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The Efficient Frontier: A Note on the Curious Difference Between Variance and Standard Deviation

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Abstract

The Markowitz Frontier of optimal portfolios is valid in both mean/variance space and in mean/standard deviation space. But there are some curious differences because lines in one space become curves in the other. This note explores and explains the curiosity.

Highlight and Key Takeaways

1. The Capital Allocation Line becomes a Curve in Mean/Variance Space
2. There is a line in Mean/Variance Space that Connects the Riskless Rate with the Tangency Portfolio, but it is not a Capital Allocation Line
3. Volatility can be either standard deviation or variance, but their efficient frontier geometry is curiously different.

Keywords

Efficient Frontier, Variance, Standard Deviation

JEL Codes

G10, G11

The venerable Markowitz Efficient Frontier of Optimal Portfolios portrays the trade-off between Mean Return and Return Volatility. A typical textbook graph plots the Mean Return on the vertical axis and Volatility on the horizontal axis¹. In many textbooks and papers, volatility is taken to be the standard deviation of returns, σ . One would presume that the variance of returns, $v = \sigma^2$, provides the same basic results, which it does, but there are curious differences that we explore in this note.

As an indicator of volatility, the standard deviation has some intuitive merits. Perhaps the most important is that a tangent line to the efficient frontier drawn from the riskless rate is a locus of all optimum portfolios. Each optimum portfolio combines the riskless rate with the unique risky portfolio that lies on the frontier at the tangency point. This tangent line, called the “Capital Allocation Line,” CAL, has the largest possible “Sharpe Ratio,”² another appealing intuitive feature. When investors can borrow or lend at the riskless rate and agree on everything, the tangency portfolio should be the market portfolio³ and the CAL becomes the “Capital Market Line,” CML.

The variance also has its merits. With the variance instead of the standard deviation, the underlying algebra of the efficient frontier is considerably more tractable. For instance, the equation of the efficient frontier is a parabola in mean/variance space which is determined by three constants, each of which is a function of the covariance matrix of returns, \mathbf{V} , and the vector of mean returns, \mathbf{R} , which are the only two inputs required to derive the (unconstrained) efficient frontier.⁴ Since the covariance matrix is populated by variances on the diagonal, (and covariances off the diagonal), standard deviations do not appear explicitly among the inputs.

The efficient frontier parabola has an optimum portfolio’s variance (on the left) as a function of its mean return,

¹ E.g., Bodie, et al., 2018, section 7.4.

² Named after William F. Sharpe. A portfolio with mean return R_p , has the Sharpe ratio $(R_p - R_f) / \sigma_p$, where R_f is the riskless rate and σ_p is the portfolio’s return standard deviation.

³ The market portfolio is the aggregation of all risky securities

⁴ Matrices and vectors will hereafter be indicated by bold face type.

$$\sigma_p^2 = \frac{a - 2bR_p + cR_p^2}{ac - b^2}, \quad (1)$$

where the P subscript indicates a portfolio on the efficient frontier, R_p is its mean return, and the three efficient set constant, a, b, and c, are $a=\mathbf{R}'\mathbf{V}^{-1}\mathbf{R}$, $b=\mathbf{R}'\mathbf{V}^{-1}\mathbf{1}$, and $c=\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}$. The bold face “ $\mathbf{1}$ ” denotes a unit vector with the same length as \mathbf{R} .⁵

Although perhaps less intuitive, there is an analogous line in mean/variance space. It extends from the riskless rate through the global minimum variance point on the parabola⁶ (1), then intersects the upper branch of the parabola exactly at the tangency portfolio, so it is akin to a capital market line. The tangency portfolio’s mean return is easily shown to be $(a - bR_F)/(b - cR_F)$; hence, by using variance, there is no need to tediously compute the slope of the mean/standard deviation curve in determining the tangency point. As we shall see, however, the variance-based line is not a capital allocation line. In variance space, optimal capital allocation is traced out by a curve!

The situation in the two spaces, the variance and standard deviation spaces, is depicted in Exhibit 1. In both spaces, the efficient frontier constants are the same. For illustrative purposes, they are specified as follows: $a=0.2$, $b=4$, and $c=100$. The illustrated riskless rate is 2%.

Using (1), the global minimum variance portfolio, located at the extreme left of each curve, is easily shown to have a mean return of 4% ($=b/c$) and a standard deviation of 10% ($=\sqrt{1/c}$). The tangency’s portfolio’s mean return is $[(.2-4(.02))/[4-100(.02)]=6\%$ and its standard deviation is approximately 14.14%.

Two lines, which will henceforth be denoted CMLV and CMLS for the variance and standard deviation spaces, respectively, are also plotted. The frontiers in the two spaces are labeled EV and ES; they look similar but, of course, are not identical since one is plotted against variance and the other against standard deviation.

⁵ This notation and associate proofs are taken from the appendix in my earlier paper, Roll (1977), which also shows that the investment proportions of any efficient portfolio are contained in $\mathbf{X} = \mathbf{B}(R_e : 1)'$ where \mathbf{X} is an (NX1) column vector, \mathbf{B} is an (NX2) matrix of constants that depends only on \mathbf{V} and \mathbf{R} , N is the number of assets and R_e is the mean return on the efficient portfolio.

⁶ This point lies at a mean return of b/c and a variance of $1/c$.

Now consider the appearances of CMLV and CMLS when plotted in the alternative space rather than in their own where they are linear. Exhibit 2 adds them onto the previous plots. They are clearly non-linear, which is perhaps no surprise, yet they still coincide with the riskless rate and the tangency portfolio on both frontiers. Note that the CMLS curve is tangent to the mean/variance frontier at the tangency portfolio's mean return of 6% while the CMLV curve in mean/standard deviation space passes through the global minimum variance point.

In its alien variance space, the capital market line from standard deviation space, CMLS, obeys the equation

$$R = R_F + \gamma_S \sqrt{v}, \quad (2)$$

where v is the variance (the horizontal variable) and $\gamma_S = (R_T - R_F) / \sigma_T$, with subscript "T" indicating the tangency portfolio.

Similarly, in its alien standard deviation space, the capital market line from variance space, CMLV, obeys the equation

$$R = R_F + \gamma_V \sigma^2, \quad (3)$$

with $\gamma_V = (R_T - R_F) / \sigma_T^2$.

Assertion: In their respective alien spaces, (2) is strictly increasing at a decreasing rate while (3) is strictly increasing at an increasing rate when $R_T > R_F$ ⁷.

Proof: The first and second derivatives of (2) are, respectively, $\partial R / \partial v = \gamma_S / (2\sqrt{v}) > 0$ and $\partial^2 R / \partial v^2 = -\gamma_S v^{-3/2} / 4 < 0$. The first and second derivatives of (3) are, respectively, $\partial R / \partial \sigma = 2\gamma_V \sigma > 0$ and $\partial^2 R / (\partial \sigma)^2 = 2\gamma_V > 0$.

These features are discernable in Exhibit 2.

Equation (2) is a Capital Allocation curve in variance space. As such, it is still the locus of optimum portfolios that combine the riskless rate with the tangency portfolio. Its non-linearity conforms to the fact that a riskless rate combined with any risky portfolio traces out a curve in the variance space. For example, if w is invested in any risky portfolio P and the complementary

⁷ Which implies that the slope coefficients, γ_S and γ_V , are both positive.

fraction $(1-w)$ is invested in a riskless asset, the mean return of the combination, C , is $R_C = wR_P + (1-w)R_F$, which is linear in w . But the variance of the combination is $\sigma_C^2 = w^2\sigma_P^2$ and the slope of the mean/variance curve is $\partial R_C / \partial \sigma_C^2 = (R_P - R_F) / 2w\sigma_P^2$, which shows that the combination is not linear in w . The second derivative is negative, so the mean/variance curve is increasing at a decreasing rate.

As it must, the curvilinear CAL (CMLS in mean/variance space) lies above and to the left of the EV frontier with a single exception at the tangency portfolio. See the top panel of Exhibit 2.

Conclusion

In the space of mean return versus standard deviation of return, a line traces out combinations of two perfectly correlated risky portfolios and another line portrays combinations of a riskless asset with any risky portfolio. However, in mean/variance space, these combinations are curves, which nonetheless share many similar properties with the lines.

There is, moreover, a line in mean/variance space that connects the riskless asset with the tangency portfolio after passing through the global minimum variance point on the efficient frontier. But this line is not the locus of optimum portfolios and hence cannot be considered a “Capital Allocation Line,” CAL. Instead, the CAL from mean/standard deviation space becomes a “Capital Allocation Curve” in mean/variance space, tracing out all optimum portfolios that are combinations of the riskless return and the tangency portfolio.

Exhibit 1. Efficient Frontiers In Variance And Standard Deviation Spaces With Their Associated Lines

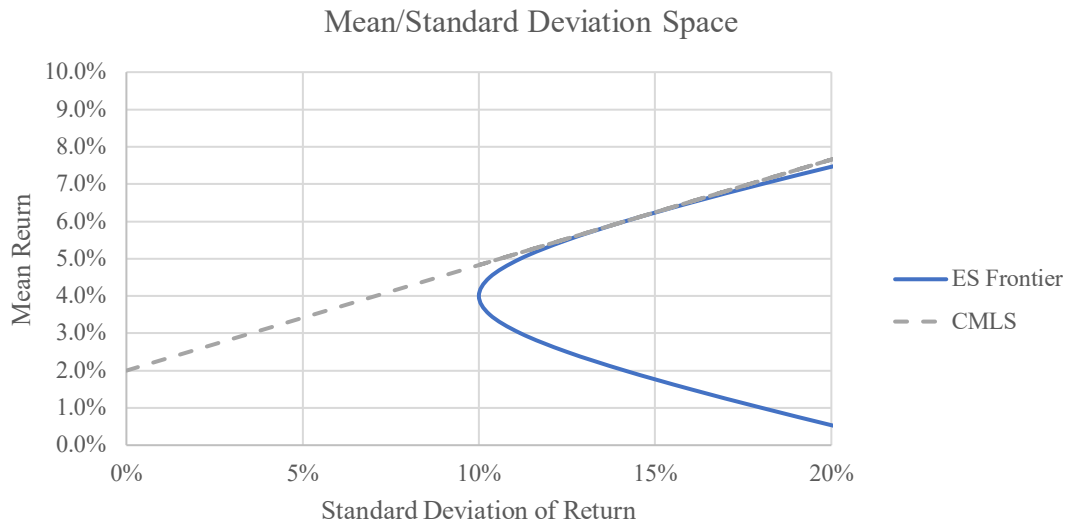
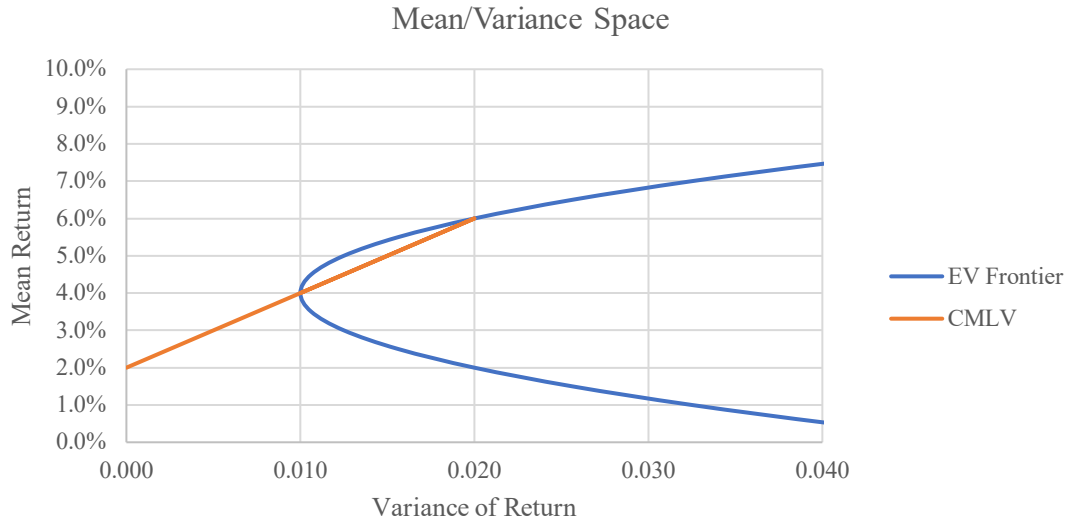
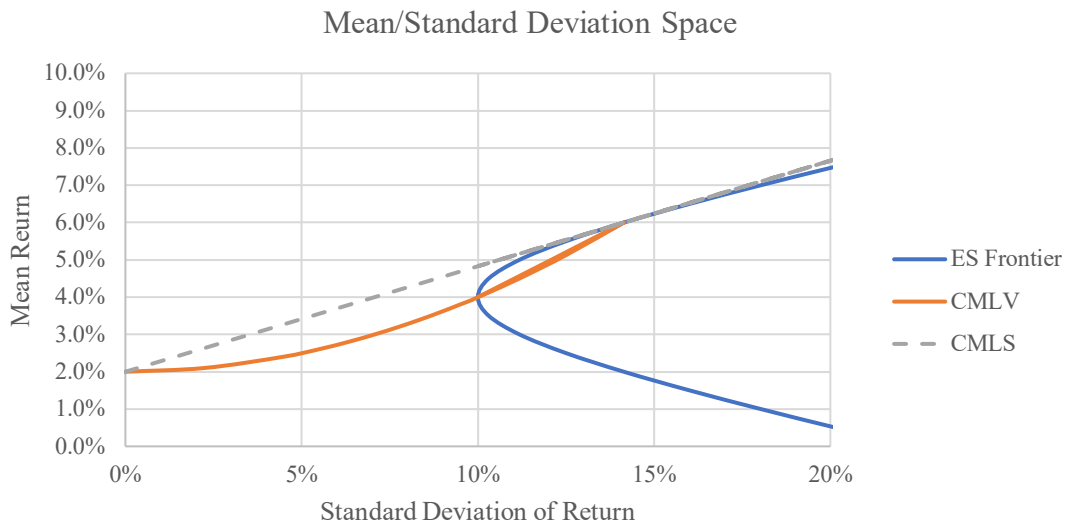
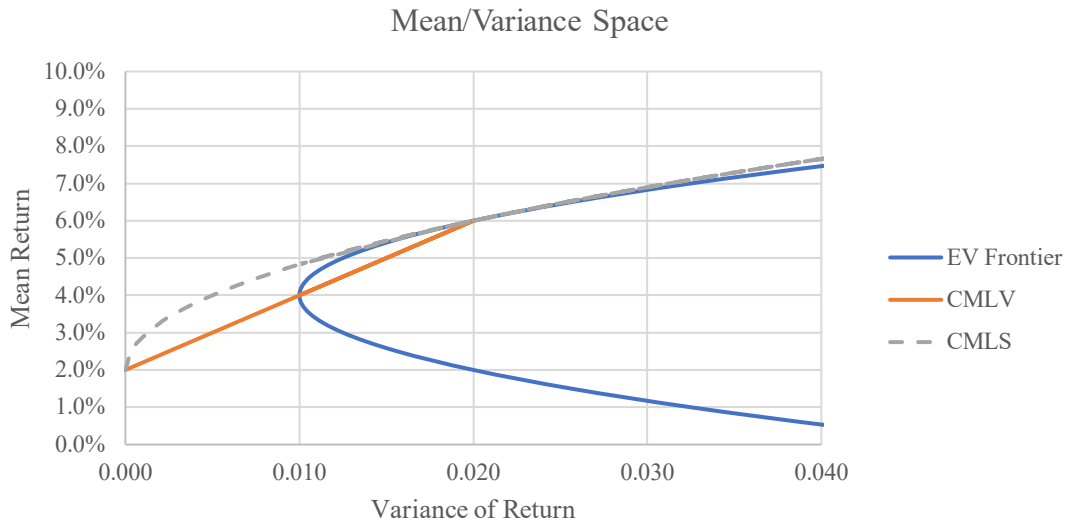


Exhibit 2. Efficient Frontiers In Variance And Standard Deviation Spaces
With Capital Market Lines and Capital Market Curves



References

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