

Appendix E

Mason's Gain Formula

Mason's gain formula is a very convenient tool to obtain transfer functions of complicated structures. It can be applied to any signal flow graph (see below), for example, the coupled form structure reproduced in Fig. E-1(a). We now state this result, and present some examples of its use. Further examples and details can be found in B. C. Kuo [1975]. For original work and theoretical developments see S. J. Mason [1953,1956].

Terminology

A signal flow graph is a collection of *nodes* interconnected by *directed branches*. In Fig. E-1(a) the nodes are indicated by circled numbers. Each node is associated with a signal, for example, node 1 is associated with

$$s_1(n) = u(n) - (R \sin \theta)y(n) + (R \cos \theta)s_2(n),$$

where $s_2(n)$ is the signal associated with node 2. Each branch has a gain, for example, the branch from node 1 to node 2 in the above example has gain z^{-1} . The gain is indicated adjacent to the branch. Whenever a branch is unlabeled, its gain is assumed to be unity.

The input and output nodes of the graph are typically labeled as $u(n)$ and $y(n)$ respectively. Notice that the signal associated with node 4 is also the output, i.e., $s_4(n) = y(n)$.

Paths. A path is merely a succession of branches, directed the same way. The *path gain* is the product of individual branch gains. (This explains why the gains are always indicated in terms of z -transforms rather than in time domain). The symbol $P_k(z)$ denotes the gain for the k th path. In Figs. E-1(b),(c) we have indicated some paths of the flow graph of Fig. E-1(a). These paths have gains

$$P_1(z) = -R \sin \theta, \quad P_2(z) = z^{-2} R \sin \theta. \quad (E.1)$$

If a path starts from the input node and ends at the output node *with no node encountered more than once*, it is said to be a *forward path*. In the above example $P_2(z)$ is a forward path. A signal flow graph can have many forward paths.

Deleting a path. The operation of 'path deletion' is central to Mason's gain formula. Deleting a path $P_k(z)$ means (i) delete all branches in the path, and (ii)

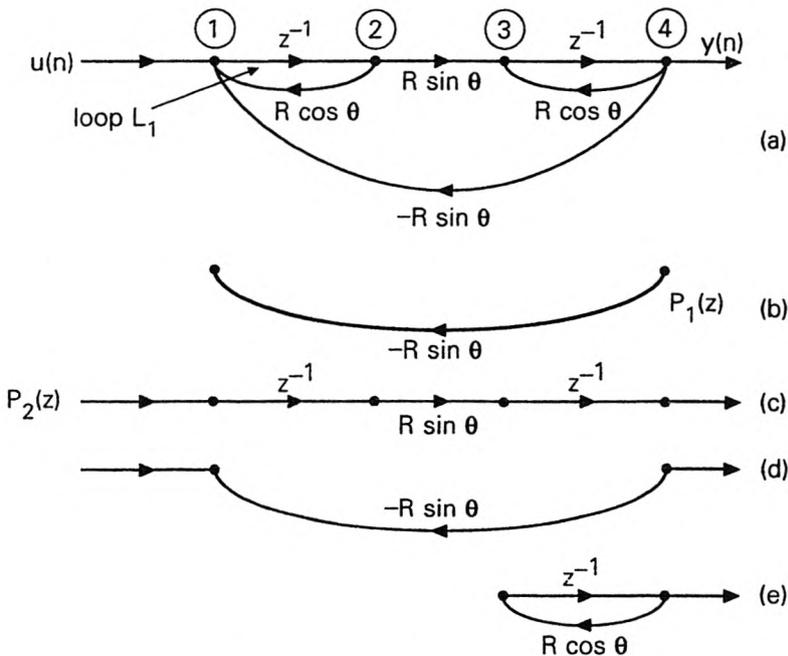


Figure E-1 (a) Signal flow graph of the coupled form structure, (b), (c) two examples of 'paths', (d) graph that remains after deleting the path ' $R \sin \theta$.' (e) graph that remains after deleting the path z^{-1} which connects nodes 1 and 2.

delete all branches that touch any node on the path. Figs. E-1(d),(e) show examples of the remaining graph after deleting certain paths from Fig. E-1(a). Note that if we delete the horizontal path which connects nodes 1 and 4 in Fig. E-1(a), the resulting graph is empty!

Loops. A path which starts and ends at the same node with no other node encountered more than once is said to be a *loop*. The product of the branch gains is said to be the *loop gain*. In the above example, the loop labeled L_1 has loop gain $z^{-1}R \cos \theta$. A signal flow graph can have any number of loops. Two loops are said to be *touching* if they share a common node, and *nontouching* if they do not. Figure E-2 shows examples of touching and nontouching loops.

Determinant of a Signal Flow Graph

The determinant of a signal flow graph is defined as

$$\begin{aligned}
 \Delta(z) = & 1 - \text{sum of all loop gains} \\
 & + \text{sum of products of gains of pairs of nontouching loops} \\
 & - \text{sum of products of gains of triples of nontouching loops} \\
 & + \dots
 \end{aligned}
 \tag{E.2}$$

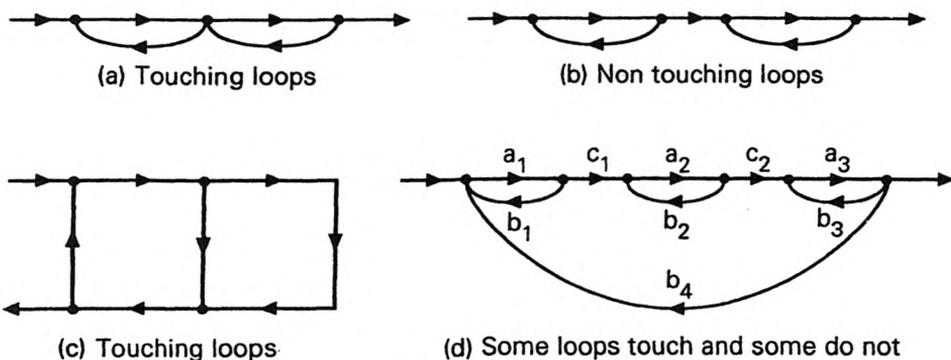


Figure E-2 Examples of touching and nontouching loops. (a) Touching, (b) nontouching, (c) touching, and (d) both types.

To demonstrate this, consider Fig. E-2(d). We have

Loops: a_1b_1 , a_2b_2 , a_3b_3 , $a_1c_1a_2c_2a_3b_4$.

Nontouching pairs: (a_1b_1, a_2b_2) , (a_1b_1, a_3b_3) , (a_2b_2, a_3b_3) .

Nontouching triples: (a_1b_1, a_2b_2, a_3b_3) .

So the determinant is

$$\begin{aligned} \Delta(z) = & 1 - (a_1b_1 + a_2b_2 + a_3b_3 + a_1c_1a_2c_2a_3b_4) \\ & + (a_1b_1a_2b_2 + a_1b_1a_3b_3 + a_2b_2a_3b_3) \\ & - a_1b_1a_2b_2a_3b_3. \end{aligned} \quad (E.3)$$

Notice that a graph with no loops has $\Delta = 1$. This is true, in particular, for the "empty graph," that is, graph with no nodes.

Mason's formula

Let $H(z)$ denote the transfer function $Y(z)/U(z)$ of a signal flow graph. Define the following notations:

1. $\Delta(z)$ = determinant of the graph.
2. L = number of forward paths, with $P_k(z)$, $1 \leq k \leq L$ denoting the forward path gains.
3. $\Delta_k(z)$ = determinant of the graph that remains after deleting the k th forward path $P_k(z)$.

Then the transfer function is

$$H(z) = \frac{\sum_{k=1}^L P_k(z)\Delta_k(z)}{\Delta(z)} \quad (\text{Mason's formula}). \quad (E.4)$$

Example E.1

Consider the direct form structure of Fig. E-3. We have only one loop, with gain az^{-1} . So the determinant $\Delta(z) = 1 - az^{-1}$. There are two forward paths, with gains $P_1(z) = b_0$ and $P_2(z) = b_1z^{-1}$. If either of these paths is deleted, the result is the empty graph, so that $\Delta_1(z) = \Delta_2(z) = 1$. Applying (E.4) we obtain

$$H(z) = \frac{Y(z)}{U(z)} = \frac{b_0 + b_1z^{-1}}{1 - az^{-1}}. \quad (E.5)$$

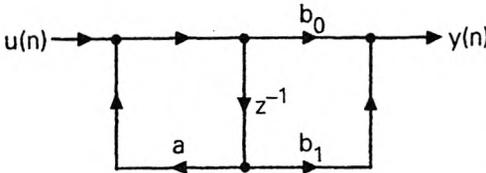


Figure E-3 First order direct form structure.

Example E.2.

Consider the coupled form structure shown in Fig. E-1(a). For this we have:

Loops: $z^{-1}R \cos \theta$, $z^{-1}R \cos \theta$, and $-z^{-2}R^2 \sin^2 \theta$.

Nontouching pairs: $z^{-1}R \cos \theta$, $z^{-1}R \cos \theta$.

There are no nontouching triples and so on. Using (E.2) the determinant of the structure is

$$\Delta(z) = 1 - 2z^{-1}R \cos \theta + z^{-2}R^2. \quad (E.6)$$

There is only one forward path, with $P_1(z) = z^{-2}R \sin \theta$. If this is deleted, the result is an empty graph, and we obtain $\Delta_1(z) = 1$. Using (E.4) we therefore obtain

$$H(z) = \frac{Y(z)}{U(z)} = \frac{z^{-2}R \sin \theta}{1 - 2z^{-1}R \cos \theta + z^{-2}R^2}. \quad (E.7)$$

Example E.3.

We now consider the lattice structure of Fig. E-4, which provides an example where $\Delta_k(z)$ are nontrivial. Here \hat{k}_i stands for $\sqrt{1 - k_i^2}$, and $-1 < k_i < 1$. First let us list the loops:

Loops: $-k_1k_2z^{-1}$, $-k_1z^{-1}$, and $-k_2\hat{k}_1^2z^{-2}$.

Nontouching pairs: $(-k_1k_2z^{-1}, -k_1z^{-1})$.

Using (E.2) we then obtain the determinant

$$\Delta(z) = 1 + k_1(k_2 + 1)z^{-1} + k_2z^{-2}, \quad \text{using } \hat{k}_i = \sqrt{1 - k_i^2}. \quad (E.8)$$

Next, we have three forward paths. The gains are

$$P_1(z) = k_2, \quad P_2(z) = k_1\hat{k}_2^2z^{-1}, \quad P_3(z) = \hat{k}_1^2\hat{k}_2^2z^{-2}.$$

The graph that remains after deletion of the forward path $P_k(z)$ is shown in Fig. E-5 for $k = 1, 2$. (If $P_3(z)$ is deleted, the remaining graph empty.) The determinants of these 'remaining' graphs are

$$\Delta_1(z) = \Delta(z), \Delta_2(z) = 1 + k_1 z^{-1}, \Delta_3(z) = 1. \quad (E.9)$$

Substituting these into (E.4) we arrive at the transfer function

$$H(z) = \frac{Y(z)}{U(z)} = \frac{k_2 + k_1(k_2 + 1)z^{-1} + z^{-2}}{1 + k_1(k_2 + 1)z^{-1} + k_2 z^{-2}}. \quad (E.10)$$

which is an allpass function, since k_i are real.

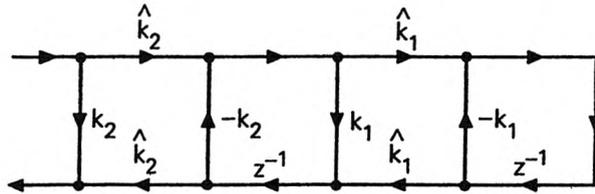


Figure E-4 The lattice structure example.

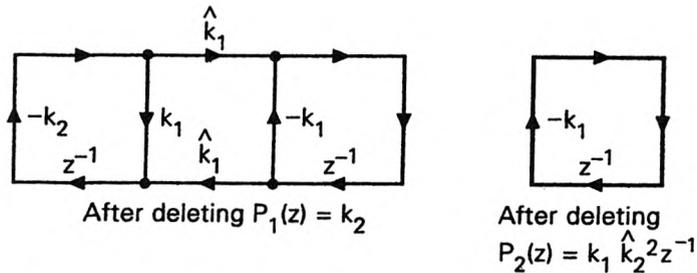


Figure E-5 Graphs that remain after deletion of certain paths from Fig. E-4.

PROBLEMS

- E.1. Find the transfer function of the following structure using Mason's formula, assuming that $y(n)$ is the output.

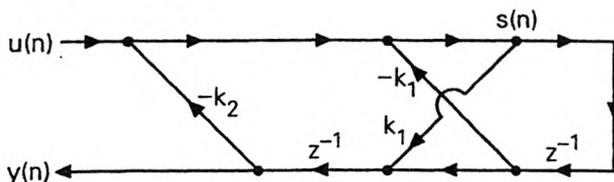


Figure PE-1

If $s(n)$ is regarded as the output instead, what will be the transfer function?

- E.2. Find the transfer function of the following structure using Mason's formula.

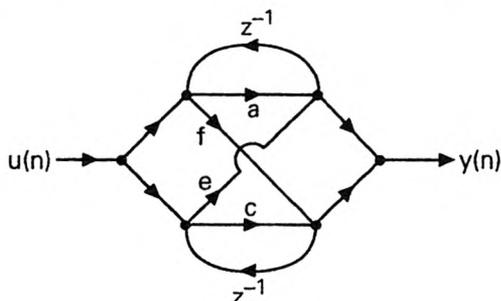


Figure PE-2

- E.3. Find the transfer function of the following structure, and show that it is allpass, whenever b_2 and b_3 are real.

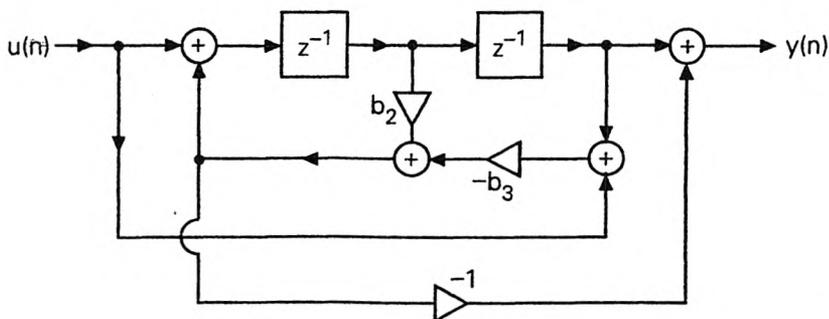


Figure PE-3

- E.4. Consider the two-input two-output system shown in Fig. PE-4. This is characterized by a 2×2 transfer matrix, $\mathbf{H}(z)$, that is,

$$\begin{bmatrix} Y_0(z) \\ Y_1(z) \end{bmatrix} = \underbrace{\begin{bmatrix} H_{00}(z) & H_{01}(z) \\ H_{10}(z) & H_{11}(z) \end{bmatrix}}_{\mathbf{H}(z)} \begin{bmatrix} U_0(z) \\ U_1(z) \end{bmatrix}.$$

Finds the elements $H_{km}(z)$ of this transfer matrix using Mason's formula.

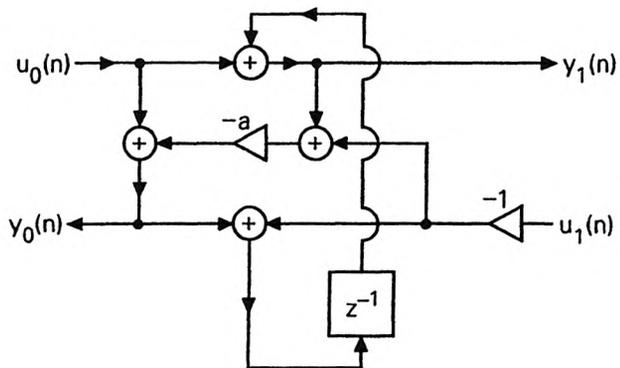


Figure PE-4