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HYPERCONTRACTIVITY OF THE SEMIGROUP OF THE FRACTIONAL LAPLACIAN ON THE n-SPHERE

RUPERT L. FRANK AND PAATA IVANISVILI

ABSTRACT. For $1 < p \leq q$ we show that the Poisson semigroup $e^{-t\sqrt{-\Delta}}$ on the n-sphere is hypercontractive from L^p to L^q in dimensions $n \leq 3$ if and only if $e^{-t\sqrt{n}} \leq \sqrt{\frac{p-1}{q-1}}$. We also show that the equivalence fails in large dimensions.

1. INTRODUCTION

1.1. Poisson semigroup on the sphere. Let

$$\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$$

be the unit sphere in $\mathbb{R}^{n+1}$, where $\|x\| = \sqrt{x_1^2 + \ldots + x_{n+1}^2}$ for $x = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}$. Let Δ be the Laplace–Beltrami operator on $\mathbb{S}^n$. We will be working with spherical polynomials $f : \mathbb{S}^n \to \mathbb{C}$, i.e., finite sums

$$f(\xi) = \sum_{d \geq 0} H_d(\xi),$$

where H_d satisfies

$$\Delta H_d = -d(d+n-1)H_d.$$

The heat semigroup $e^{t\Delta}$ is defined by $e^{t\Delta} f = \sum_{d \geq 0} e^{-d(d+n-1)t} H_d$. The hypercontractivity result for the heat semigroup on $\mathbb{S}^n$ states that for any $1 \leq p \leq q < \infty$, any integer $n \geq 1$, and any $t \geq 0$ we have

$$\|e^{t\Delta} f\|_q \leq \|f\|_p \quad \text{for all } f \qquad \text{if and only if} \qquad e^{-tn} \leq \sqrt{\frac{p-1}{q-1}}, \tag{1}$$

where $\|f\|_p^p = \|f\|_{L^p(\mathbb{S}^n, d\sigma_n)}^p = \int_{\mathbb{S}^n} |f|^p d\sigma_n$, and $d\sigma_n$ is the normalized surface area measure of $\mathbb{S}^n$. The case $n = 1$ was solved independently in [9] and [10], and the general case $n \geq 2$ was settled in [7]. We remark that the condition $e^{-tn} \leq \sqrt{\frac{p-1}{q-1}}$ in (1) is different from the classical hypercontractivity condition $e^{-t} \leq \sqrt{\frac{p-1}{q-1}}$ in Gauss space due to Nelson [8], and on the hypercube due to Bonami [2]. The appearance of the extra factor n in (1) can be explained from the fact that the spectral gap (the smallest nonzero eigenvalue) of $-\Delta$ equals n.

In [7] the authors ask what the corresponding hypercontractivity estimates are for the Poisson semigroup on $\mathbb{S}^n$. As pointed out in [7], there are two natural

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Poisson semigroups on $\mathbb{S}^n$ one can consider: 1) $e^{-t\sqrt{-\Delta}}f$, and 2) $P_r f = \sum r^d H_d$, $r \in [0,1]$. Notice that when $n=1$ both of these semigroups coincide (with $r = e^{-t}$). It was conjectured by E. Stein that

$$\|P_r f\|_q \le \|f\|_p \quad \text{if and only if} \quad r \le \sqrt{\frac{p-1}{q-1}}$$

holds on $\mathbb{S}^n$ for all $n \ge 1$. Besides the case $n=1$ mentioned above, the case $n=2$ was confirmed in [4], and the general case $n \ge 2$ in [1].

The question of hypercontractivity for the semigroup $e^{-t\sqrt{-\Delta}}$ on $\mathbb{S}^n$ for $n \ge 2$, however, has remained open. Since the spectral gap of $\sqrt{-\Delta}$ equals $\sqrt{n}$, it is easy to see that a necessary condition for the estimate $\|e^{-t\sqrt{-\Delta}}f\|_q \le \|f\|_p$ is $e^{-t\sqrt{n}} \le \sqrt{\frac{p-1}{q-1}}$; see Section 2.1. One might conjecture that this necessary condition is also sufficient. Surprisingly, it turns out the answer is positive in small dimensions and negative in large dimensions.

Theorem 1.1. *Let $1 < p < q$, $n \ge 1$, and $t \ge 0$. Then*

$$\text{(i) } \|e^{-t\sqrt{-\Delta}}f\|_q \le \|f\|_p \quad \textit{for all } f \qquad \textit{implies} \qquad \text{(ii) } e^{-t\sqrt{n}} \le \sqrt{\frac{p-1}{q-1}}. \tag{2}$$

Moreover, (ii) implies (i) in dimensions $n \le 3$. Finally, for any $q > \max\{2,p\}$, there exists $n_0 = n_0(p,q) \ge 4$ such that (ii) *does not imply* (i) *in dimensions n with $n \ge n_0$.*

It remains an open problem to find a necessary and sufficient condition on $t > 0$ in dimensions $n \ge 4$ for which the semigroup $e^{-t\sqrt{-\Delta}}$ is hypercontractive from $L^p(\mathbb{S}^n)$ to $L^q(\mathbb{S}^n)$.

2. Proof of Theorem 1.1

2.1. **The necessity part (i) ⇒ (ii).** We recall this standard argument for the sake of completeness. Let $f(\xi) = 1 + \varepsilon H_1(\xi)$ where H_1 is any (real) spherical harmonic of degree 1, i.e., $\Delta H_1 = -nH_1$. Then $e^{-t\sqrt{-\Delta}}f(\xi) = 1 + \varepsilon e^{-t\sqrt{n}}H_1(\xi)$. As $\varepsilon \to 0$, we obtain

$$\begin{aligned}
&\int_{\mathbb{S}^n} |1 + \varepsilon e^{-t\sqrt{n}} H_1(\xi)|^q \, d\sigma_n \\
&= \int_{\mathbb{S}^n} \left(1 + q\varepsilon e^{-t\sqrt{n}} H_1(\xi) + \frac{q(q-1)}{2}\varepsilon^2 e^{-2t\sqrt{n}} H_1^2(\xi) + O(\varepsilon^3)\right) d\sigma_n \\
&= 1 + \frac{q(q-1)}{2}\varepsilon^2 e^{-2t\sqrt{n}} \|H_1\|_2^2 + O(\varepsilon^3).
\end{aligned}$$

Thus,

$$\|e^{-t\sqrt{-\Delta}}f\|_q = 1 + \frac{q-1}{2}\varepsilon^2 e^{-2t\sqrt{n}} \|H_1\|_2^2 + O(\varepsilon^3). \tag{3}$$

Similarly, we have

$$\|f\|_p = 1 + \frac{p-1}{2}\varepsilon^2 \|H_1\|_2^2 + O(\varepsilon^2). \tag{4}$$

Substituting (3) and (4) into the inequality $\|e^{-t\sqrt{-\Delta}}f\|_q \le \|f\|_p$, and taking $\varepsilon \to 0$ we obtain the necessary condition $e^{-2t\sqrt{n}} \le \frac{p-1}{q-1}$ which coincides with (ii) in (2).

2.2. **The sufficiency part (ii) ⇒ (i) in dimensions** $n = 1,2,3$**.** Our goal is to show that if $1 < p < q$ and if $t \geq 0$ is such that $e^{-t2\sqrt{n}} \leq \frac{p-1}{q-1}$, then

$$\|e^{-t\sqrt{-\Delta}} f\|_q \leq \|f\|_p \quad \text{in dimensions} \quad n = 1,2,3. \tag{5}$$

The case $n = 1$ was confirmed in [10]. In what follows we assume $n \in \{2,3\}$. First we need the fact that the heat semigroup $e^{t\Delta}$ has a nonnegative kernel. Indeed, for each $t > 0$ there exists $K_t : [-1,1] \to [0,\infty)$ such that

$$e^{t\Delta} f(\xi) = \int_{\mathbb{S}^n} K_t(\xi \cdot \eta) f(\eta) d\sigma_n(\eta),$$

where $\xi \cdot \eta = \sum_{j=1}^{n+1} \xi_j \eta_j$ for $\xi = (\xi_1, \ldots, \xi_{n+1})$ and $\eta = (\eta_1, \ldots, \eta_{n+1})$, see, for example, Proposition 4.1 in [7]. Next, we recall the subordination formula

$$e^{-x} = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-y-x^2/(4y)} \frac{dy}{\sqrt{y}} \quad \text{valid for all } x \geq 0, \tag{6}$$

By the functional calculus, we deduce that the Poisson semigroup $e^{-t\sqrt{-\Delta}}$ has a positive kernel with total mass 1. The latter fact together with the convexity of the map $x \mapsto |x|^p$ for $p \geq 1$ implies that $\|e^{-t\sqrt{-\Delta}}\|_p \leq \|f\|_p$ for all $t \geq 0$. Thus, it suffices to verify (5) for those $t \geq 0$ for which $e^{-2t\sqrt{n}} = \frac{p-1}{q-1}$.

Next we claim that it suffices to verify (5) only for the powers p, q such that $2 \leq p \leq q$. Indeed, assume (5) holds for $2 \leq p \leq q$. By duality and the symmetry of the semigroup $e^{-t\sqrt{-\Delta}}$ we obtain $\|e^{-t\sqrt{-\Delta}} f\|_{p'} \leq \|f\|_{q'}$ where $p' = \frac{p}{p-1}$, $q' = \frac{q}{q-1}$, $1 < q' \leq p' \leq 2$. Notice that $\frac{p-1}{q-1} = \frac{q'-1}{p'-1}$, thus we extend (5) to all p, q such that $1 < p \leq q \leq 2$. It remains to extend (5) for those powers p, q when $p \leq 2 \leq q$. To do so, let $p \leq 2 \leq q$, and let $t \geq 0$ be such $e^{-2t\sqrt{n}} = \frac{p-1}{q-1}$. Choose $t_1, t_2 \geq 0$ so that $t = t_1 + t_2$ and $e^{-2t_1\sqrt{n}} = p - 1$ and $e^{-2t_2\sqrt{n}} = \frac{1}{q-1}$. Then we have

$$\|e^{-t\sqrt{-\Delta}} f\|_q = \|e^{-t_2\sqrt{-\Delta}}(e^{-t_1\sqrt{-\Delta}} f)\|_q \leq \|e^{-t_1\sqrt{-\Delta}} f\|_2 \leq \|f\|_p.$$

In what follows we assume $2 \leq p \leq q$. We will use a standard argument to deduce the validity of the hypercontractivity estimate from a log Sobolev inequality. Nonnegativity of the kernel for the Poisson semigroup combined with the triangle inequality implies $|e^{-t\sqrt{-\Delta}} f| \leq e^{-t\sqrt{-\Delta}}|f|$ for any f. Thus by continuity and standard density arguments we can assume that $f \geq 0$, f is not identically zero, and f is smooth in (5).

The equality $e^{-2t\sqrt{n}} = \frac{p-1}{q-1}$ implies $q = 1 + e^{2t\sqrt{n}}(p-1)$. Fix $p \geq 2$ and consider the map

$$\varphi(t) = \|e^{-t\sqrt{-\Delta}} f\|_{q(t)} > 0, \quad t \geq 0,$$

where $q(t) = 1 + e^{2t\sqrt{n}}(p-1)$. If we show $\varphi'(t) \leq 0$, then we obtain $\varphi(t) \leq \varphi(0) = \|f\|_p$, and this proves the sufficiency part. Let $\psi(t) = \ln \varphi(t)$. We have

$$\frac{q^2}{q'} \psi'(t) = -\ln\left(\int_{\mathbb{S}^n} (e^{-t\sqrt{-\Delta}} f)^q d\sigma_n\right) + \frac{\int_{\mathbb{S}^n} (e^{-t\sqrt{-\Delta}} f)^q \left(\ln(e^{-t\sqrt{-\Delta}} f)^q + \frac{q^2}{q'} \frac{\partial_t e^{-t\sqrt{-\Delta}} f}{e^{-t\sqrt{-\Delta}} f}\right) d\sigma_n}{\int_{\mathbb{S}^n} (e^{-t\sqrt{-\Delta}} f)^q d\sigma_n}.$$

Clearly $\psi' \leq 0$ if and only if

$$\int_{\mathbb{S}^n} (e^{-t\sqrt{-\Delta}} f)^q \ln(e^{-t\sqrt{-\Delta}} f)^q d\sigma_n - \int_{\mathbb{S}^n} (e^{-t\sqrt{-\Delta}} f)^q d\sigma_n \ln\left(\int_{\mathbb{S}^n} (e^{-t\sqrt{-\Delta}} f)^q d\sigma_n\right)$$
$$\leq \frac{q^2}{q'} \int_{\mathbb{S}^n} (e^{-t\sqrt{-\Delta}} f)^{q-1} \sqrt{-\Delta}(e^{-t\sqrt{-\Delta}} f) d\sigma_n.$$

Let $g = e^{-t\sqrt{-\Delta}} f \geq 0$. Then we can rewrite the previous inequality as

$$\int_{\mathbb{S}^n} g^q \ln g^q d\sigma_n - \int_{\mathbb{S}^n} g^q d\sigma_n \ln\left(\int_{\mathbb{S}^n} g^q d\sigma_n\right) \leq \frac{q^2}{2(q-1)\sqrt{n}} \int_{\mathbb{S}^n} g^{q-1} \sqrt{-\Delta} g d\sigma_n, \tag{7}$$

where we used the fact that $q' = 2(q-1)\sqrt{n}$. Since $e^{-t\sqrt{-\Delta}}$ is contractive in $L^\infty(\mathbb{S}^n)$ with a nonnegative, symmetric kernel, it follows that the validity of the estimate (7) for $q = 2$ implies (7) for all $q \in [2, \infty)$; see, e.g., Theorem 4.1 in [3].

Let $g = \sum_{k\geq 0} H_d$ be the decomposition of g into its spherical harmonics. Then the estimate (7) for $q = 2$ takes the form

$$\int_{\mathbb{S}^n} g^2 \ln g^2 d\sigma_n - \int_{\mathbb{S}^n} g^2 d\sigma_n \ln\left(\int_{\mathbb{S}^n} g^2 d\sigma_n\right) \leq \sum_{k\geq 0} 2\sqrt{\frac{k(k+n-1)}{n}} \|H_k\|_2^2.$$

It follows from Beckner's conformal log Sobolev inequality [1] (which is a consequence of Lieb's sharp Hardy–Littlewood–Sobolev inequality [6]) that for any smooth nonnegative $g = \sum_{k\geq 0} H_k$ we have

$$\int_{\mathbb{S}^n} g^2 \ln g^2 d\sigma_n - \int_{\mathbb{S}^n} g^2 d\sigma_n \ln\left(\int_{\mathbb{S}^n} g^2 d\sigma_n\right) \leq \sum_{k\geq 0} \Delta_n(k) \|H_k\|_2^2$$

with $\Delta_n(k) = 2n \sum_{m=0}^{k-1} \frac{1}{2m+n}$. Thus, the estimate (5) is a consequence of the following lemma.

Lemma 2.1. *Let $n \in \{2, 3\}$. Then for all integers $k \geq 1$ one has*

$$n \sum_{m=0}^{k-1} \frac{1}{2m+n} \leq \sqrt{\frac{k(k+n-1)}{n}}.$$

Proof. We first check the inequality for $k \leq 3$ by direct computation. Indeed, the case $k = 1$ is an equality. The case $k = 2$ can be checked as follows,

$$1 + \frac{n}{2+n} = \frac{2+2n}{2+n} \leq \sqrt{\frac{2+2n}{n}},$$

which is true because $n(2+2n) \leq (2+n)^2$ holds for $n = 2, 3$. The case $k = 3$ can be checked similarly:

$$\frac{2+2n}{2+n} + \frac{n}{4+n} \leq \sqrt{\frac{6+3n}{n}}$$

holds for $n = 2, 3$ (notice that this inequality fails for $n = 4$).

Next, we assume $k \geq 4$. We have

$$\sum_{m=0}^{k-1} \frac{1}{m+\frac{n}{2}} = \frac{2}{n} + \sum_{m=1}^{k-1} \frac{1}{m+\frac{n}{2}} \leq \frac{2}{n} + \int_0^{k-1} \frac{1}{x+\frac{n}{2}} dx = \frac{2}{n} + \ln\left(\frac{k+\frac{n}{2}-1}{\frac{n}{2}}\right).$$

Thus it suffices to show

$$\frac{2}{n}+\ln\left(\frac{k+\frac{n}{2}-1}{\frac{n}{2}}\right)-\frac{2}{n}\sqrt{\frac{k(k+n-1)}{n}}\leq 0.$$

Notice that the left hand side, call it $h(k)$, is decreasing in k. Indeed, we have

$$h'(k)=\frac{1}{\frac{n}{2}+k-1}-\frac{2k+n-1}{n\sqrt{kn(k+n-1)}}\leq\frac{1}{\frac{n}{2}+k-1}-\frac{1}{\sqrt{kn}}\leq\frac{1}{2\sqrt{\frac{n}{2}(k-1)}}-\frac{1}{\sqrt{kn}}\leq 0.$$

On the other hand, we have for $n=2,3$,

$$h(4)=\frac{2}{n}+\ln\left(\frac{6+n}{n}\right)-\frac{2}{n}\sqrt{\frac{12+4n}{n}}\leq 0.$$

Indeed, if $n=2$, $h(4)=1+2\ln 2-\sqrt{10}<0$, and if $n=3$, $h(4)=\frac{2+3\ln 3-4\sqrt{2}}{3}<0$. $\square$

2.3. **Counterexample to (ii) $\Rightarrow$ (i) in high dimensions.** Let $\lambda:=\frac{n-1}{2}$, and let $C_d^{(\lambda)}(x)$ be the Gegenbauer polynomial

$$C_d^{(\lambda)}(x)=\sum_{j=0}^{\lfloor\frac{d}{2}\rfloor}(-1)^j\frac{\Gamma(d-j+\lambda)}{\Gamma(\lambda)j!(d-2j)!}(2x)^{d-2j}, \tag{8}$$

where $\lfloor\frac{d}{2}\rfloor$ denotes the largest integer m such that $m\leq\frac{d}{2}$, and $\Gamma(x)$ is the Gamma function. Notice that if we let $Y_d(\xi)=C_d^{(\lambda)}(\xi\cdot e_1)$, where $e_1=(1,0,\ldots,0)\in\mathbb{R}^{n+1}$, then $Y_d(\xi)$ is a spherical harmonic of degree d on $\mathbb{S}^n$. In particular, for $t\geq 0$ such that $e^{-2t\sqrt{n}}=\frac{p-1}{q-1}$, the estimate $\|e^{-t\sqrt{-\Delta}}f\|_{L^q(\mathbb{S}^n)}\leq\|f\|_{L^p(\mathbb{S}^n)}$ applied to $f=Y_d(\xi)$ is equivalent to the estimate

$$\frac{\|Y_d\|_q}{\|Y_d\|_p}\leq e^{t\sqrt{d(d+n-1)}}=\left(\frac{q-1}{p-1}\right)^{\frac{1}{2}\sqrt{\frac{d(d+n-1)}{n}}}. \tag{9}$$

Next, we need

Lemma 2.2. *For any $d\geq 0$ we have*

$$\lim_{n\to\infty}\frac{\|Y_d\|_{L^q(\mathbb{S}^n,d\sigma_n)}}{\|Y_d\|_{L^p(\mathbb{S}^n,d\sigma_n)}}=\frac{\|h_d\|_{L^q(\mathbb{R},d\gamma)}}{\|h_d\|_{L^p(\mathbb{R},d\gamma)}}, \tag{10}$$

where $d\gamma(y)=\frac{e^{-y^2/2}}{\sqrt{2\pi}}dy$ is the standard Gaussian measure on the real line, and $h_d(x)$ is the probabilistic Hermite polynomial

$$h_d(x)=\sum_{j=0}^{\lfloor\frac{d}{2}\rfloor}\frac{(-1)^j d!}{j!(d-2j)!}\frac{x^{d-2j}}{2^j}. \tag{11}$$

Proof. Indeed, notice that

$$\|Y_d\|_p^p=\int_{\mathbb{S}^n}|C_d^{(\lambda)}(\xi\cdot e_1)|^p d\sigma_n(\xi)=\int_{-1}^1|C_d^{(\lambda)}(t)|^p c_\lambda(1-t^2)^{\lambda-\frac{1}{2}}dt, \tag{12}$$

where $c_\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\frac{1}{2})\Gamma(\lambda+\frac{1}{2})}$. In particular, after the change of variables $t = \frac{s}{\sqrt{2\lambda}}$ in (12), and multiplying both sides in (12) by $(d!/(2\lambda)^{d/2})^p$ we obtain

$$\left(\frac{d!}{(2\lambda)^{d/2}}\right)^p \|Y_d\|_p^p = \int_{\mathbb{R}} \left|\frac{d!}{(2\lambda)^{d/2}} C_d^{(\lambda)}\left(\frac{s}{\sqrt{2\lambda}}\right)\right|^p \frac{c_\lambda}{\sqrt{2\lambda}}\left(1-\frac{s^2}{2\lambda}\right)^{\lambda-\frac{1}{2}} \mathbb{1}_{[-\sqrt{2\lambda},\sqrt{2\lambda}]}(s)ds,$$

where $\mathbb{1}_{[-\sqrt{2\lambda},\sqrt{2\lambda}]}(s)$ denotes the indicator function of the set $[-\sqrt{2\lambda},\sqrt{2\lambda}]$. Notice that by Stirling's formula for any $j \geq 0$, and any $d \geq 0$ we have

$$\lim_{\lambda\to\infty} \frac{1}{\lambda^{d-j}} \frac{\Gamma(d-j+\lambda)}{\Gamma(\lambda)} = 1. \tag{13}$$

Therefore, (11) and (8) together with (13) imply that for all $s \in \mathbb{R}$ we have

$$\lim_{\lambda\to\infty} \frac{d!}{(2\lambda)^{d/2}} C_d^{(\lambda)}\left(\frac{s}{\sqrt{2\lambda}}\right) = h_d(s).$$

Invoking Stirling's formula again we have

$$\lim_{\lambda\to\infty} \frac{c_\lambda}{\sqrt{2\lambda}}\left(1-\frac{s^2}{2\lambda}\right)^{\lambda-\frac{1}{2}} \mathbb{1}_{[-\sqrt{2\lambda},\sqrt{2\lambda}]}(s) = \frac{e^{-s^2/2}}{\sqrt{2\pi}} \quad \text{for all} \quad s \in \mathbb{R}.$$

Finally, to apply Lebesgue's dominated convergence theorem it suffices to verify that for all $s \in \mathbb{R}$ and all $\lambda \geq \lambda_0$ we have the following pointwise estimates

$$a) \quad \frac{c_\lambda}{\sqrt{2\lambda}}\left(1-\frac{s^2}{2\lambda}\right)^{\lambda-\frac{1}{2}} \mathbb{1}_{[-\sqrt{2\lambda},\sqrt{2\lambda}]}(s) \leq Ce^{-s^2/2}$$

$$b) \quad \frac{d!}{(2\lambda)^{d/2}} C_d^{(\lambda)}\left(\frac{s}{\sqrt{2\lambda}}\right) \leq c_1(d)(1+|s|)^{c_2(d)},$$

where $\lambda_0, C, c_1(d), c_2(d)$ are some positive constants independent of λ and s.

To verify a) it suffices to consider the case $s \in [-\sqrt{2\lambda},\sqrt{2\lambda}]$. Since $\lim_{\lambda\to\infty} \frac{c_\lambda}{\sqrt{2\lambda}} = \frac{1}{\sqrt{2\pi}}$ it follows that $\frac{c_\lambda}{\sqrt{2\lambda}} \leq C$ for all $\lambda \geq \lambda_0$, where λ_0 is a sufficiently large number. Next, the estimate $(1-\frac{s^2}{2\lambda})^{\lambda-1/2} \leq C'e^{-s^2/2}$ for $s \in [-\sqrt{2\lambda},\sqrt{2\lambda}]$ follows if we show that $(1-\frac{1}{2\lambda})\ln(1-t) \leq C''/\lambda - t$ for all $t := \frac{s^2}{2\lambda} \in [0,1]$ where C'' is a universal positive constant. The latter inequality follows from $\ln(1-t) \leq -t$ for $t \in [0,1]$.

To verify b) it suffices to show that for all $\lambda \geq \lambda_0 > 0$ and all integers j such that $d \geq j \geq 0$ one has

$$\frac{1}{\lambda^{d-j}} \frac{\Gamma(d-j+\lambda)}{\Gamma(\lambda)} \leq C(d-j),$$

where $C(d-j)$ depends only on $d-j$. The latter inequality follows from (13) provided that $\lambda \geq \lambda_0$ where λ_0 is a sufficiently large number.

Thus, it follows from the Lebesgue's dominated convergence theorem that

$$\lim_{n\to\infty} \frac{d!}{(n-1)^{d/2}} \|Y_d\|_{L^p(\mathbb{S}^n, d\sigma_n)} = \|h_d\|_{L^p(\mathbb{R}, d\gamma)}.$$

The lemma is proved. □

Now we fix $q > \max\{p, 2\}$ and, in order to prove the failure of (ii) $\Rightarrow$ (i) for all sufficiently large n, we argue by contradiction and assume that there is a sequence of dimensions $\{n_j\}_{j\geq 1}$ going to infinity such that (ii) $\Rightarrow$ (i) in Theorem 1.1 does hold. Then, by combining (9) and (10) we have

$$\frac{\|h_d\|_{L^q(\mathbb{R},d\gamma)}}{\|h_d\|_{L^p(\mathbb{R},d\gamma)}} \leq \left(\frac{q-1}{p-1}\right)^{\frac{\sqrt{d}}{2}}. \tag{14}$$

On the other hand, a consequence of the main result in [5] and the assumption $q > \max\{p, 2\}$ is that

$$\lim_{d\to\infty}\left(\frac{\|h_d\|_{L^q(\mathbb{R},d\gamma)}}{\|h_d\|_{L^p(\mathbb{R},d\gamma)}}\right)^{1/d} = \left(\frac{q-1}{\max\{p,2\}-1}\right)^{\frac{1}{2}},$$

which is in contradiction with (14).

Remark 2.1. *Let $B(x, y)$ be the Beta function. The estimate (9) for $p = 2$ and $q = 4$ takes the form*

$$\int_{-1}^{1} |C_d^{(\frac{n-1}{2})}(t)|^4 (1-t^2)^{\frac{n-2}{2}}\,dt \leq 9^{\sqrt{\frac{d(d+n-1)}{n}}}\,\frac{(n-1)^2 B(1/2, n/2)}{d^2(2d+n-1)^2 B^2(n-1,d)}, \tag{15}$$

where we used the fact that $\|Y_d\|^2_{L^2(\mathbb{S}^n)} = \frac{n-1}{d(2d+n-1)B(n-1,d)}$. The numerical computations show that the inequality (15) already fails for $d = 7$ and $n = 13$.

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