Appendix D

Spectral factorization techniques

D.0 INTRODUCTION

In Sec. 3.2.5 we defined the spectral factor $H_0(z)$ of a FIR transfer function $H(z)$ with nonnegative frequency response $H(e^{j\omega})$. This section should be reviewed at this time. The spectral factor $H_0(z)$ is not unique, and we can find all possible solutions by finding the zeros of $H(z)$ and grouping them appropriately. We will now describe an algorithm, reported in Mian and Nainer [1982], for computing a spectral factor $H_0(z)$ *without finding the zeros* of $H(z)$. The spectral factor given by this algorithm has minimum phase (and is therefore unique). Moreover, the method works even if $H(z)$ has zeros *on the unit circle,* unlike some other techniques that have been reported in the literature.

D.1 THE COMPLEX CEPSTRUM

The technique to be described here uses some fundamental properties of the so-called complex cepstrum of a sequence. A detailed treatment of this topic, along with references to poineering work in this area, can be found in Chap. 12 of Oppenheim and Schafer [1989]. Here we will present the definition and only those properties that are relevant for the spectral factorization algorithm.

Let $x(n)$ be a sequence with z-transform $X(z)$. In general this converges in some annulus in the z-plane. We shall specify this region explicitly only when it becomes relevant later on. Consider the function $\widehat{X}(z) \stackrel{\Delta}{=} [\ln X(z)]$. This is in general complex valued [as is $X(z)$]. Assuming that this has a nontrivial region of convergence, we can find its inverse z-transform $\widehat{x}(n)$. This sequence $\widehat{x}(n)$ is said to be the *complex cepstrum* of $x(n)$. † We also say "complex cepstrum of $X(z)$" when that is more convenient.

We will now consider some examples to bring out a few key properties.

† $\widehat{x}(n)$ is not necessarily a complex sequence; the term 'complex' is used to emphasize the fact that a complex logarithm is involved.

Example D.1.1

Let $X(z) = 1 - \alpha_k z^{-1}$ which is the z-transform of the causal FIR sequence $x(n)$ whose nonzero samples are $\{1, -\alpha_k\}$. Then $\widehat{X}(z) = \ln(1 - \alpha_k z^{-1})$. By using the expansion

$$\ln(1+a) = -\sum_{n=1}^{\infty} \frac{(-a)^n}{n}, \qquad |a| < 1, \tag{D.1.1}$$

we obtain the power series expansion

$$\widehat{X}(z) = -\sum_{n=1}^{\infty} \frac{\alpha_k^n z^{-n}}{n}, \qquad |z| > |\alpha_k|. \tag{D.1.2}$$

Note that the region of convergence is $|z| > |\alpha_k|$. The complex cepstrum $\widehat{x}(n)$ of the sequence $x(n)$ is the inverse z-transform of $\widehat{X}(z)$ i.e.,

$$\widehat{x}(n) = \frac{-\alpha_k^n}{n} \mathcal{U}(n-1), \tag{D.1.3}$$

where $\mathcal{U}(n)$ denotes the unit-step function. In this example $\widehat{x}(n)$ is a *causal* sequence. If $|\alpha_k| \leq 1$, then this sequence is bounded, and approaches zero for $n \to \infty$. Note that $\widehat{x}(n)$ is infinitely long, even though $x(n)$ has finite duration.

Next consider the example $Y(z) = 1 - \beta_k z$. This is the z-transform of the noncausal FIR sequence $y(n)$ whose nonzero sample values are $\{-\beta_k, 1\}$. The quantity $\widehat{Y}(z) \stackrel{\Delta}{=} [\ln Y(z)]$ can be expressed as

$$\widehat{Y}(z) = \sum_{n=-\infty}^{-1} \frac{\beta_k^{-n} z^{-n}}{n}, \qquad |z| < \frac{1}{|\beta_k|}. \tag{D.1.4}$$

The region of convergence is $|z| < |\beta_k^{-1}|$, and the complex cepstrum of $y(n)$ is

$$\widehat{y}(n) = \frac{\beta_k^{-n}}{n} \mathcal{U}(-n-1), \tag{D.1.5}$$

which is anticausal. If $|\beta_k| \leq 1$, $\widehat{y}(n)$ is bounded, and approaches zero for $n \to -\infty$.

D.1.1 Minimum and Maximum Phase Filters

Let $H_0(z) = \sum_{n=0}^{N} h_0(n) z^{-n}$ be a causal minimum phase FIR filter (i.e., no zeros outside the unit circle). Let a_k, $1 \leq k \leq n_a$ be the zeros on the unit circle, and let b_k, $1 \leq k \leq n_b$ be those inside. Then,

$$H_0(z) = \prod_{k=1}^{n_a} (1 - a_k z^{-1}) \prod_{k=1}^{n_b} (1 - b_k z^{-1}), \quad |a_k| = 1, |b_k| < 1, \tag{D.1.6}$$

where we have assumed $h_0(0) = 1$ for simplicity. Since $\ln XY = \ln X + \ln Y$, the quantity $[\ln H_0(z)]$ is the sum of the logarithms of the first order factors in (D.1.6).

Using the ideas in Example D.1.1, we see that the function $\widehat{H}_0(z) \stackrel{\Delta}{=} [\ln H_0(z)]$ converges in the region $|z| > 1$, and that the complex cepstrum of $h_0(n)$ is the causal sequence

$$\widehat{h}_0(n) = \underbrace{-\sum_{k=1}^{n_a} \frac{a_k^n}{n} \mathcal{U}(n-1)}_{c_a(n)} \underbrace{-\sum_{k=1}^{n_b} \frac{b_k^n}{n} \mathcal{U}(n-1)}_{c_b(n)}. \qquad (D.1.7)$$

If $H_0(z)$ has real coefficients, then complex zeros are accompanied by their conjugates so that the summations above are real.

As a second example, consider the filter

$$H_1(z) = \prod_{k=1}^{n_a}(1 - a_k z^{-1}) \prod_{k=1}^{n_b}(1 - b_k z), \quad |a_k| = 1, |b_k| < 1. \qquad (D.1.8)$$

This has the zeros a_k on unit circle, and the zeros $1/b_k$ *outside* the unit circle. This is a maximum phase filter. Its logarithm can be expressed as

$$\ln H_1(z) = \sum_{k=1}^{n_a} \ln(1 - a_k z^{-1}) + \sum_{k=1}^{n_b} \ln(1 - b_k z). \qquad (D.1.9)$$

Using Example D.1.1 we see that the first summation converges for $|z| > 1$ and the second summation for $|z| < \min_k |b_k^{-1}|$. So the above z-transform has region of convergence $1 < |z| < \min_k |b_k^{-1}|$. Its inverse transform, that is, the cepstrum of $h_1(n)$ is given by

$$\widehat{h}_1(n) = \underbrace{-\sum_{k=1}^{n_a} \frac{a_k^n}{n} \mathcal{U}(n-1)}_{c_a(n)} + \underbrace{\sum_{k=1}^{n_b} \frac{b_k^{-n}}{n} \mathcal{U}(-n-1)}_{c_b(-n)}. \qquad (D.1.10)$$

Thus the complex cepstrum of $h_0(n)$ is causal (see (D.1.7)) whereas that of $h_1(n)$ has an anticausal part $c_b(-n)$ (contributed by the zeros $1/b_k$ outside the unit circle).

An intriguing relation. Since $\ln XY = \ln X + \ln Y$, the complex cepstrum of $G(z) \stackrel{\Delta}{=} H_1(z)H_0(z)$ is given by

$$\widehat{g}(n) = 2c_a(n) + c_b(n) + c_b(-n). \qquad (D.1.11)$$

On the other hand, the complex cepstrum of $H_0^2(z)$ is equal to $2\widehat{h}_0(n)$. From (D.1.7) we have

$$2\widehat{h}_0(n) = 2c_a(n) + 2c_b(n) \qquad (D.1.12)$$

We can therefore obtain $\widehat{h}_0(n)$ from $\widehat{g}(n)$ as follows: simply fold the anticausal part $c_b(-n)$, add to the causal part, and divide by two. We will use this idea later.

D.1.2 Subtleties in the Computation

There are some subtle issues which must be carefully considered when computing the complex cepstrum. Consider $\widehat{X}(z) = [\ln X(z)]$. If the region of convergence of this quantity includes the unit circle, we can evaluate $\widehat{X}(e^{j\omega}) = \ln X(e^{j\omega})$, and then find $\widehat{x}(n)$ by performing an inverse Fourier transformation.

If, on the other hand, $\widehat{X}(z)$ does not converge on the unit circle, we can always take care of this as follows: define $x_1(n) = \rho^{-n}x(n)$ so that $X_1(z) = X(\rho z)$. For appropriate choice of ρ, the quantity $\widehat{X}_1(z) \stackrel{\Delta}{=} [\ln X_1(z)]$ will converge on the unit circle. Having computed its inverse Fourier transform $\widehat{x}_1(n)$, we can then obtain the complex cepstrum of the original sequence $x(n)$ using the relation $\widehat{x}(n) = \rho^n \widehat{x}_1(n)$. In what follows, we shall therefore assume that $X(z)$ converges on the unit circle, that is, that $X(e^{j\omega})$ exists.

Subtleties About the Phase Response

Let $X(e^{j\omega}) = |X(e^{j\omega})|e^{j\phi(\omega)}$. Then its complex logarithm can be expressed as

$$\widehat{X}(e^{j\omega}) = \ln X(e^{j\omega}) = \ln|X(e^{j\omega})| + j\phi(\omega). \qquad (D.1.13)$$

We know that the phase function $\phi(\omega)$ is not uniquely determined by $X(e^{j\omega})$ since we can replace $\phi(\omega)$ with $\phi(\omega) + 2\pi k$ for integer k, without changing $X(e^{j\omega})$. However, $[\ln X(e^{j\omega})]$ is affected by the choice of k.

In Sec. 2.4.1 we stated that there are two kinds of phase responses, unwrapped and wrapped, denoted respectively as $\phi_u(\omega)$ and $\phi_w(\omega)$. For many applications, we need not distinguish between these, and the subscript is omitted. But in the computation of $[\ln X(e^{j\omega})]$, one uses the unwrapped phase $\phi_u(\omega)$ (which is free from discontinuities in $\phi_w(\omega)$ caused by the modulo 2π operation on phase). The reason for this is the following: whenever we consider the complex cepstrum of a product [e.g., equation (D.1.6)], we would like the result to be the sum of individual complex cepstra (e.g., as in (D.1.7)). From (D.1.13) we see that this will happen only if we consistently use unwrapped phases everywhere. (The sum of unwrapped phases gives the total unwrapped phase.)

The unwrapped phase $\phi_u(\omega)$ can be computed using the algorithm in Tribolet [1977] (which we will not repeat here). Furthermore, a computer program for computation of the complex cepstrum, which incorporates phase unwrapping, is available in Tribolet and Quatieri [1979].

When is the unwrapped phase periodic? The wrapped phase response $\phi_w(\omega)$ is periodic with period 2π whereas the unwrapped phase response $\phi_u(\omega)$ is not necessarily so. (Example: let $H(z) = z^{-K}$, then $\phi_u(\omega) = -K\omega$). This will create some difficulties in (D.1.13) because the Fourier transform $\widehat{X}(e^{j\omega})$ of the sequence $\widehat{x}(n)$ is supposed to be periodic.

It can be shown (Problems D.1 and D.2) that if we have a transfer function of the form $(1 - \alpha z^{-1})$ where α is a possibly complex number with $|\alpha| < 1$, then the unwrapped phase is still periodic with period 2π. This is also true of $(1 - \alpha z)$ for $|\alpha| < 1$. More generally, this is true for products of the form

$$\prod(1 - \alpha_k z^{-1}) \prod(1 - \beta_k z), \quad |\alpha_k|, |\beta_k| < 1. \qquad (D.1.14)$$

Thus, the factors corresponding to the zeros inside the unit circle should be represented as $(1 - \alpha z^{-1}), |\alpha| < 1$ rather than as $(1 - \alpha' z)$, $|\alpha'| > 1$. Similarly the zeros outside the unit circle must be represented by $(1 - \beta z), |\beta| < 1$ rather than as $(1 - \beta' z^{-1}), |\beta'| > 1$. This will ensure that the unwrapped phase is periodic.

Elimination of the nonperiodic component. Notice that any FIR transfer function with *no zeros on the unit circle* can be written as

$$H(z) = cz^{-K} \prod(1 - \alpha_k z^{-1}) \prod(1 - \beta_k z), \quad |\alpha_k|, |\beta_k| < 1 \tag{D.1.15}$$

where K is an integer, α_k are the zeros inside the unit circle and $(1/\beta_k)$ are the zeros outside. The unwrapped phase $\phi_u(\omega)$ of this system is not periodic, because of the factor z^{-K}. So the Fourier transform $[\ln H(e^{j\omega})]$ is not periodic if we use the unwrapped phase. On the other hand, if we use the wrapped phase, then the imaginary part of $[\ln H(e^{j\omega})]$ has discontinuities (jumps of 2π).

in practice, this situation is avoided by estimating K and then replacing $H(z)$ with $z^K H(z)$. This eliminates the nonperiodic component of the unwrapped phase. Given the quantity $H(z)$, the algorithm in Tribolet and Quatieri [1979] first obtains the unwrapped phase, and then estimates K, and actually computes the complex cepstrum of $z^K H(z)$. As a result, the output of this algorithm is unaffected if the input to the algorithm [i.e., $H(z)$] is replaced with $z^{-L}H(z)$ for arbitrary integer L.

The computation of K in (D.1.15) is performed as follows: suppose the unwrapped phase $\phi_u(\omega)$ has been found. Compute the difference $\phi_u(\pi) - \phi_u(-\pi)$. Since the nonperiodicity comes only from $e^{-j\omega K}$, this difference is equal to $-2K\pi$. From this K can be found.

D.2 A CEPSTRAL INVERSION ALGORITHM

It is possible to recover $x(n)$ uniquely from $\widehat{x}(n)$. Techniques for such inversion can be found in Chap. 12 of Oppenheim and Schafer [1989]. We now present a procedure for the case where $X(z)$ is rational, and $x(n)$ satisfies the following properties.

1. $x(n)$ is real, causal, and stable (i.e., all poles of $X(z)$ are inside the unit circle).
2. $X(z)$ has no zeros outside the unit circle (i.e., it is a mimimum phase function).
3. $x(0) > 0$.

In this case the cepstral sequence $\widehat{x}(n)$ has the form (D.1.7), and is causal, as well as real. (The poles of $X(z)$ give rise to terms similar to those in (D.1.7), except for sign). Both $X(z)$ and $\widehat{X}(z)$ converge for $|z| > 1$. In this region, we can write

$$X(z) = e^{\widehat{X}(z)} = 1 + \widehat{X}(z) + \frac{\widehat{X}^2(z)}{2!} + \ldots \tag{D.2.1}$$

where $\widehat{X}(z) = \widehat{x}(0) + \widehat{x}(1)z^{-1} + \widehat{x}(2)z^{-2} + \ldots$ Equating the constant coefficients on both sides of (D.2.1), we get

$$x(0) = 1 + \widehat{x}(0) + \frac{\widehat{x}^{\,2}(0)}{2!} \ldots \; = e^{\widehat{x}(0)}. \tag{D.2.2}$$

Suppose we are given the causal sequence $\widehat{x}(n)$. Then we can compute $x(0)$ from the above equation, and obtain the remaining samples of $x(n)$ by using the recursion

$$x(n) = \sum_{k=1}^{n} \frac{k}{n} \widehat{x}(k) x(n-k), \quad n > 0. \tag{D.2.3}$$

To see this note that the equation $X(z) = e^{\widehat{X}(z)}$ implies $X'(z) = X(z)\widehat{X}'(z)$ (where prime denotes derivative with respect to z). Using the fact that $zX'(z)$ is the

z-transform of $-nx(n)$, we obtain $nx(n) = x(n) * n\widehat{x}(n)$, where $*$ stands for convolution. Eqn. (D.2.3) follows from this. This equation also shows that if $x(n)$ has finite duration (say L), we can compute it from the first L samples of $\widehat{x}(n)$ even though $\widehat{x}(n)$ itself does not have finite duration.

D.3 A SPECTRAL FACTORIZATION ALGORITHM

Suppose we are given the coefficients of the FIR filter $H(z)$, with frequency response $H(e^{j\omega}) \geq 0$ (which, in particular, has zero-phase). We will assume that the coefficients are real i.e., $H(e^{j\omega})$ is even (as in most filter bank applications). Thus if z_k is a zero of $H(z)$, then so is z_k^*. Recall that the minimum-phase spectral factor $H_0(z)$ is obtained by retaining all the zeros inside the unit circle, and one out of every double zero on the unit circle. (See Sec. 3.2.5.) So $H_0(z)$ has real zeros and complex-conjugate pairs of zeros, and therefore has real coefficients. We now present a method to extract this real-coefficient minimum-phase spectral factor $H_0(z)$.

$H(z)$ is a zero phase filter, and has the form $H(z) = \sum_{n=-N}^{N} h(n)z^{-n}$. We can write this function as

$$H(z) = Az^K \prod_{k=1}^{n_a}(1 - a_k z^{-1})^2 \prod_{k=1}^{n_b}(1 - b_k z^{-1}) \prod_{k=1}^{n_b}(1 - b_k z), \qquad (D.3.1)$$

with $|a_k| = 1$ and $|b_k| < 1$. In writing (D.3.1) we have used the following facts: (a) if α is a zero of $H(z)$ then so is $1/\alpha^*$, (b) the zeros on the unit circle have even multiplicity (since $H(e^{j\omega}) \geq 0$), and (c) if α is a zero then so is α^* (since the coefficients are real).

The minimum phase spectral factor has the form

$$H_0(z) = B \prod_{k=1}^{n_a}(1 - a_k z^{-1}) \prod_{k=1}^{n_b}(1 - b_k z^{-1}). \qquad (D.3.2)$$

From this we have $h_0(0) = B$ so that B is real. It can be shown that $A = B^2 > 0$. (*Proof.* Evaluate $H(z)$ and $H_0(z)$ at $z = 1$ and use $H(1) = H_0^2(1)$. This gives $A = B^2$ as long as $H(e^{j0}) \neq 0$, which is a valid assumption for the lowpass case).

Our aim is to compute $H_0(z)$ [i.e., its coefficients $h_0(n)$] from $H(z)$, by first computing the complex cepstrum of $h(n)$. The program in Tribolet and Quatieri [1979] can be adapted for this purpose. As explained at the end of Sec. D.1, this program will automatically estimate and eliminate z^K. In this sense the program is insensitive to a time-shift of the input data. So we will focus our attention only on the effective input function

$$F(z) = A \prod_{k=1}^{n_a}(1 - a_k z^{-1})^2 \prod_{k=1}^{n_b}(1 - b_k z^{-1}) \prod_{k=1}^{n_b}(1 - b_k z). \qquad (D.3.3)$$

From this we first compute the complex cepstrum $\widehat{f}(n)$. This can be done by inverse transformation of $[\ln F(z)]$. But we have to be careful: $[\ln F(z)]$ does not converge on the unit circle (since $F(z)$ has zeros there). Fortunately, however, $F(z)$ does not have any zeros in the region $1 < |z| < \min_k \frac{1}{|b_k|}$, so that we can choose this to be the region of convergence of $[\ln F(z)]$. Then, the function $F_1(z) \triangleq F(\rho z)$ which is the z-transform of

$$f_1(n) \triangleq \rho^{-n} f(n), \tag{D.3.4}$$

converges on the unit circle, for ρ in the range

$$1 < \rho < \min_k \frac{1}{|b_k|}$$

We can therefore work with $[\ln F_1(e^{j\omega})]$ and compute the inverse Fourier transform $\widehat{f}_1(n)$, and then obtain $\widehat{f}(n) = \rho^n \widehat{f}_1(n)$. This has the form

$$\widehat{f}(n) = (\ln A)\delta(n) + 2c_a(n) + c_b(n) + c_b(-n), \tag{D.3.5}$$

where $c_a(n)$ and $c_b(n)$ are real valued causal sequences defined in (D.1.7) and $c_b(-n)$ is anticausal. (Clearly $\widehat{f}(n)$ is real as long as $A > 0$. See comments below.) By folding the anticausal part $c_b(-n)$, adding to the causal part, and dividing by two, we therefore obtain

$$\Big(0.5 \ln A\Big)\delta(n) + c_a(n) + c_b(n). \tag{D.3.6}$$

But the complex cepstrum $\widehat{h}_0(n)$ of the minimum phase spectral factor $H_0(z)$ [eqn. (D.3.2)] is also given by this expression (using $A = B^2$). Thus if we use the causal sequence (D.3.6) as $\widehat{x}(n)$ in (D.2.3), then the inverted sequence $x(n)$ will be equal to $h_0(n)$. In other words, the spectral factor $H_0(z)$ has been determined.

D.3.1 Computational Issues

The program in Tribolet and Quatieri [1979] computes the complex cepstrum $\widehat{f}_1(n)$ of $f_1(n)$. The user has the responsibility of choosing ρ and supplying $f_1(n)$ defined in (D.3.4). The program first evaluates $F_1(e^{j\omega})$ for a finite number (say L) of values of ω. This is done by computing the DFT of the sequence $f_1(n)$. This gives the sequence

$$F_1(e^{j2\pi k/L}), \quad 0 \leq k \leq L-1. \tag{D.3.7}$$

The inverse DFT of $\left[\ln F_1(e^{j2\pi k/L})\right]$ is then obtained [by using the unwrapped phase of $F_1(e^{j\omega})$ for the purpose]. This gives an approximation of $\widehat{f}_1(n)$ [complex cepstrum of $f_1(n)$]. The accuracy of this approximation depends on L. Once $\widehat{f}_1(n)$ is thus obtained, the user proceeds as described above, in order to complete the spectral factorization.

The DFT and IDFT are typically done by using the fast Fourier transform (FFT). For further discussion on the choice of L and ρ, see [Mian and Nainer, 1982]. Usually the choice $L = 8N$ gives acceptable results. The choice of ρ is tricky because the zeros b_k are not given. In practice, this choice might require trial and error. Once the spectral factor $H_0(z)$ is identified, one can plot $|H_0(e^{j\omega})|^2$ and compare with the plot of $H(e^{j\omega})$, to check the accuracy as well as to make sure that there have been no fatal errors.

D.3.2 Summary of the Spectral-Factorization Procedure

In the above derivation, we have provided several computational details, so that the user is well-informed about these. However, the implementaion itself is very simple, and does not require many of the above details. Here is a summary of the procedure.

1. Given the function $H(z) = \sum_{n=-N}^{N} h(n)z^{-n}$ with $H(e^{j\omega}) \geq 0$, define the causal sequence $g(n) = \rho^{-n}h(n-N)$, $0 \leq n \leq 2N$, where $1 < \rho < \min_k 1/|b_k|$. Since b_k are unknown, this choice of ρ requires some guess work. In the author's experience, the value $\rho = 1.02$ produced excellent results for most of half band filters $H(z)$ used in the two-channel QMF problem. Note that the user need not know the quantity K in (D.3.1), since the program in Tribolet and Quatieri [1979] is insensitive to any time-shift of its input.
2. With $g(n)$ used as the input to the program in Tribolet and Quatieri [1979], the output is $\widehat{g}(n)$, which is the complex cepstrum of $g(n)$. Notice that the length of $\widehat{g}(n)$ is typically much longer than $2N+1$, since a much longer DFT is used in the computation. (Unlike $g(n)$, the sequence $\widehat{g}(n)$ is noncausal, and the values for negative n should be carefully identified from the output; see instruction in the program listing of Tribolet and Quatieri [1979].)
3. Fold the anticausal part of $\widehat{g}(n)$, add it to the causal part, and divide by two. This gives the causal sequence (D.3.6). By using this causal sequence as $\widehat{x}(n)$ in (D.2.2) and (D.2.3), evaluate the sequence $x(n)$. This sequence is equal to $h_0(n)$. The spectral factor $H_0(z) = \sum_{n=0}^{N} h_0(n)z^{-n}$ has therefore been determined.

Design Example D.3.1.

We now consider an example where $H(z)$ is a zero phase FIR equiripple half-band filter with order $2N = 178$. The response $H(e^{j\omega})$ has been ensured to be nonnegative by first designing a causal linear phase filter $G(e^{j\omega})$ by use of the McClellan-Parks program, and then defining $H(z) = z^N G(z)+\delta$ where δ is the peak stopband ripple of $G(e^{j\omega})$. Figure D.3-1 shows the response $H(e^{j\omega})$. The stopband edge is at $\omega_S = 0.527\pi$. The figure also shows the response $|H_0(e^{j\omega})|$ of the minimum phase spectral factor $H_0(z)$, computed using the above technique. The stopband attenuation of $H(e^{j\omega})$ is about 80 dB and that of $H_0(e^{j\omega})$ is about 40 dB. The value $\rho = 1.02$ was used in the computation.

There have been other approaches for spectral factorization of FIR filters. See for example Friedlander [1983]. While these have some advantages over the one described above, they work only if there are no zeros on the unit circle.

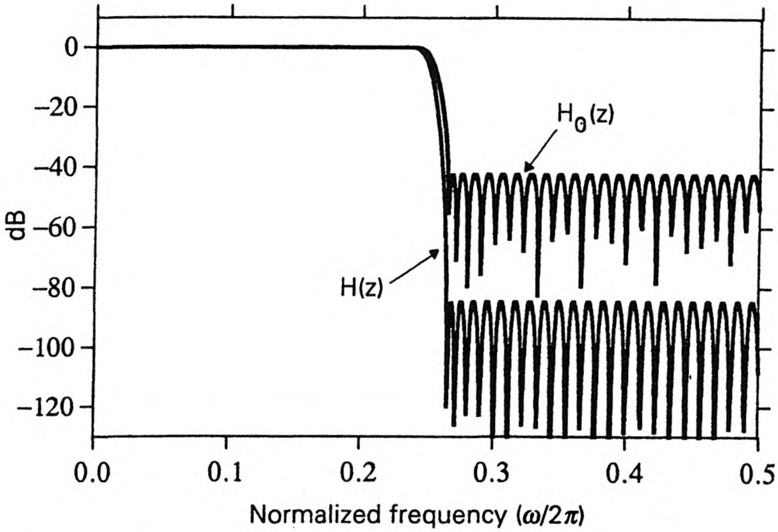

Figure D.3-1 Design example D.3.1. Magnitude responses of $H(z)$ and its spectral factor $H_0(z)$.

PROBLEMS

D.1. Consider the transfer function $H(z) = 1 - \alpha z$, with $0 < \alpha < 1$. This can be written as $H(z) = \alpha(\frac{1}{\alpha} - z)$ and the phase response $\phi(\omega)$ is equal to the phase of

$$G(e^{j\omega}) = \frac{1}{\alpha} - e^{j\omega}.$$

It is convenient to use a vector diagram to trace the behavior of $\phi(\omega)$. This is shown below.

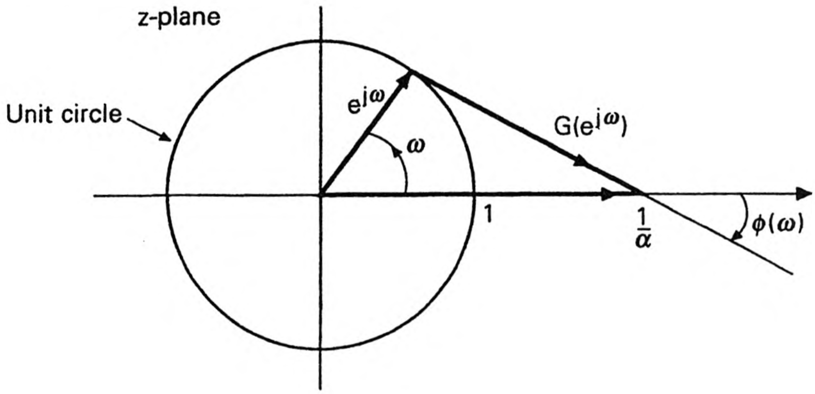

Figure PD-1

Similar diagrams can be drawn for arbitray rational transfer functions. (For example, see pp. 220–221 of Oppenheim and Schafer [1989].) As ω varies from 0 to 2π, the vector $G(e^{j\omega})$ changes both in magnitude and angular orientation. Its angle $\phi(\omega)$ can span a range exceeding 2π. So, the unwrapped phase $\phi_u(\omega)$ can be conceptually obtained by tracing the variation of $\phi(\omega)$, but carefully avoiding any "modulo 2π reduction." (Thus, if $\phi(\omega)$ is plotted with respect to ω, there will be no abrupt jumps of 2π.) In what follows, we therefore have $\phi(\omega) = \phi_u(\omega)$. By using this diagram, we can also find out whether the unwrapped phase is periodic. For this note that $\phi(0) - \phi(2\pi) = 2\pi m$ for some integer m, since $G(e^{j\omega})$ is periodic. $\phi(\omega)$ is periodic with period 2π if and only if $m = 0$.

a) For $H(z) = 1 - \alpha z$, with $0 < \alpha < 1$, show that $\phi_u(\omega)$ is periodic with period 2π.
b) Hence show that the same is true for $H(z) = 1 - \alpha z$, where α is possibly complex with $|\alpha| < 1$. (*Hint.* Try a frequency shift.)
c) Hence show that the same is true for $H(z) = 1 - \alpha z^{-1}$, where α is possibly complex with $|\alpha| < 1$.

Note. Part (b) takes care of zeros outside the unit circle, whereas part (c) takes care of zeros inside.

D.2. Let $H(z) = 1 - \alpha z$ with $\alpha > 1$. Show that the unwrapped phase does not satisfy $\phi_u(0) = \phi_u(2\pi)$. By a simple argument, extend this to the case where α is complex with $|\alpha| > 1$. Show finally that the same is true for $H(z) = 1 - \alpha z^{-1}$, $|\alpha| > 1$. (*Note.* It helps to first read Problem D.1).