Supplemental Material for: Skyrmion Qubits: A New Class of Quantum Logic Elements Based on Nanoscale Magnetization

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I. THE MODEL

We consider a thin magnetic insulator with normalized magnetization $\mathbf{m} = [\sin\Theta\cos\Phi, \sin\Theta\sin\Phi, \cos\Theta]$, described by the real time action

$$\mathcal{S} = \bar{S}\int d\tilde{t}\int d\tilde{\mathbf{r}}[\frac{SN_A}{\alpha}\dot{\Phi}(\Pi-1) - N_A\mathcal{F}(\Phi,\Pi)]\,, \quad (1)$$

where $\bar{S}$ is the magnitude of the effective spin and a is the lattice spacing. $\dot{\Phi} = \partial_{\tilde{t}}\Phi$ denotes the real time derivative, and $\Pi = \cos\Theta$ is canonically conjugate to Φ. We consider the inversion-symmetric classical Heisenberg model with competing interactions, originally introduced in Ref. 1

$$\mathcal{F} = -\frac{J_1}{2}(\nabla\mathbf{m})^2 - \frac{J_2 a^2}{2}(\nabla^2\mathbf{m})^2 - \frac{\mathbf{H}}{a^2}\cdot\mathbf{m} + \frac{K}{a^2}m_z^2\,, \quad (2)$$

where J_1, J_2, H and K are in units of [eV]. We introduce dimensionless variables $\mathbf{r} = \tilde{\mathbf{r}}/(\ell a)$, $t = \tilde{t}/\varepsilon_\Lambda$ and $\beta = \tilde{\beta}\varepsilon_\Lambda$, with ℓ and ε_Λ a characteristic length and energy scale respectively. Stationary configurations of the action (1), denoted as Φ_{cl} and Π_{cl}, are found by minimizing the energy functional, *i.e* by solving the equations $\delta\mathcal{F}/\delta\Phi_{\text{cl}} = 0 = \delta\mathcal{F}/\delta\Pi_{\text{cl}}$. This class of solutions are characterized by a finite topological charge $Q = \frac{1}{4\pi}\int d\mathbf{r}\mathbf{m}\cdot(\partial_x\mathbf{m}\times\partial_y\mathbf{m})$, and a magnetization profile that decays to zero at spatially infinity. Choosing $\ell = \sqrt{J_2/J_1} = 1$, $\varepsilon_\Lambda = J_1$, and $\mathbf{H} = H\hat{z}$, the action of Eq. (1) in reduced units is written as follows $\mathcal{S} = \bar{S}\int dt\int dr[\dot{\Phi}(\Pi-1) - \mathcal{F}(\Phi,\Pi)]$, with

$$\mathcal{F} = -(\nabla\mathbf{m})^2/2 - (\nabla^2\mathbf{m})^2/2 - hm_z + \tilde{\kappa}m_z^2\,, \quad (3)$$

with $\tilde{\kappa} = K/J_1$ and $h = H/J_1$.

Rotationally symmetric solutions of the model (3) are described by $\Phi(\mathbf{r}) = -Q\phi$ and $\Theta(\mathbf{r}) = \Theta(\rho)$ with boundary conditions $\Theta_{\text{cl}}(0) = \pi$ and $\Theta_{\text{cl}}(\rho\to\infty) = 0$. Upon minimization of the energy $\int d\mathbf{r}\mathcal{F}$ we obtain a nonlinear Euler equation for the skyrmion profile $\Theta(\mathbf{r})$, which generally cannot be solved analytically. In the $\rho \gg 1$ limit we find the approximate solution $\Theta(\rho) = \Re[c_+ e^{-\gamma_+\rho} + c_- e^{-\gamma_-\rho}]$, with $\gamma_\pm = \sqrt{-1\pm\tilde{\gamma}}/\sqrt{2}$ with $\tilde{\gamma} = \sqrt{1-4(h+2\tilde{\kappa})}$ and $c_\pm$ constants. The skyrmion size is defined as $\lambda \equiv 1/\Re[\gamma_\pm]$. (see Ref. 1 for more details on the model). For example for $K=0$ and $H=0.86$ T, the skyrmion size is $\lambda = 10a$. The rotationally symmetric skyrmion profile is depicted in Fig. 2-d).

To generate the double-well potential term, essential for the construction of the helicity-qubit, we consider skyrmions characterized by elliptical profiles, which can be the result of defect engineering. The skyrmion profile is parametrized as $\Theta_\ell = \Theta(\rho) + g(\rho)\cos 2\phi$, where $g(\rho)$ is dictated by the microscopic mechanism responsible for skyrmion deformation. Here we use the phenomenological function $g(\rho) = \text{sech}[(\rho-\lambda)/\Delta_0]$, and the elliptical skyrmion is depicted in Fig. 3-d).

Energy and Hamiltonian Terms. We summarize the energy terms on the classical level in physical units, the definition of dimensionless parameters and the corresponding quantized Hamiltonian terms. The various interactions of interest introduced in the main part of the manuscript are

$$\mathcal{F}' = \bar{S}\int d\mathbf{r}[\frac{K}{a^2}m_z^2 - \frac{H}{a^2}m_z + \frac{K_x}{a^2}m_x^2 + \frac{H_\perp}{a^2}ym_x - \mathbf{E}\cdot\mathbf{P} + \frac{B}{a^2}f(t)\cos(\omega t+\phi_{\text{ext}})xm_x]\,, \quad (4)$$

with $\mathbf{P} = [\hat{e}_x\times(\mathbf{m}\times\partial_x\mathbf{m}) + \hat{e}_y\times(\mathbf{m}\times\partial_y\mathbf{m})]$ the electric polarization, and $\mathbf{E} = EP_E a\hat{z}$ the electric field. K, H, and K_x are in units of [eV], $H_\perp$ and B in units of [eV/m], P_E in units of [C/m^2], E in units of V/m, and a the lattice constant in units of [m]. Choosing a as a typical length and J_1 as a typical energy scale, we arrive at

$$\mathcal{F}' = \int d\mathbf{r}[\tilde{\kappa}m_z^2 - hm_z + \tilde{\kappa}_x m_x^2 + \tilde{h}_\perp ym_x - \varepsilon_z\hat{z}\cdot\mathbf{P} + \tilde{b}f(t)\cos(\omega t+\phi_{\text{ext}})xm_x]\,, \quad (5)$$

given now in dimensionless units $\tilde{\kappa} = \bar{S}K/J_1$, $\tilde{\kappa}_x = \bar{S}K_x/J_1$, $h = \bar{S}H/J_1$, $\tilde{h}_\perp = \bar{S}H_\perp a/J_1$, $\tilde{b} = \bar{S}Ba/J_1$, and $\varepsilon_z = \bar{S}EP_E a^3/J_1$. Following the quantization procedure described in detail below, the quantum Hamiltonian in terms of φ_0 and S_z reads,

$$\tilde{H} = \kappa S_z^2 - hm_z + \kappa_x\cos 2\varphi_0 - E_z\cos\varphi_0 + h_\perp\sin\varphi_0 + bf(t)\cos(\omega t+\phi_{\text{ext}})\,, \quad (6)$$

where $\kappa = \tilde{\kappa}\int_{\mathbf{r}}(1-\cos\Theta)^2/[\int_{\mathbf{r}}(1-\cos\Theta)]^2$, $\kappa_x = (\tilde{\kappa}_x/4)\int_{\mathbf{r}}\sin 2\Theta g(\rho)$, $E_z = \varepsilon_z\int_{\mathbf{r}}[\sin 2\Theta/2\rho + \Theta']$, $h_\perp = (\tilde{h}_\perp/4)\int_{\mathbf{r}}\rho\sin\Theta$, and $b = (\tilde{b}/4)\int_{\mathbf{r}}\rho\sin\Theta$.

II. SKYRMION QUANTIZATION

To investigate the quantum effects, we employ a functional integral formulation, in which the partition function is given by $Z = \int \mathcal{D}\mathbf{m} e^{i\mathcal{S}(\mathbf{m},\dot{\mathbf{m}})}$. Here $\mathcal{S} = \int dtL$ is the action, with

$$L = \bar{S} \int d\mathbf{r}[\mathcal{A}(\mathbf{m}) \cdot \dot{\mathbf{m}} - N_A \mathcal{F}] , \qquad (7)$$

where $\mathcal{A}(\mathbf{m}) = [1 - \tilde{e}_\Phi \cdot (e_\Phi \times \mathbf{m})] e_\Phi / (\tilde{e}_\Phi \cdot \mathbf{m}))$ is the gauge potential. We use $\tilde{e}_\Phi = (\cos\Phi, \sin\Phi, 0)$ and $e_\Phi = (-\sin\Phi, \cos\Phi, 0)$, and we also note that $\mathcal{A} \cdot \dot{\mathbf{m}} = (1 - \cos\Theta)\dot{\Phi}$. The commutation relations are $\{m_i(\mathbf{r}), m_j(\mathbf{r}')\} = \epsilon_{ijk} m_k(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}')$, where $\{A, B\}$ is the Poisson bracket satisfying

$$\{A(\mathbf{r}), B(\mathbf{r}')\} = \int d\mathbf{r}''[\frac{\delta A(\mathbf{r})}{\delta\Phi(\mathbf{r}'')} \frac{\delta B(\mathbf{r}')}{\delta\Pi(\mathbf{r}'')} - \frac{\delta A(\mathbf{r})}{\delta\Pi(\mathbf{r}'')} \frac{\delta B\mathbf{r}'}{\delta\Phi(\mathbf{r}'')}] \qquad (8)$$

provided that $\{\Phi(\mathbf{r}), \Pi(\mathbf{r}')\} = \delta(\mathbf{r} - \mathbf{r}')$. The model $\mathcal{F}$ is characterized by an unbroken global symmetry, $\mathbf{m} \to \mathcal{M}(\varphi_0(t))\mathbf{m}$, with

$$\mathcal{M} = \begin{bmatrix} \cos\varphi_0 & -\sin\varphi_0 & 0 \\ \sin\varphi_0 & \cos\varphi_0 & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

Instead of the original magnetization vector $\mathbf{m}$, it appears convenient to introduce $\mathbf{n} = \sqrt{1 - \cos\Theta}/\sin\Theta\ \mathbf{m}$ and the corresponding gauge vector $\mathcal{A}_\mathbf{n} = \partial_\Phi \mathbf{n}$, such that the Wess Zumino term of the action remains unchanged, $\mathcal{A}_\mathbf{n} \cdot \dot{\mathbf{n}} = (1 - \cos\Theta)\dot{\Phi}$. It is also important to note that the zero mode of the skyrmion associated with infinitesimal rotations is equal to $\mathcal{A}_{\mathbf{n}_0} = \partial_\Phi \mathbf{n}_0$, where $\mathbf{n}_0$ describes the skyrmion profile. In the naive perturbation expansion around the skyrmion, the zero mode leads to infrared divergences unless it is removed from the path integral by imposing proper constraints.

Here we use a path integral quantization method according to which the collective coordinates are introduced by performing a canonical transformation of the dynamical variables in the phase space path integral[2,3]. The zero mode is removed by introducing a δ-function constraint of the form

$$1 = \int \mathcal{D}\varphi_0(t) J_{\varphi_0} \delta(F_1) , \qquad (9)$$

with $F_1 = \int d\mathbf{r} \mathcal{A}_{\mathbf{n}_0} \cdot (\tilde{\mathbf{n}} - \tilde{\mathbf{n}}_0)$, $J_{\varphi_0} = \delta F_1 / \delta\varphi_0$ is the Jacobian of the transformation, and we use the tilde notation for rotated vectors, $\tilde{B} = \mathcal{M}B$. The constraint ensures fluctuations of the magnetization field around the skyrmion are orthogonal to the rotational zero mode. We introduce an additional constraint related to the conservation of the conjugate to φ_0 momentum,

$$1 = \int \mathcal{D}S_z(t) J_{S_z} \delta(F_2) , \qquad (10)$$

with $F_2 = (1/\Lambda) \int d\mathbf{r} \mathcal{A}_{\mathbf{n}_0} \cdot (\tilde{\mathcal{A}}_\mathbf{n} - \tilde{\mathcal{A}}_{\mathbf{n_0}})$, $J_{S_z} = \delta F_2 / \delta S_z$, and $\Lambda = \int d\mathbf{r} \mathcal{A}_{\mathbf{n}_0} \cdot \mathcal{A}_{\mathbf{n}_0}$ a normalization constant. To ensure that the above change of variables constitutes a canonical transformation we introduce the following variables in the integration, $\mathbf{n} = \tilde{\mathbf{n}}_0 + \boldsymbol{\gamma}$ and $\mathcal{A}_\mathbf{n} = c\tilde{\mathcal{A}}_{\mathbf{n}_0} + \boldsymbol{\zeta}$, with c a constant to be specified from the momentum conservation constraint, $P - S_z = F_2 = 0$ with $P = \int d\mathbf{r} \mathcal{A}_\mathbf{n} \cdot \partial_{\varphi_0} \mathbf{n} = \int d\mathbf{r} (1 - \cos\Theta) \partial_{\varphi_0} \Phi$ the total momentum. We note that it holds $\{P, \Phi\} = -\partial_\phi \Phi$, confirming that P plays the role of infinitesimal generator of rotations. After some straightforward calculation we find

$$c = \frac{S_z - \int \boldsymbol{\zeta} \cdot \partial_{\varphi_0} \mathbf{n}}{\int \mathcal{A}_{\mathbf{n}_0} \cdot \partial_\Phi \mathbf{n}} . \qquad (11)$$

We confirm that the phase-space path integral retain its canonical form in terms of the new variables $\int_{\mathbf{r},t} \mathcal{A}_\mathbf{n} \cdot \dot{\mathbf{n}} = \int_t [S_z \dot{\varphi}_0 + \int_\mathbf{r} \boldsymbol{\zeta} \cdot \dot{\boldsymbol{\gamma}}]$, and also that the two Jacobian factors cancel $J_{\varphi_0} J_{S_z} = 1$, with $J_{\varphi_0} = \int \mathcal{A}_{\mathbf{n}_0} \cdot \partial_{\Phi_0} \mathbf{n}$. We note that $\boldsymbol{\zeta}$ and $\boldsymbol{\gamma}$ denote fluctuations around the gauge and magnetization vectors correspondingly, and can be associated to the fluctuations around fields Φ, Θ.

The partition function is now written in terms of the new variables,

$$Z = \int \mathcal{D}\varphi_0 \mathcal{D}S_z \mathcal{D}\boldsymbol{\zeta} \mathcal{D}\boldsymbol{\gamma} \delta(F_1) \delta(F_2) e^{i\mathcal{S}(\varphi_0, S_z, \boldsymbol{\zeta}, \boldsymbol{\gamma})} \qquad (12)$$

with

$$\mathcal{S} = \bar{S} \int_{t,\mathbf{r}} [S_z \dot{\varphi}_0 + \boldsymbol{\zeta} \cdot \dot{\boldsymbol{\gamma}}) - \mathcal{F}(\varphi_0, S_z, \boldsymbol{\zeta}, \boldsymbol{\gamma})] . \qquad (13)$$

Our current task is to analyze the energy functional $\mathcal{F}(\Pi, \Phi)$ of Eq. (2) in terms of the new variables $\varphi_0, S_z, \boldsymbol{\zeta}$ and $\boldsymbol{\gamma}$. Since $P(t) = \int_\mathbf{r} \tilde{\Pi} \partial_\phi \Phi$, with $\tilde{\Pi} = 1 - \Pi$, we can apply the following transformation $\tilde{\Pi}(\mathbf{r}, t) = [S_z(t) - \int_\mathbf{r} \eta(\mathbf{r}, t) \partial_\phi \Phi] \tilde{\Pi}_{\text{cl}} / \Lambda + \eta(\mathbf{r}, t)$, with $\Lambda = \int_\mathbf{r} \tilde{\Pi}_{\text{cl}} \partial_\phi \Phi$, while η corresponds to quantum fluctuations around the classical configuration. Fluctuations around the field Φ are denoted as $\Phi = \Phi_{\text{cl}}[\mathbf{r}, \varphi_0(t)] + \xi(\mathbf{r}, t)$. It is easy to verify that with the above definitions the Wess-Zumino term maintains its canonical form $\int_{\mathbf{r},t} \tilde{\Pi} \dot{\Phi} = \int_t S_z \dot{\varphi}_0 + \int_{\mathbf{r},t} \eta \dot{\xi}$. To prove this relation we used the constraint F_1 of Eq. (9), which in terms of the new variables takes the form $F_1 = \int (1 - \Pi_{\text{cl}}) \xi = 0$.

By implementing these changes of variables, the partition function is given as,

$$Z = \int \mathcal{D}\varphi_0 \mathcal{D}S_z e^{i\int_t [\bar{S} S_z \dot{\varphi}_0 - \tilde{H}(\varphi_0, S_z)]} \tilde{Z}[\varphi_0, S_z] , \qquad (14)$$

where $\tilde{H} = \kappa S_z^2 - hS_z - E_z \cos\Phi_0$ is the quantum Hamiltonian with $\kappa = \tilde{\kappa} \int_\mathbf{r} \tilde{\Pi}_{\text{cl}}^{\ 2} / \Lambda_0^2$, $\Lambda_0 = \int \tilde{\Pi}_{\text{cl}}$, and $E_z = \varepsilon_z \int d\mathbf{r} [\sin 2\Theta_{\text{cl}} / 2\rho + \Theta'_{\text{cl}}]$. We note that we retain leading terms in powers of $1/\bar{S}$, and up to quadratic in S_z and η, ξ. The fluctuating part of the partition function equals,

$$\tilde{Z} = \int \mathcal{D}\chi \mathcal{D}\chi^\dagger \delta(F_1) \delta(F_2) e^{i\int_{\mathbf{r},t} \chi^\dagger [\mathcal{G} + \mathcal{K}] \chi} , \qquad (15)$$

with $\chi = (\eta, \xi)$. Here the operator $\mathcal{G}$ describes the magnon spectrum around the skyrmion, while $\mathcal{K}$ is responsible for the dynamical coupling of the skyrmion with the surrounding magnons. The fluctuating part of the partition function is written in a Gaussian form and can be manipulated within the real-time Keldysh functional integral. The analysis will generate a dissipative term as well as a Langevin random noise, which will play a role in the estimation of the skyrmion qubit decoherence time. These terms are omitted from the present analysis and are left for the future.

Using standard equivalence between the path integral and canonical quantization, we introduce a collective coordinate operator $\hat{\varphi}_0$ and its conjugate momentum $\hat{S}_z = (-i/\bar{S})\partial_{\hat{\varphi}_0}$ with $[\hat{\varphi}_0, S_z] = i/\bar{S}$. If $E_z = 0$, the momentum operator commutes with the corresponding Hamiltonian and stationary states are labeled by a conserved charge s constrained to be an integer with $S_z|s\rangle = s/\bar{S}|s\rangle$. The phase space associated to $\hat{\varphi}_0|\varphi_0\rangle = \varphi_0|\varphi_0\rangle$ has a circular topology $|\varphi_0\rangle = |\varphi_0 + 2\pi\rangle$. We also note that it holds $e^{\pm i\hat{\varphi}_0}|s\rangle = |s \pm 1\rangle$.

III. BASIC QUBIT TYPES

A. S_z-qubit

In the presence of an out-of-plane uniform magnetic field, an easy-axis anisotropy and an out-of-plane electric field, the Hamiltonian for the helicity degrees of freedom reads,

$$H_{S_z} = \kappa(\hat{S}_z - h/\kappa)^2 - E_z \cos\hat{\varphi}_0 \,. \tag{16}$$

We note that Eq. (16) resembles the circuit Hamiltonian of the Cooper pair box, with κ the charging energy, h/κ the offset charge, and E_z the Josephson energy. To determine the state $|s\rangle$ and eigenenergies $\mathcal{E}_s$, we solve the corresponding Schrödinger equation for states $\Psi_s(\varphi_0) = \langle\varphi_0|s\rangle$,

$$\kappa[-\frac{i}{\bar{S}}\partial_{\varphi_0} - h/\kappa]^2\Psi_s(\varphi_0) - E_z\cos\varphi_0\Psi_s(\varphi_0) = \mathcal{E}_s\Psi_s(\varphi_0)\,, \tag{17}$$

with boundary conditions $\Psi_s(\varphi_0) = \Psi_s(\varphi_0 + 2\pi)$. For $\bar{h} \in [0, 1/2]$, with $\bar{h} = \bar{S}h/\kappa$, functions $\Psi_s(\varphi_0)$ can be written using the Mathieu functions as

$$\Psi_s(\varphi) = e^{i\bar{h}\varphi_0} c_j \mathcal{M}_j\left(\frac{4\mathcal{E}_s}{\kappa}, \frac{2E_z}{\kappa}, \frac{\varphi_0}{2}\right)\,, \tag{18}$$

where the index $j = C(S)$ represents the even (odd) solutions and $\mathcal{E}_s = (\kappa/4)\mathcal{M}_A[2(\kappa - \bar{h}), -2E_z/\kappa]$.

We now focus on the $E_z \ll \kappa$ and $\bar{h} = 1/2$ regime, where the lowest two levels are almost degenerate and seperated by a small energy internal controlled by the electric field. The skyrmion qubit states $|0\rangle$ and $|1\rangle$, with $\hat{S}_z|s\rangle = s/\bar{S}|s\rangle$, represent deviations of the z component of the magnetization from equilibrium. For $E_z > 0$, the degeneracy is lifted and the energy eigenstates are symmetric and antisymmetric superpositions of skyrmion qubit states,$(|0\rangle \pm |1\rangle)/\sqrt{2}$.

Truncating the full Hilbert space to the subspace spanned by these two states, one can write the qubit reduced Hamiltonian as

$$\hat{H}_q = \frac{H_0}{2}\hat{\sigma}_z - \frac{X_c}{2}\hat{\sigma}_x\,, \tag{19}$$

with $H_0 = \kappa(1 - 2\bar{h})/\bar{S}$, and $X_c = E_z$ while the qubit energy levels are given by $\mathcal{E}_\pm = \pm\sqrt{H_0^2 + X_c^2}/2$.

B. Helicity-qubit

In this section we discuss how an elementary qubit described by a double-well potential profile can be constructed. The helicity-qubit is a good example of how one can engineer the qubit properties through the choice of suitable parameters. To proceed, we consider a material with an easy-plane anisotropy, $\mathcal{F}_x = \bar{S}K_x/a^2\int_{\mathbf{r}} m_x^2$, and a skyrmion with an elliptical profile, such as the one depicted in Fig. 3-d). The Hamiltonian reads $H_{\varphi_0} = \kappa\hat{S}_z^2 - h\hat{S}_z + V(\hat{\varphi}_0)$, with the double-well potential given by

$$V(\hat{\varphi}_0) = \kappa_x\cos 2\hat{\varphi}_0 - E_z\cos\hat{\varphi}_0 + h_\perp\sin\hat{\varphi}_0\,. \tag{20}$$

The elliptical deformation is parametrized as $\Theta_{\rm cl}(\rho,\phi) = \Theta_{\rm cl}(\rho) + g(\rho)\cos 2\phi$, with $g(\rho)$ dictated by the microscopic mechanism responsible for deforming the skyrmion. Here we use $g(\rho) = \operatorname{sech}[(\rho - \lambda)/\Delta_0]$, and $\kappa_x = (\bar{S}K_x/4J_1)\int \sin 2\Theta_{\rm cl} f(\rho)$. We note that a circular skyrmion profile with $g(\rho)$ fails to reshape potential landscape and produce the required $\cos 2\hat{\varphi}_0$ term. A depth difference between the wells can be created by an in plane magnetic field gradient of the form $\mathbf{H}_\perp(\mathbf{r}) = H_\perp y\hat{x}$. We then find $h_\perp = -(\bar{S}H_\perp/4J_1)\int \rho\sin\Theta_{\rm cl}$. An application of a magnetic field gradient of the form $\mathbf{H}_\perp(\mathbf{r}) = H_\perp x\hat{y}$, results an asymmetric potential term of opposite sign $h_\perp = (H_\perp/4J_1)\int \rho\sin\Theta_{\rm cl}$.

We seek eigenfunctions of H_{φ_0} as a linear combinations of the 2π-periodic basis functions

$$\Psi_n(\Phi_0) = \frac{1}{\sqrt{2\pi}}\sum_m c_m^n e^{im\varphi_0}\,, \tag{21}$$

which maintain the required 2π-periodicity in φ_0, and n labels the nth eigenstate. The corresponding set of equations for coefficients c_m^n reads,

$$\begin{aligned}(\frac{\kappa}{\bar{S}^2}m^2 - E_n - \frac{h}{\bar{S}}m)c_m^n + \frac{\kappa_x}{2}(c_{m+2}^n + c_{m-2}^n)\\ -\frac{E_z}{2}(c_{m+1}^n + c_{m-1}^n) + \frac{h_\perp}{2}(c_{m+1}^n - c_{m-1}^n) = 0\,.\end{aligned} \tag{22}$$

A numerical diagonalization keeping up to $|m| = 50$ terms yields the eigenenergies and the corresponding coefficients c_m^n. In Fig. 3-b) we plot the lowest three eigenvalues together with the potential $V(\varphi_0)$, with a minimum at $\varphi_m = \tan^{-1}(\sqrt{16\kappa_x^2 - E_z^2}/E_z)$. All three E_z, h_y, and h are external parameters to adjust the energy levels. For $E_z = 0$ there is a degeneracy point at $\bar{h} = \bar{S}h/\kappa = 1$. For $h_\perp = 0$, the two lowest energy functions $\Psi_{0,1}$ are symmetric and antisymmetric combinations of the two wavefunctions localized in each well, while for $h_\perp > 0$, they are localized in different wells.

In the limit $\bar{h} - 1 \ll 1$, we can reduce the analysis to the two lowest levels and derive the qubit Hamiltonian,

$$\hat{H}'_q = \frac{H_0}{2}\hat{\sigma}_z - \frac{X_c}{2}\hat{\sigma}_x\,, \tag{23}$$

provided that $H_0 = (\mathcal{E}_1 - \mathcal{E}_0)$ and $X_c = g_e E_z$ for $h_\perp = 0$, or $X_c = g_b h_\perp$ for $Ez = 0$. Constants H_0, g_e and g_b are found numerically.

IV. QUBIT CONTROL

In this section we discuss how skyrmion qubits can be manipulated to implement quantum algorithms, with the techniques introduced here being applicable to both skyrmion types. The predominant protocol is via microwave magnetic field gradients with frequencies at the qubit transition ω_q. A magnetic field gradient couples with the magnetization as $\mathcal{F}_{\text{ext}} = \int d\mathbf{r}\mathbf{B}(\mathbf{r},t)\cdot\mathbf{m}(\mathbf{r},t)$, with $\mathbf{B}(\mathbf{r},t) = B/a^2\cos(\omega t + \phi)f(t)x\hat{e}_x$. Additional Hamiltonian terms appear, $H_{\text{ext}}(t) = b\cos(\omega t + \phi)f(t)\cos\varphi_0$, with $b = (B/4J_1)\int_{\mathbf{r}}\rho\sin\Theta_{\text{cl}}$. In terms of the reduced quantum Hamiltonian we find,

$$H^q_{\text{ext}} = b_x(t)\hat{\sigma}_x\,, \tag{24}$$

where $b_x(t) = b_0\cos(\omega t + \phi)f(t)$ and $b_0 = b\Re(\langle 1|\cos\varphi_0|0\rangle + \langle 0|\cos\varphi_0|1\rangle)/2$. Here $|0\rangle$ and $|1\rangle$ denote the lowest two qubit states.

The eigenvectors of the unperturbed qubit Hamiltonian H_q of Eq. (19) are

$$\begin{aligned}|\Psi_-\rangle &= \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle\\ |\Psi_+\rangle &= \sin\frac{\theta}{2}|0\rangle - \cos\frac{\theta}{2}|1\rangle\,,\end{aligned} \tag{25}$$

with $\tan\theta = X_c/H_0$, while the corresponding eigenvalues are $\mathcal{E}_\pm = \pm\sqrt{H_0^2 + X_c^2}/2$ and energy splitting $\omega_q = (\mathcal{E}_+ - \mathcal{E}_-)$. In the $|\Psi_\pm\rangle$ basis, the driven qubit Hamiltonian takes the form

$$H_q = \frac{\omega_q}{2}\hat{\sigma}_z + b_x(t)[\cos\theta\hat{\sigma}_x + \sin\theta\hat{\sigma}_z]\,. \tag{26}$$

We note that close to the degeneracy point $X_c \ll H_0$ (so called "sweet spot"), it holds $\sin\theta \ll \cos\theta$. To elucidate the role of the drive, we transform H_q into the rotating frame at a frequency ω,

$$H_{\text{rot}} = \frac{\Delta\omega}{2}\hat{\sigma}_z + \frac{\Omega f(t)}{2}[\cos\phi\hat{\sigma}_x + \sin\phi\hat{\sigma}_y]\,, \tag{27}$$

with $\Delta\omega = \omega_q - \omega$ the detuning frequency and $\Omega = b_0\cos\theta$. When $f(t) = 1$, the eigenvectors of the rotated Hamiltonian are of the form

$$\begin{aligned}|\tilde{\Psi}_-\rangle &= \cos\frac{\tilde{\theta}}{2}|\Psi_-\rangle + \sin\frac{\tilde{\theta}}{2}|\Psi_+\rangle\\ |\tilde{\Psi}_+\rangle &= \sin\frac{\tilde{\theta}}{2}|\Psi_-\rangle - \cos\frac{\tilde{\theta}}{2}|\Psi_+\rangle\,,\end{aligned} \tag{28}$$

where $\tan\tilde{\theta} = \Omega/\Delta\omega$. The corresponding energy levels are $\tilde{\mathcal{E}}_\pm = \pm\sqrt{\Delta\omega^2 + \Omega^2}/2$. If at initial time we start from the state $|\tilde{\Psi}_-\rangle$, the probability to find the system at the state $|\tilde{\Psi}_+\rangle$ is given by $P(t) = \Omega^2/\tilde{\Omega}^2\sin^2(\tilde{\Omega}t/2)$, with $\tilde{\Omega} = \sqrt{\Delta\omega^2 + \Omega^2}$ the Rabi frequency. Single-qubit operations correspond to rotations of the qubit state by a certain angle about a particular axis, as the result of a unitary operator applied to the target qubit. As an example, for *in-phase* pulses $\phi_{\text{ext}} = 0$, and resonant driving on the qubit energy splitting, $\Delta\omega = 0$, the unitary operator $U_x(t) = e^{-\frac{i}{2}\vartheta(t)\hat{\sigma}_x}$ corresponds to rotations around the x-axis by an angle $\vartheta(t) = -\Omega\int_0^t f(t')dt'$[4]. *Out-of-phase* pulses $\phi_{\text{ext}} = \pi/2$ correspond to rotations of the qubit state about the y axis.

V. RELAXATION MECHANISMS

The scope of this section is to calculate the relaxation and decoherence rates, which as we demonstrate, are directly proportional to the spectral densities of the random noises acting on the qubit. To include classical noise sources we consider the magnetization dynamics encoded in the Landau-Lifshitz-Gilbert equation (LLG), $\dot{\mathbf{m}} = \gamma\mathcal{F}_{\text{eff}}\times\mathbf{m} + \alpha\mathbf{m}\times\dot{\mathbf{m}}$, where $\mathcal{F}_{\text{eff}} = -\delta\mathcal{F}/\delta\mathbf{m}$, and α is the Gilbert damping constant. Starting from the LLG equation, it is possible to derive the equation of motion for the collective coordinates $\zeta = \varphi_0, S_z$[5],

$$G_{ij}\dot{\zeta}_j + F_i - \alpha_i\dot{\zeta}_i = 0\,, \tag{29}$$

where $G_{ij} = 1 = -G_{ji}$ is the gyrotropic tensor, $F_i = \partial\mathcal{F}/\partial\zeta_i$ is the generalized force, and α_i is the damping tensor given by $\alpha_i = \alpha\bar{S}\int\partial\mathbf{m}/\partial\zeta_i\cdot\partial\mathbf{m}/\partial\zeta_i$. In particular we find, $\alpha_{\varphi_0} = \bar{S}\alpha\int_{\mathbf{r}}\sin\Theta$ and $\alpha_{S_z} = \bar{S}\alpha\int_{\mathbf{r}}(1-\cos\Theta)^2/\Lambda_0^2$, with $\Lambda_0 = \int_{\mathbf{r}}(1-\cos\Theta)$. Thus, the motion of both the helicity φ_0 and the conjugate momentum S_z is governed by Ohmic dissipation terms with corresponding dissipation constants α_{φ_0} and α_{S_z} respectively. They are accompanied by random fluctuating forces, which enter the quantum Hamiltonian as $\hat{H} \to \hat{H} + \xi_{\varphi_0}\varphi_0 + \xi_{S_z}\hat{S}_z$, with ξ_i fully characterized by the classical ensemble averages $\langle\xi_i(t)\rangle = 0$ and $\langle\xi_i(t)\xi_j(t')\rangle = \delta_{ij}S_i(t-t')$. The dissipative kernel α_i and the correlator $S_i(t-t')$ are related via the fluctuation-dissipation theorem,

$$S_i(\omega) = \alpha_i \omega \coth(\frac{\beta\omega}{2}), \quad (30)$$

with $S_i(t) = \int d\omega/2\pi e^{-i\omega t} S_i(\omega)$.

We now seek the reduced form the fluctuating fields enter the qubit Hamiltonian, a procedure that is identical for both skyrmion qubit categories. In the subspace spanned by states $|0\rangle$ and $|1\rangle$ we find

$$H_q = \frac{H_0}{2}\hat{\sigma}_z - \frac{X_c}{2}\hat{\sigma}_x + \xi_\zeta(t)\gamma_{\zeta,i}\hat{\sigma}_i, \quad (31)$$

with $i = x, y, z$ and $\zeta = \varphi_0, S_z$. Constants $\gamma_{\zeta,i}$ are $\gamma_{\zeta,x} = \Re[\langle 1|\hat{\zeta}|0\rangle + \langle 0|\hat{\zeta}|1\rangle]/2$, $\gamma_{\zeta,y} = \Im[\langle 1|\hat{\zeta}|0\rangle - \langle 0|\hat{\zeta}|1\rangle]/2$, and $\gamma_{\zeta,z} = \Re[\langle 1|\hat{\zeta}|1\rangle - \langle 0|\hat{\zeta}|0\rangle]/2$.

Typically, the dynamics of two-level systems are expressed in terms of two rates: the longitudinal relaxation rate $\Gamma_1 = T_1^{-1}$ and the dephasing rate $\Gamma_2 = T_2^{-1}$. The latter is a combination of effects of the depolarization Γ_1 and of the pure dephasing Γ_φ, combined to a rate $\Gamma_2 = \Gamma_1/2 + \Gamma_\varphi$. According to the Bloch-Redfield theory it holds,

$$\begin{aligned}\Gamma_1 &= [\alpha_{\varphi_0}(\gamma^{\perp}_{\varphi_0})^2 + \alpha_{S_z}(\gamma^{\perp}_{S_z})^2]\omega_q \coth(\frac{\beta\omega_q}{2}) \\ \Gamma_\varphi &= [\alpha_{\varphi_0}(\gamma^{\parallel}_{\varphi_0})^2 + \alpha_{S_z}(\gamma^{\parallel}_{S_z})^2]2/\beta, \end{aligned} \quad (32)$$

provided that $\gamma^{\perp}_{\zeta} = \cos\theta\gamma_{\zeta,x} + \gamma_{\zeta,y} + \cos\theta\gamma_{\zeta,z}$ and $\gamma^{\parallel}_{\zeta} = \cos\theta\gamma_{\zeta,z} + \sin\theta\gamma_{\zeta,x}$. An estimate of T_1 and T_2 in physical units is given in Table II of the main manuscript.

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