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# On a conjecture of Widom

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**Abstract**
We prove a conjecture of Widom (2002 *Int. Math. Res. Not.* 455–64 (*Preprint* math/0108008)) about the reality of eigenvalues of certain infinite matrices arising in asymptotic analysis of large Toeplitz determinants. As a byproduct, we obtain a new proof of Okounkov's formula for the (determinantal) correlation functions of the Schur measures on partitions.

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## Introduction

Consider an operator $T$ in $L^2(n, n+1, \ldots)$, $n = 0, 1, \ldots,$ with matrix elements

$$T_{pq} = \sum_{k\geqslant 1} \left(\frac{\phi_-}{\phi_+}\right)_{p+k} \left(\frac{\phi_+}{\phi_-}\right)_{-q-k}$$

where $(f)_k$ denotes the $k$th Fourier coefficient of a function $f(z)$ on the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ and the functions

$$\phi_+(z) = \mathrm{e}^{\gamma^+ z} \prod_{i\geqslant 1} \frac{1+\beta_i^+ z}{1-\alpha_i^+ z}, \qquad \phi_-(z) = \mathrm{e}^{\gamma^-/z} \prod_{i\geqslant 1} \frac{1+\beta_i^-/z}{1-\alpha_i^-/z}$$

are determined by certain nonnegative parameters $\{\alpha_i^\pm\}$, $\{\beta_i^\pm\}$ and $\gamma^\pm$ such that $\sum_i \left(\alpha_i^\pm + \beta_i^\pm\right) < \infty$ and $\alpha_i^\pm, \beta_i^\pm \leqslant \text{const} < 1$ for all $i \geqslant 1$. The main goal of this paper is to prove that the spectrum of this operator is real and lies between 0 and 1.

This property of the operator $T$ turns out to be useful in the asymptotic analysis of growing Toeplitz determinants with symbol $\phi = \phi_+\phi_-$ or, more generally, of the distribution functions of the largest parts of random partitions distributed according to the associated Schur measure. The statement was conjectured by Widom in [W] in the case when the parameters $\alpha_i^+, \beta_i^-, \gamma^\pm$ vanish and the number of nonzero $\alpha_i^-$s and $\beta_i^+$s is finite. Some more special cases were proved in [BDR, W]. We refer to [W] for details and further references.

The key fact which allows us to prove the reality of the spectrum is that the resolvent $T(1-T)^{-1}$ of matrix $T$ after conjugation by a diagonal matrix with plus–minus 1s becomes totally positive. While proving this fact, we obtain as a byproduct a new proof of Okounkov's formula for the correlation functions of the Schur measures, see [O] and also [J, R] for other proofs. More specifically, we show that the Schur measure may be viewed as an $L$-ensemble and explicitly compute the correlation kernel $K = L(1+L)^{-1}$.

## An operator identity

In what follows, $\mathbb{Z}_+$ denotes the set of nonnegative integers and all contour integrals are taken over circles centred at the origin with radii close enough to 1.

Let $\phi_+$ be a holomorphic function in the disc

$$D_r = \{z \in \mathbb{C} : |z| \leqslant r\}$$

with radius $r > 1$ and let $\phi_-$ be a holomorphic function outside the disc $D_{1/r}$. In other words,

$$\phi_\pm(z) = \sum_{n=0}^{\infty} (\phi_\pm)_n z^{\pm n}, \qquad (\phi_\pm)_n = O(r^{-n}) \quad \text{as} \quad n \to \infty.$$

Also assume that $\phi_+$ and $\phi_-$ do not vanish on $D_r$ and $\overline{\mathbb{C}} \setminus D_{1/r}$, respectively.

Consider an operator $L$ in $L^2(\mathbb{Z}_+) \oplus L^2(\mathbb{Z}_+)$ with matrix

$$L = \begin{bmatrix} 0 & A^t \\ -B & 0 \end{bmatrix},$$

where the generating functions of matrix elements of $A$ and $B$ are

$$\sum_{p,q\geqslant 0} A_{pq} u^p v^q = \frac{1}{u+v}\left(\frac{\phi_+(u)}{\phi_+(-v)} - 1\right), \qquad \sum_{p,q\geqslant 0} B_{pq} u^p v^q = \frac{1}{u+v}\left(\frac{\phi_-(u^{-1})}{\phi_-(-v^{-1})} - 1\right).$$

One can also write the matrix elements in terms of the contour integrals

$$A_{pq} = \frac{1}{(2\pi \mathrm{i})^2} \oint\oint \left(\frac{\phi_+(u)}{\phi_+(-v)} - 1\right) \frac{\mathrm{d}u\,\mathrm{d}v}{(u+v)u^{p+1}v^{q+1}},$$
$$B_{pq} = \frac{1}{(2\pi \mathrm{i})^2} \oint\oint \left(\frac{\phi_-(u^{-1})}{\phi_-(-v^{-1})} - 1\right) \frac{\mathrm{d}u\,\mathrm{d}v}{(u+v)u^{p+1}v^{q+1}}.$$

These integral representations imply that $|A_{pq}|$ and $|B_{pq}|$ decay faster than $\text{const} \cdot x^{-p-q}$ for any $1 < x < r$ as $p+q \to \infty$. Thus, the sum of absolute values of matrix elements of $L$ is finite, and $L$ is a trace class operator.

Note that the change $\phi_+(z) \leftrightarrow \phi_+^{-1}(-z)$ replaces $A$ by $A^t$, the change $\phi_-(z) \leftrightarrow \phi_-^{-1}(-z)$ replaces $B$ by $B^t$ and the change

$$(\phi_+(u), \phi_-(v)) \longleftrightarrow (\phi_-(u^{-1}), \phi_+(v^{-1}))$$

switches $A$ and $B$. Thus, the switch $(A, B) \leftrightarrow (B^t, A^t)$ is achieved by

$$(\phi_+(u), \phi_-(v)) \longleftrightarrow \left(\phi_-^{-1}(-u^{-1}), \phi_+^{-1}(-v^{-1})\right).$$

**Theorem 1.** *Assume that the operator $1 + A^tB$ is invertible (equivalently, $1 + BA^t$ is invertible). Then the operator $1 + L$ is invertible, and the matrix of the operator $K = L(1+L)^{-1} = 1 - (1+L)^{-1}$ has the form*

$$K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} = \begin{bmatrix} 1 - (1+A^tB)^{-1} & (1+A^tB)^{-1}A^t \\ -(1+BA^t)^{-1}B & 1-(1+BA^t)^{-1} \end{bmatrix}$$

*where*

$$(K_{11})_{pq} = \delta_{pq} - \frac{(-1)^{p+q}}{(2\pi \mathrm{i})^2} \oint\oint_{|zw|<1} \frac{\Phi(z,w)\,\mathrm{d}z\,\mathrm{d}w}{(1-zw)z^{p+1}w^{q+1}},$$

$$(K_{12})_{pq} = \frac{(-1)^{p}}{(2\pi \mathrm{i})^2} \oint\oint_{|zw|<1} \frac{\Phi(z,w)w^q\,\mathrm{d}z\,\mathrm{d}w}{(1-zw)z^{p+1}},$$

$$(K_{21})_{pq} = \frac{(-1)^{q+1}}{(2\pi \mathrm{i})^2} \oint\oint_{|zw|<1} \frac{\Phi(z,w)w^p\,\mathrm{d}z\,\mathrm{d}w}{(1-zw)z^{q+1}},$$

$$(K_{22})_{pq} = \frac{1}{(2\pi \mathrm{i})^2} \oint\oint_{|zw|<1} \frac{\Phi(z,w)z^p w^q\,\mathrm{d}z\,\mathrm{d}w}{1-zw},$$

*and*

$$\Phi(z,w) = \frac{\phi_-(z)\phi_+(w^{-1})}{\phi_+(z)\phi_-(w^{-1})}.$$

**Comment.** *The proof of this statement given below is a verification of the relation* $(1-K)(1+L) = 1$ *rather than the derivation of the inverse of the operator* $1+L$, *so the reader might wonder where the formulae for the operator* $K$ *above came from. The answer to this question is hidden in theorem 3.*

*Symmetric minors of the matrix* $L$ *are the weights of partitions with respect to the so-called Schur measure. Simple linear algebra implies that, consequently, minors of the matrix* $K = L(1+L)^{-1}$ *must be the correlation functions of the Schur measure. Another matrix* $\widetilde{K}$ *whose minors give these correlation functions has been previously computed in [O], and the formulae for* $K$ *above were obtained from the known formula for* $\widetilde{K}$ *by conjugation by a diagonal matrix with plus–minus 1s on the diagonal. Some details can be found in the next section and references therein.*

**Proof.** The proof of the first equality is straightforward. Thanks to the symmetries mentioned before the statement of the theorem it suffices to prove the integral formulae for $K_{11}$ and $K_{12}$. Let us start with $K_{11}$.

We need to show that $(1 + A^t B)(1 - K_{11}) = 1$ or, equivalently, $A^t B(1 - K_{11}) = K_{11}$. Explicit computation gives

$$\begin{aligned}
(A^t B(1-K_{11}))_{pq} &= \sum_{l,m\geqslant 0} A_{lp}B_{lm}(1-K_{11})_{mq} = \frac{1}{(2\pi\mathrm{i})^6}\oint\cdots\oint_{|zw|<1}\sum_{l,m\geqslant 0} \\
&\quad\times\left(\frac{\phi_+(u_1)}{\phi_+(-v_1)}-1\right)\frac{\mathrm{d}u_1\,\mathrm{d}v_1}{(u_1+v_1)u_1^{l+1}v_1^{p+1}}\left(\frac{\phi_-\left(u_2^{-1}\right)}{\phi_-\left(-v_2^{-1}\right)}-1\right) \\
&\quad\times\frac{\mathrm{d}u_2\,\mathrm{d}v_2}{(u_2+v_2)u_2^{l+1}v_2^{m+1}}\frac{\phi_-(z)\phi_+(w^{-1})}{\phi_+(z)\phi_-(w^{-1})}\frac{\mathrm{d}z\,\mathrm{d}w}{(1-zw)(-z)^{m+1}(-w)^{q+1}} \\
&= \frac{1}{(2\pi\mathrm{i})^6}\oint\cdots\oint_{|zw|<1,|u_1u_2|>1,|v_2z|>1}\left(\frac{\phi_+(u_1)}{\phi_+(-v_1)}-1\right)\frac{\mathrm{d}u_1\,\mathrm{d}v_1}{(u_1+v_1)v_1^{p+1}} \\
&\quad\times\left(\frac{\phi_-\left(u_2^{-1}\right)}{\phi_-\left(-v_2^{-1}\right)}-1\right)\frac{\mathrm{d}u_2\,\mathrm{d}v_2}{(u_2+v_2)}\frac{\phi_-(z)\phi_+(w^{-1})}{\phi_+(z)\phi_-(w^{-1})} \\
&\quad\times\frac{\mathrm{d}z\,\mathrm{d}w}{(-w)^{q+1}}\frac{(-1)}{(u_1u_2-1)(v_2z+1)(1-zw)}
\end{aligned}$$

where we imposed additional conditions $|u_1u_2| > 1$, $|v_2z| > 1$ on the integration contours to ensure the convergence of two geometric series under the integral.

We can immediately perform the integration over $u_2$ and $v_2$. Indeed, there is only one simple pole $u_2 = u_1^{-1}$ inside the $u_2$-contour and there is only one simple pole $v_2 = -z^{-1}$ inside the $v_2$-contour. Evaluating the residues, we obtain that the integral above equals

$$\frac{1}{(2\pi i)^4}\oint\cdots\oint_{|zw|<1}\left(\frac{\phi_+(u_1)}{\phi_+(-v_1)}-1\right)\frac{du_1\,dv_1}{(u_1+v_1)v_1^{p+1}} \times\left(\frac{\phi_-(u_1)}{\phi_-(z)}-1\right)\frac{1}{(z-u_1)}\frac{\phi_-(z)\phi_+(w^{-1})}{\phi_+(z)\phi_-(w^{-1})}\frac{dz\,dw}{(-w)^{q+1}}\frac{(-1)}{(1-zw)}.$$

Let us choose the contours so that $|u_1| < |z|$ and open the parentheses $\left(\frac{\phi_-(u_1)}{\phi_-(z)}-1\right)$ in the integral above. The second term vanishes because it has no singularities inside the $u_1$-contour. The first term has only one simple pole $z = u_1$ inside the $z$-contour, and the integration over $z$ gives

$$\frac{1}{(2\pi i)^3}\oint\oint\oint_{|u_1w|<1}\left(\frac{\phi_+(u_1)}{\phi_+(-v_1)}-1\right)\frac{du_1\,dv_1}{(u_1+v_1)v_1^{p+1}}\frac{\phi_-(u_1)\phi_+(w^{-1})}{\phi_+(u_1)\phi_-(w^{-1})}\frac{dw}{(-w)^{q+1}}\frac{(-1)}{(1-u_1w)}.$$

Now let us choose the contours so that $|u_1| > |v_1|$ and open the parentheses $\left(\frac{\phi_+(u_1)}{\phi_+(-v_1)}-1\right)$. The first term can be integrated over $u_1$—there is only one pole $u_1 = -v_1$ inside the $u_1$-contour. Thus, the first term equals the corresponding residue, that is

$$\frac{-1}{(2\pi i)^2}\oint\oint_{|v_1w|<1}\frac{\phi_+(w^{-1})}{\phi_+(-v_1)}\frac{dv_1\,dw}{(1+v_1w)v_1^{p+1}(-w)^{q+1}}.$$

By deforming the $w$-contour to $\infty$ and picking up the residue at $w = -v_1^{-1}$, we immediately see that this integral is equal to $\delta_{pq}$.

Now the second term is equal to

$$\frac{1}{(2\pi i)^3}\oint\oint\oint_{|u_1w|<1,|u_1|>|v_1|}\frac{du_1\,dv_1}{(u_1+v_1)v_1^{p+1}}\frac{\phi_-(u_1)\phi_+(w^{-1})}{\phi_+(u_1)\phi_-(w^{-1})}\frac{dw}{(1-u_1w)(-w)^{q+1}}.$$

Deforming the $v_1$-contour to $\infty$, we pick up the residue at $v_1 = -u_1$ which gives

$$\frac{-1}{(2\pi i)^2}\oint\oint_{|u_1w|<1}\frac{\phi_-(u_1)\phi_+(w^{-1})}{\phi_+(u_1)\phi_-(w^{-1})}\frac{du_1\,dw}{(1-u_1w)(-u_1)^{p+1}(-w)^{q+1}},$$

and this is exactly $(K_{11})_{pq} - \delta_{pq}$. The proof of the formula for $K_{11}$ is complete.

In order to prove the formula for $K_{12}$, we need to show that $(1 + A^tB)K_{12} = A^t$ or $A^tBK_{12} = A^t - K_{12}$. The computation of $A^tBK_{12}$ literally follows the above arguments for $K_{11}$ and leads to the sum of two terms:

$$\frac{1}{(2\pi i)^2}\oint\oint_{|v_1w|<1}\frac{\phi_+(w^{-1})}{\phi_+(-v_1)}\frac{w^q\,dv_1\,dw}{(1+v_1w)v_1^{p+1}} + \frac{1}{(2\pi i)^2}\oint\oint_{|u_1w|<1}\frac{\phi_-(u_1)\phi_+(w^{-1})}{\phi_+(u_1)\phi_-(w^{-1})}\frac{w^q\,du_1\,dw}{(1-u_1w)(-u_1)^{p+1}}.$$

The first term is immediately seen to be equal to $A_{qp}$ and the second one is exactly $-(K_{12})_{pq}$. □

### Schur functions and Schur measures

We refer the reader to [M, S] for general information on partitions and symmetric functions.

Let $\Lambda$ be the algebra of symmetric functions. It can be viewed as the algebra of polynomials in countably many indeterminates $\Lambda = \mathbb{C}[h_1, h_2, \ldots]$ where the indeterminates $h_k$ are the complete homogeneous symmetric functions of degree $k$. We also agree that $h_0 = 1$ and $h_{-k} = 0$ for $k < 0$.

The Schur symmetric functions $s_\lambda$ are parameterized by partitions $\lambda$ and are expressed through $h_k$ s by the Jacobi–Trudi formula

$$s_\lambda = \det\left[h_{\lambda_i - i + j}\right]_{i,j=1}^{N}$$

where $N$ is any number greater or equal to the number of nonzero parts of $\lambda$. The Schur functions form a linear basis in $\Lambda$.

An algebra homomorphism $\pi : \Lambda \to \mathbb{C}$ is uniquely determined by its values on $h_k$s or by generating series of these values

$$H^\pi(z) = \sum_{z=0}^{\infty} \pi(h_n) z^n.$$

Recall that a sequence $\{a_n\}_{n=0}^{\infty}$ is called *totally positive* if all minors of the matrix $[a_{i-j}]_{i,j\geqslant 0}$ are nonnegative. Here, all $a_{-k}$ for $k > 0$ are assumed to be equal to zero. We will only consider totally positive sequences with $a_0 = 1$; clearly, multiplication of all members of a sequence by the same positive number does not affect total positivity.

The following statement was independently proved by Aissen–Edrei–Schoenberg–Whitney in 1951 [AESW, E] and by Thoma in 1964 [T]. An excellent exposition of deep relations of this result to representation theory of the infinite symmetric group can be found in Kerov's book [K].

**Theorem 2.** *A sequence $\{a_n\}_{n=0}^{\infty}$, $a_0 = 1$, is totally positive if and only if its generating series has the form*

$$\sum_{n=0}^{\infty} a_n z^n = \mathrm{e}^{\gamma z} \frac{\prod_{i\geqslant 1}(1 + \beta_i z)}{\prod_{i\geqslant 1}(1 - \alpha_i z)} =: F(\alpha, \beta, \gamma)$$

*for certain nonnegative parameters $\{\alpha_i\}$, $\{\beta_i\}$ and $\gamma$ such that $\sum_i(\alpha_i + \beta_i) < \infty$.*

*Equivalently, an algebra homomorphism $\pi : \Lambda \to \mathbb{C}$ takes nonnegative values on all Schur functions if and only if the sequence $\{\pi(h_n)\}_{n\geqslant 0}$ is totally positive, that is, $H^\pi(z) = F(\alpha, \beta, \gamma)$ for a suitable choice of parameters $(\alpha, \beta, \gamma)$.*

We will call a specialization $\pi : \Lambda \to \mathbb{C}$ *positive* if $\pi(s_\lambda) \geqslant 0$ for all partitions $\lambda$. Thus, the theorem above may be viewed as a classification of all positive specializations of the algebra of symmetric functions.

Let us now consider specializations $\pi_+$ and $\pi_-$ of $\Lambda$ such that

$$H^{\pi_+}(z) = \phi_+(z), \qquad H^{\pi_-}(z) = \phi_-(z^{-1})$$

for the holomorphic functions $\phi^{\pm}$ of the previous section, and set $s_\lambda^{\pm} := \pi_{\pm}(s_\lambda)$.

Following Okounkov [O] assign to any partition $\lambda$, the following (generally speaking, complex) weight:

$$P\{\lambda\} = \frac{s_\lambda^+ s_\lambda^-}{Z}, \qquad Z = \exp \sum_{k\geqslant 1} k(\ln\phi_+(z))_k(\ln\phi_-(z))_k.$$

One can show that $\sum_\lambda P\{\lambda\}$ is an absolutely convergent series with sum equal to 1. The distribution $P$ is called the *Schur measure*.

Theorem 1 proved in the previous section yields a new proof of the determinantal formula for the correlation functions of the Schur measure.

The following statement was proved in [O]; other proofs can be found in [J, R].

For any partition $\lambda$, denote by $\mathcal{L}(\lambda)$ the infinite subset $\{\lambda_i - i\}_{i=1}^{\infty}$ of $\mathbb{Z}$.

**Theorem 3.** *For any* $x_1, \ldots, x_n \in \mathbb{Z}$

$$\sum_{\lambda:\mathcal{L}(\lambda)\supset\{x_1,\ldots,x_n\}} P\{\lambda\} = \det[\mathcal{K}(x_i, x_j)]_{i,j=1}^n$$

*where*

$$\mathcal{K}(x, y) = \frac{1}{(2\pi \mathrm{i})^2} \oint\oint_{|zw|<1} \frac{\phi_-(z)\phi_+(w^{-1})}{\phi_+(z)\phi_-(w^{-1})} \frac{z^p w^q \,\mathrm{d}z \,\mathrm{d}w}{1 - zw}.$$

**Proof.** We will use the material of the appendix in [BOO]. Let $\lambda$ be a partition and $(p_1, \ldots, p_d \mid q_1, \ldots, q_d)$ be its Frobenius coordinates[1]. Using the well-known Giambelli formula $s_\lambda = \det\left[s_{(p_i|q_j)}\right]_{i,j=1}^d$, see, e.g., [M, example I.3.9], we obtain

$$P\{\lambda\} = \frac{s_\lambda^+ s_\lambda^-}{Z} = Z^{-1} \det\left[s^+_{(p_i|q_j)}\right]_{i,j=1}^d \det\left[s^-_{(p_i|q_j)}\right]_{i,j=1}^d.$$

Comparing the following formula for the generating series of the hook Schur functions

$$1 + (u + v) \sum_{p,q\geqslant 0} s_{(p|q)} u^p v^q = \frac{H(u)}{H(-v)}, \qquad H(z) = \sum_{n=0}^{\infty} h_n z^n,$$

see, e.g., [M, example I.3.14], with the definition of matrices $A$ and $B$ in the previous section we see that

$$s^+_{(p|q)} = A_{pq}, \qquad s^-_{(p|q)} = B_{pq}, \qquad p, q \geqslant 0.$$

Thus,

$$P\{\lambda\} = Z^{-1} \cdot \det\left[A_{p_i q_j}\right]_{i,j=1}^d \det\left[B_{p_i q_j}\right]_{i,j=1}^d$$

is, up to a constant, the value of the symmetric minor of the matrix $L$ from the previous section with $2d$ rows and $2d$ columns marked by $(q_1, \ldots, q_d \mid p_1, \ldots, p_d)$. This means that the Schur measure interpreted through the Frobenius coordinates of partitions defines a determinantal point process on $\mathbb{Z}_+ \sqcup \mathbb{Z}_+$, and its correlation kernel is given by $K = L(1 + L)^{-1}$. (The operator $1 + L$ is invertible because $\det(1 + L) = Z \neq 0$.)

Finally, to pass from Frobenius coordinates $(p_1, \ldots, p_d \mid q_1, \ldots, q_d) \subset \mathbb{Z}_+ \sqcup \mathbb{Z}_+$ to $\mathcal{L}(\lambda) \subset \mathbb{Z}$ we can use the complementation principle, see [appendix A.3, BOO], thanks to the Frobenius lemma

$$\mathcal{L}(\lambda) = \{p_1, \ldots, p_d\} \sqcup (\{-1, -2, -3, \ldots\} \backslash \{-q_1 - 1, \ldots, -q_d - 1\}),$$

see, e.g., [M, I.1(1.7)]. It is readily seen that the operation $\triangle$ of [appendix A.3, BOO] transforms the kernel $K$ of theorem 1 to the kernel $\mathcal{K}$ of theorem 3 up to a conjugation by a diagonal matrix of plus–minus 1s. Since such conjugation does not change the determinants $\det[\mathcal{K}(x_i, x_j)]_{i,j=1}^n$, the proof is complete. □

[1] Frobenius coordinate $p_i$ measures the number of boxes in the $i$th row of the corresponding Young diagram to the right of the diagonal. Similarly, $q_j$ is the number of boxes in the $j$th column below the diagonal. Formally, $p_i = \lambda_i - i$, $q_i = \lambda_i' - i$, $i = 1, \ldots, d$, where $d$ is the length of the diagonal of the Young diagram corresponding to $\lambda$.

**Total positivity and Widom's conjecture**

In this section, we assume that the functions $\phi_+(z)$ and $\phi_-(z^{-1})$ are generating functions of totally positive sequences, that is,

$$\phi_+(z) = \mathrm{e}^{\gamma^+ z} \frac{\prod_{i\geqslant 1}\left(1+\beta_i^+ z\right)}{\prod_{i\geqslant 1}\left(1-\alpha_i^+ z\right)}, \qquad \phi_-(z) = \mathrm{e}^{\gamma^-/z} \frac{\prod_{i\geqslant 1}(1+\beta_i^-/z)}{\prod_{i\geqslant 1}(1-\alpha_i^-/z)}$$

for certain nonnegative parameters $\{\alpha_i^\pm\}$, $\{\beta_i^\pm\}$ and $\gamma^\pm$ such that $\sum_i\left(\alpha_i^\pm+\beta_i^\pm\right) < \infty$ and $\alpha_i^\pm, \beta_i^\pm < r^{-1} < 1$ for all $i \geqslant 1$.

Let $P_n$ be the projection operator in $L^2(\mathbb{Z}_+)$ which projects onto $n$ first basis vectors:

$$P_n : (x_0, x_1, x_2, \ldots) \mapsto (x_0, x_1, \ldots, x_{n-1}, 0, 0, \ldots).$$

**Theorem 4.** *Under the above assumption, for any $n = 1, 2, \ldots$ the spectra of the operators $K_{11}^{(n)} = (1-P_n)K_{11}(1-P_n)$ and $K_{22}^{(n)} = (1-P_n)K_{22}(1-P_n)$ are real and lie between 0 and 1.*

Let us start with a lemma.

**Lemma 5.** *Let $T_n$ be a sequence of trace class operators in a Hilbert space which converges to a (trace class) operator $T$ in trace norm. Assume that $Sp(T_n) \subset S$ for a closed set $S \subset \mathbb{C}$ and all large enough $n$. Then, $Sp(T) \subset S \cup \{0\}$.*

**Proof.** By virtue of (proposition A.11, [BOO]), trace norm convergence implies the convergence of entire functions in $z$

$$\det(1+zT_n) \to \det(1+zT)$$

uniformly on compact subsets of $\mathbb{C}$. Therefore, any zero of $\det(1+zT)$ is a limit point of zeros of $\det(1+zK_n)$. □

**Proof of theorem 4.** Let us first show that $Sp(K_{11})$ and $Sp(K_{22})$ lie inside the interval [0, 1).

Since the specializations $\pi_\pm$ associated with $\phi_\pm$ are positive, the matrices $A$ and $B$ are totally positive. Indeed, by Giambelli's formula their minors are values of $\pi_\pm$ on appropriate Schur functions. Furthermore, recall that $A_{pq}$ and $B_{pq}$ decay faster than const $\cdot\, x^{-p-q}$ for any $1 < x < r$ as $p+q \to \infty$. This means that the matrix $A^t B$ is also totally positive and its matrix elements satisfy similar estimates. Consequently, $A^t B$ is a trace class operator, and $\det(1+A^t B) - 1$ is equal to the sum of all symmetric minors of $A^t B$, which is nonnegative. Thus, the operator $1 + A^t B$ is invertible.

The decay of the matrix elements of $A^t B$ shows that $P_n A^t B P_n$ converges to $A^t B$ in trace norm as $n \to \infty$ (indeed, the trace norm of a matrix does not exceed the sum of the absolute values of the matrix elements). It is well known that the eigenvalues of a totally positive matrix are nonnegative, see, e.g., [An, corollary 6.6]. Hence, $Sp(P_n A^t B P_n) \subset \mathbb{R}_{\geqslant 0}$, and by lemma 5, $Sp(A^t B) \subset \mathbb{R}_{\geqslant 0}$. Therefore, $Sp(K_{11}) = A^t B(1+A^t B)^{-1} \subset [0, 1)$. The argument for $K_{22}$ is very similar.

To extend the argument to $K_{11}^{(n)}, K_{22}^{(n)}$, we need a linear algebraic lemma.

**Lemma 6.** *Let $C$ be an $(m+n)\times(m+n)$ matrix such that $1 + C$ is invertible and let $D$ be the $n \times n$ lower right corner of $C(1+C)^{-1}$. Then,*

$$\det(1-D) = \sum_{X\subset\{1,\ldots,m\}} \frac{C\binom{X}{X}}{\det(1+C)}$$

*(the sum above includes* $X = \varnothing$ *and* $C\binom{\varnothing}{\varnothing} = 1$*). Assume further that* $\det(1 - D) \neq 0$ *and set* $E = D(1 - D)^{-1}$*. Then any minor of E is, up to a constant, a sum of certain minors of the initial matrix C:*

$$\frac{E\binom{X}{Y}}{\det(1+E)} = \sum_{Z\subset\{1,\ldots,m\}} \frac{C\binom{Z\sqcup X}{Z\sqcup Y}}{\det(1+C)}.$$

**Proof.** The first identity follows from the formula for the minors of the inverse

$$\det(1-D) = \frac{\det G}{\det(1+C)} = \sum_{X\subset\{1,\ldots,m\}} \frac{C\binom{X}{X}}{\det(1+C)},$$

where $G$ is the $m \times m$ upper-left corner of $1 + C$.

There are two ways to compute minors of $1 - D$, one of them using $1 + E$ and the other using $1 + C$. This yields

$$(1+E)\begin{pmatrix} X \\ Y \end{pmatrix} = \frac{\det(1+E)}{\det(1+C)}(1+C)\begin{pmatrix} 1,\ldots,m,X \\ 1,\ldots,m,Y \end{pmatrix}.$$

In the case when $X$ and $Y$ do not intersect, the last identity implies the second identity of the lemma. When $X$ and $Y$ intersect, expansion of the left-hand side and the right-hand side into a linear combination of minors of $E$ and $C$, respectively, yields the second identity of the lemma by induction. □

Now we can apply this statement to the totally positive matrices $C = P_m A^t B P_m$, $m > n$. Then, clearly, $\det(1 - D) > 0$, and $E = D(1 - D)^{-1}$ with

$$D = (1-P_n)\frac{P_m A^t B P_m}{1 + P_m A^t B P_m}(1-P_n),$$

is totally positive. Since eigenvalues of totally positive matrices are nonnegative, we obtain $Sp(D) \subset [0, 1)$. As was mentioned earlier, $P_m A^t B P_m \to A^t B$ in trace norm as $m \to \infty$. Hence, $D \to K_{11}^{(n)}$ in trace norm as $m \to \infty$, and lemma 5 implies that $Sp\big(K_{11}^{(n)}\big) \subset [0, 1]$. The case of $K_{22}^{(n)}$ is handled similarly. □

Note now that by expanding $(1-zw)^{-1} = 1+zw+(zw)^2+\cdots$ in the integral representation for $K_{11}$, we obtain

$$(K_{11})_{pq} = \delta_{pq} - (-1)^{p+q}\sum_{k\geqslant 0}\left(\frac{\phi_-}{\phi_+}\right)_{p-k}\left(\frac{\phi_+}{\phi_-}\right)_{-q+k} = (-1)^{p+q}\sum_{k\geqslant 1}\left(\frac{\phi_-}{\phi_+}\right)_{p+k}\left(\frac{\phi_+}{\phi_-}\right)_{-q-k}.$$

Widom in [W] conjectured that the eigenvalues of the operator in $L^2(n, n+1, \ldots)$ with $(p, q)$-matrix element

$$\sum_{k\geqslant 1}\left(\frac{\phi_-}{\phi_+}\right)_{p+k}\left(\frac{\phi_+}{\phi_-}\right)_{-q-k}$$

lie between 0 and 1 for the specific choice of

$$\phi_+(z) = \prod_{i\geqslant 1}(1 + r_i z), \qquad \phi_-(z) = \prod_{i\geqslant 1}(1 - s_i z^{-1})^{-1}$$

with nonnegative $r_i$ and $s_i$ only finitely many of which are nonzero; $r_i, s_i < 1$. Clearly, this statement is an immediate corollary of theorem 4.

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