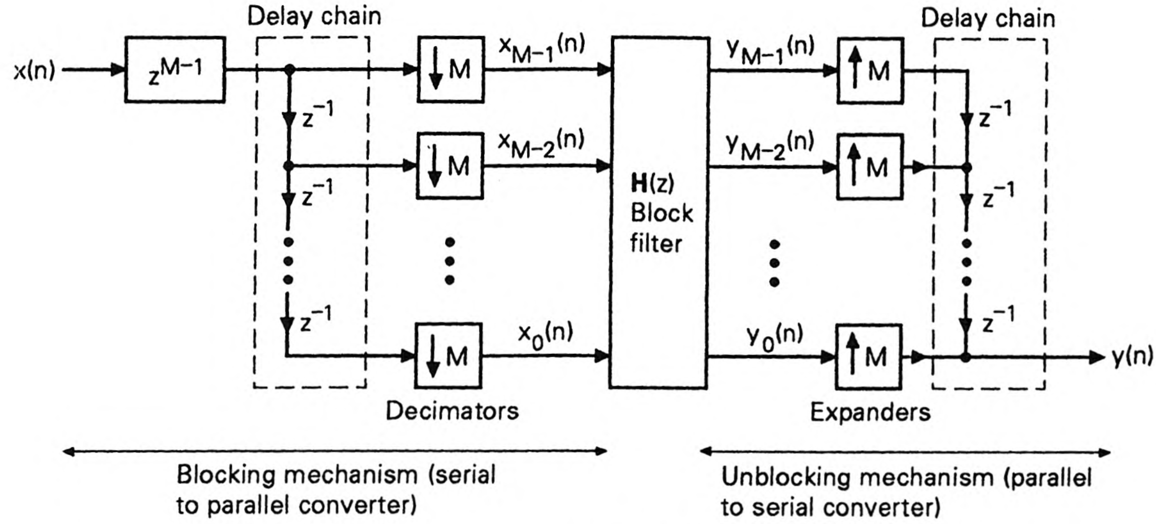


Figure 10.1-2 Representation of block digital filtering in terms of multirate building blocks.

Increased Parallelism Offered by the Block Filter

The structure of Fig. 10.1-2 also shows the "speed advantage" of blocking. The system $\mathbf{H}(z)$ is operating at a rate which is M times lower than the input rate. So the sampling rate of the input signal $x(n)$ can be M times larger than the speed of the basic computational unit. This advantage, which depends on the block size M, can be made arbitrarily large by increasing M. However, there is a price paid for this: since $\mathbf{H}(z)$ is an $M \times M$ system, it requires larger number of computational units (multipliers and adders) than the original scalar system. Summarizing, we have obtained increased computational parallelism in the blocked implementation, by increasing the number of computational units. As a result we are able to process signals which arrive at M times higher rate (than the rate that can normally be handled by *one* computational unit).

Relation to Alias-Free Filter Banks

The decimators and expanders in the representation of Fig. 10.1-2 produce the alias components $X(zW_M^k)$, just as in a filter bank. However, magically, $Y(z)$ is free from these alias-components because, by definition of $\mathbf{H}(z)$, the overall system in Fig. 10.1-2 is still a linear time invariant system with transfer function $H(z)$. The explanation of this is that, alias components are somehow canceled.

Returning now to filter banks, we know that any M-channel maximally decimated filter bank (Fig. 5.4-1) can be redrawn as in Fig. 5.5-3(c), where $\mathbf{P}(z)$ is the product $\mathbf{R}(z)\mathbf{E}(z)$ of polyphase matrices. The structures of Figs. 5.5-3(c) and 10.1-2 are identical (except for the advance operator z^{M-1}, which will not affect any significant conclusions.) Since the structure of Fig. 10.1-2 is indeed alias-free by definition of $\mathbf{H}(z)$, we conclude that this structure is equivalent to an M-channel alias-free maximally decimated filter bank.

We know from Sec. 5.7.2 that the filter bank is alias-free *if, and only if,* $\mathbf{P}(z)$ is a pseudocirculant. This shows that the blocked version $\mathbf{H}(z)$ of a scalar transfer function $H(z)$ is *necessarily* a pseudocirculant. The pseudocirculant property has been observed implicitly in Barnes and Shinnaka [1980], and also in Marshall [1982]. It has been further studied in Vaidyanathan and Mitra [1988].

Next, how can we determine the elements of $\mathbf{H}(z)$? The pseudocirculant property means that all rows are determined by the elements $H_{0,k}(z)$ of the 0th row. We also know that the transfer function of the alias-free filter bank is given by (5.7.13) where $P_\ell(z)$ are the elements of the 0th row of $\mathbf{P}(z)$. From this we conclude that the scalar filter $H(z)$ is related to the blocked version as

$$H(z) = \sum_{\ell=0}^{M-1} z^{-\ell} H_{0,\ell}(z^M). \tag{10.1.5}$$

So the elements of the 0th row of $\mathbf{H}(z)$ are the Type 1 polyphase components (usually denoted $E_\ell(z)$) of $H(z)$:

$$H_{0,\ell}(z) = E_\ell(z). \tag{10.1.6}$$

Example 10.1.1

Let $H(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$. We can rewrite

$$H(z) = [1 + 4z^{-3}] + 2z^{-1} + 3z^{-2} \quad \text{(scalar filter)}, \tag{10.1.7}$$

from which we identify the Type 1 polyphase components (for $M = 3$) as $E_0(z) = 1 + 4z^{-1}$, $E_1(z) = 2$, and $E_2(z) = 3$. So the 3×3 blocked version is the pseudocirculant

$$\begin{bmatrix} 1+4z^{-1} & 2 & 3 \\ 3z^{-1} & 1+4z^{-1} & 2 \\ 2z^{-1} & 3z^{-1} & 1+4z^{-1} \end{bmatrix} \quad \text{(blocked filter)}. \tag{10.1.8}$$

Next consider an IIR example; let $H(z) = (-a + z^{-1})/(1 - az^{-1})$. This can be written as

$$H(z) = \frac{-a(1 - z^{-2})}{1 - a^2 z^{-2}} + \frac{(1 - a^2)z^{-1}}{1 - a^2 z^{-2}} \quad \text{(scalar filter)}, \tag{10.1.9}$$

so that the Type 1 polyphase components (for $M = 2$) are $E_0(z) = -a(1 - z^{-1})/(1 - a^2 z^{-1})$, and $E_1(z) = (1 - a^2)/(1 - a^2 z^{-1})$. So the 2×2 blocked version is the pseudocirculant

$$\mathbf{H}(z) = \frac{1}{1 - a^2 z^{-1}} \begin{bmatrix} -a(1 - z^{-1}) & 1 - a^2 \\ (1 - a^2)z^{-1} & -a(1 - z^{-1}) \end{bmatrix} \quad \text{(blocked filter)}. \tag{10.1.10}$$

For real a, the scalar IIR filter $H(z)$ is allpass, that is, $\widetilde{H}(z)H(z) = 1$. How did this allpass property reflect into the blocked version? In Problem 10.2 we request the reader to verify the interesting fact that $\widetilde{\mathbf{H}}(z)\mathbf{H}(z) = \mathbf{I}$. In other words, $\mathbf{H}(z)$ is paraunitary!

More generally, we can summarize the above results as follows.

♠**Theorem 10.1.1. On blocked version of a scalar filter.** Let $\mathbf{H}(z)$ represent the $M \times M$ blocked version of a scalar transfer function $H(z)$. Then $\mathbf{H}(z)$ is a pseudocirculant, and its 0th row is given by

$$[\, E_0(z) \quad E_1(z) \quad \ldots \quad E_{M-1}(z) \,], \tag{10.1.11}$$

where $E_\ell(z)$ are the Type 1 polyphase components of $H(z)$ [i.e., $H(z) = \sum_{\ell=0}^{M-1} z^{-\ell} E_\ell(z^M)$]. Moreover $\mathbf{H}(z)$ is paraunitary if and only if $H(z)$ is allpass, that is, $\widetilde{H}(z)H(z) = c$ if and only if $\widetilde{\mathbf{H}}(z)\mathbf{H}(z) = c\mathbf{I}$. ◊

Proof. It only remains to prove the part of the statement involving the paraunitary property. With the 0th row of pseudocirculant $\mathbf{H}(z)$ given by (10.1.11), the kth row ($k > 0$) is

$$[\, z^{-1}E_{M-k}(z) \quad \ldots \quad z^{-1}E_{M-1}(z) \quad E_0(z) \quad \ldots \quad E_{M-k-1}(z)\,]. \tag{10.1.12}$$

Since $E_\ell(z)$ are the Type 1 polyphase components of $H(z)$, it is easily verified (Problem 10.4) that the polyphase components of $z^{-k}H(z)$ are the elements of the kth row above. So we can express

$$z^{-k}H(z) = \sum_{\ell=0}^{M-1} z^{-\ell}H_{k\ell}(z^M). \tag{10.1.13}$$

By writing this for all values of k ($0 \le k \le M-1$) we obtain the matrix equation

$$\begin{bmatrix} H(z) \\ z^{-1}H(z) \\ \vdots \\ z^{-(M-1)}H(z) \end{bmatrix} = \mathbf{H}(z^M)\mathbf{e}(z), \tag{10.1.14}$$

where $\mathbf{e}(z) = [\,1 \quad z^{-1} \quad \ldots \quad z^{-(M-1)}\,]^T$. Since (10.1.14) holds for all z, it holds if we replace z with zW^{-k}, where $W = e^{-j2\pi/M}$. By doing this for $k = 0, \ldots M-1$, we arrive at M equations which can be collected together as follows:

$$\mathbf{\Lambda}(z)\mathbf{W}\mathbf{Q}(z) = \mathbf{H}(z^M)\mathbf{\Lambda}(z)\mathbf{W}, \tag{10.1.15}$$

where

$$\begin{aligned} \mathbf{\Lambda}(z) &= \text{diag}\,[\,1 \quad z^{-1} \quad \ldots \quad z^{-(M-1)}\,], \\ \mathbf{Q}(z) &= \text{diag}\,[\,H(z) \quad H(zW^{-1}) \quad \ldots \quad H(zW^{-(M-1)})\,], \end{aligned} \tag{10.1.16}$$

and $\mathbf{W}$ is the $M \times M$ DFT matrix. Clearly $\mathbf{W}$ and $\mathbf{\Lambda}(z)$ are paraunitary. So $\mathbf{Q}(z)$ is paraunitary if and only if $\mathbf{H}(z)$ is paraunitary. But since $\mathbf{Q}(z)$ is diagonal with elements $H(zW^{-k})$, it is paraunitary if and only if $H(z)$ is allpass. This completes the proof. $\nabla\nabla\nabla$

Application to alias-free filter banks. For an alias-free filter bank, $\mathbf{P}(z)$ is pseudocirculant and the distortion function $T(z)$ is given by (5.7.13). From the above theorem we conclude that $T(z)$ is allpass (i.e., the filter bank is free from amplitude distortion) if and only if $\mathbf{P}(z)$ is paraunitary.

10.1.2 Linear Periodically Time Varying (LPTV) Systems

In this text, we have seen linear time varying (LTV) systems on many occassions. The decimator and expander, defined in Chap. 4, are examples of such systems. The fractional sampling rate changer (Fig. 4.1-10(b)) is another such example. In these examples the input and output signals have different rates.

For a more sophisticated example, consider the M-channel maximally decimated filter bank (Fig. 5.4-1). In Chap. 5 we found that this is an LTV system, characterized by the input output relation (5.4.5). This relation reduces to $\widehat{X}(z) = T(z)X(z)$, i.e., the filter bank becomes an LTI system, if and only if it is alias-free.

Recall that an LTI system is characterized by an impulse response $h(n)$ such that the output $y(n)$ is computed by convolution:

$$y(n) = \sum_m h(m)x(n-m). \tag{10.1.17}$$

For an LTV system (with input rate = output rate), $y(n)$ is still a linear combination of the samples $x(n-m)$ as above, but $h(m)$ is not fixed; it depends on the output time index n. The relation is of the form

$$y(n) = \sum_m a_n(m)x(n-m). \tag{10.1.18}$$

This idea is best understood by drawing a schematic structure, as shown in Fig. 10.1-3. Here we have an Nth order FIR filter, whose impulse response coefficients are not fixed (as for LTI systems) but varies with output time index n.

An LPTV system (with period M) has the further property that $a_n(m)$ is a periodic function (period M) of the output time index n [which is *also* the subscript on $a(m)$ in (10.1.18)]. In other words,

$$a_n(m) = a_{n+M}(m), \quad \text{for all } n, m. \tag{10.1.19}$$

Figure 10.1-4 demonstrates an LPTV system with period = 2. Whenever n is even, the output is taken to be that of the filter with impulse response $a_0(m)$. When n is odd, the ouput is taken to be that of $a_1(m)$. This behavior can be compactly represented using multirate notation as shown in Fig. 10.1-5(a). Here the filter $A_n(z)$ is given by

$$A_n(z) \stackrel{\Delta}{=} \sum_m a_n(m)z^{-m}. \tag{10.1.20}$$

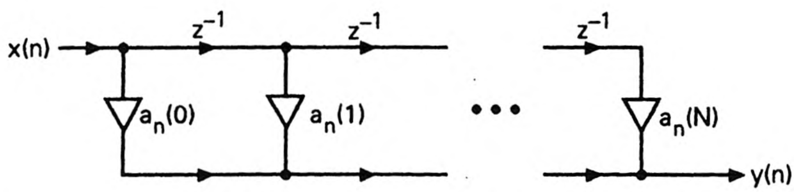


Figure 10.1-3 An LTV FIR filter. Here $a_n(m)$ is a function of the output time index n.

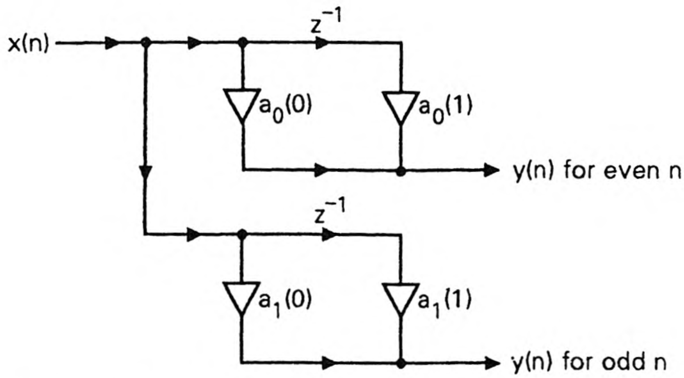
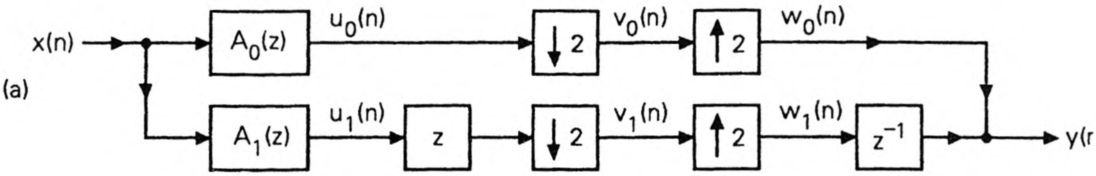


Figure 10.1-4 An FIR LPTV system with period two.



(b)

n	0	1	2	3	4	5	...
$u_0(n)$	y(0)	x	y(2)	x	y(4)	x	...
$u_1(n)$	x	y(1)	x	y(3)	x	y(5)	...
$v_0(n)$	y(0)	y(2)	y(4)	...			
$v_1(n)$	y(1)	y(3)	y(5)	...			
$w_0(n)$	y(0)	0	y(2)	0	y(4)	0	...
$w_1(n)$	y(1)	0	y(3)	0	y(5)	0	...
y(n)	y(0)	y(1)	y(2)	y(3)	y(4)	y(5)	...

Figure 10.1-5 (a) Representation of an LPTV system in terms of multirate notations. (b) Explaining the operation.

Relation to Filter Banks

Extending the above discussion, the more general case where the LPTV system has period M can similarly be represented by the structure of Fig. 10.1-6. (This representation is restricted to systems where the input and output have equal rates; but the system can be FIR or IIR.) The system output $y(n)$ at time n is equal to the output of $A_k(z)$ at time n, where $k = n \bmod M$.

We can think of this system as a M channel filter bank with analysis and synthesis filters

$$H_n(z) = z^{M-1-n} A_{M-1-n}(z), \quad F_n(z) = z^{-(M-1-n)}. \tag{10.1.21}$$

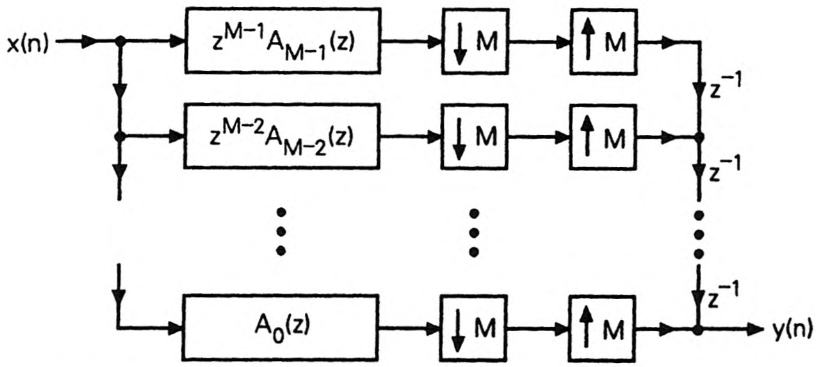


Figure 10.1-6 Representation of an arbitrary (FIR or IIR) LPTV system of period M, using multirate building blocks. (Input rate = output rate).

By using the polyphase decomposition on the filters $A_n(z)$, we gain further insight. Thus, let the filters $A_n(z)$ be represented as

$$A_n(z) = \sum_{\ell=0}^{M-1} z^{-\ell} G_{n,\ell}(z^M), \quad 0 \le n \le M-1. \tag{10.1.22}$$

Let $\mathbf{E}(z)$ be the Type 1 polyphase component of the analysis bank, and $\mathbf{R}(z)$ the Type 2 polyphase matrix of synthesis bank. Clearly $\mathbf{R}(z) = \mathbf{I}$ in this case. The quantity $\mathbf{E}(z)$, on the other hand depends on $G_{n,\ell}(z)$. As a demonstration, for $M = 3$ we can verify that

$$\mathbf{E}(z) = \begin{bmatrix} G_{2,2}(z) & zG_{2,0}(z) & zG_{2,1}(z) \\ G_{1,1}(z) & G_{1,2}(z) & zG_{1,0}(z) \\ G_{0,0}(z) & G_{0,1}(z) & G_{0,2}(z) \end{bmatrix}. \tag{10.1.23}$$

For arbitrary M, the form of $\mathbf{E}(z)$ can be written in a similar way (Problem 10.5). Thus the LPTV system is equivalent to Fig. 10.1-7 where $\mathbf{P}(z) = \mathbf{R}(z)\mathbf{E}(z)$.

We know from filter bank theory that this system is alias-free (hence time invariant) if and only if $\mathbf{P}(z)$ is pseudocirculant. Now let us see what happens to $G_{n,\ell}(z)$ when $\mathbf{P}(z)$ [i.e., $\mathbf{E}(z)$] is pseudocirculant. By insepection of (10.1.23) we conclude that under this condition $G_{n,\ell}(z)$ is independent of

n. This means that $A_n(z)$ is same for all n in Fig. 10.1-6, so that the original LPTV system becomes *time invariant!* This is consistent with the fact that the maximally decimated filter bank is time invariant if and only if it is alias free.

The main points of this section are summarized in Table 10.1.1.

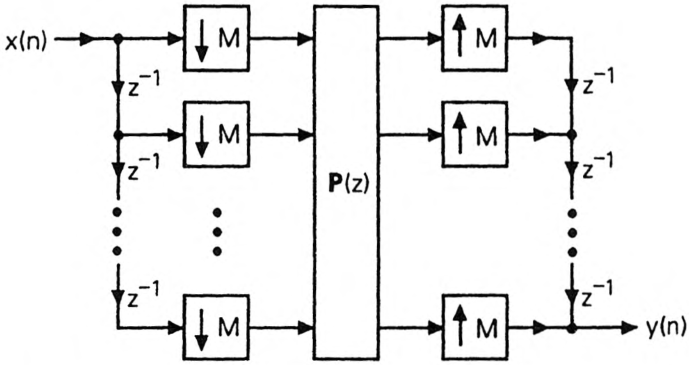


Figure 10.1-7 Equivalent representation of an arbitrary LPTV system of period M (input rate = output rate). The matrix $\mathbf{P}(z)$ is a pseudocirculant if and only if the LPTV system degenerates into an LTI system.

10.2 UNCONVENTIONAL SAMPLING THEOREMS

Let $x_a(t)$ be a continuous-time signal and define

$$x_a^{(s)}(t) = \sum_{n=-\infty}^{\infty} x_a(nT)\delta_a(t - nT), \tag{10.2.1}$$

where $\delta_a(t)$ is the impulse function defined in Chap. 2. $x_a^{(s)}(t)$ is the uniformly sampled version of $x_a(t)$, with sample spacing equal to T (Fig. 10.2-1). Equivalently, the sampling frequency (or rate) is $2\pi/T$. The Fourier transform of $x_a^{(s)}(t)$ is given by the relation

$$X_a^{(s)}(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_a\Big(j(\Omega - \frac{2\pi k}{T})\Big), \tag{10.2.2}$$

provided this summation converges [Oppenheim and Schafer, 1989]. Thus $X_a^{(s)}(j\Omega)$ is obtained by adding to $X_a(j\Omega)$ an infinite number of shifted copies (images), the shift being in uniform integer multiples of $2\pi/T$.

Figure 10.2-2 is a demonstration of this effect. In this figure we have assumed that $x_a(t)$ is σ-BL (defined in Sec. 2.1.4), that is, $|X_a(j\Omega)| = 0$ for $|\Omega| \geq \sigma$. From Sec. 2.1.4 we know that if the sampling rate $2\pi/T$ exceeds the Nyquist rate $\Theta = 2\sigma$, then none of the images has an overlap with the

TABLE 10.1.1 Block filtering, LPTV systems, and filter banks.

1. **Block digital filters.** Given a scalar transfer function $H(z)$ with input $x(n)$ and output $y(n)$, define the vector sequences $\mathbf{x}_B(n)$ and $\mathbf{y}_B(n)$ as in (10.1.3). Then these are related by an $M \times M$ transfer function $\mathbf{H}(z)$, called the blocked version of $H(z)$. Fig. 10.1-2 shows this block implementation of $H(z)$.
   a) The blocked version $\mathbf{H}(z)$ is pseudocirculant. The scalar function $H(z)$ can be obtained from the 0th row of $\mathbf{H}(z)$ as
   $$H(z) = \sum_{\ell=0}^{M-1} z^{-\ell} H_{0,\ell}(z^M). \qquad (T10.1)$$
   b) Conversely, if $\mathbf{H}(z)$ is pseudocirculant, the structure of Fig. 10.1-2 is an LTI system (this being not true for arbitrary $\mathbf{H}(z)$) and $\mathbf{H}(z)$ represents the blocked version of the scalar transfer function given by (T10.1).
2. **Linear periodically time varying systems.** A linear periodically time varying (LPTV) system with period M (and same input and output rates) is characterized by a set of M transfer functions $A_n(z)$. The system can be represented by the structure of Fig. 10.1-6.
   a) The output at time n is equal to the output of $A_k(z)$ at time n, where $k = n \bmod M$.
   b) An LPTV system with period M (with equal input and output rates) can *always* be represented by the equivalent structure of Fig. 10.1-7, where $\mathbf{P}(z)$ is an $M \times M$ transfer matrix.
   c) Conversely, for arbitrary transfer matrix $\mathbf{P}(z)$, Fig. 10.1-7 represents an LPTV system (with equal input and output rates) of period M.
3. **Relation to filter banks.** An M-channel maximally decimated filter bank (Fig. 5.4-1) can always be represented by the structure of Fig. 10.1-7 where $\mathbf{P}(z)$ is the product $\mathbf{R}(z)\mathbf{E}(z)$ of the polyphase matrices of the analysis and synthesis banks.
   a) This representation closely resembles the block implementation of a scalar transfer function $H(z)$ (Fig. 10.1-2).
   b) The representation also resembles the general representation of an LPTV system (with equal input and output rates).
   c) The blocked version $\mathbf{H}(z)$ of a scalar $H(z)$ is always pseudocirculant; the filter bank is alias-free if and only if $\mathbf{P}(z)$ in Fig. 10.1-7 is pseudocirculant; the LPTV system is actually time *invariant* if and only if $\mathbf{P}(z)$ in Fig. 10.1-7 is pseudocirculant.
   d) Let $\mathbf{H}(z)$ be $M \times M$ pseudocirculant. Consider the transfer function $H(z)$ defined in (T10.1) above. $H(z)$ is allpass if and only if $\mathbf{H}(z)$ is paraunitary. This means two things: (i) the distortion function $T(z)$ of an alias-free filter bank is allpass if and only if $\mathbf{P}(z)$ is paraunitary, and (ii) a scalar transfer function is allpass if and only if its blocked version is paraunitary.

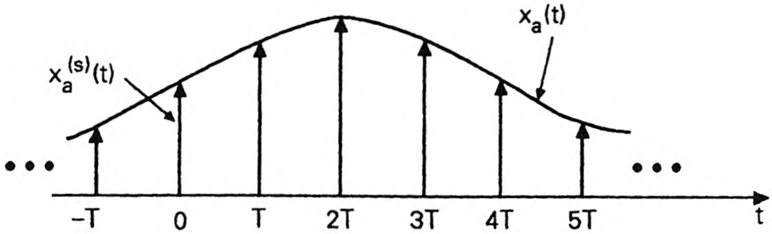


Figure 10.2-1 A continuous-time signal $x_a(t)$ and the uniformly sampled version $x_a^{(s)}(t)$.

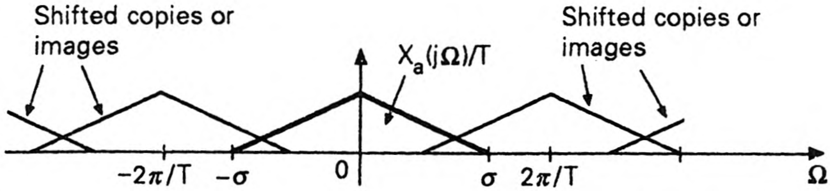


Figure 10.2-2 Fourier transform of the sampled version $x_a^{(s)}(t)$ of a bandlimited signal $x_a(t)$. Sampling rate $= 2\pi/T$.

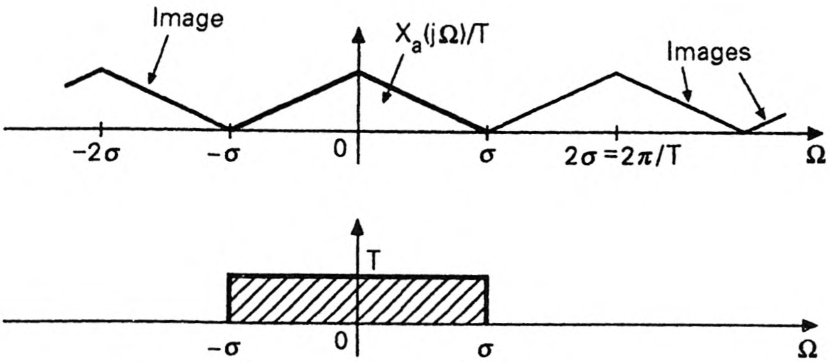


Figure 10.2-3 (a) Fourier transform of the sampled version $x_a^{(s)}(t)$ of a σ-bandlimited signal $x_a(t)$ sampled at Nyquist rate 2σ (i.e. $2\pi/T = 2\sigma$). (b) The ideal lowpass filter which reconstructs $x_a(t)$ from $x_a^{(s)}(t)$.

original version $X_a(j\Omega)$. In this case we can reconstruct $x_a(t)$ from $x_a^{(s)}(t)$ by removing the images with an ideal lowpass filter (Fig. 10.2-3). This filter has impulse response

$$h(t) = \frac{\sin(\pi t/T)}{(\pi t/T)}. \tag{10.2.3}$$

In the time domain, the recovered signal $x_a(t)$ is therefore the convolution of

$x_a^{(s)}(t)$ with $h(t)$. This simplifies to the well-known reconstruction formula,

$$x_a(t) = \sum_{n=-\infty}^{\infty} x_a(nT) \frac{\sin(\pi/T)(t - nT)}{(\pi/T)(t - nT)}, \tag{10.2.4}$$

which is also called the "interpolation" formula. The Nyquist frequency $\Theta = 2\sigma$ is the minimum rate at which $x_a(t)$ should be sampled so that it can be recovered from these samples.† Sampling at the rate Θ is called Nyquist sampling.

The above result is the *uniform sampling* theorem [Nyquist, 1928], [Whittaker, 1929], and [Shannon, 1949]: if we "uniformly sample" the σ-BL signal $x_a(t)$ at the Nyquist rate Θ, then we do not lose any information, and can reconstruct $x_a(t)$ from these samples using (10.2.4). It should be noticed, however, that (10.2.4) is equivalent to passing $x_a^{(s)}(t)$ through an ideal lowpass filter (10.2.3). This filter is noncausal and unstable [since $h(t)$ is not absolutely integrable]. In practice, we have to live with an approximation of $h(t)$, and the reconstruction is not exact. If the sampling rate $2\pi/T$ exceeds the minimum required rate Θ by a significant margin, then the images in Fig. 10.2-3 are more widely separated from the main term. So the lowpass filter can have a wider transition bandwidth and the reconstruction can be done more accurately with a practical filter.

Unconventional Sampling

Instead of sampling the σ-BL signal $x_a(t)$ at the Nyquist rate Θ, suppose we sample $x_a(t)$ and its derivative $\dot{x}_a(t)$ at *half* the Nyquist rate. It is possible to recover $x_a(t)$ from these two undersampled signals. This was actually shown in Problem 5.13, by formulating this as a two-channel analog QMF bank problem. By using an M-channel analog QMF formulation, it is possible to derive other extensions of the sampling theorem. For example, if we sample $x_a(t)$ and its $M-1$ derivatives at the rate Θ/M, we can recover $x_a(t)$ from this information (Problem 10.7). As seen from these Problems, the reconstruction filters are unrealizable, and should be replaced with practical approximations. (To be fair, the lowpass reconstruction filter (10.2.3) used in the case of uniform sampling is also unrealizable.) In the next subsection, we will obtain the discrete-time analog of this result, called the difference-sampling theorem. In contrast to the continuous-time case, this theorem involves practical (in fact FIR) reconstruction filters.

Another generalization of sampling is the so-called nonuniform sampling, demonstrated in Fig. 10.2-4. Here the samples are spaced 'too far

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† This assumes, of course, that no further information is available about $x_a(t)$ except that it is σ-BL. If this is not true, then the situation is different. For example, if $x_a(t)$ is *known* to be a sinusoid $A\sin(\omega_0 t + \beta)$, then it can be recovered from a *finite number* of samples since we need to extract only three pieces of information (A, ω_0, and β) from the samples! Similar comment holds if $x_a(t)$ is known to be a sum of finite number of sinusoids.

apart' (compared to Nyquist rate Θ) in some regions and 'too close' in some regions. Yet, theory has it [Jerri, 1977] that we can recover $x_a(t)$ from such samples as long as the 'average sampling rate' $\geq \Theta$. (A special case is the situation when only the *past values* of $x_a(t)$ are sampled at the rate 2Θ!) We will not prove this general result, as it does not place in evidence practical reconstruction techniques. In Sec. 10.2.2 we prove more practical special cases, for which reconstruction techniques can be found based on a FIR digital filter bank approach.

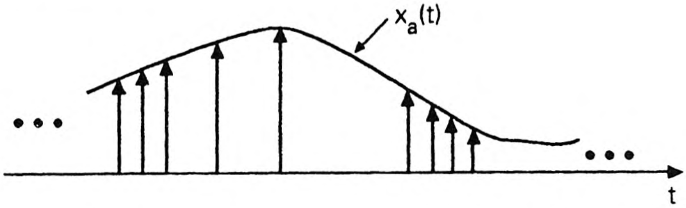

Figure 10.2-4 Nonuniform sampling of $x_a(t)$.

10.2.1 Difference Sampling Theorems for Sequences

We will now discuss difference sampling theorems, which can be considered as discrete time counterparts of derivative sampling theorems. We begin with an example.

Example 10.2.1

Let $x(n)$ be an arbitrary sequence, and let $x_1(n)$ denote its first difference, i.e.,

$$x_1(n) = x(n) - x(n-1). \tag{10.2.5}$$

Consider the two sequences

$$y_0(n) = x(2n), \quad y_1(n) = x_1(2n). \tag{10.2.6}$$

These are the two-fold decimated versions of $x(n)$ and its first-difference. Can we recover $x(n)$ from these two undersampled sequences? [Evidently, the number of samples per unit time, counting both the signals $y_0(n)$ and $y_1(n)$, is the same as that for $x(n)$.] The even numbered samples of $x(n)$ are already available in $y_0(n)$. It only remains to see if the odd numbered samples can be recovered from $y_1(n)$. Now $y_1(n)$ has samples

$$\ldots x(-2) - x(-3), \quad x(0) - x(-1), \quad x(2) - x(1), \quad x(4) - x(3) \ldots \tag{10.2.7}$$

From this it is clear that we can recover all odd-numbered samples of $x(n)$ by subtracting out the even-numbered samples from these differences.

A more systematic approach will help us to extend this idea to the case of higher order differences. For this we view the problem as a two channel QMF problem, as shown in Fig. 10.2-5. The analysis filters are $H_0(z) = 1$, and $H_1(z) = 1 - z^{-1}$ (representing the first-difference operation). The aim is to find synthesis filters $F_0(z), F_1(z)$ such that we have perfect reconstruction. This is an easy problem and is made easier by use of the polyphase approach. Thus, we can redraw the system as in Fig. 10.2-6, where

$$\mathbf{E} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}. \qquad (10.2.8)$$

It is clear that the choice $\mathbf{R} = \mathbf{E}^{-1}$ results in perfect recovery, that is, $\widehat{x}(n) = x(n-1)$. It is readily verified that the matrix $\mathbf{E}$ is its own inverse, so we take

$$\mathbf{R} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}. \qquad (10.2.9)$$

The synthesis filters are now computed according to

$$[\,F_0(z) \quad F_1(z)\,] = [\,z^{-1} \quad 1\,]\,\mathbf{R} = [\,z^{-1} \quad 1\,]\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}. \qquad (10.2.10)$$

This simplifies to $F_0(z) = 1+z^{-1}$ and $F_1(z) = -1$. If these filters are used in Fig. 10.2-5, we have perfect reconstruction that is, $\widehat{x}(n) = x(n-1)$.

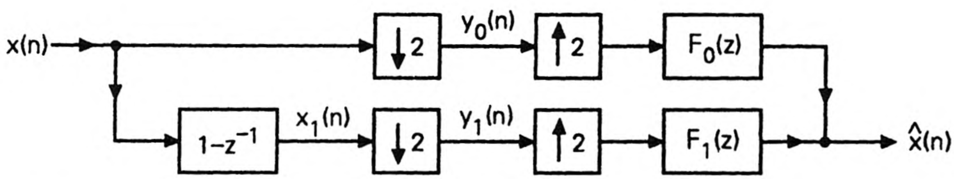

Figure 10.2-5 The difference sampling and reconstruction, viewed as a QMF bank problem.

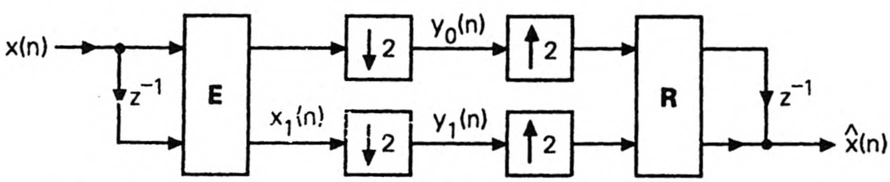

Figure 10.2-6 Redrawing Fig. 10.2-5 using the polyphase notation.

The above example can be considered to be the discrete-time equivalent of the derivative sampling theorem discussed above for continuous-time signals. (Instead of derivatives, we have differences.) Notice however, that the reconstruction scheme is very simple (and realizable), involving only FIR filters.

One motivation for thinking about such 'sampling theorems' is demonstrated in Fig. 10.2-7, where $x(n)$ is a slowly varying sequence, i.e., the adjacent samples differ by a 'very small' amount. Assume that each sample $x(n)$ requires 16 bits for its representation. Let us say that the differences $x(n) - x(n-1)$, being very small, require only 8 bits for their representation. Now instead of 'storing' or 'transmitting' all samples of $x(n)$ with 16 bits per sample, we can store (two-fold) decimated versions of $x(n)$ (16 bits per sample) and the first difference (8 bits per sample). This reduces the data rate to an average of 12 bits per sample. This is similar in principle to subband coding (Sec. 4.5.2).

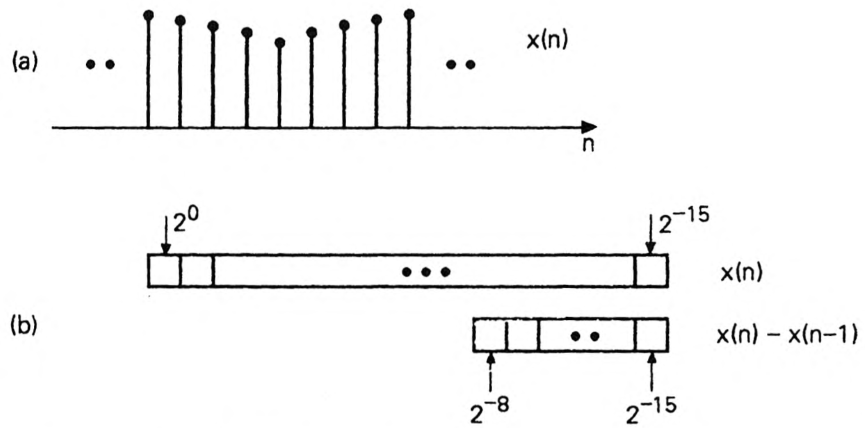

Figure 10.2-7 (a) A slowly varying signal $x(n)$, and (b) binary representations for $x(n)$ and its first difference.

Extension to Higher Order Differences

Can we extend this idea for higher differences? We can define the second difference in terms of the first difference $x_1(n)$ as $x_2(n) = x_1(n) - x_1(n-1)$, and so on. Thus, let $x_k(n), 1 \leq k \leq M-1$ denote the first $M-1$ differences of the signal $x(n)$. We wish to recover $x(n)$ from the M-fold decimated versions

$$y_k(n) = x_k(Mn), \quad 0 \leq k \leq M-1. \tag{10.2.11}$$

[For $k = 0$ we define $x_0(n) = x(n)$, i.e., the original sequence.] Once again, this problem can be handled using the filter-bank approach. For this note that the kth difference $x_k(n)$ has the z-transform

$$X_k(z) = (1 - z^{-1})X_{k-1}(z) = (1 - z^{-1})^k X(z), \tag{10.2.12}$$

so that the kth difference operator is the transfer function

$$H_k(z) = (1 - z^{-1})^k. \tag{10.2.13}$$

Thus the difference sampling scheme can be represented by a maximally decimated analysis bank (Fig. 10.2-8). If this is redrawn using the polyphase notation (Sec. 5.5), the $\mathbf{E}(z)$ matrix is a constant given by

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & -1 & 0 & \dots & 0 \\ 1 & -2 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{bmatrix} \qquad (10.2.14)$$

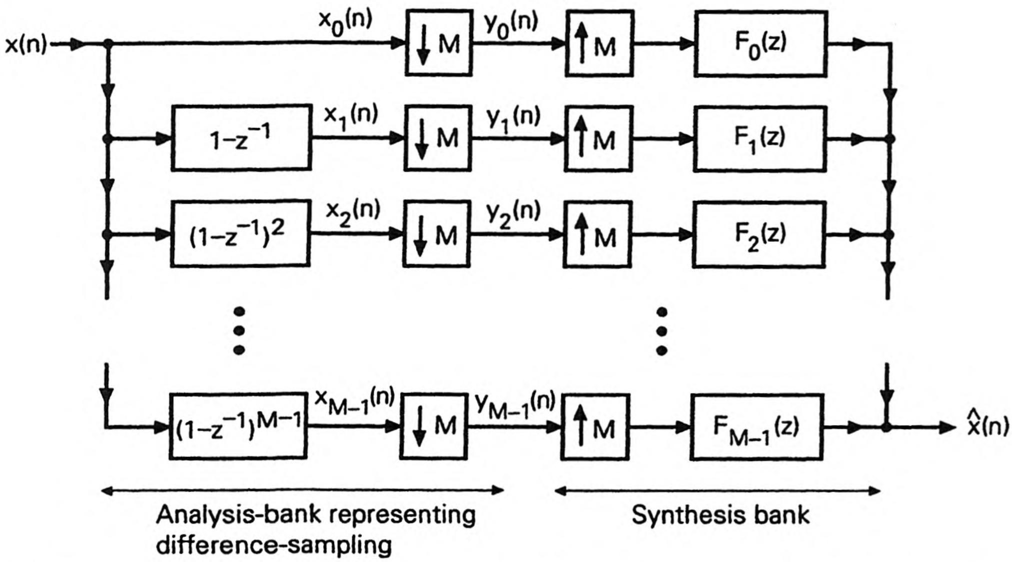

Figure 10.2-8 The difference sampling and reconstruction, posed as a QMF problem.

The rows of this $M \times M$ matrix are coefficients of $(1 - z^{-1})^k$, which are the binomial coefficients with alternating sign. It can be shown (see below) that this matrix is its own inverse so that we can take $\mathbf{R}(z) = \mathbf{E}$ in Fig. 5.5-3(b), for perfect reconstruction. In other words, if the synthesis filters are chosen as

$$[\,F_0(z) \quad F_1(z) \quad \dots \quad F_{M-1}(z)\,] = [\,1 \quad z^{-1} \quad \dots \quad z^{-(M-1)}\,]\,\mathbf{E} \qquad (10.2.15)$$

the reconstructed signal is given by $\widehat{x}(n) = x(n - M + 1)$.

Proof that E is its own inverse. By definition, $\mathbf{E}$ has the following property:

$$\mathbf{E}\mathbf{v}(x) = \mathbf{v}(1 - x), \qquad (10.2.16)$$

where $\mathbf{v}(x) \triangleq [\,1 \quad x \quad \dots \quad x^{M-1}\,]^T$. From this we obtain

$$\mathbf{E}^2\mathbf{v}(x) = \mathbf{v}(x). \qquad (10.2.17)$$

Since this holds for all x, we conclude, in particular, that

$$\mathbf{E}^2 \underbrace{[\mathbf{v}(x_0) \quad \mathbf{v}(x_1) \ldots \quad \mathbf{v}(x_{M-1})]}_{\mathbf{V}} = [\mathbf{v}(x_0) \quad \mathbf{v}(x_1) \ldots \quad \mathbf{v}(x_{M-1})]. \tag{10.2.18}$$

The matrix indicated as $\mathbf{V}$ is an $M \times M$ Vandermonde matrix and is non-singular if x_i are distinct (Appendix A). So it can be canceled in the above equation, yielding $\mathbf{E}^2 = \mathbf{I}$, i.e., $\mathbf{E}^{-1} = \mathbf{E}$. $\nabla\nabla\nabla$

Example 10.2.2

Consider the case when $M = 3$. The difference signals are

$$x_1(n) = x(n) - x(n-1), \quad x_2(n) = x_1(n) - x_1(n-1). \tag{10.2.19}$$

The decimated signals from which we wish to recover $x(n)$ are $y_0(n) = x(3n), y_1(n) = x_1(3n), y_2(n) = x_2(3n)$. The reconstruction is done using the synthesis bank

$$[F_0(z) \quad F_1(z) \quad F_2(z)] = [z^{-2} \quad z^{-1} \quad 1] \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & -2 & 1 \end{bmatrix}. \tag{10.2.20}$$

so that $F_0(z) = 1 + z^{-1} + z^{-2}, F_1(z) = -2 - z^{-1}$, and $F_2(z) = 1$.

Note that the synthesis filters are FIR, and that the highest required order is equal to $M - 1$. Compare this with the case of derivative sampling [e.g., Problem 5.13(d)] of continuous-time signals, where the synthesis filters are unrealizable. To be fair, it should be mentioned that $(1 - z^{-1})$ is only an approximate equivalent of the derivative operation; in fact if we perform bilinear transformation of $H(s) = s$ (which is a differentiator), we obtain $(1 - z^{-1})/(1 + z^{-1})$, which (is unstable and) represents the exact discrete-time equivalent of differentiation.

10.2.2 Nonuniform Sampling Theorems for Sequences

We now explain the concept of nonuniform "sampling" of sequences with an example. Consider a σ-BL sequence (i.e., a sequence $x(n)$ such that $X(e^{j\omega}) = 0$ for $\sigma \leq |\omega| \leq \pi$) with $\sigma = 2\pi/3$. Fig. 10.2-9(a) shows an example. In Sec. 4.1.1 we showed how we can decimate such a sequence by the noninteger quantity 3/2 to obtain the full band signal $Y(e^{j\omega})$ (Fig. 4.1-10). Fig. 4.1.-11 also shows the two signals $x(n)$ and $y(n)$ in the time domain. This can be considered to be uniform decimation by a factor of 3/2, since the samples $y(n)$ are still uniformly spaced in time.

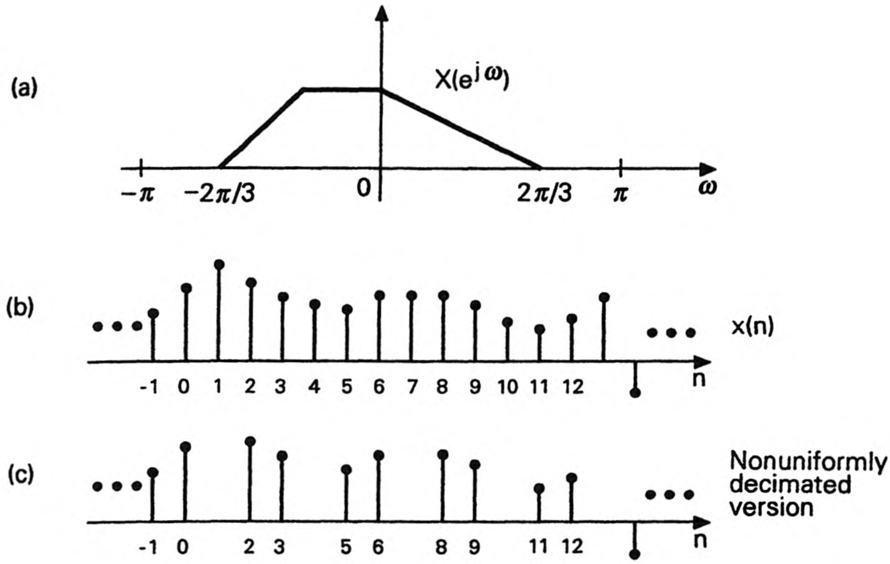

Figure 10.2-9 (a) Fourier transform of a bandlimited sequence, (b) the sequence $x(n)$, and (c) a nonuniformly decimated version.

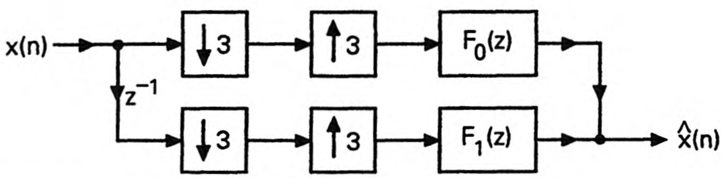

Figure 10.2-10 The nonuniform decimation and reconstruction, posed as a QMF problem.

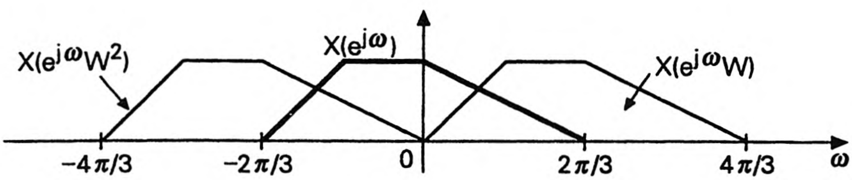

Figure 10.2-11 Demonstration of alias components of $X(e^{j\omega})$, which appear in nonuniform decimation.

But there is another (simpler) way to 'decimate' $x(n)$ by a factor $3/2$, which can be described as follows: (i) divide the time axis into intervals of length three, (ii) retain the first two samples in each interval, and discard the third. This is demonstrated in Fig. 10.2-9. The resulting sequence is a *nonuniformly* decimated version of $x(n)$. Can we recover $x(n)$ from this version?

. The answer is in the affirmative. We will show that this problem can be formulated as a multirate digital filter bank problem. The reconstruction of $x(n)$ from the nonuniformly decimated version is equivalent to finding a set of *synthesis filters* for perfect reconstruction. (The analysis filters are predetermined and are not under our control; see below). These synthesis filters are ideal (unrealizable) filters, but can actually be approximated using linear phase FIR filters, as we will demonstrate with practical designs.

Filter Bank Model for Nonuniform Sampling

Consider Fig. 10.2-10 which is a 3-channel maximally decimated filter bank, in which only two of the analysis filters are nonzero. More precisely we have $H_0(z) = 1, H_1(z) = z^{-1}$, and $H_2(z) = 0$. The analysis bank can be considered to be a nonuniform decimator, retaining only the samples indicated in Fig. 10-2-9(c). Our aim is to find synthesis filters such that $\widehat{x}(n) = x(n)$, under the assumption that $x(n)$ is bandlimited to $|\omega| < 2\pi/3$. (Without the bandlimited constraint we cannot do this because we have only two-thirds of the original number of samples per unit time.)

Solving for Synthesis Filters

First recall that the most general equations for perfect reconstruction are given by the AC matrix formulation (Sec. 5.4.3). From this we obtain

$$\begin{aligned} H_0(z)F_0(z) + H_1(z)F_1(z) &= 3, \\ H_0(zW)F_0(z) + H_1(zW)F_1(z) &= 0 \quad [\text{to eliminate } X(zW)], \\ H_0(zW^2)F_0(z) + H_1(zW^2)F_1(z) &= 0 \quad [\text{to eliminate } X(zW^2)], \end{aligned} \tag{10.2.21}$$

with $W = e^{-j2\pi/3}$. Substituting $H_0(z) = 1, H_1(z) = z^{-1}$, and $z = e^{j\omega}$, this gives rise to

$$\begin{bmatrix} 1 & 1 \\ 1 & W^{-1} \\ 1 & W^{-2} \end{bmatrix} \begin{bmatrix} F_0(e^{j\omega}) \\ \\ e^{-j\omega}F_1(e^{j\omega}) \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}. \tag{10.2.22}$$

In general we cannot solve for the two filters $F_0(z), F_1(z)$ to satisfy the *three* conditions (10.2.22). We can, however, make further progress by using the bandlimited property of $x(n)$. First, we constrain $F_0(z)$ and $F_1(z)$ such that

$$F_0(e^{j\omega}) = 0, \quad F_1(e^{j\omega}) = 0, \quad \frac{2\pi}{3} \le |\omega| \le \pi. \tag{10.2.23}$$

(We are discussing only ideal filters for the moment). This eliminates any alias components which occupy the region outside the band of $X(e^{j\omega})$. It only remains to cancel aliasing in the region $|\omega| < 2\pi/3$.

Figure 10.2-11 demonstrates typical plots of $X(e^{j\omega}W^k)$, for $k = 0, 1$ and 2. In the frequency region $0 \le \omega < 4\pi/3$, the quantity $X(e^{j\omega}W^2)$ is zero, so that the third equation in (10.2.21) need not be satisfied. So we have to choose $F_0(z)$ and $F_1(z)$ such that

$$\begin{bmatrix} 1 & 1 \\ 1 & W^{-1} \end{bmatrix} \begin{bmatrix} F_0(e^{j\omega}) \\ e^{-j\omega}F_1(e^{j\omega}) \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \qquad (10.2.24)$$

for $0 \leq \omega < 2\pi/3$. Similarly the alias component $X(e^{j\omega}W)$ is zero in the region $-4\pi/3 < \omega \leq 0$ so that the middle equation in (10.2.21) need not be considered in this region. So $F_0(z)$ and $F_1(z)$ have to satisfy

$$\begin{bmatrix} 1 & 1 \\ 1 & W^{-2} \end{bmatrix} \begin{bmatrix} F_0(e^{j\omega}) \\ e^{-j\omega}F_1(e^{j\omega}) \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \qquad (10.2.25)$$

for $-2\pi/3 < \omega \leq 0$. Summarizing, we choose $F_0(z)$ and $F_1(z)$ to satisfy the two equations (10.2.24) if $0 \leq \omega < 2\pi/3$, and the two equations (10.2.25) if $-2\pi/3 < \omega \leq 0$. For ω outside either of these regions, we set $F_0(e^{j\omega}) = 0$, and $F_1(e^{j\omega}) = 0$ as stated earlier.

After solving the above two sets of equations we arrive at the following results:

$$F_0(e^{j\omega}) = \begin{cases} e^{-j\omega}(1-c+js), & 0 \leq \omega < 2\pi/3 \\ e^{-j\omega}(1-c-js), & -2\pi/3 < \omega \leq 0 \\ 0 & \text{otherwise,} \end{cases} \qquad (10.2.26)$$

and

$$F_1(e^{j\omega}) = \begin{cases} (1-c-js), & 0 \leq \omega < 2\pi/3 \\ (1-c+js), & -2\pi/3 < \omega \leq 0 \\ 0 & \text{otherwise.} \end{cases} \qquad (10.2.27)$$

Here c and s are defined as $c = \cos(2\pi/3)$, $\quad s = -\sin(2\pi/3)$. Notice that the responses $F_0(e^{j\omega})$ and $F_1(e^{j\omega})$ are piecewise constants. Essentially, the frequency axis has been divided into three regions of equal widths (Fig. 10.2-12), and $F_k(e^{j\omega})$ takes on a fixed (complex) value in each of these regions. We say that $F_k(z)$ is a *multilevel* filter (Sec. 4.6.5.)

Implementing the Multilevel Synthesis Filters

The above solutions can be expressed neatly in terms of an ideal lowpass filter $G_L(e^{j\omega})$ and an ideal Hilbert transformer $G_H(e^{j\omega})$ (defined below). The ideal lowpass filter is

$$G_L(e^{j\omega}) = \begin{cases} 1 & |\omega| < 2\pi/3 \\ 0 & \text{otherwise,} \end{cases} \qquad (10.2.28)$$

and the ideal Hilbert transformer [Rabiner and Gold, 1975] is

$$G_H(e^{j\omega}) = \begin{cases} j & 0 < \omega < \pi \\ -j & -\pi < \omega < 0 \end{cases} \qquad (10.2.29)$$

(see Fig. 10.2-13). It is then clear that the above synthesis filters can be expressed as

$$F_0(z) = z^{-1}\Big(1 - c + sG_H(z)\Big)G_L(z),$$
$$F_1(z) = \Big(1 - c - sG_H(z)\Big)G_L(z). \tag{10.2.30}$$

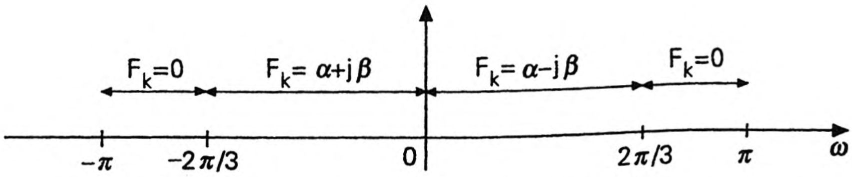

Figure 10.2-12 The interval $-\pi \leq \omega \leq \pi$ is divided into three equal regions. $F_k(e^{j\omega})$ is constant in each region. Also it has conjugate symmetry with respect to zero-frequency.

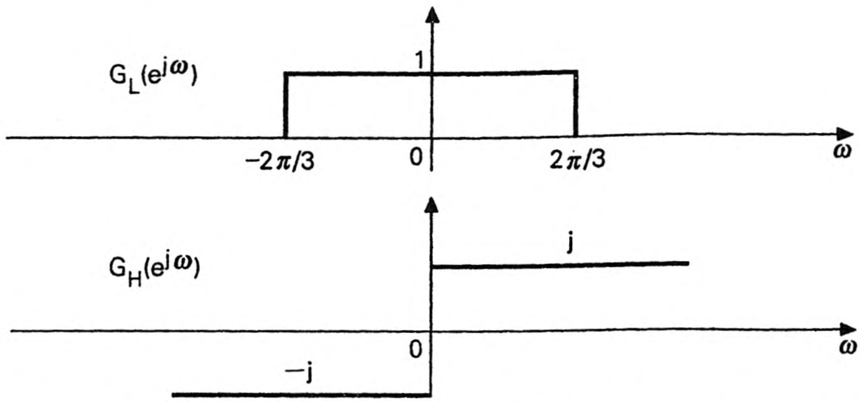

Figure 10.2-13 Definition of the ideal filters $G_L(e^{j\omega})$ and $G_H(e^{j\omega})$.

In practice, we can approximate the zero-phase filters $G_L(z)$ and $G_H(z)$ with real coefficient linear phase FIR filters $\widehat{G}_L(z)$ and $\widehat{G}_H(z)$ by using the McClellan-Parks algorithm as elaborated in Rabiner and Gold [1975]. Assuming that $\widehat{G}_H(z)$ has order $2K$, we have $\widehat{G}_H(e^{j\omega}) \approx e^{-j\omega K} \times G_H(e^{j\omega})$. So we can implement the synthesis bank as in Fig. 10.2-14. The extra delay z^{-K} in the top branch compensates for the group delay due to $\widehat{G}_H(z)$. The synthesis filters in this practical structure are

$$F_0(z) = z^{-1}\Big((1 - c)z^{-K} + s\widehat{G}_H(z)\Big)\widehat{G}_L(z),$$
$$F_1(z) = \Big((1 - c)z^{-K} - s\widehat{G}_H(z)\Big)\widehat{G}_L(z). \tag{10.2.31}$$

Reconstruction Error Created by Filter Approximation

The practical approximations (10.2.31) to the ideal solution evidently result in reconstruction error, so that the filter bank in Fig. 10.2-14 is not a perfect reconstruction system. For any maximally decimated filter bank, we know that

$$\widehat{X}(z) = T(z)X(z) + \text{alias terms.} \tag{10.2.32}$$

If aliasing terms have been sufficiently attenuated we have $\widehat{X}(z) \approx T(z)X(z)$, and the distortion function $T(z)$ reduces to

$$T(z) = (1-c)z^{-(K+1)}\widehat{G}_L(z). \tag{10.2.33}$$

This distortion is free from $\widehat{G}_H(z)$! This shows that the approximation error involved in the design of the Hilbert transformer does not affect $T(z)$; it affects only the extent to which alias-terms have been canceled. The lowpass filter $\widehat{G}_L(z)$ completely determines the amplitude and phase distortions in the reconstructed signal $\widehat{x}(n)$.

Summarizing, the Hilbert transformer $\widehat{G}_H(z)$ controls the extent to which aliasing has been canceled in the range $|\omega| < 2\pi/3$, whereas the lowpass filter $\widehat{G}_L(z)$ suppresses aliasing components in the range $|\omega| > 2\pi/3$. The passband ripple of $\widehat{G}_L(z)$ completely determines the amplitude distortion in $\widehat{x}(n)$. There is no phase distortion if $\widehat{G}_L(z)$ (hence $T(z)$) has linear phase.

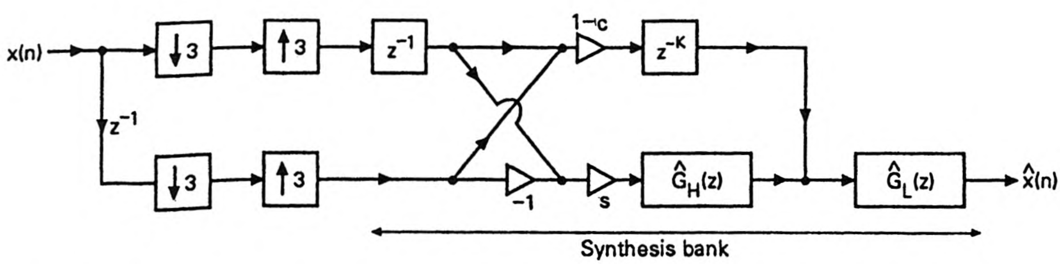

Figure 10.2-14 The complete analysis/synthesis system representing nonuniform decimation and reconstruction.

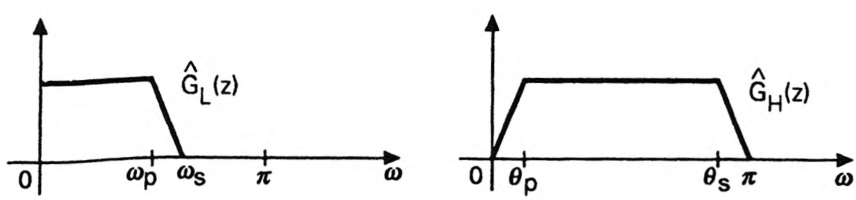

Figure 10.2-15 Defining bandedges for the real coefficient linear phase filters $\widehat{G}_L(z)$ and $\widehat{G}_H(z)$.

Design Example 10.2.1

We now demonstrate these ideas with an example. Figure 10.2-15 shows the definitions of the bandedges for $\widehat{G}_L(z)$ and $\widehat{G}_H(z)$, both of which are real coefficient linear phase filters. In Fig. 10.2-16(a) we show the magnitude $|X(e^{j\omega})|$ for our test sequence $x(n)$, which is a real finite length sequence of length 71. The plot shows that $x(n)$ is (approximately) bandlimited to $|\omega| < 2\pi/3$. We will reconstruct $x(n)$ from the nonuniform subset of samples indicated in Fig. 10.2-9(c), by using the synthesis bank in Fig. 10.2-14. $\widehat{G}_L(z)$ is taken to be of order 72, and has following features: $\omega_p = 0.58\pi, \omega_S = 0.70\pi$, and stopband attenuation > 55dB. The Hilbert transformer $\widehat{G}_H(z)$ has order 50, with $\theta_p = 0.04\pi$ and $\theta_S = 0.96\pi$. The magnitude responses of $\widehat{G}_L(z)$ and $\widehat{G}_H(z)$ are shown in the figure. With these filters used in the structure of Fig. 10.2-14, the quantity $|\widehat{X}(e^{j\omega})|$ for the reconstructed signal is shown in Fig. 10.2-16(c), which is in good agreement with $|X(e^{j\omega})|$. Since $T(z)$ has linear phase, there is no phase distortion, so we conclude that $\widehat{x}(n)$ is indeed a good approximation of $x(n)$.

As explained above, the Hilbert transformer serves to eliminate aliasing in the signal band. But a practical Hilbert transformer has to have a transition bandwidth around zero-frequency (because $G_H(1) = 0$; see Rabiner and Gold [1975]). If this bandwidth around zero-frequency is large, it results in poor reconstruction.

Efficieny of Reconstruction

The above reconstruction scheme is not the most efficient technique for the purpose, and is meant only to demonstrate the fundamental principles underlying the recovery of a signal from nonuniform samples. For more efficient (polyphase) techniques, and for further detailed comparison between various techniques, the reader is referred to Vaidyanathan and Liu [1990].

Generalizations

Several generalizations of the above approach are available. Thus consider the system of Fig. 10.2-17. Here we have an M-channel filter bank in which only the first L analysis filters are nonzero. And these nonzero filters are delay elements of the form $z^{-n_k}, 0 \leq k \leq L-1$, with

$$0 \leq n_0 < n_1 \ldots < n_{L-1} \leq M-1. \tag{10.2.34}$$

The effect of the analysis bank is merely to retain the subset of samples

$$x(nM - n_k), \quad 0 \leq k \leq L-1. \tag{10.2.35}$$

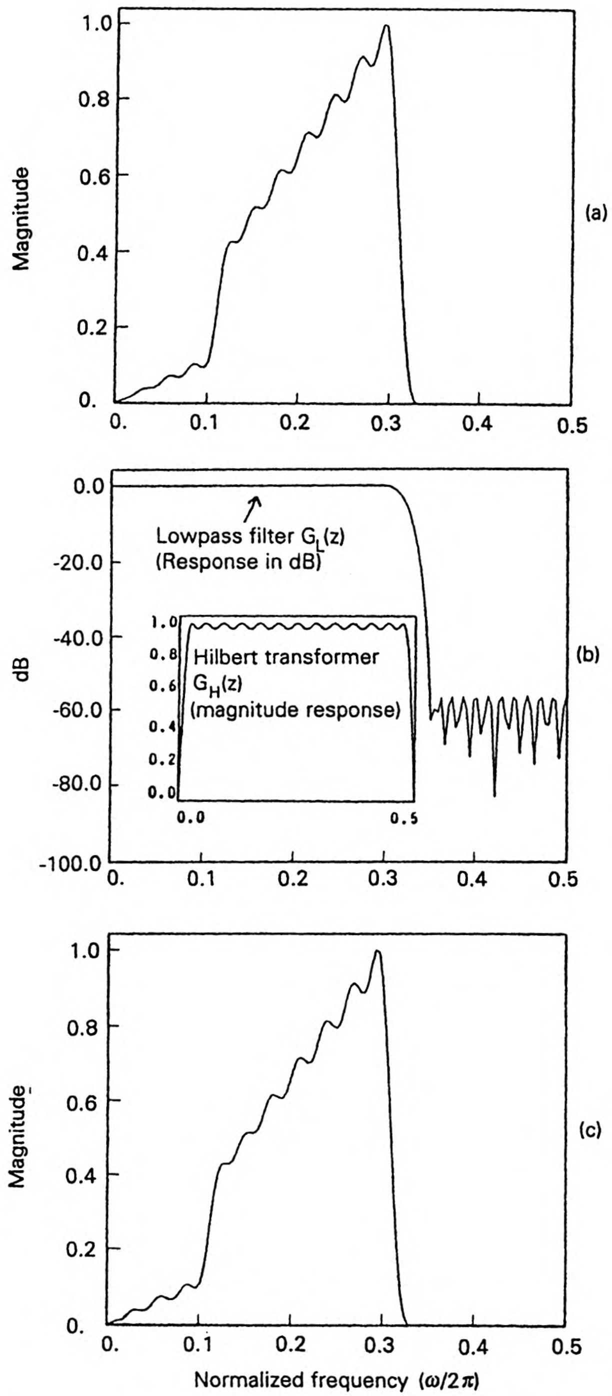

Figure 10.2-16 Design example 10.2.1. (a) $|X(e^{j\omega})|$, (b) filter responses, and (c) $|\widehat{X}(e^{j\omega})|$.

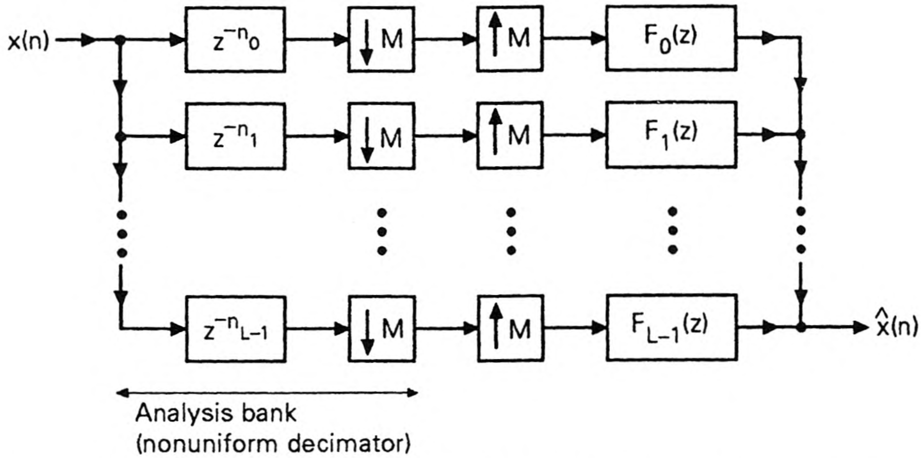

Figure 10.2-17 Generalization of nonuniform decimation and reconstruction.

This means that the time-index is divided into intervals of length M, and L samples are retained in each interval. (The time indices for these L samples are equal to $-n_k$ modulo M.) The analysis bank, therefore, is a nonuniform decimator. If the signal $x(n)$ is bandlimited to the region $|\omega| < L\pi/M$, we can find the L synthesis filters $F_k(z)$ such that there is perfect recovery of $x(n)$ from the nonuniformly decimated version! These synthesis filters, once again, turn out to be multilevel filters. (More precisely, if we imagine the frequency region $0 \leq \omega < 2\pi$ to be divided into M contiguous intervals, then $F_k(e^{j\omega})$ is a (complex) constant in each interval.) So these are ideal filters and must be approximated by practical designs.

A further generalization arises when we consider multiband signals, that is, signals that are not bandlimited, but limited to a union of bands in the frequency domain. (For example, $X(e^{j\omega})$ could be zero everywhere except in $0 \leq \omega \leq 0.1\pi$ and $0.4\pi \leq \omega \leq 1.3\pi$.) The nonuniform decimation/reconstruction process works in this case as well. Many of these ideas also generalize to the case of two dimensional signals [Vaidyanathan and Liu, 1990].

Relation to Sampling of a Continuous-Time Signal

The signal $x(n)$, which is bandlimited to $L\pi/M$, can be considered to be the oversampled version of a continuous-time signal $x_a(t)$. Here $x_a(t)$ is sampled at the rate $2\pi/T$, which is M/L times the Nyquist rate Θ. The set of nonuniformly decimated samples (10.2.35) can be considered to be a set of nonuniformly sampled values of $x_a(t)$, (which is a subset of the original oversampled values). The nonuniformity is such that the *average* number of samples per unit time is reduced by the factor M/L, so that it becomes equal to the Nyquist rate Θ.

Evidently the nonuniform pattern repeats periodically after every L

samples (see Fig. 10.2-9(c)). In this sense, the above results address only a special case of nonuniform sampling, namely the case of *recurring* or *periodic* nonuniformity. It should also be noted that, in our special case, the locations of the nonuniformly spaced samples are not permitted to be arbitrary; if $x_a(t_1)$ and $x_a(t_2)$ are two samples in the nonuniformly spaced system, then t_1/t_2 is required to be rational.

10.1. Consider the scalar system $H(z) = 1/(1 - az^{-1})$. Write down the general form of the blocked version for arbitrary block size M.

10.2. Verify that the 2×2 matrix in (10.1.10) is paraunitary.

10.3. Let $\mathbf{P}_0(z)$ and $\mathbf{P}_1(z)$ be $M \times M$ pseudocirculants. Prove that the product $\mathbf{P}_0(z)\mathbf{P}_1(z)$ is pseudocirculant, and that $\mathbf{P}_0(z)\mathbf{P}_1(z) = \mathbf{P}_1(z)\mathbf{P}_0(z)$. (*Hint.* A pseudocirculant is related to the blocked version of an LTI system.)

10.4. Let $E_\ell(z), 0 \leq \ell \leq M-1$ be the Type 1 polyphase components of $H(z)$. Derive an expression for the polyphase components of $z^{-k}H(z)$. Assume $0 \leq k \leq M-1$ for simplicity.

10.5. Show that the polyphase matrix $\mathbf{E}(z)$ for the analysis bank of Fig. 10.1-6 has the form (10.1.23) for $M = 3$, where $G_{n,\ell}(z)$ are the polyphase components defined in (10.1.22). Also, obtain the form of $\mathbf{E}(z)$ for arbitrary M.

10.6. A sequence $x(n)$ is said to be bandlimited if $X(e^{j\omega}) = 0$ for $\sigma \leq |\omega| \leq \pi$ for some $\sigma < \pi$. Show that $x(n)$ cannot be bandlimited if it is causal (unless $x(n) = 0$ for all n). *Hint.* Assume $x(n)$ is causal and bandlimited. Construct a sequence $y(n)$ from $x(n)$ such that it is also causal, but bandlimited to a *narrower* band. Keep repeating till ...

Another hint. First try with the assumption $\sigma < \pi/2$ if you wish!

10.7. Consider the following system which is the continuous-time analog of the M-channel filter bank.

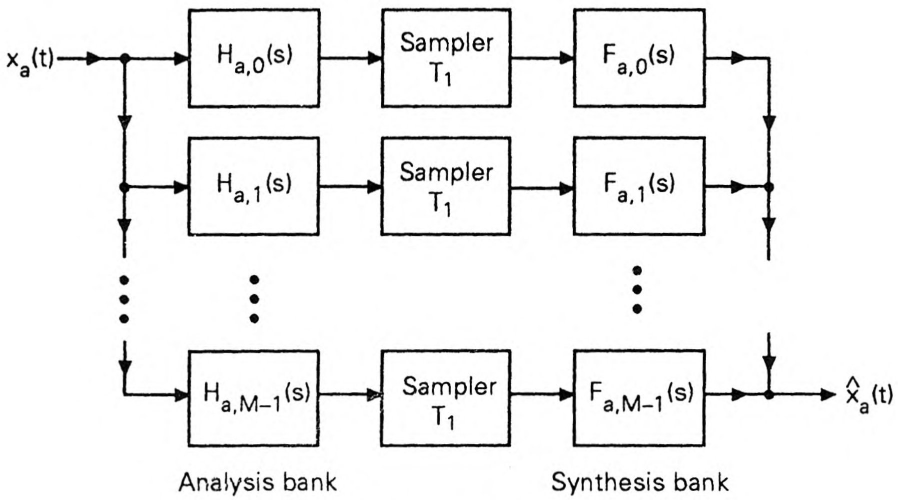

Figure P10-7

The continuous-time signal $x_a(t)$, which we assume to be σ-BL, is passed through M analog filters $H_{a,k}(s)$, and then sampled at the rate $2\pi/T_1$. (The sampler is defined precisely as in Problem 5.13.) The rate $2\pi/T_1$ is equal to

Θ/M, where $\Theta \triangleq 2\sigma$ is the Nyquist rate. Each of the sampled signals is in general subject to aliasing since it is not necessarily bandlimited. We assume $F_{a,k}(j\Omega) = 0$ for $|\Omega| \geq \sigma$ so that aliasing terms which fall outside the band of $x_a(t)$ are automatically eliminated.

a) Express $\widehat{X}_a(j\Omega)$ in terms of $X_a(j\Omega)$ and the filters in the structure.

b) Suppose we are given the set of M analysis filters $H_{a,k}(j\Omega)$, and wish to find the M synthesis filters for perfect reconstruction. Show that the frequency region $-\sigma \leq \Omega < \sigma$ can be divided into M intervals such that in each interval we have to solve M equations for the M unknowns $F_{a,k}(j\Omega)$.

c) Consider the special case where $H_{a,k}(s) = s^k$. This means that the output of $H_{a,k}(s)$ is the kth derivative of $x_a(t)$. Show that the $M \times M$ matrix which should (in principle) be inverted to obtain the M synthesis filters for perfect reconstruction is nonsingular. This proves the existence (but does not assert realizability!) of these synthesis filters, and gives a proof of the generalized derivative sampling theorem. (*Hint.* Review Vandermonde matrices from Appendix A).

10.8. Consider a sequence $x(n)$ with $|X(e^{j\omega})| = 0$ for $3\pi/4 \leq |\omega| \leq \pi$. It is clear that we can decimate it by 4/3 without losing information. Suppose we wish to perform *nonuniform* decimation as in Sec. 10.2.2. We can do so by using the analysis bank shown in Fig. P10-8(a).

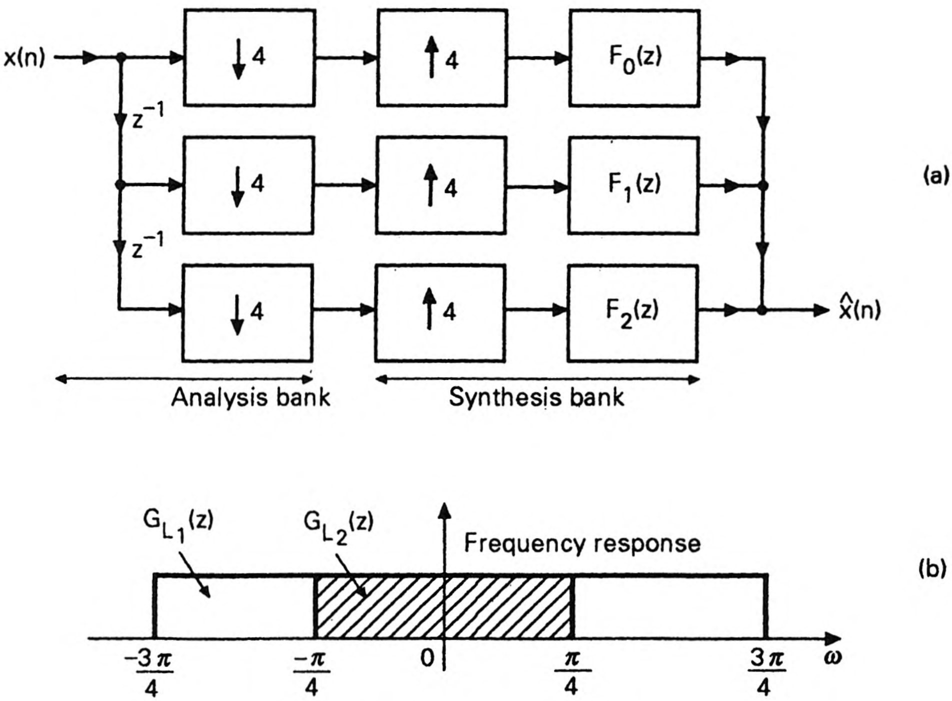

Figure P10-8 (a), (b)

We wish to reconstruct $x(n)$ by using the synthesis bank system shown. Assume

$F_k(e^{j\omega}) = 0$ for $3\pi/4 \le |\omega| \le \pi$ so that alias components outside the signal band are automatically removed.

a) Show how $F_0(z), F_1(z)$ and $F_2(z)$ can be found, in order to provide perfect reconstruction. (Divide the frequency region into appropriate number of intervals, and solve 3×3 equations in each interval. *Hint.* even though there are several sets of 3×3 equations, you need to invert *only one* constant matrix, if you do things right). Show that $F_k(e^{j\omega})$ can be expressed as

$$e^{-jk\omega}F_k(e^{j\omega}) = \begin{cases} a_k & \frac{-3\pi}{4} < \omega \le \frac{-\pi}{4} \\ b_k & \frac{-\pi}{4} < \omega \le \frac{\pi}{4} \\ c_k & \frac{\pi}{4} \le \omega < \frac{3\pi}{4} \end{cases} \qquad (P10.8a)$$

where a_k, b_k, c_k are (possibly complex) constants. Identify the constants a_k, b_k, c_k for $k = 0, 1, 2$.

b) Show that the filters $F_k(z)$ have real-valued impulse response.

c) Show that these filters can be expressed as

$$\begin{aligned} F_0(z) &= \Big(1 + G_{L_2}(z) - (1 - G_{L_2}(z))G_H(z)\Big)G_{L_1}(z), \\ F_1(z) &= 2z\Big(1 - G_{L_2}(z)\Big)G_{L_1}(z), \\ F_2(z) &= z^2\Big(1 + G_{L_2}(z) + (1 - G_{L_2}(z))G_H(z)\Big)G_{L_1}(z), \end{aligned} \qquad (P10.8b)$$

where $G_{L_1}(z), G_{L_2}(z)$ are ideal lowpass filters with response shown in Fig. P10-8(b), and $G_H(z)$ is an ideal Hilbert transformer. Draw a structure for the synthesis bank using the filters $G_{L_1}(z), G_{L_2}(z)$ and $G_H(z)$ as building blocks. In practice we would like to replace $G_{L_1}(z), G_{L_2}(z)$ and $G_H(z)$ with causal FIR linear phase approximations $\widehat{G}_{L_1}(z), \widehat{G}_{L_2}(z)$ and $\widehat{G}_H(z)$ of orders N_{L_1}, N_{L_2} and N_H. Show how the expressions given in (P10.8b) should be modified to incorporate the group delays of these filters. (Assume filter orders are even where necessary.)