QUANTUM STATISTICAL MECHANICS AND THE BOUNDARY OF MODULAR CURVES

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ABSTRACT. The theory of limiting modular symbols provides a noncommutative geometric model of the boundary of modular curves that includes irrational points in addition to cusps. A noncommutative space associated to this boundary is constructed, as part of a family of noncommutative spaces associated to different continued fractions algorithms, endowed with the structure of a quantum statistical mechanical system. Two special cases of this family of quantum systems can be interpreted as a boundary of the system associated to the Shimura variety of GL_2 and an analog for SL_2. The structure of KMS states for this family of systems is discussed. In the geometric cases, the ground states evaluated on boundary arithmetic elements are given by pairings of cusp forms and limiting modular symbols.

1. INTRODUCTION

Since the introduction of the Bost–Connes system in the mid '90s, [1], a very rich interplay between number theory and quantum statistical mechanics developed, involving the Galois theory of abelian and non-abelian extensions of number fields ([6], [7], [13], [16], [25], [32]), Shimura varieties ([4], [13], [17]), L-series and zeta functions ([3], [8]), etc. On the other hand, the work of Manin and Marcolli, [23], and the subsequent work [14], [24], [26], developed a theory of limiting modular symbols, and a related noncommutative geometry model of the boundary of modular curves. It was remarked in §7.9 of [5] that the noncommutative compactification of modular curves of [23] should fit as a "boundary stratum" of the quantum statistical mechanical system of [4] that generalized the Bost–Connes system from the case of GL_1 to the case of GL_2 and has the geometry of modular curves directly built into its construction through the Shimura variety of GL_2. However, the relation between the limiting modular symbols of [23] and the GL_2 quantum statistical mechanical system of [4] has not been fully analyzed. It is the purpose of this note to outline a construction of a "boundary algebra" for the GL_2 system, endowed with an induced time evolution, that is built on the noncommutative boundary of modular curves described in [23] and on limiting modular symbols.

The GL_2 system of [4] is based on a convolution algebra involving Hecke operators and possibly degenerate level structures on modular curves. More precisely, in [4] one considers functions on the set

$$\{(g, \rho, z) \in GL_2^+(\mathbb{Q}) \times M_2(\hat{Z}) \times \mathbb{H} \,|\, g\rho \in M_2(\hat{\mathbb{Z}})\}$$

that are *invariant* under the action of $\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$ by

$$(\gamma_1, \gamma_2) : (g, \rho, z) \mapsto (\gamma_1 g \gamma_2^{-1}, \gamma_2 \rho, \gamma_2 z).$$

This algebra is then endowed with a time evolution and covariant representations. A first difficulty in extending the system constructed in this way to the boundary is that on $\mathbb{P}^1(\mathbb{R}) = \partial \mathbb{H}$ the action of $\mathrm{SL}_2(\mathbb{Z})$ by fractional linear transformation has dense orbits, hence it no longer makes sense to consider functions that are invariant under this action. Thus, what we consider here as an alternative is to retain the $\mathrm{SL}_2(\mathbb{Z})$ invariance as above in the variables (g, ρ), while replacing the action of $\mathrm{SL}_2(\mathbb{Z})$ on $z \in \mathbb{H}$ by a different action for which, instead of requiring invariance, we introduce as part of the algebra generators that implement that action (which amounts to taking a quotient in the noncommutative sense). The action considered is built out of the partial inverses of the shift of the continued fraction expansion. (Regarding the resulting decoupling of this action and the $\mathrm{GL}_2^+(\mathbb{Q})$-action, see the comments below about isogenies.)

We first extend the original definition of the GL_2 system of [4] to other subgroups of $\mathrm{GL}_2(\mathbb{Q})$ that include the case of $\mathrm{SL}_2(\mathbb{Z})$ and $\mathrm{GL}_2(\mathbb{Z})$. The main requirement for such subgroups $\Gamma \subset \mathrm{GL}_2(\mathbb{Q})$ is to have an associated Hecke algebra $\Xi = \Gamma \backslash \mathrm{GL}_2(\mathbb{Q}) / \Gamma$. In order to account for some additional structure considered in the setting of limiting modular symbols in [23], we also introduce the choice of a finite index subgroup $G \subset \Gamma$ and the coset spaces $\mathbb{P}_\alpha = \Gamma \alpha G / G$.

We focus in particular on a choice of $\Gamma = \Gamma_N$, dependent on an integer $N \in \mathbb{Z} \smallsetminus \{0\}$, consisting of matrices in $\mathrm{GL}_2(\mathbb{Q})$ with determinant in the subgroup of $\mathrm{GL}_1(\mathbb{Q})$ generated by the prime factors of N and -1. The main motivation for this choice is that these subgroups contain certain semigroups associated to an N-dependent family of continued fraction algorithms that we will use in the construction of a "boundary systems". This family includes a $\mathrm{GL}_2(\mathbb{Z})$-version of the original GL_2 system as a special case. Moreover, the partition function for these systems has a natural interpretation as the zeta function of the GL_2 system (the zeta function of $\mathbb{P}^1$) with a finite number of Euler factors removed.

We then construct two families of noncommutative algebras (both dependent on the integer parameter N). The first is a family of "bulk algebras" that generalize the GL_2 system of [4], involving the Hecke algebra Ξ_N with a (partially defined) action on the level structure $\rho \in M_2(\hat{\mathbb{Z}})$, the union $\mathcal{P}_N$ of coset spaces $\mathbb{P}_\alpha$, and the half planes $\mathbb{H}^\pm$. The other is a corresponding family of "boundary algebras" that are semigroup crossed products of the algebra of bounded functions from the interval $[0, 1]$ to the restriction of the bulk algebra to the coordinates $(g, \rho, s) \in \mathrm{GL}_2(\mathbb{Q}) \times M_2(\hat{\mathbb{Z}}) \times \mathcal{P}_N$, by a semigroup $\mathrm{Red}_N \subset \Gamma_N$ that implements a family of continued fraction algorithms parameterized by N.

In this construction, in the special case where $N = 1$, the action of $\mathrm{GL}_2(\mathbb{Z})$ on $\mathbb{H}^\pm$ of the first system is replaced in the second one by the action of the shift T of the usual GL_2-continued fraction algorithm on $[0, 1]$. This action on $[0, 1]$ is equivalent to the action of $\mathrm{GL}_2(\mathbb{Z})$ on $\mathbb{P}^1(\mathbb{R})$, so it is interpreted here as a way to describe pushing

the action of $\mathrm{GL}_2(\mathbb{Z})$ on $\mathbb{H}^\pm$ to the boundary $\mathbb{P}^1(\mathbb{R})$. As mentioned above, we no longer require invariance with respect to this action, and we introduce isometries S_k, associated to the partial inverses of T, to implement the action at the level of the algebra. Note that, by exchanging the $\mathrm{GL}_2(\mathbb{Z})$ action with the semigroup action implemented by the S_k, this action on $[0,1]$ becomes decoupled from the partial action of $\mathrm{GL}_2(\mathbb{Q})$, unlike what happens on the upper half plane. In terms of the original interpretation of the GL_2-system as implementing the commensurability relation on lattices with possibly degenerate level structure, in this boundary setting what remains of the commensurability relation affects the level structures (both through the action on $M_2(\mathbb{Z})$ and on the coset space $\mathcal{P}$) but does not change the pseudolattice $\mathbb{Z}\theta + \mathbb{Z} \subset \mathbb{R}$. The reason behind this choice is the lack (at present) of a good theory of isogeny for noncommutative tori, unlike the notion of isomorphism realized by the bimodules implementing Morita equivalence. This means that, at the level of the noncommutative boundary of the modular curve (which should be thought of as the moduli space of noncommutative tori with level structure), we see the commensurability relation as a relation on level structures. Both the semigroup and the Hecke operators simultaneously act on the cosets in $\mathcal{P}$, with commuting actions.

A more elaborate model of the boundary algebra would require developing a good setting for nontrivial isogenies of noncommutative tori. This can in principle be done by considering a larger class of bimodules that are not imprimitivity bimodules associated to Morita equivalences (for instance, the bimodules constructed in Proposition 5.7 of [19]), and select among these the ones that correspond to a good notion of isogenies. While this approach is certainly feasible, it is outside of the narrower scope of the present paper, and will be considered elsewhere.

The case $N = -1$ corresponds to the $\mathrm{SL}_2(\mathbb{Z})$-continued fraction algorithm. The other algebras in our family, for other values of N, do not have the same direct interpretation in terms of the geometry of modular curves as the $N = \pm 1$ cases, because the semigroup Red_N of the continued fraction algorithm sits inside the group Γ_N but will no longer necessarily have, in general, the same orbit structure. The main reason to consider this entire family of algebras is because the structure of KMS states of the resulting quantum statistical mechanical systems becomes more transparent when viewed over this whole family with varying parameter N.

We show that the structure of KMS states depends on that of the Cuntz–Krieger–Toeplitz type system generated by the isometries implementing the continued fraction algorithm and on the generalization of the GL_2-system. For all $N \neq 1$ there is a critical inverse temperature $\beta_{N,c}$ in the interval $(1,2)$ with the property that no KMS exist for $\beta < \beta_{N,c}$. Above this critical temperature there are as many extremal KMS states as there are for our generalized GL_2-system. In particular, for all $\beta > 2$ all the KMS states are Gibbs states and are parameterized by the representations $\pi_{\rho,x,s}$ with an invertible $\rho \in \mathrm{GL}_2(\hat{\mathbb{Z}})$. For $\beta \to \infty$ these Gibbs states converge weakly to KMS_∞ states given by evalutation at a point $(1,\rho,x,s)$. In the special case $N = 1$ the KMS states at finite β disappear entirely, due to the fact that in this case the Cuntz–Krieger–Toeplitz part has no KMS states, while only the ground states remain and

satisfy a weaker form of the KMS condition. The SL_2-case with $N = -1$ is, in this respect, the nicer in this family of algebras, as it has both the geometric interpretation in terms of modular curves and a nicer structure of KMS states with a convergent partition function in the low-temperature range $\beta > 2$ and Gibbs states converging to the ground states as the temperature goes to zero.

The ground states are the only ones that we need in order to investigate the pairing with limiting modular symbols. So for that purpose we can restrict to the case $N = 1$, which more closely reflects the setting of [23]. We introduce a class of "boundary arithmetic elements". These are obtained by first constructing a "boundary value map" which is a linear map from the bulk to the boundary algebra associated to the choice of a cusp form. The same map can be applied to the arithmetic algebra of the bulk system (which as in the original GL_2-case is an algebra of unbounded multipliers consisting of modular functions and Hecke operators). The subalgebra generated by the images is the arithmetic algebra of the boundary system. The image of the boundary value map consists of elements obtained by a procedure of averaging along geodesics. The evaluation of the ground states on these boundary values therefore agrees with the pairing of cusp forms with limiting modular symbols. This construction reflects the fact that, while the abelian class field theory of imaginary quadratic fields arises from evaluation of modular functions at complex multiplication points in the upper half plane, the corresponding geometry of real multiplication is expected to depend on a suitable averaging along geodesics with endpoints at conjugate quadratic irrationalities in a real quadratic field.

2. The GL_2-system

In this section we recall the basic properties of the GL_2-quantum statistical mechanical system of [4], in a version that accounts for the choice of a finite index subgroup of $\mathrm{GL}_2(\mathbb{Z})$ and for a more general class of subgroups of $\mathrm{GL}_2(\mathbb{Q})$ that include $\mathrm{GL}_2(\mathbb{Z})$ and $\mathrm{SL}_2(\mathbb{Z})$ as special cases.

2.1. **The GL_2-quantum statistical mechanical system.** The GL_2-quantum statistical mechanical system constructed in [4] as a generalization of the Bost–Connes system of [1] has algebra of observables given by the non-commutative C^*-algebra describing the "bad quotient" of the space of 2-dimensional $\mathbb{Q}$-lattices up to scaling by the equivalence relation of commensurability. This algebra is made dynamical by a time evolution defined by the determinant of the $\mathrm{GL}_2^+(\mathbb{Q})$ matrix that implements commensurability. There is an arithmetic algebra of unbounded multipliers on the C^*-algebra of observable, which is built in a natural way out of modular functions and Hecke operators (see [4] and Chapter 3 of [5]). The KMS-states for the time evolution have an action of $\mathbb{Q}^* \backslash \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},f})$ by symmetries, which include both automorphisms and endomorphisms of the C^*-dynamical system. The KMS states at zero temperature, defined as weak limits of KMS-states at positive temperature, are evaluations of modular functions at points in the upper half plane and the induced action of symmetries on KMS-states recovers the Galois action of the automorphisms

of the modular field. In the case of imaginary quadratic fields, the associated Bost–Connes system, constructed in [13] and [16], can be seen as a specialization of the GL_2-quantum statistical mechanical system of [4] at 2-dimensional $\mathbb{Q}$-lattices that are 1-dimensional $\mathbb{K}$-lattices, with $\mathbb{K}$ the imaginary quadratic field, and to CM points in the upper half plane.

Our goal here is to adapt this construction to obtain a specialization of the GL_2 system to the boundary $\mathbb{P}^1(\mathbb{R})$ and a further specialization for real quadratic fields. Our starting point will be the same algebra of the GL_2 system of [4], hence we start by reviewing briefly that construction in order to use it in our setting. It is convenient, for the setting we consider below, to extend the construction of the GL_2 system recalled above to the case where we replace $GL_2^+(\mathbb{Q})$ acting on the upper half plane $\mathbb{H}$ with $GL_2(\mathbb{Q})$ acting on $\mathbb{H}^{\pm}$ (the upper and lower half planes) and we consider a fixed finite index subgroup G of $GL_2(\mathbb{Z})$, where the latter replaces $\Gamma = SL_2(\mathbb{Z}) = GL_2^+(\mathbb{Q}) \cap GL_2(\hat{\mathbb{Z}})$ in the construction of the GL_2-system. We can formulate the resulting quantum statistical mechanical system in the following way. We can consider two slightly different versions of the convolution algebra.

Definition 2.1. *The involutive algebra $\mathcal{A}_G^c$ is given by complex valued functions on*

(2.1) $$\mathcal{U}^{\pm} = \{(g, \rho, z) \in GL_2(\mathbb{Q}) \times M_2(\hat{Z}) \times \mathbb{H}^{\pm} \,|\, g\rho \in M_2(\hat{\mathbb{Z}})\}$$

that are invariant under the action of $G \times G$ by $(g, \rho, z) \mapsto (\gamma_1 g \gamma_2^{-1}, \gamma_2 \rho, \gamma_2 z)$, and that have finite support in $g \in G \backslash GL_2(\mathbb{Q})$, depend on the variable $\rho \in M_2(\hat{\mathbb{Z}})$ through the projection onto some finite level $p_N : M_2(\hat{\mathbb{Z}}) \to M_2(\mathbb{Z}/N\mathbb{Z})$ and have compact support in the variable $z \in \mathbb{H}^{\pm}$. The convolution product on $\mathcal{A}_G^c$ is given by

(2.2) $$(f_1 \star f_2)(g, \rho, z) = \sum_{h \in G \backslash GL_2(\mathbb{Q}) \,:\, h\rho \in M_2(\hat{\mathbb{Z}})} f_1(gh^{-1}, h\rho, h(z)) g_2(h, \rho, z)$$

and the involution is $f^*(g, \rho, z) = \overline{f(g^{-1}, g\rho, g(z))}$. The algebra $\mathcal{A}_G^c$ is endowed with a time evolution given by $\sigma_t(f)(g, \rho, z) = |\det(g)|^{it} f(g, \rho, z)$.

Let Ξ denote the coset space $\Xi = GL_2(\mathbb{Z}) \backslash GL_2(\mathbb{Q}) / GL_2(\mathbb{Z})$ and let $\mathbb{Z}\Xi$ denote the free abelian group generated by the elements of Ξ. For simplicity of notation we write $\Gamma = GL_2(\mathbb{Z})$. The following facts are well known from the theory of Hecke operators. For any double coset $T_\alpha = \Gamma\alpha\Gamma$ in Ξ, there are finitely many $\alpha_i \in \Gamma\alpha\Gamma$ such that $\Gamma\alpha\Gamma = \sqcup_i \Gamma\alpha_i$. Thus, one can define a product on Ξ by setting

(2.3) $$T_\alpha T_\beta = \sum_\gamma c_{\alpha\beta}^\gamma T_\gamma$$

where for $\Gamma\alpha\Gamma = \sqcup_i \Gamma\alpha_i$ and $\Gamma\beta\Gamma = \sqcup_j \Gamma\beta_j$, the coefficient $c_{\alpha\beta}^\gamma$ counts the number of pairs (i, j) such that $\Gamma\alpha_i\beta_j = \Gamma\gamma$. The ring structure on $\mathbb{Z}\Xi$ determined by the product (2.3) can equivalently be described by considering finitely supported functions $f : \Xi \to \mathbb{Z}$ with the associative convolution product

(2.4) $$(f_1 \star f_2)(g) = \sum_h f_1(gh^{-1}) f_2(h)$$

where the sum is over the cosets Γh with $h \in \mathrm{GL}_2(\mathbb{Q})$ or equivalently over $\Gamma\backslash\mathrm{GL}_2(\mathbb{Q})$. The Hecke operators are built in this form into the algebra of the GL_2-system, through the dependence on the variable $g \in \Gamma\backslash\mathrm{GL}_2(\mathbb{Q})/\Gamma$, see the discussion in [5], Proposition 3.87.

2.1.1. *Coset spaces.* We introduce here a variant $\mathcal{A}^c_{\mathrm{GL}_2(\mathbb{Z}),G,\mathcal{P}}$ of the GL_2-algebra, where an additional variable is introduced that accounts for the choice of the finite index subgroup $G \subset \mathrm{GL}_2(\mathbb{Z})$ through the coset spaces $\mathbb{P}_\alpha = \mathrm{GL}_2(\mathbb{Z})\alpha G/G$, for $\alpha \in \mathrm{GL}_2(\mathbb{Q})$, which include for $\alpha = 1$ the coset space $\mathbb{P} = \mathrm{GL}_2(\mathbb{Z})/G$.

Lemma 2.2. *Let $G \subset \mathrm{GL}_2(\mathbb{Z})$ be a finite index subgroup such that $\alpha G\alpha^{-1} \cap \mathrm{GL}_2(\mathbb{Z})$ is also a finite index subgroup, for all $\alpha \in \mathrm{GL}_2(\mathbb{Q})$. Consider the double coset $\mathrm{GL}_2(\mathbb{Z})\alpha G$, with the left action of $\mathrm{GL}_2(\mathbb{Z})$ and the right action of G. The orbit spaces $\mathbb{P}_\alpha = \mathrm{GL}_2(\mathbb{Z})\alpha G/G$ are finite. The algebra $\mathbb{Z}\Xi$ of Hecke operators acts on the module $\mathbb{Z}\mathcal{P}$ with $\mathcal{P} = \cup_\alpha \mathbb{P}_\alpha$.*

Proof. We show that the map $\mathrm{GL}_2(\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{Z})\alpha G$ given by multiplication $\gamma \mapsto \gamma\alpha$ induces a bijection between $\mathrm{GL}_2(\mathbb{Z})/(\alpha G\alpha^{-1} \cap \mathrm{GL}_2(\mathbb{Z}))$ and $\mathrm{GL}_2(\mathbb{Z})\alpha G/G$, hence the orbit space $\mathbb{P}_\alpha = \mathrm{GL}_2(\mathbb{Z})\alpha G/G$ is finite. For $\ell = \alpha g\alpha^{-1} \in \alpha G\alpha^{-1} \cap \mathrm{GL}_2(\mathbb{Z})$ and $\gamma \in \mathrm{GL}_2(\mathbb{Z})$ we have $\gamma\ell\alpha = \gamma\alpha g \sim \gamma\alpha$ in $\Gamma\alpha G/G$ so the map is well defined on equivalence classes. It is injective since two $\gamma, \gamma' \in \Gamma$ with the same image differ by $\gamma'\gamma^{-1} = \alpha\ell\alpha^{-1}$ in $\alpha G\alpha^{-1} \cap \Gamma$ and it is also surjective by construction. The action of the algebra $\mathbb{Z}\Xi$ of Hecke operators is given by the usual multiplication of cosets. We write $\Gamma\alpha\Gamma = \sqcup_i \Gamma\alpha_i$ for finitely many $\alpha_i \in \Gamma\alpha\Gamma$ and $\Gamma\beta G = \sqcup_j \Gamma\beta_j$ for finitely many $\beta_j \in \Gamma\beta G$. The product is then given by

$$\Gamma\alpha\Gamma \cdot \Gamma\beta G = \sum_\gamma c^\gamma_{\alpha\beta}\Gamma\gamma G$$

where c_γ counts the number of pairs (i,j) such that $\Gamma\alpha_i\beta_j = \Gamma\gamma$. $\qquad\square$

The condition that $\alpha G\alpha^{-1} \cap \mathrm{GL}_2(\mathbb{Z})$ is also a finite index subgroup, for all $\alpha \in \mathrm{GL}_2(\mathbb{Q})$ is certainly satisfied, for instance, when G is a congruence subgroup.

In terms of generators T_α in the Hecke algebra $\mathbb{Z}\Xi$ and an element $\sum_s a_s\delta_s$ in $\mathbb{Z}\mathcal{P}$, we write the action of $\mathbb{Z}\Xi$ on the module $\mathbb{Z}\mathcal{P}$ as

$$(2.5) \qquad T_\alpha \sum_s a_s\delta_s = \sum_s a_s T_\alpha\delta_s = \sum_s a_s \sum_i c^i_{\alpha,s}\delta_{s_i} = \sum_i (\sum_s a_s c^i_{\alpha,s})\delta_{s_i}.$$

We can equivalently write elements of $\mathbb{Z}\mathcal{P}$ as finitely supported functions $\xi : \mathcal{P} \to \mathbb{Z}$, and elements of $\mathbb{Z}\Xi$ as finitely supported functions $f : \Xi \to \mathbb{Z}$, and write the action in the equivalent form

$$(2.6) \qquad\qquad (f \star \xi)(s) = \sum_h f(gh^{-1})\xi(hs)$$

where the sum is over cosets Γh and for $\xi(s) = \sum_\sigma a_\sigma\delta_\sigma(s)$ we write $\xi(hs)$ as

$$(2.7) \qquad\qquad \xi(hs) := \sum_i (\sum_\sigma a_\sigma c^i_{h,\sigma})\delta_{s_i}(s).$$

2.1.2. *More general subgroups and coset spaces.* We will also want to consider a more general family of double coset spaces in order to consider all possible N-continued fraction expansions as described at the beginning of Section 3. For $N \in \mathbb{Z}\backslash\{0\}$, Let Ξ_N denote the coset space $\Xi_N = \Gamma_N\backslash\mathrm{GL}_2(\mathbb{Q})/\Gamma_N$, where for $|N| > 1$,

$$\Gamma_N = \{g \in \mathrm{GL}_2(\mathbb{Q})|\det(g) \in \mathcal{G}_N\}$$

where $\mathcal{G}_N$ is the subgroup of $\mathbb{Q}^\times$ generated by -1 and the prime factors of N.

In the case of $N = 1$ we take $\Gamma_1 := \mathrm{GL}_2(Z)$ and $\Xi_1 = \Xi$, as before. When $N = -1$, we take $\Gamma_{-1} := SL_2(\mathbb{Z})$.

We may also consider a finite index subgroup G of Γ_N and associated orbit spaces $\mathbb{P}_{N,\alpha} = \Gamma_N\alpha G/G$ for $\alpha \in \mathrm{GL}_2(\mathbb{Q})$. The discussion in Lemma 2.2 remains the same, where now $c^\gamma_{\alpha\beta}$ counts the number of pairs (i,j) such that $\Gamma_N\alpha_i\beta_j = \Gamma_N\gamma$. We suppress the N subscript when it is clear from context.

To be more concrete, we illustrate here some explicit examples.

2.1.3. *The* $\mathrm{SL}_2(\mathbb{Z})$ *case.* In the case of the algebra of the GL_2-system of [4], for an invertible $\rho \in \mathrm{GL}_2(\hat{\mathbb{Z}})$, the relevant Hecke algebra is $\mathbb{Z}\Xi_{-1} = \mathcal{H}(\Gamma_{-1},\mathcal{M})$ where $\Gamma_{-1} = \mathrm{SL}_2(\mathbb{Z})$ and the subsemigroup $\mathcal{M} = M_2^+(\mathbb{Z})$ of $\mathrm{GL}_2^+(\mathbb{Q})$.

In this case (see Chapter 4 of [15]) $\mathcal{H}(\Gamma_{-1},\mathcal{M})$, as an algebra over $\mathbb{Z}$, is generated by the Hecke operators

$$T(\ell) = \sum_{\alpha\in\Gamma_{-1}\backslash\mathcal{M}(\ell)/\Gamma_{-1}} \Gamma_{-1}\alpha\Gamma_{-1} = \sum_{ad=\ell,\,a|d} T(a,d),$$

with $\mathcal{M}(\ell) = \{\alpha \in \mathcal{M} \mid \det(\alpha) = \ell\}$ and

$$T(a,d) = \Gamma_{-1}\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\Gamma_{-1},$$

subject to the relations, for $k,\ell \in \mathbb{N}$,

$$T(\ell)T(k) = \sum_{d|\gcd\{k,\ell\}} d\,T(d,d)\,T(\frac{k\ell}{d^2}).$$

Equivalently, the Hecke algebra $\mathcal{H}(\Gamma_{-1},\mathcal{M})$ splits into primary components

$$\mathcal{H}(\Gamma_{-1},\mathcal{M}) = \otimes_p\mathcal{H}(\Gamma_{-1},\mathcal{M})_p,$$

over the set of primes p, where $\mathcal{H}(\Gamma_{-1},\mathcal{M})_p = \mathbb{Z}[T(p),T(p,p)]$.

This description of the Hecke algebra is obtained directly from the following properties of right cosets and double cosets (Chapter 4 of [15]). Given $\alpha \in \mathcal{M}$, the right coset $\Gamma_{-1}\alpha$ contains a unique representative of the form

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \quad \text{with } a,d \in \mathbb{N},\ 0 \le b < d.$$

The set $\mathcal{M}(\ell)$ decomposes as a disjoint union of $\sigma_1(\ell) = \sum_{d|\ell} d$ right Γ_{-1}-cosets, with a set of representatives given by the matrices

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \quad \text{with } d \in \mathbb{N}, \ 0 \le b < d, \ a = \ell/d.$$

Given $\alpha \in \mathcal{M}$ there are $\gamma_1, \gamma_2 \in \Gamma_{-1}$ and $a, d \in \mathbb{N}$ with $a|d$ such that

$$\gamma_1 \alpha \gamma_2 = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}.$$

2.1.4. *The* $\mathrm{GL}_2(\mathbb{Z})$ *case.* Here we consider also the case where $\Gamma = \mathrm{GL}_2(\mathbb{Z})$ and $\mathcal{M} = M_2(\mathbb{Z}) \cap \mathrm{GL}_2(\mathbb{Q})$ is the subsemigroup of $\mathrm{GL}_2(\mathbb{Q})$. In this case the explicit description of $\mathbb{Z}\Xi = \mathcal{H}(\Gamma, \mathcal{M})$ is similar to the previous case (see Chapter 5 of [15]), and $\mathcal{H}(\Gamma, \mathcal{M})$ is generated by the Hecke operators

$$T(\ell) = \sum_{\alpha \in \Gamma \backslash \mathcal{M}(\ell)/\Gamma} \Gamma \alpha \Gamma,$$

where here $\mathcal{M}(\ell) = \{\alpha \in M_2(\mathbb{Z}) \,|\, |\det \alpha| = \ell\}$. The Hecke algebra splits into primary components as in the previous case. We refer the reader to [15] for more details.

2.1.5. *The case with congruence subgroups.* In the case where we also consider a choice of a non-trivial congruence subgroup $G \subset \Gamma$, with a Hecke algebra $\mathbb{Z}\Xi = \mathcal{H}(\Gamma, \mathcal{M})$ as in the previous cases (see Section 2.7 of [28]), we can identify the $\mathbb{Z}$-module $\mathbb{Z}\mathcal{P}$ with

$$\mathbb{Z}\mathcal{P} = \mathbb{Z}[\Gamma \backslash \mathcal{M}]^G,$$

with the identification induced by the injective homomorphism of $\mathbb{Z}$-modules

$$\phi : \mathbb{Z}\mathcal{P} \to \mathbb{Z}[\Gamma \backslash \mathcal{M}], \quad \phi(\Gamma \alpha G) = \sum_i \Gamma \alpha_i,$$

for $\Gamma \alpha G = \sqcup_i \Gamma \alpha_i$ a decomposition into right-cosets. We can then write the action of $\mathbb{Z}\Xi$ on $\mathbb{Z}\mathcal{P}$ described above in (2.6), (2.7) in the form

$$\Gamma \beta \Gamma \cdot \xi = \sum_\alpha a_\alpha \, \Gamma \alpha \beta_j,$$

where

$$\xi = \sum_\alpha a_\alpha \, \Gamma \alpha$$

is a G-invariant element in $\mathbb{Z}[\Gamma \backslash \mathcal{M}]$ and $\Gamma \beta \Gamma = \sqcup_j \Gamma \beta_j$. The action is independent of the choice of representatives (Lemma 2.7.3 of [28]).

Thus, for general elements $h \in \mathbb{Z}\Xi$ and $\xi \in \mathbb{Z}\mathcal{P}$ with $h = \sum_\beta b_\beta \, \Gamma \beta \Gamma$ and $\xi = \sum_\alpha a_\alpha \Gamma \alpha G$, we reformulate (2.6), (2.7) as

$$h \star \xi = \sum_{\alpha, \beta, \gamma} b_\beta \, a_\alpha \, c^\gamma_{\alpha, \beta} \Gamma \gamma G,$$

see (2.7.3) of [28].

2.1.6. *The bulk algebra.* We now proceed to the construction of a "bulk algebra" (namely, the algebra associated to the bulk space $\mathbb{H}$), which includes the choice of a finite index subgroup $G \subset \Gamma_N$.

Definition 2.3. *Let $G \subset \Gamma_N$ be a finite index subgroup and let $\mathcal{P}_N = \cup_\alpha \mathbb{P}_{N,\alpha}$ with $\mathbb{P}_{N,\alpha} = \Gamma_N \alpha G / G$ for $\alpha \in \mathrm{GL}_2(\mathbb{Q})$. The involutive algebra $\mathcal{A}^c_{\Gamma_N, G, \mathcal{P}_N}$ is given by complex valued functions on the space*

$$(2.8) \qquad \mathcal{U}^\pm_{G, \mathcal{P}_N} = \{(g, \rho, z, \xi) \in \mathrm{GL}_2(\mathbb{Q}) \times M_2(\hat{\mathbb{Z}}) \times \mathbb{H}^\pm \times \mathcal{P}_N \,|\, g\rho \in M_2(\hat{\mathbb{Z}})\},$$

that are invariant under the action of $\Gamma_N \times \Gamma_N$ by

$$(g, \rho, z, s) \mapsto (\gamma_1 g \gamma_2^{-1}, \gamma_2 z, \gamma_2 s).$$

Moreover, functions in $\mathcal{A}^c_{\Gamma_N, G, \mathcal{P}_N}$ have finite support in $\Gamma_N \backslash \mathrm{GL}_2(\mathbb{Q})$ and in $\mathcal{P}_N$, compact support in $z \in \mathbb{H}^\pm$, and they depend on the variable $\rho \in M_2(\hat{\mathbb{Z}})$ through the projection onto some finite level $p_n : M_2(\hat{\mathbb{Z}}) \to M_2(\mathbb{Z}/n\mathbb{Z})$. The convolution product of $\mathcal{A}^c_{\Gamma_N, G, \mathcal{P}_N}$ is given by

$$(2.9) \qquad (f_1 \star f_2)(g, \rho, z, s) = \sum_{h \in \mathcal{S}_{\rho, N}} f_1(gh^{-1}, h\rho, h(z), hs) f_2(h, \rho, z, s),$$

where we are using the notation (2.6), (2.7) for the action of Hecke operators on functions of $\mathcal{P}_N$ and $\mathcal{S}_{\rho,N}$ is the collection of Γ_N-cosets that have some representative element in the set $\{g \in \mathrm{GL}_2(\mathbb{Q}) | g\rho \in M_2(\hat{\mathbb{Z}})\}$. In particular, when $N = 1$, $\mathcal{S}_\rho = \mathrm{GL}_2(\mathbb{Z}) \backslash \{g \in \mathrm{GL}_2(\mathbb{Q}) : g\rho \in M_2(\hat{\mathbb{Z}})\}$.

The involution is $f^*(g, \rho, z, s) = \overline{f(g^{-1}, g\rho, g(z), gs)}$. The algebra $\mathcal{A}^c_{\Gamma_N, G, \mathcal{P}_N}$ is endowed with a time evolution given by

$$\sigma_t(f)(g, \rho, z, s) = |\det(g)|^{it} f(g, \rho, z, s).$$

We focus here on the algebra $\mathcal{A}^c_{\Gamma_N, G, \mathcal{P}_N}$ and we construct Hilbert space representations analogous to the ones considered for the original GL_2-system.

Consider then the Hilbert space $\mathcal{H}_{\rho, N} = \ell^2(\mathcal{S}_{\rho, N})$, and the representations $\pi_{(\rho, z, s)} : \mathcal{A}^c_{\Gamma_N, G, \mathcal{P}_N} \to \mathcal{B}(\mathcal{H}_{\rho, N})$

$$\pi_{(\rho, z, s)}(f)\xi(g) = \sum_{h \in \mathcal{S}_{\rho, N}} f(gh^{-1}, h\rho, h(z), hs)\xi(h).$$

We can complete the algebra $\mathcal{A}^c_{\Gamma_N, G, \mathcal{P}_N}$ to a C^*-algebra $\mathcal{A}_{\Gamma_N, G, \mathcal{P}_N}$ in the norm $\|f\| = \sup_{(\rho, z, s)} \|\pi_{(\rho, z, s)}(f)\|_{\mathcal{B}(\mathcal{H}_{\rho, N})}$. The time evolution is implemented in the representation $\pi_{(\rho, z, s)}$ by the Hamiltonian $H\xi(g) = \log|\det(g)|\xi(g)$.

2.2. **The arithmetic algebra.** We proceed exactly as in the case of the GL_2-system of [4] to construct an arithmetic algebra associated to $\mathcal{A}_{\Gamma_N, G, \mathcal{P}_N}$. As in [4] this will not be a subalgebra but an algebra of unbounded multipliers.

The arithmetic algebra $\mathcal{A}^{ar}_{\Gamma_N, G, \mathcal{P}_N}$ is the algebra over $\mathbb{Q}$ obtained as follows. We consider functions on $\mathcal{U}^\pm_{G, \mathcal{P}_N}$ of (2.8) that are invariant under the action of $\Gamma_N \times \Gamma_N$

by $(g, \rho, z) \mapsto (\gamma_1 g \gamma_2^{-1}, \gamma_2 \rho, \gamma_2 z)$ and that are finitely supported in $g \in G \backslash \mathrm{GL}_2(\mathbb{Q})$ and in $\mathcal{P}_N$, that depend on ρ through some finite level projection $p_n(\rho) \in M_2(\mathbb{Z}/n\mathbb{Z})$ and that are holomorphic in the variable $z \in \mathbb{H}$ and satisfy the growth condition that $|f(g, \rho, z, s)|$ is bounded by a polynomial in $\max\{1, \Im(z)^{-1}\}$ when $\Im(z) \to \infty$. The resulting algebra $\mathcal{A}_{\Gamma_N, G, \mathcal{P}_N}^{ar}$ acts, via the convolution product (2.9), as unbounded multipliers on the algebra $\mathcal{A}_{\Gamma_N, G, \mathcal{P}_N}$. This construction and its properties are completely analogous to the original case of the GL_2-system and we refer the reader to [4], [5] for details.

The invariance property ensures that these are modular functions for G (written as Γ_N-invariant functions on $\mathbb{H} \times \mathbb{P}_N$ rather than as G-invariant functions on $\mathbb{H}$). These functions are endowed with the same convolution product (2.9).

3. Boundary GL_2-system

We now consider how to extend this setting to incorporate the boundary $\mathbb{P}^1(\mathbb{R})$ of the upper half-plane $\mathbb{H}$ and $\mathbb{Q}$-pseudo-lattices generalizing the $\mathbb{Q}$-lattices of [4]. A brief discussion of the boundary compactification of the GL_2-system was given in §7.9 of [5], but the construction of a suitable quantum statistical mechanical system associated to the boundary was never worked out in detail.

We replace the full $\mathbb{P}^1(\mathbb{R})$ boundary of $\mathbb{H}^\pm$ with the smaller interval $[0, 1]$. In the $N = 1$ case this choice is natural as this interval meets every orbit of the $\mathrm{GL}_2(\mathbb{Z})$ action and the equivalence relation given by this action can be described equivalently through the shift T of the continued fraction expansion. This action can be implemented via the semigroup of reduced matrices in the form of a crossed product algebra. Inspired by this case, we adopt the same setting for the whole family of algebras parameterized by the nontrivial integer N, with corresponding continued fraction algorithms on the interval $[0, 1]$ and associated semigroups. We analyze Hilbert space representations, time evolution, Hamiltonian, partition function and KMS states.

3.1. **Continued fraction algorithms.** We consider the countable family of N-continued fraction expansions given by

$$(3.1) \qquad [a_0; a_1, a_2, a_3, ...]_N = a_0 + \cfrac{N}{a_1 + \cfrac{N}{a_2 + \cfrac{N}{a_3 + ...}}}$$

with $a_i \geq N$ when $N \geq 1$ and $a_i \geq |N| + 1$ when $N \leq -1$. We denote the set of allowed digits of the N-continued fraction expansion by Φ_N,

$$\Phi_N = \begin{cases} \mathbb{N}_{\geq N} & \text{when } N \geq 1 \\ \mathbb{N}_{\geq |N|+1} & \text{when } N \leq -1 \end{cases}.$$

For each N-continued fraction expansion, we introduce an algebra associated to the boundary $\mathbb{P}^1(\mathbb{R})$ with the action of a certain subsemigroup of $\mathrm{GL}_2(\mathbb{Q})$, called the semigroup of reduced matrices, depending on the choice of N. In the case that

$N = 1$ this semigroup of reduced matrices is contained in $\mathrm{GL}_2(\mathbb{Z})$ and in the case that $N = -1$ it is contained in $\mathrm{PSL}_2(\mathbb{Z})$. In the $N = \pm 1$ cases, the associated algebra can be interpreted as a boundary algebra of the GL_2 system. While we have no similar direct geometric interpretation when $|N| > 1$, considering the whole family of systems leads to some interesting observations about the structure of the KMS states.

3.2. **Boundary dynamics and coset spaces.** The N-continued fraction expansion of a real number x can be retrieved via the shift operator $T_N : [0, 1] \to [0, 1]$ given by

$$(3.2) \qquad T_N(x) = \frac{N}{x} - \left\lfloor \frac{N}{x} \right\rfloor, \quad x \neq 0; \quad T_N(0) = 0.$$

For $x \in [0, 1)$, one has that $a_0 = 0$ and $a_i = \left\lfloor \frac{N}{T_N^{i-1}(x)} \right\rfloor$ in the case that $N \geq 1$, and $a_0 = 1$ and $a_i = -\left\lfloor \frac{N}{T_N^{i-1}(1-x)} \right\rfloor$ in the case that $N \leq -1$.

We extend T_N to a map on $[0, 1] \times \mathbb{P}$ by

$$(3.3) \qquad T_N : (x, s) \mapsto \left(\frac{N}{x} - \left\lfloor \frac{N}{x} \right\rfloor, \begin{pmatrix} -\lfloor N/x \rfloor & N \\ 1 & 0 \end{pmatrix} \cdot s \right).$$

We remark that in the geometrically meaningful case of $N = 1$, the set $[0, 1] \times \mathbb{P}$ meets every orbit of the action of $\mathrm{GL}_2(\mathbb{Z})$ on $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}$, acting on $\mathbb{P}^1(\mathbb{R})$ by fractional linear transformations and on $\mathbb{P} = \mathrm{GL}_2(\mathbb{Z})/G$ by the left-action of $\mathrm{GL}_2(\mathbb{Z})$ on itself. Moreover, two points (x, s) and (y, t) in $[0, 1] \times \mathbb{P}$ are in the same $\mathrm{GL}_2(\mathbb{Z})$-orbit iff there are integers $n, m \in \mathbb{N}$ such that $T_1^n(x, s) = T_1^m(y, t)$.

Lemma 3.1. *The action of the shift map (3.3) on $[0, 1] \times \mathbb{P}$ extends to an action on $[0, 1] \times \mathbb{P}_{N,\alpha}$, with $\mathbb{P}_{N,\alpha} = \Gamma_N \alpha G / G$, for any given $\alpha \in \mathrm{GL}_2(\mathbb{Q})$, hence to an action on $[0, 1] \times \mathcal{P}_N$ with $\mathcal{P}_N = \cup_\alpha \mathbb{P}_{N,\alpha}$.*

Proof. The action of T_N on $(x, s) \in [0, 1] \times \mathbb{P}$ is implemented by the action of the matrix

$$\begin{pmatrix} -\lfloor N/x \rfloor & N \\ 1 & 0 \end{pmatrix} \in \Gamma_N.$$

The same matrix acts by left multiplication on $\mathbb{P}_{N,\alpha} = \Gamma_N \alpha G / G$, hence it determines a map $T_N : [0, 1] \times \mathbb{P}_{N,\alpha} \to [0, 1] \times \mathbb{P}_{N,\alpha}$. $\square$

Lemma 3.2. *Let $\mathbb{C}\mathcal{P}_N$ denote the ring of finitely supported complex valued functions on $\mathcal{P}_N$ and let $\mathcal{B}_N = \mathcal{B}([0, 1], \mathbb{C}\mathcal{P}_N)$ denote the ring of bounded functions from $[0, 1]$ to $\mathbb{C}\mathcal{P}_N$. Consider the action of the semigroup $\mathbb{Z}_+$ on $\mathcal{B}_N$ determined by the action of T_N on $[0, 1] \times \mathcal{P}_N$ of Lemma 3.1 and the action on $\mathcal{B}_N$ by Hecke operators acting on $\mathbb{C}\mathcal{P}_N$. These two actions commute.*

Proof. We write functions $f \in \mathcal{B}_N$ in the form $\sum_\alpha f_\alpha(x, s_\alpha)\delta_\alpha$ where δ_α is the characteristic function of $\mathbb{P}_{N,\alpha} = \Gamma_N \alpha G / G$ and $s_\alpha \in \mathbb{P}_{N,\alpha}$. The action of $\mathbb{Z}_+$ is given by

$$T_N^n : \sum_\alpha f_\alpha(x, s_\alpha)\delta_\alpha \mapsto \sum_\alpha f_\alpha(T_N^n(x, s_\alpha))\delta_\alpha,$$

with $T_N(x, s_\alpha)$ as in (3.3), while the action of a Hecke operator T_β is given by

$$(3.4) \qquad T_\beta : \sum_\alpha f_\alpha(x, s_\alpha)\delta_\alpha \mapsto \sum_\gamma (\sum_\alpha c_{\beta,\alpha}^\gamma f_\alpha(x, s_\alpha))\delta_\gamma,$$

with $c_{\beta\alpha}^\gamma$ defined as in (2.5), modified appropriately for the choice of N. It is then clear that these two actions commute. $\square$

Lemma 3.3. *Let $X_{N,k} \subset [0,1]$ be the subset of points $x \in [0,1]$ with N-continued fraction expansion starting with the digit $k \in \Phi_N$. Let $\mathcal{B}_N$ denote the algebra of bounded complex valued functions on $[0,1] \times \mathcal{P}_N$. Let $\tau_N(f) = f \circ T_N$ denote the action of the shift $T_N : [0,1] \to [0,1]$ of the N-continued fraction expansion on $f \in \mathcal{B}_N$.*
  Let $\mathcal{B}_N$ act as multiplication operators on $L^2([0,1], d\mu_N)$ with $d\mu_N$ the T_N-invariant measures on $[0,1]$,

$$d\mu_N(x) = \begin{cases} \left(\log \frac{N+1}{N}\right)^{-1} (N+x)^{-1}\, dx & \text{if } N \in \mathbb{Z} \backslash \{0, -1\} \\ (1-x)^{-1}\, dx & \text{if } N = -1 \end{cases}.$$

With the notation

$$(3.5) \qquad g_{N,k} = \begin{pmatrix} 0 & N \\ 1 & k \end{pmatrix} \quad \text{and} \quad g_{N,k}^{-1} = \begin{pmatrix} -\frac{k}{N} & 1 \\ \frac{1}{N} & 0 \end{pmatrix},$$

consider the operators

$$(3.6) \qquad S_{N,k}\xi(x) = \chi_{X_{N,k}}(x) \cdot \xi(g_{N,k}^{-1}x) \quad \text{and} \quad \tilde{S}_{N,k}\xi(x) = \xi(g_{N,k}\, x),$$

for $\xi \in L^2([0,1], d\mu_N)$, with $\chi_{X_{N,k}}$ the characteristic function of the subset $X_{N,k} \subset [0,1]$.
  These satisfy $\tilde{S}_{N,k} = S_{N,k}^*$ with $S_{N,k}^* S_{N,k} = 1$ and $\sum_k S_{N,k} S_{N,k}^* = 1$. They also satisfy the relation

$$(3.7) \qquad \sum_k S_{N,k}\, f\, S_{N,k}^* = f \circ T_N.$$

Proof. The shift map of the continued fraction expansion, given by $T_N(x) = N/x - [N/x]$, acts on $x \in X_{N,k}$ as $x \mapsto g_{N,k}^{-1}x$ with the matrix $g_{N,k}^{-1}$ acting by fractional linear transformations. The operators $S_{N,k}$ defined as in (3.6) are not isometries on $L^2([0,1], dx)$ with respect to the Lebesgue measure dx. However, if we consider the T_N-invariant probability measures $d\mu_N$, then we have $d\mu_N \circ g_{N,k}^{-1}|_{X_{N,k}} = d\mu|_{X_{N,k}}$ for

all $k \in \mathbb{N}$, hence

$$\langle S_{N,k}\xi_1, S_{N,k}\xi_2 \rangle = \int_{X_{N,k}} \bar{\xi}_1 \circ g_{N,k}^{-1} \xi_2 \circ g_{N,k}^{-1} \, d\mu_N$$

$$= \int_{X_{N,k}} \bar{\xi}_1 \circ g_{N,k}^{-1} \xi_2 \circ g_{N,k}^{-1} \, d\mu_N \circ g_{N,k}^{-1} = \int_{[0,1]} \bar{\xi}_1 \, \xi_2 \, d\mu_N = \langle \xi_1, \xi_2 \rangle.$$

We have $\tilde{S}_{N,k} S_{N,k}\xi(x) = \xi(x)\chi_{X_{N,k}}(g_{N,k}x) = \xi(x)$. Moreover, $\tilde{S}_{N,k} = S_{N,k}^*$ in this inner product since we have

$$\langle \xi_1, S_{N,k}\xi_2 \rangle = \int_{X_{N,k}} \bar{\xi}_1 \, \xi_2 \circ g_{N,k}^{-1} \, d\mu_N$$

$$= \int_{X_{N,k}} \bar{\xi}_1 \, \xi_2 \circ g_{N,k}^{-1} \, d\mu_N \circ g_{N,k}^{-1} = \int_{[0,1]} \bar{\xi}_1 \circ g_{N,k} \, \xi_2 \, d\mu_N = \langle \tilde{S}_{N,k}\xi_1, \xi_2 \rangle.$$

We also have $\sum_k S_{N,k} f \, S_{N,k}^* \xi(x) = \sum_k f(g_{N,k}^{-1}x)\chi_{X_{N,k}}(x)\xi(x) = f(T_N(x))$. Thus we obtain $\sum_k S_{N,k} f \, S_{N,k}^* = f \circ T_N$, which in particular also implies $\sum_k S_{N,k} S_{N,k}^* = 1$. $\quad\square$

3.3. Semigroups.

Consider the set of matrices in $\mathrm{GL}_2(\mathbb{Q})$

$$(3.8) \qquad \mathrm{Red}_{N,n} := \begin{cases} \left\{ \begin{pmatrix} 0 & N \\ 1 & k_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & N \\ 1 & k_n \end{pmatrix} \Big| k_i \in \mathbb{Z}_{\geq N} \right\} & \text{if } N \geq 1 \\[2em] \left\{ \begin{pmatrix} 0 & N \\ 1 & k_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & N \\ 1 & k_n \end{pmatrix} \Big| k_i \in \mathbb{Z}_{\geq |N|+1} \right\} & \text{if } N \leq -1 \end{cases}$$

Note that $\mathrm{Red}_{N,n} \subset \Gamma_N$ and in particular when $N = 1$, $\mathrm{Red}_{1,n} \subset \mathrm{GL}_2(\mathbb{Z})$, and when $N = -1$, $\mathrm{Red}_{-1,n} \subset \mathrm{SL}_2(\mathbb{Z})$.

The semigroups of reduced matrices are defined as

$$(3.9) \qquad\qquad\qquad \mathrm{Red}_{\mathrm{N}} := \cup_{n \geq 1} \mathrm{Red}_{N,n}.$$

An equivalent description of the Red_1 semigroup is given by ([18])

$$\mathrm{Red}_1 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}) \,\Big|\, 0 \leq a \leq b, \, 0 \leq c \leq d \right\}.$$

Lemma 3.4. *Assigning to a matrix*

$$(3.10) \qquad\qquad\qquad \gamma = \begin{pmatrix} 0 & N \\ 1 & n_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & N \\ 1 & n_k \end{pmatrix}$$

in $\mathrm{Red}_{\mathrm{N}}$ the product $n_1 \cdots n_k \in \mathbb{N}$ is a well defined semigroup homomorphism.

Proof. We only need to check that the representation of a matrix γ in $\mathrm{Red}_{\mathrm{N}}$ as a product (3.10) is unique so that the map is well defined. It is then by construction a semigroup homomorphism.

First we consider the $N = 1$ case. The group $\mathrm{GL}_2(\mathbb{Z})$ has generators

$$\sigma = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \rho = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

with relations $(\sigma^{-1}\rho)^2 = (\sigma^{-2}\rho^2)^6 = 1$. The semigroup Red_1 can be equivalently described as the subsemigroup of the semigroup generated by σ and ρ made of all the words in σ, ρ that end in ρ, so elements are products of matrices of the form $\sigma^{n-1}\rho = \begin{pmatrix} 0 & 1 \\ 1 & n \end{pmatrix}$. We have

$$\gamma = \begin{pmatrix} 0 & 1 \\ 1 & n_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & n_{\ell(\gamma)} \end{pmatrix}$$

where $\ell(\gamma)$ is the number of ρ's in the word in σ and ρ representing γ. The semigroup generated by σ and ρ is a free semigroup, as the only relations in $\mathrm{GL}_2(\mathbb{Z})$ between these generators involve the inverse σ^{-1}. If an element $\gamma \in \mathrm{Red}_1$ had two different representations (3.10), for two different ordered sets $\{n_1, \ldots, n_k\}$ and $\{m_1, \ldots, m_l\}$ then we would have a relation

$$\sigma^{n_1-1}\rho\sigma^{n_2-1}\rho\cdots\sigma^{n_k-1}\rho = \sigma^{m_1-1}\rho\sigma^{m_2-1}\rho\cdots\sigma^{m_l-1}\rho$$

involving the generators σ and ρ but not their inverses, which would contradict the fact that σ and ρ generate a free semigroup.

Next we consider the case $N \in \mathbb{Z}\backslash\{-1, 0, 1\}$. We observe that we can decompose elements of Red_N in terms of ρ, σ and $\eta_N = \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}$, a diagonal matrix depending on N since

$$\begin{pmatrix} 0 & N \\ 1 & n \end{pmatrix} = \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & n \end{pmatrix} = \eta_N \sigma^{n-1}\rho.$$

If an element $\gamma \in \mathrm{Red}_N$ for $|N| > 1$ had two different representations (3.10), for two different ordered sets $\{n_1, \ldots, n_k\}$ and $\{m_1, \ldots, m_l\}$ then we would have a relation

$$\eta_N\sigma^{n_1-1}\rho\eta_N\sigma^{n_2-1}\rho\cdots\eta_N\sigma^{n_k-1}\rho = \eta_N\sigma^{m_1-1}\rho\eta_N\sigma^{m_2-1}\rho\cdots\eta_N\sigma^{m_l-1}\rho.$$

As before, there are no relations between ρ and σ. There cannot be a relation involving η_N and ρ and σ. If we had $\mathrm{word}(\eta_N, \rho, \sigma) = 1$ then the determinant of the left-hand side would be $\pm N^r$ where r is the number of times η_N appears in the word, while the determinant of the right-hand side would be 1. Since we are in the case $|N| > 1$, this is a contradiction.

Finally we consider the $N = -1$ case. $\mathrm{PSL}_2(\mathbb{Z})$ can be written as a free product of cyclic groups

$$\mathrm{PSL}_2(\mathbb{Z}) \simeq C_2 \star C_3$$

with generators

(3.11) $$B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

of degree 2 and 3 respectively ($B^2 = 1$ and $C^3 = 1$). We can write a matrix in Red_{-1}

$$\gamma = \begin{pmatrix} 0 & -1 \\ 1 & n_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & -1 \\ 1 & n_{\ell(\gamma)} \end{pmatrix}$$

in terms of these generators by noting that in $\mathrm{PSL}_2(\mathbb{Z})$,

$$\begin{pmatrix} 0 & -1 \\ 1 & n \end{pmatrix} = B(CB^{-1})^n$$

and hence

$$\gamma = B(CB^{-1})^{n_1} \cdots B(CB^{-1})^{n_{\ell(\gamma)}}$$

(3.12) $$= B(CB^{-1})^{n_1-1}C^2B^{-1}(CB^{-1})^{n_2-1}C^2B^{-1}\ldots C^2B^{-1}(CB^{-1})^{n_{\ell(\gamma)}-1}.$$

Since each $n_i \geq 2$, this is a reduced sequence of words in C_2 and C_3. Every element in a free product can be written uniquely as a reduced sequence of words. Furthermore, each element of the cyclic groups C_2 and C_3 can be written uniquely as B^k or C^k where k is required to be either positive or negative. The form (3.12) is unique. If an element $\gamma \in \mathrm{Red}_{-1}$ had two different expressions of the form (3.10), it would contradict this uniqueness. $\qquad\square$

Lemma 3.5. *Let $\mathcal{B}_N$ denote the algebra of bounded complex valued functions on $[0,1] \times \mathcal{P}_N$ that are finitely supported in $\mathcal{P}_N$. The transformations $\alpha_\gamma(f) = \chi_{X_\gamma} \cdot f \circ \gamma^{-1}$ for $\gamma \in \mathrm{Red}_N$ define a semigroup action of Red_N on $\mathcal{B}_N$. This action commutes with the action of Hecke operators.*

Proof. We check that $\alpha_\gamma(f) = \chi_{X_\gamma} \cdot f \circ \gamma^{-1}$ is a well defined semigroup action of Red_N on $\mathcal{B}_N$. For γ of the form (3.10) we have $\alpha_\gamma = \alpha_{g_1} \cdots \alpha_{g_n}$ with the factors $g_i = g_{N,k_i}$ as in (3.5), since for two matrices γ, γ' in Red_N related by $\gamma' = g_{N,k}\gamma$ for some $g_{N,k}$ as in (3.5) we have $\chi_{X_k} \cdot \chi_{X_{\gamma'}} \circ g_{N,k}^{-1} = \chi_{X_\gamma}$.

The commutation with the action of Hecke operators can be checked as in the case of the shift T_N in Lemma 3.2. We write elements of the algebra in the form $\sum_\alpha f_\alpha(x, s_\alpha)\delta_\alpha$ where δ_α is the characteristic function of $\mathbb{P}_{N,\alpha} = \Gamma_N\alpha G/G$ and $s_\alpha \in \mathbb{P}_{N,\alpha}$, with the action of Hecke operators as in (3.4). The action of $\gamma \in \mathrm{Red}_N$ on the other hand is given by $\alpha_\gamma \sum_\alpha f_\alpha(x, s_\alpha)\delta_\alpha = \sum_\alpha \chi_{X_\gamma}(x) f(\gamma^{-1}(x, s_\alpha))\delta_\alpha$. These actions commute, as in the case of Lemma 3.2. $\qquad\square$

3.4. **A boundary algebra .** We now introduce an algebra associated to the boundary of the bulk-system. In order to explain the reason behind our construction, consider first again the bulk space, namely the upper-half-plane $\mathbb{H}$ or $\mathbb{H} \times \mathcal{P}$ in the case where we fix a choice of a finite index subgroup $G \subset \mathrm{GL}_2(\mathbb{Z})$.

In the algebra of the $\Gamma = \mathrm{GL}_2$-system on the bulk space, we consider functions $f(g, \rho, z)$ that are invariant under the action of $\Gamma \times \Gamma$ mapping $(g, \rho, z) \mapsto (\gamma_1 g \gamma_2^{-1}, \gamma_2\rho, \gamma_2 z)$ (and similarly for the $\mathbb{H} \times \mathcal{P}$ case). This same prescription cannot be used to define a boundary algebra, since the action of $\Gamma = \mathrm{GL}_2(\mathbb{Z})$ (or $\mathrm{SL}_2(\mathbb{Z})$)

on the boundary $\mathbb{P}^1(\mathbb{R}) = \partial\mathbb{H}$ has dense orbits, hence requiring this $\Gamma \times \Gamma$-invariance would force continuous functions to be constant.

One possible way around this problem would be to replace invariance under the $\Gamma \times \Gamma$-action (in fact, invariance under the second copy of Γ, as that is the one acting on the z variable in the bulk, hence on the boundary variable in $\mathbb{P}^1(\mathbb{R})$) by taking an algebra given by a crossed product with Γ. A similar kind of boundary algebra was considered in Section 4 of [23]. Using a crossed product with Γ would imply dealing with a boundary algebra that contains a copy of $C^*(\Gamma)$. Invertible ρ's would determine, as in the GL$_2$-system, representations on the Hilbert space $\mathcal{H} = \ell^2(M_2^+(\mathbb{Z}))$ and in such representation the algebra $C^*(\Gamma)$ generates a type II_1 factor in $\mathcal{H}$. This affects the construction of KMS states for this algebra. Gibbs-type states with respect to the trace Tr_Γ can be evaluated on elements in the commutant of this factor, as discussed in Section 7 of [4]. However, here we do not make this choice in the construction of the boundary algebra, and we leave this to separate future work. This is tied up to the question mentioned in the introduction, of developing a good theory of isogeny for noncommutative tori.

The point of view we follow here on constructing a boundary algebra is based instead on a different observation, namely on the fact that the orbits of the action of $\Gamma = \mathrm{GL}_2(\mathbb{Z})$ on $\mathbb{P}^1(\mathbb{R})$ can be equivalently described as the orbits of a discrete dynamical system T acting on the interval $[0, 1]$. Thus, we will replace the crossed product by G with a semigroup crossed product that implements this equivalence relation as part of the algebra. The reason why we prefer this approach to the crossed product by G is because the dynamical system T used here is the same generalized shift of the continuous fractions expansion used in [23] to construct limiting modular symbols, and one of our main goals in this paper is obtaining a boundary algebra that is especially suited to relate to limiting modular symbols, hence this viewpoint is more natural here.

Moreover, as already discussed, this viewpoint allows us to see our boundary algebra as one case ($N = \pm 1$ for $\Gamma = \mathrm{GL}_2(\mathbb{Z})$ and $\mathrm{SL}_2(\mathbb{Z})$, respectively) of a countable family of algebras labelled by an integer N, associated to a family of different continued fraction algorithms. Considering this whole family of algebras will help us illustrate some interesting phenomena in the structure of KMS states, even though only the $N = \pm 1$ cases have a direct interpretation as boundary algebras of the respective bulk system and related to the geometry of modular curves.

Thus, in the following we first restrict the boundary variable $\theta \in \mathbb{P}^1(\mathbb{R})$ to the interval $[0, 1]$, because of the prior observation that this set meets every $\mathrm{GL}_2(\mathbb{Z})$-orbit, and then we implement the action of the shift operator T in the form of a semigroup crossed product algebra. This corresponds to taking the quotient by the action of T (hence by the action of $\mathrm{GL}_2(\mathbb{Z})$) in a noncommutative way, by considering a crossed product algebra instead of an algebra of functions constant along the orbits. This will be a semigroup crossed product with respect to the semigroup Red_N discussed above, and in a form that will implement the action of the shift operator T as in Lemma 3.3. We will also be considering bounded functions of $x \in [0, 1]$, rather than continuous

functions. The reason will be clear when we consider such boundary functions that are obtained as integration on certain configurations of geodesics in the bulk space.

Note that if we write, as before, Ξ for the set of cosets $\Gamma\alpha\Gamma$ and $\mathcal{P}$ for the set of cosets $\Gamma\alpha G$, we can identify the sets $\mathcal{P} \simeq \Xi \times \mathbb{P}$, with the finite coset space $\mathbb{P} = \Gamma/G$. It is convenient to use this identification, so that, when we consider the shift operator T (in the case $N = 1$) acting on $[0,1] \times \mathcal{P}$, this can be viewed as the action of T on $[0,1] \times \mathbb{P}$ as in [23], with T acting trivially on Ξ

Definition 3.6. *Let $\mathcal{A}^c_{\partial,N}$ denote the associative algebra of continuous complex valued functions on*

(3.13) $$\mathcal{U}_{\partial,G,N} = \{(g,\rho,s) \in \mathrm{GL}_2(\mathbb{Q}) \times M_2(\hat{\mathbb{Z}}) \times \mathcal{P}_N \mid g\rho \in M_2(\hat{\mathbb{Z}})\}$$

that are invariant with respect to the action of $\Gamma_N \times \Gamma_N$ by $(g,\rho) \mapsto (\gamma_1 g \gamma_2^{-1}, \gamma_2 \rho)$ and are finitely supported in $\mathcal{P}_N$ and in $\Gamma_N \backslash \mathrm{GL}_2(\mathbb{Q})$ with the dependence on ρ through a finite level projection $p_n(\rho) \in M_2(\mathbb{Z}/n\mathbb{Z})$, endowed with the convolution product

(3.14) $$(f_1 \star f_2)(g,\rho,s) = \sum_{h \in \mathcal{S}_{\rho,N}} f_1(gh^{-1}, h\rho, hs) f_2(h,\rho,s)$$

and with the involution $f^*(g,\rho,s) = \overline{f(g^{-1}, g\rho, gs)}$. Let $\pi_{\rho,s} : \mathcal{A}^c_{\partial,N} \to \mathcal{B}(\mathcal{H}_{\rho,N})$ be the representation $\pi_{\rho,s}(f)\xi(g) = \sum_h f(gh^{-1}, h\rho, hs)\xi(h)$ for $h \in \mathcal{S}_{\rho,N}$. Let $\mathcal{A}_{\partial,N}$ denote the C^*-algebra completion of $\mathcal{A}^c_{\partial,N}$ with respect to $\|f\| = \sup_{(\rho,s)} \|\pi_{\rho,s}(f)\|_{\mathcal{H}_{\rho,N}}$. Let $\mathcal{B}_{\partial,N} = \mathcal{B}([0,1], \mathcal{A}_{\partial,N})$ be the algebra of bounded functions from $[0,1]$ to $\mathcal{A}_{\partial,N}$, with pointwise product

$$(f_1 \star f_2)(g,\rho,x,s) = \sum_h f_1(gh^{-1}, h\rho, x, hs) f_2(h,\rho,x,s)$$

and involution $f^*(g,\rho,x,s) = \overline{f(g^{-1}, g\rho, x, gs)}$.

Definition 3.7. *Let $\mathcal{A}_{\partial,G,\mathcal{P}_N}$ be the involutive associative algebra generated by $\mathcal{B}_{\partial,N}$ and by isometries $S_{N,k}$, with $k \in \Phi_N$. It has relations $S^*_{N,k} S_{N,k} = 1$ and $\sum_k S_{N,k} S^*_{N,k} = 1$ and relations of the form*

(3.15) $$S_{N,k} f = \chi_{X_{N,k}} \cdot f \circ g_{N,k}^{-1} \cdot S_{N,k} \quad and \quad S^*_{N,k} f = f \circ g_{N,k} \cdot S^*_{N,k},$$

where $\chi_{X_{N,k}}$ is the characteristic function of the subset $X_{N,k} \subset [0,1]$ of points with N-continued fraction expansion starting with k. The matrices $g_{N,k}, g_{N,k}^{-1}$ in $\mathrm{GL}_2(\mathbb{Q})$ are as in (3.5), with $f \circ g_{N,k}^{\pm}(g,\rho,x,s) = f(g,\rho,g_{N,k}^{\pm}(x,s))$. They also satisfy the relation

(3.16) $$\sum_{k \in \mathbb{N}} S_k f S^*_k = f \circ T_N$$

for all $f \in B_{\partial,N}$. Here, for $f = f(g,\rho,x,s)$, we have

$$(f \circ T_N)(g,\rho,x,s) = f(g,\rho,T_N(x,s)),$$

with the action of T_N on $[0,1] \times \mathcal{P}_N$ as in Lemma 3.1. The involution on $\mathcal{A}_{\partial,G,\mathcal{P}_N}$ is given by the involution on $\mathcal{B}_{\partial,N}$ and by $S_{N,k} \mapsto S^*_{N,k}$.

In fact, the relation (3.16) follows from the relations (3.15) as in Lemma 3.3, but we write it explicitly as it is the relation that implements the dynamical system T_N. Note that we are implicitly using in the construction of the algebra the fact that the semigroup action of Red_N and the action of Hecke operators (that is built into the convolution product of $\mathcal{B}_{\partial,N}$) commute as in Lemma 3.5.

3.5. **Semigroup crossed product.** Several examples of semigroup crossed product algebras have been considered in relation to quantum statistical mechanical system, especially in various generalizations of the Bost–Connes system. However, there is no completely standard definition of semigroup crossed product algebra in the literature. For our purposes here, the following setting suffices.

Definition 3.8. *Let $\mathcal{A}$ be a C^*-algebra, and let $\mathcal{S}$ be a countable semigroup together with a semigroup homomorphism $\beta : \mathcal{S}^{op} \to \mathrm{End}(\mathcal{A})$. For $\ell \in \mathcal{S}$, let $\beta_\ell(1) = e_\ell$ be an idempotent in $\mathcal{A}$ and let α_ℓ denote a partial inverse of β_ℓ on $e_\ell \mathcal{A} e_\ell$. The (algebraic) semigroup crossed product algebra $\mathcal{A} \rtimes \mathcal{S}$ is the involutive $\mathbb{C}$-algebra generated by $\mathcal{A}$ and elements S_ℓ, S_ℓ^*, for all $\ell \in \mathcal{S}$ with the relations*

$$S_\ell S_{\ell'} = S_{\ell\ell'}, \quad S_\ell^* S_\ell = 1, \quad S_\ell S_\ell^* = e_\ell, \quad \sum_\ell S_\ell S_\ell^* = 1,$$

$$S_\ell \, X \, S_\ell^* = \alpha_\ell(X), \quad S_\ell^* \, X \, S_\ell = \beta_\ell(X).$$

If $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a representation as bounded operators on a Hilbert space, and the S_ℓ act as isometries on $\mathcal{H}$, compatibly with the relations above, then semigroup crossed product C^*-algebra (which will also be denoted by $\mathcal{A} \rtimes \mathcal{S}$) is the C^*-completion in $\mathcal{B}(\mathcal{H})$ of the above algebraic crossed product.

Lemma 3.9. *The algebra $\mathcal{A}_{\partial,G,\mathcal{P}_N}$ can be identified with the semigroup crossed product $\mathcal{B}_{\partial,N} \rtimes \mathrm{Red}_N$ of the algebra $\mathcal{B}_{\partial,N}$ of Definition 3.6 and the semigroup of reducible matrices, with respect to the action $\alpha : \mathrm{Red}_N \to \mathrm{Aut}(B_{\partial,N})$ by $\alpha_\gamma(f) = \chi_{X_\gamma} \cdot f \circ \gamma^{-1}$, where for γ of the form (3.10), the set $X_\gamma \subset [0,1]$ is the cylinder set consisting of points with N-continued fraction expansion starting with the sequence $n_1, \ldots, n_N$.*

Proof. We see as in Lemma 3.5 that $\alpha_\gamma(f) = \chi_{X_\gamma} \cdot f \circ \gamma^{-1}$ defines a semigroup action of Red_N on $\mathcal{B}_{\partial,N}$. The semigroup crossed product algebra is generated by $B_{\partial,N}$ and by isometries μ_γ for $\gamma \in \mathrm{Red}_N$ satisfying $\mu_\gamma \mu_{\gamma'} = \mu_{\gamma\gamma'}$ for all $\gamma, \gamma' \in \mathrm{Red}_N$, $\mu_\gamma^* \mu_\gamma = 1$ for all $\gamma \in \mathrm{Red}_N$ and $\mu_\gamma f \mu_\gamma^* = \alpha_\gamma(f)$. It suffices to consider isometries $\mu_{g_{N,k}} =: S_{N,k}$ associated to the elements $g_{N,k} \in \mathrm{Red}_N$ as in (3.5) with $\mu_\gamma = S_{n_1} \cdots S_{n_N}$ for $\gamma \in \mathrm{Red}_N$ as in (3.10). We then see that the generators and relations of the algebras $B_{\partial,N} \rtimes \mathrm{Red}_N$ agree with those of the algebra $\mathcal{A}_{\partial,G,\mathcal{P}_N}$ of Definition 3.7. $\square$

Corollary 3.10. *Let $E \subset [0,1] \times \mathcal{P}_N$ be a subset invariant under the action of the shift T_N of the N-continued fraction expansion. Then E determines an algebra $\mathcal{A}_{\partial,G,\mathcal{P}_N,E}$ given by $B_{\partial,N,E} \rtimes \mathrm{Red}_N$ where $B_{\partial,N,E} = \mathcal{B}(E, \mathcal{A}_{\partial,N})$ is the algebra of bounded functions from E to $\mathcal{A}_{\partial,N}$ as in Definition 3.6. Extending the functions by zero outside of E realizes $\mathcal{A}_{\partial,G,\mathcal{P}_N,E}$ as a subalgebra of $\mathcal{A}_{\partial,G,\mathcal{P}_N}$.*

Proof. If the set E is T_N-invariant then the algebra $B_{\partial,N,E}$ is invariant under the action of the semigroup Red_N by $\alpha_\gamma(f) = \chi_{X_\gamma} \cdot f \circ \gamma^{-1}$, since for $\gamma = g_{N,k_1} \cdots g_{N,k_n}$, the matrix γ^{-1} acts on X_γ as the shift T_N^n. Thus, we can form the semigroup crossed product algebra $B_{\partial,N,E} \rtimes \mathrm{Red}_N$ as in Lemma 3.9. Since $B_{\partial,N,E}$ and $B_{\partial,N}$ are algebras of bounded functions that are not required to be continuous, extending the functions to zero outside of E realizes $B_{\partial,N,E}$ as a subalgebra of $B_{\partial,N}$ compatibly with the action of the semigroup Red_N. $\qquad\square$

3.6. **Representations and time evolution.** Let $\mathcal{H}_{\rho,N}$ be the same Hilbert space considered above, $\mathcal{H}_{\rho,N} = \ell^2(\mathcal{S}_{\rho,N})$ with $\mathcal{S}_{\rho,N}$ the collection of Γ_N-cosets that have some representative element in the set $\{g \in \mathrm{GL}_2(\mathbb{Q}) | g\rho \in M_2(\hat{\mathbb{Z}})\}$. We will consider the case of an invertible $\rho \in \mathrm{GL}_2(\hat{\mathbb{Z}})$. This choice is made to guarantee non-negative spectrum of the Hamiltonian of Proposition 3.12 and is also geometrically motivated by the $\mathrm{GL}_2(\mathbb{Z})$ (i.e. $N = 1$) setting as discussed in Section 3.4. When $N = 1$ and $\rho \in \mathrm{GL}_2(\hat{\mathbb{Z}})$ is invertible we can write

$$\mathcal{S}_{\rho,1} = \mathrm{GL}_2(\mathbb{Z})\backslash\{g \in \mathrm{GL}_2(\mathbb{Q}) | g\rho \in M_2(\hat{\mathbb{Z}})\} = \mathrm{GL}_2(\mathbb{Z})\backslash M_2^\times(\mathbb{Z}),$$

with $M_2^\times(\mathbb{Z}) = \{M \in M_2(\mathbb{Z}) \,|\, \det(M) \neq 0\}$.

Let $\mathcal{W}_N = \cup_n \mathcal{W}_{N,n}$ denote the set of all finite sequences $k_1, \ldots, k_n$ with $k_i \in \Phi_N$, including an element $\emptyset$ corresponding to the empty sequence. Consider the Hilbert spaces $\ell^2(\mathcal{W}_N)$ and $\tilde{\mathcal{H}}_{N,\rho} = \ell^2(\mathcal{W}_N) \otimes \mathcal{H}_{N,\rho}$.

Lemma 3.11. *The algebra $\mathcal{A}_{\partial,G,\mathcal{P}_N} = \mathcal{B}_{\partial,N} \rtimes \mathrm{Red}_N$ acts on the Hilbert space $\tilde{\mathcal{H}}_{N,\rho}$ through the representations*

$$(3.17) \qquad \pi_{\rho,x,s}(f)\,(\xi(g) \otimes \epsilon_{k_1,\ldots,k_n}) = \sum_{h \in \mathcal{S}_{\rho,N}} f(gh^{-1}, h\rho, g_\gamma(x, hs))\,\xi(h) \otimes \epsilon_{k_1,\ldots,k_n}$$

for $f \in B_{\partial,N}$ and with $g_\gamma = g_{N,k_1} \cdots g_{N,k_n}$, with g_{N,k_i} as in (3.5), and

$$\pi_{\rho,x,s}(S_{N,k})\,(\xi \otimes \epsilon_{k_1,\ldots,k_n}) = \xi \otimes \epsilon_{k,k_1,\ldots,k_n},$$

$$(3.18) \qquad \pi_{\rho,x,s}(S_{N,k}^*)(\xi \otimes \epsilon_{k_1,\ldots,k_n}) = \begin{cases} \xi \otimes \epsilon_{k_2,\ldots,k_n} & k_1 = k \\ 0 & otherwise. \end{cases}$$

In what follows we sometimes write $\pi_{\rho,x,s}(S_{N,k})$ as $S_{N,k}$ because the mapping of these operators does not depend on the choice of (ρ, x, s).

Proof. We check that (3.17) gives a representation of the subalgebra $B_{\partial,N}$ and that the operators $\pi_{\rho,x,s}(f)$, $S_{N,k}$, $S_{N,k}^*$ of (3.17) and (3.18) satisfy the relations $S_{N,k}^* S_{N,k} = 1$, $\sum_k S_{N,k} S_{N,k}^* = 1$, $S_{N,k} \pi_{\rho,x,s}(f) = \pi_{\rho,x,s}(\chi_{X_{N,k}} f \circ g_{N,k}^{-1})S_{N,k}$ and $S_{N,k}^* \pi_{\rho,x,s}(f) = \pi_{\rho,x,s}(f \circ g_{N,k})S_{N,k}^*$. For the first property it suffices to see that $\pi_{\rho,x,s}(f_1 \star f_2) = \pi_{\rho,x,s}(f_1) \circ \pi_{\rho,x,s}(f_2)$. We have

$$\pi_{\rho,x,s}(f_1 \star f_2)\,(\xi(g) \otimes \epsilon_{k_1,\ldots,k_n}) = \sum_{h \in \mathcal{S}_{\rho,N}} (f_1 \star f_2)(gh^{-1}, h\rho, g_\gamma(x, hs))\,\xi(h) \otimes \epsilon_{k_1,\ldots,k_n} =$$

$$\sum_{h \in \mathcal{S}_{\rho,N}} \sum_{\ell \in \mathcal{S}_{\rho,N}} f_1(gh^{-1}\ell^{-1}, \ell h\rho, g_\gamma(x, \ell h s)) f_2(\ell, h\rho, g_\gamma(x, h s)) \xi(h) \otimes \epsilon_{k_1,\dots,k_n},$$

where we used Lemma 3.5. This is then equal to

$$\sum_{\ell \in \mathcal{S}_{\rho,N}} f_1(gh^{-1}\ell^{-1}, \ell h\rho, g_\gamma(x, \ell h s))(\pi_{\rho,x,s}(f_2)\xi)(\ell) \otimes \epsilon_{k_1,\dots,k_n} = \pi_{\rho,x,s}(f_1)\pi_{\rho,x,s}(f_2)\xi \otimes \epsilon_{k_1,\dots,k_n}.$$

The relations $S^*_{N,k}S_{N,k} = 1$ and $\sum_k S_{N,k}S^*_{N,k} = 1$ follow directly from (3.18). For relations between the $S_{N,k}, S^*_{N,k}$ and the $\pi_{\rho,x,s}(f)$, we have

$$S_{N,k}\pi_{\rho,x,s}(f)\,\xi \otimes \epsilon_{k_1,\dots,k_n} = S_{N,k} \sum_{h \in \mathcal{S}_{\rho,N}} f(gh^{-1}, h\rho, g_k^{-1}g_k g_\gamma(x, h s))\,\xi(h) \otimes \epsilon_{k_1,\dots,k_n}$$

$$= \sum_{h \in \mathcal{S}_{\rho,N}} f(gh^{-1}, h\rho, g_{N,k}^{-1}g_{N,k}g_\gamma(x, h s))\,\chi_{X_{N,k}}(g_{N,k}g_\gamma x)\,\xi(h) \otimes \epsilon_{k,k_1,\dots,k_n},$$

for $g_\gamma = g_{N,k_1} \cdots g_{N,k_n}$, with $\chi_{X_{N,k}}(g_{N,k}g_\gamma x) = 1$, so we get

$$\pi_{\rho,x,s}(\chi_{X_{N,k}} \cdot f \circ g_{N,k}^{-1})\,S_{N,k}\,\xi \otimes \epsilon_{k,k_1,\dots,k_n}.$$

The second relation is similar: we have

$$S^*_{N,k}\pi_{\rho,x,s}(f)\,\xi \otimes \epsilon_{k_1,\dots,k_n} = S^*_{N,k} \sum_{h \in \mathcal{S}_{\rho,N}} f(gh^{-1}, h\rho, g_k^{-1}g_k g_\gamma(x, h s))\,\xi(h) \otimes \epsilon_{k_1,\dots,k_n}$$

$$= \sum_{h \in \mathcal{S}_{\rho,N}} f(gh^{-1}, h\rho, g_{N,k}g_{\gamma'}(x, h s))\,S^*_{N,k}\xi(h) \otimes \epsilon_{k_1,\dots,k_n},$$

with $g_{\gamma'} = g_{N,k_2} \cdots g_{N,k_n}$, so that we obtain

$$\pi_{\rho,x,s}(f \circ g_{N,k})\,S^*_{N,k}\,\xi \otimes \epsilon_{k_1,\dots,k_n}.$$

Thus, (3.17) and (3.18) determine a representation of $\mathcal{A}_{\partial,G,\mathcal{P}_N} = \mathcal{B}_{\partial,N} \rtimes \mathrm{Red}_N$ by bounded operators on the Hilbert space $\widetilde{\mathcal{H}}_{\rho,N}$. $\square$

Proposition 3.12. *The transformations $\sigma_{N,t}(f)(g, \rho, x, s) = |\det(g)|^{it} f(g, \rho, x, s)$ and $\sigma_{N,t}(S_{N,k}) = k^{it}S_{N,k}$ define a time evolution $\sigma_N : \mathbb{R} \to \mathrm{Aut}(\mathcal{A}_{\partial,G,\mathcal{P}_N})$. In the representations of Lemma 3.11 on $\widetilde{\mathcal{H}}_{\rho,N}$ with $\rho \in \mathrm{GL}_2(\hat{\mathbb{Z}})$ the time evolution is implemented by the Hamiltonian*

$$(3.19) \qquad H_N\,\xi(g) \otimes \epsilon_{k_1,\dots,k_n} = \log(|\det(g)| \cdot k_1 \cdots k_n)\,\xi(g) \otimes \epsilon_{k_1,\dots,k_n},$$

with partition function
(3.20)

$$Z_N(\beta) = \mathrm{Tr}(e^{-\beta H_N}) = \begin{cases} \dfrac{\zeta(\beta)\zeta(\beta-1) \prod_{p\,prime\,:\,p|N}\left(1 - p^{-\beta}\right)\left(1 - p^{-(\beta-1)}\right)}{1 + \sum_{n=1}^{N-1} n^{-\beta} - \zeta(\beta)} & \text{if } N > 1 \\[3ex] \dfrac{\zeta(\beta)\zeta(\beta-1) \prod_{p\,prime\,:\,p|N}\left(1 - p^{-\beta}\right)\left(1 - p^{-(\beta-1)}\right)}{1 + \sum_{n=1}^{|N|} n^{-\beta} - \zeta(\beta)} & \text{if } N \leq -1 \end{cases}$$

with $\zeta(\beta)$ the Riemann zeta function. In the $N = 1$ case there is no partition function.

Proof. We have

$$\sigma_{N,t}(f_1 \star f_2)(g, \rho, x, s) = \sigma_{N,t}(\sum_h f_1(gh^{-1}, h\rho, x, hs) f_2(h, \rho, x, s)) =$$

$$\sum_h |\det(gh)^{-1}|^{it} |\det(h)|^{it} f_1(gh^{-1}, h\rho, x, hs) f_2(h, \rho, x, s) = \sigma_{N,t}(f_1) \star \sigma_t(f_2).$$

We also have $\sigma_{N,t}(S_{N,k}^*) = k^{-it} S_{N,k}^*$ and we see that the action of $\sigma_{N,t}$ is compatible with the relations in $\mathcal{A}_{\partial,G,\mathcal{P}_N}$ and defines a 1-parameter family of algebra homomorphisms. By direct comparison between (3.19) and (3.17) and (3.18) we also see that

$$\pi_{\rho,x,s}(\sigma_{N,t}(f)) = e^{itH} \pi_{\rho,x,s}(f) e^{-itH} \quad \text{and} \quad \sigma_{N,t}(S_{N,k}) = e^{itH} S_{N,k} e^{-itH}.$$

We have

$$Z_N(\beta) = \sum_{g \in \mathcal{S}_{\rho,N}} |\det(g)|^{-\beta} \cdot \sum_{k=k_1 \cdots k_n : k_i \in \Phi_N} k^{-\beta}$$

where Φ_N is the set of possible digits in the N-continued fraction expansion.

For the first sum, we begin by considering the $N = 1$ case. We now have that $\mathcal{S}_\rho = \mathrm{GL}_2(\mathbb{Z}) \backslash M_2^\times(\mathbb{Z})$, where $M_2^\times(\mathbb{Z}) = \{M \in M_2(\mathbb{Z}) \,|\, \det(M) \neq 0\}$. Thus, we are counting $\{M \in M_2^\times(\mathbb{Z}) \,|\, |\det(M)| = n\}$ modulo $\mathrm{GL}_2(\mathbb{Z})$. Up to a change of basis in $\mathrm{GL}_2(\mathbb{Z})$ we can always write a sublattice of $\mathbb{Z}^2$ in the form

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mathbb{Z}^2$$

with $a, d \geq 1$ and $0 \leq b < d$, [30]. Thus, we are equivalently counting such matrices with determinant n. This counting is given by $\sigma(n) = \sum_{d|n} d$ so the first sum is $\sum_{n \geq 1} \sigma(n) \, n^{-\beta} = \zeta(\beta)\zeta(\beta - 1)$ as in the original GL_2-system, and converges on $\beta \in (2, \infty)$.

In the general case, we again consider $\rho \in \mathrm{GL}_2(\hat{\mathbb{Z}})$, and we now have that $\mathcal{S}_{\rho,N}$ is the set of matrices in $M_2^\times(\mathbb{Z})$ with determinant not divisible by any prime factor of N, up to the equivalence relation defined by $\mathrm{GL}_2(\mathbb{Z})$. The first sum is then given by

$$\sum_{g \in \mathcal{S}_{\rho,N}} |\det(g)|^{-\beta} = \sum_{n \geq 1 : (N,n)=1} \sigma(n) n^{-\beta}$$

$$= \left(\sum_{n \geq 1 : (N,n)=1} n^{-\beta} \right) \left(\sum_{n \geq 1 : (N,n)=1} n^{-(\beta-1)} \right)$$

$$= \zeta(\beta)\zeta(\beta - 1) \prod_{p \, prime \, : \, p|N} \left(1 - p^{-\beta}\right) \left(1 - p^{-(\beta-1)}\right)$$

where the counting $\sigma(n) = \sum_{d|n} d$ is identical to the $N = 1$ case. Again, this series converges on $\beta \in (2, \infty)$.

To compute the second sum, let $P_{N,n}$ denote the total number of ordered factorizations of n into positive integer factors in α_N. In the $N \geq 1$ case, the sum we are

considering is

$$\sum_{n \geq 1} P_{N,n}\, n^{-\beta} = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} n^{-\beta} \sum_{n=n_1 \cdots n_k : n_i \geq N} 1$$

$$= \sum_{k \geq 1} \prod_{i=1}^{k} \left(\sum_{n_i \geq N} n_i^{-\beta} \right)$$

$$= \begin{cases} \displaystyle\sum_{k=1}^{\infty} (\zeta(\beta))^k = \frac{1}{1 - \zeta(\beta)} & \text{if } N = 1 \\[2em] \displaystyle\sum_{k=1}^{\infty} (\zeta(\beta) - \sum_{n=1}^{N-1} n^{-\beta})^k = \frac{1}{1 + \sum_{n=1}^{N-1} n^{-\beta} - \zeta(\beta)} & \text{if } N > 1. \end{cases}$$

In the $N = 1$ case, note that the series $\sum_{k=1}^{\infty} (\zeta(\beta))^k$ converges when $|\zeta(\beta)| < 1$. However, when $\beta > 1$, $\zeta(\beta) > 1$ and the series does not converge there. The first series $\sum_{n \geq 1} \sigma_1(n)\, n^{-\beta} = \zeta(\beta)\zeta(\beta - 1)$ converges for $\beta > 2$. Since the second series does not converge anywhere in the region $(2, \infty)$, there is no partition function.

In the $N > 1$ case, the relevant series converges when

$$\left| \zeta(\beta) - \sum_{n=1}^{N-1} n^{-\beta} \right| = |\zeta(\beta) - (1 + \xi(\beta))| < 1$$

where $\xi(\beta) = 0$ when $N = 2$ and $\xi(\beta) = \sum_{n=2}^{N-1} n^{-\beta}$ when $N > 2$. In the range $\beta \in (1, \infty)$ the ζ-function is decreasing to 1 and it crosses the value $\zeta(\beta) = 2$ at a point $\beta_{2,c} \sim 1.728647$. When $N = 2$, the series converges on $(\beta_{2,c}, \infty)$. When $N > 2$ we consider the function $\zeta(\beta) - \xi(\beta)$ where $\xi(\beta)$ consists of a finite sum of terms of the form $n^{-\beta}$ each of which are continuous, decreasing to 0 as $\beta \to \infty$ and have some finite value at $\beta = 1$. Since $\lim_{\beta \to \infty} \zeta(\beta) - \xi(\beta) = 1$ and $\lim_{\beta \to 1^+} \zeta(\beta) - \xi(\beta) = \infty$, there will be some point $\beta_{N,c} > 1$ at which $\zeta(\beta_{N,c}) - \xi(\beta_{N,c}) = 2$. The corresponding series then converges on $(\beta_{N,c}, \infty)$. Since each $n^{-\beta}$ term is decreasing, we also know that $\beta_{N+1,c} < \beta_{N,c}$.

Similarly, in the $N \leq -1$ case we have

$$\sum_{n \geq 1} P_{N,n}\, n^{-\beta} = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} n^{-\beta} \sum_{n=n_1 \cdots n_k : n_i \geq |N|+1} 1$$

$$= \sum_{k \geq 1} \prod_{i=1}^{k} \left(\sum_{n_i \geq |N|+1} n_i^{-\beta} \right)$$

$$= \sum_{k=1}^{\infty} (\zeta(\beta) - \sum_{n=1}^{|N|} n^{-\beta})^k = \frac{1}{1 + \sum_{n=1}^{|N|} n^{-\beta} - \zeta(\beta)}.$$

As before, this series converges on $(\beta_{N,c}, \infty)$ where for $N \leq -1$, $\beta_{N,c} = \beta_{1-N,c}$. In particular, $\beta_{-1,c} = \beta_{2,c} \sim 1.728647$. $\qquad \square$

Lemma 3.13. *For $N \geq 2$ and $N \leq -1$, the partition function $Z_N(\beta)$ of Proposition 3.12 is defined by an absolutely convergent series*

$$Z_N(\beta) = \mathrm{Tr}(e^{-\beta H_N}) = \sum_{\lambda \in \mathrm{Spec}(H_N)} e^{-\beta \lambda}$$

for $\beta > 2$. Its analytic continuation (3.20) has poles at $\beta \in \{1, \beta_{N,c}, 2\}$, for a point $1 < \beta_{N,c} < 2$. In the geometrically relevant case of $N = -1$, $\beta_{-1,c} \sim 1.728647$.

Proof. As argued in the proof of 3.12, the denominator of $Z_N(\beta)$ has a single zero at a point $1 < \beta_{N,c} < 2$. The Riemann zeta function $\zeta(\beta)$ has a pole at $\beta = 1$. Therefore, the sum $\sum_{n \geq 1\,:\,(N,n)=1} \sigma_1(n) n^{-\beta}$ is convergent for $\beta > 2$ and its analytic continuation $\zeta(\beta)\zeta(\beta - 1) \prod_{p\,prime\,:\,p|N} \left(1 - p^{-\beta}\right) \left(1 - p^{-(\beta-1)}\right)$ has poles at $\beta = 2$ and $\beta = 1$.   □

3.7. **KMS states.** We classify the KMS states for the family of dynamical systems $(\mathcal{A}_{\partial,G,\mathcal{P}_N}, \sigma_{N,t})$. Since we have $\mathcal{A}_{\partial,G,\mathcal{P}_N} = \mathcal{B}_{\partial,N} \rtimes \mathrm{Red}_N$, we consider the KMS states for the modified GL_2 part of the system ($\mathcal{B}_{\partial,N}$), and the part of the system generated by the isometries $S_{N,k}$ (Red_N), which is a Cuntz-Krieger-Toeplitz type algebra, separately. The KMS states of the Cuntz-Krieger-Toeplitz type algebras have been studied by [10], and we draw on their main results. We show that in the $N = 1$ case, corresponding the the standard GL_2 system, there are no KMS states at any temperature, though we may still define a ground state. In all other cases, the system has two critical temperatures at $\beta = \beta_{N,c} < 2$ and $\beta = 2$. When $\beta < \beta_{N,c}$ there are no KMS_β states. When $\beta_{N,c} < \beta < 2$, the structure of the KMS_β states will be identical to the structure on the modified GL_2 part of the system alone, though we have not computed this explicitly. When $\beta > 2$, the KMS_β states are given by Gibbs states, whose limit as $\beta \to \infty$ gives the ground states.

Lemma 3.14. *The subalgebra of $\mathcal{A}_{\partial,G,\mathcal{P}_N}$ generated by the family $\{S_{N,k}\}_{k \in \alpha_N}$ of isometries is a Cuntz-Krieger-Toeplitz algebra, denoted by $\mathcal{O}_A$ in the setting of [10] with the infinite matrix $A = \{A(x,y)\}_{x,y \in \alpha_N}$ given by $A(x,y) = 1$ for all $x, y \in \Phi_N$. In other words, it satisfies the following properties on its initial and final projections $q_k = S_{N,k}^* S_{N,k}$ and $p_k = S_{N,k} S_{N,k}^*$. For all $k, j \in \Phi_N$:*

  (1) $q_k q_j = q_j q_k$,
  (2) $S_{N,k}^* S_{N,j} = 0$ if $j \neq k$,
  (3) $q_k S_{N,j} = S_{N,j}$,
  (4) *and* $\prod_{k \in X} q_k \prod_{j \in Y} (1 - q_j) = 0$ *for X, Y finite subsets of Φ_N.*

Proof. Conditions (1), (3), and (4) follow from the fact that $S_{N,k}^* S_{N,k} = 1$ (Lemma 3.3). Condition (2) is easily verified. If $k \neq j$ then

$$S_{N,k}^* S_{N,j} \xi(x) = S_{N,k}^* (\chi_{X_{N,j}}(x) \xi(g_{N,j}^{-1} x))$$
$$= \chi_{X_{N,j}}(g_{N,k} x) \xi(g_{N,k} g_{N,j}^{-1} x) = 0.$$

□

Proposition 3.15. *The KMS_β states of the dynamical system $(\mathcal{A}_{\partial, G, \mathcal{P}_N}, \sigma_{N,t})$ can be characterized as follows. When $N = 1$, there are no KMS_β states for any temperature β. When $N \leq -1$ or $N > 1$, the system has a critical temperature $\beta_{N,c} \in (1,2)$. We then have the following.*

(1) *When $\beta < \beta_{N,c}$ there are no KMS_β states.*
(2) *When $\beta > \beta_{N,c}$, there is one KMS_β state for every KMS_β state of the modified GL_2 system corresponding to $\mathcal{B}_{\partial,N}$.*
(3) *In particular, when $\beta > 2$, the KMS_β states correspond to a unique KMS_β on the Red_N part of the system paired with one of a set of Gibbs states on the $\mathcal{B}_{\partial,N}$ part of the system, parameterized by (ρ, x, s) with $\rho \in M_2(\hat{\mathbb{Z}})$ invertible.*

Proof. If there is a KMS_β state on $(\mathcal{A}_{\partial, G, \mathcal{P}_N}, \sigma_{N,t})$, it must restrict to a KMS_β state on the subalgebra of $\mathcal{A}_{\partial, G, \mathcal{P}_N}$ generated by the family of isometries $\{S_{N,k}\}_{k \in \alpha_N}$. We will first characterize the KMS_β states of this subalgebra, which we have established in Lemma 3.14 is a Cuntz-Krieger-Toeplitz algebra. We also note that since Φ_N is a countable set and in our case the matrix A is simply a matrix with every entry set to 1, Standing Hypothesis 8.1 (i), and (ii) of [10] are satisfied. The dynamics $\sigma_{N,t}$ on the subalgebra take the form $\sigma_{N,t}(S_{N,k}) = k^{it} S_{N,k}$ for each $k \in \Phi_N$, and since Φ_N is a set of real numbers in the interval $(1, \infty)$ the rest of Standing Hypothesis 8.1 of [10] is also satisfied.

We now draw on the main results of [10]. Corollary 9.7 states that there is a critical temperature $\dot{\beta}_{N,c}$ above which there is a single KMS_β state at each temperature β. Theorem 14.5 states that there is a second critical temperature $\ddot{\beta}_{N,c}$ below which there are no KMS_β states at all. These critical temperatures are defined as follows. Let Ω_N be the set of words in Φ_N and $\Omega_{N,j,k}$ be the set of words in Φ_N beginning with j and ending with k. Then $\dot{\beta}_{N,c}$ and $\ddot{\beta}_{N,c}$ are the abscissas of convergence of the series

$$Z(\beta) = \sum_{\mu \in \Omega_N} (\mu)^{-\beta} \quad \text{and} \quad Z_{jk}(\beta) = \sum_{\mu \in \Omega_{N,j,k}} (\mu)^{-\beta}$$

respectively. Note that the abscissa of convergence of the second series is independent of the choice of j and k. We will now calculate these critical temperatures.

The partition function $Z(\beta)$ has already been calculated in the second part of Proposition 3.12 and is given by

$$Z(\beta) = \begin{cases} \displaystyle\sum_{k=1}^{\infty} (\zeta(\beta))^k & \text{if } N = 1 \\[2.5em] \displaystyle\sum_{k=1}^{\infty} (\zeta(\beta) - \sum_{n=1}^{N-1} n^{-\beta})^k & \text{if } N > 1 \\[2.5em] \displaystyle\sum_{k=1}^{\infty} (\zeta(\beta) - \sum_{n=1}^{|N|} n^{-\beta})^k & \text{if } N \leq -1. \end{cases}$$

Modifying this calculation slightly we find that

$$Z_{jk}(\beta) = (jk)^{-\beta} + \sum_{n=1}^{\infty} \sum_{\mu \in \Omega_N : |\mu|=n} j^{-\beta}\mu^{-\beta}k^{-\beta}$$

$$= (jk)^{-\beta}\left(1 + \sum_{n=1}^{\infty}\prod_{i=1}^{n}\sum_{k_i \in \alpha_N} k_i^{-\beta}\right) = (jk)^{-\beta}Z(\beta).$$

Clearly $Z(\beta)$ and $Z_{jk}(\beta)$ have the same abscissa of convergence, $\ddot{\beta}_{N,c} = \dot{\beta}_{N,c}$. In fact in the $N \neq 1$ case, this abscissa of convergence is $\beta_{N,c}$ of 3.12. When $N = 1$, neither series converges for any value of β, $\dot{\beta}_{N,c} = \ddot{\beta}_{N,c} = \infty$. Hence there are no KMS$_\beta$ states for any finite inverse temperature β.

Now we focus our attention on the subalgebra corresponding to $\mathcal{B}_{\partial,N}$ in the $N \neq 1$ case. In the range $\beta > 2$, $e^{-\beta H_N}$ is trace class, by Lemma 3.13. We therefore have Gibbs states of the form, for $X \in \mathcal{A}_{\partial,G,\mathcal{P}_N}$

$$\varphi_{\beta,N}(X) = \frac{\text{Tr}(\pi_{\rho,x,s}(X)e^{-\beta H_N})}{\text{Tr}(e^{-\beta H_N})}$$

$$= Z_N(\beta)^{-1}\sum_{k_i \in \alpha_N, g \in \mathcal{S}_{\rho,N}} |\det(g)|^{-\beta}(k_1\cdots k_n)^{-\beta}\langle \delta_g \otimes \epsilon_{k_1,\ldots,k_n}, \pi_{\rho,x,s}(X)\delta_g \otimes \epsilon_{k_1,\ldots,k_n}\rangle.$$

depending on our choice of representation $\pi_{\rho,x,s}$.

We now restrict to the subalgebra $\mathcal{B}_{\partial,N}$. The Gibbs states above give, for $f \in \mathcal{B}_{\partial,N}$

$$\varphi_{\beta,\rho,x,s}(f) = \frac{\text{Tr}(\pi_{\rho,x,s}(f)e^{-\beta H_N})}{\text{Tr}(e^{-\beta H_N})}$$

$$= Z(\beta)^{-1}\sum_{k_i \in \alpha_N, g \in \mathcal{S}_{\rho,N}} |\det(g)|^{-\beta}(k_1\cdots k_n)^{-\beta}f(1, g\rho, g_\gamma(x, gs))$$

where $\gamma \in \text{Red}_N$ is determined by $k_1, \ldots, k_n$. These are parameterized by the choice of $\rho \in M_2(\hat{\mathbb{Z}})$ invertible. $\square$

Remark 3.16. *The standard* GL$_2$ *system (when* $\Gamma = \text{SL}_2(\mathbb{Z})$*) has been studied in* [17]. *Their analysis of the behaviour when* $\beta \in (1,2)$ *is not directly applicable in our case, because for* $N = 1$ *(*$\Gamma = \text{SL}_2(\mathbb{Z})$*), the Red$_N$ part of the system has no KMS states at any temperature* β. *However, it would be interesting to see whether a similar analysis can be applied when* $\Gamma = \Gamma_N$ *for some* $N > 1$ *or* $N \leq -1$.

As in [4] we consider the ground states at zero temperature as the weak limit of the Gibbs states for $\beta \to \infty$

$$\varphi_{\infty,\rho,x,s}(f) = \lim_{\beta \to \infty}\varphi_{\beta,\rho,x,s}(f).$$

Corollary 3.17. *When* $N \neq 1$, *the ground states are given by*

$$\varphi_{\infty,\rho,x,s}(f) = f(1, \rho, x, s).$$

Proof. Observe that whenever $N \neq 1$, $\lim_{\beta \to \infty} Z_N(\beta) = 1$. Furthermore, the only $g \in \mathcal{S}_{\rho,N}$ with $|\det(g)| = 1$ is the identity element, and hence the only terms for which $\lim_{\beta \to \infty} |\det(g)|^{-\beta}$ does not vanish are those for which $g = 1$. A word in α_N satisfies $k_1 ... k_n \geq |N|^n$ if $N > 1$ and $k_1 ... k_n \geq (|N| + 1)^n$ if $N \leq 1$. Hence $\lim_{\beta \to \infty} |k_1 ... k_n|^{-\beta}$ vanishes unless $k_1 ... k_n$ is the empty word. We have that the ground states are

$$(3.21) \qquad \varphi_{\infty,\rho,x,s}(f) = \langle \delta_1 \otimes \epsilon_\emptyset, \pi_{\rho,x,s}(f) \, \delta_1 \otimes \epsilon_\emptyset \rangle = f(1, \rho, x, s),$$

the evaluation of the function f at the point $g = 1$ and (ρ, x, s) that determines the representation $\pi_{\rho,x,s}$. $\qquad \square$

Although in the $N = 1$ case there are no low-temperature KMS states, and hence the weak limit does not exist, we can still define the projection onto the kernel of the Hamiltonian as in 3.21. This satisfies the weak KMS condition in the sense that function

$$F(t) = \varphi_{\infty,\rho,x,s}(f \sigma_t(f')) = f(1, \rho, x, s) f'(1, \rho, x, s)$$

has a bounded holomorphic extension to the upper half plane.

4. Averaging on geodesics and boundary values

In this section we construct boundary values of the observables of the GL_2-system of §2. We use the theory of limiting modular symbols of [23]. We show that the resulting boundary values localize nontrivially at the quadratic irrationalities and at the level sets of the multifractal decomposition considered in [14]. We discuss in particular the case of quadratic irrationalities.

4.1. **Geodesics between cusps.** Let $C_{\alpha,\epsilon,s}$ with $\alpha \in \mathbb{P}^1(\mathbb{Q}) \smallsetminus \{0\}$, $\epsilon \in \{\pm\}$, and $s \in \mathbb{P}$ denote the geodesic in $\mathbb{H}^\epsilon \times \mathbb{P}$ with endpoints at the cusps $(0, s)$ and (α, s) in $\mathbb{P}^1(\mathbb{Q}) \times \mathbb{P}$.

Let $p_k(\alpha), q_k(\alpha)$ be the successive numerators and denominators of the $\mathrm{GL}_2(\mathbb{Z})$-continued fraction expansion of $\alpha \in \mathbb{Q}$ with $p_n(\alpha)/q_n(\alpha) = \alpha$ and let

$$g_k(\alpha) := \begin{pmatrix} p_{k-1}(\alpha) & p_k(\alpha) \\ q_{k-1}(\alpha) & q_k(\alpha) \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}).$$

We denote by $C_{\alpha,\epsilon,s}^k$ the geodesic in $\mathbb{H}^\epsilon \times \mathbb{P}$ with endpoints at the cusps

$$\frac{p_{k-1}(\alpha)}{q_{k-1}(\alpha)} = g_k^{-1}(\alpha) \cdot 0 \quad \text{and} \quad \frac{p_k(\alpha)}{q_k(\alpha)} = g_k^{-1}(\alpha) \cdot \infty,$$

where $g \cdot z$ for $g \in \mathrm{GL}_2(\mathbb{Z})$ and $z \in \mathbb{H}^\pm$ is the action by fractional linear transformations.

For C a geodesic in $\mathbb{H}^\pm$ let ds_C denote the geodesic length element. In the case of $C_{\infty,\epsilon,s}$ we have $ds_{C_{\infty,\epsilon,s}}(z) = dz/z$.

We use the notation $\{g \cdot 0, g \cdot \alpha\}_G$ to denote the homology class determined by the image in the quotient $X_G = G \backslash \mathbb{H} = \mathrm{GL}_2(\mathbb{Z}) \backslash (\mathbb{H}^\pm \times \mathbb{P})$ of the geodesic $C_{\alpha,\epsilon,s}$, for g a representative of $s \in \mathbb{P}$. Similarly we write $\{\alpha, \beta\}_G$ for homology classes determined

by the images in the quotient X_G of geodesics in $\mathbb{H}^{\pm} \times \mathbb{P}$ with endpoints α, β at cusps in $\mathbb{P}^1(\mathbb{Q}) \times \mathbb{P}$.

4.2. Limiting Modular Symbols.

We recall briefly the construction of limiting modular symbols from [23]. We consider here some finite index subgroup $G \subset \Gamma$ of $\Gamma = \mathrm{PGL}_2(\mathbb{Z})$. We denote by $\mathbb{P} = \Gamma/G$ the finite coset space of this subgroup. We also write the quotient modular curve as $X_G = G\backslash\mathbb{H} = \Gamma\backslash(\mathbb{H} \times \mathbb{P})$.

Recall that the classical modular symbols $\{\alpha, \beta\}_G$, with $\alpha, \beta \in \mathbb{P}^1(\mathbb{Q})$ are defined as the homology classes in $H_1(X_G, \mathbb{R})$ defined as functionals that integrate lifts to $\mathbb{H}$ of differentials on X_G along the geodesic arc in $\mathbb{H}$ connecting α and β, see [21]. They satisfy additivity and invariance: for all $\alpha, \beta, \gamma \in \mathbb{P}^1(\mathbb{Q})$

$$\{\alpha, \beta\}_G + \{\beta, \gamma\}_G = \{\alpha, \gamma\}_G \quad \text{and} \quad \{g\alpha, g\beta\}_G = \{\alpha, \beta\}_G \ \ \forall g \in G.$$

Thus, it suffices to consider the modular symbols of the form $\{0, \alpha\}_G$ with $\alpha \in \mathbb{Q}$. These satisfy the relation

$$\{0, \alpha\}_G = -\sum_{k=1}^{n} \{\frac{p_{k-1}(\alpha)}{q_{k-1}(\alpha)}, \frac{p_k(\alpha)}{q_k(\alpha)}\}_G = -\sum_{k=1}^{n} \{g_k^{-1}(\alpha) \cdot 0, g_k^{-1}(\alpha) \cdot \infty\}_G,$$

where $p_k(\alpha), q_k(\alpha)$ are the successive numerators and denominators of the $\mathrm{GL}_2(\mathbb{Z})$-continued fraction expansion of $\alpha \in \mathbb{Q}$ with $p_n(\alpha)/q_n(\alpha) = \alpha$ and

$$g_k(\alpha) = \begin{pmatrix} p_{k-1}(\alpha) & p_k(\alpha) \\ q_{k-1}(\alpha) & q_k(\alpha) \end{pmatrix}.$$

(There is an analogous formula for the $\mathrm{SL}_2(\mathbb{Z})$-continued fraction.)

Limiting modular symbols were introduced in [23], to account for the noncommutative compactification of the modular curves X_G by the boundary $\mathbb{P}^1(\mathbb{R})$ with the G action. One considers the infinite geodesics $L_\theta = \{\infty, \theta\}$ given by the vertical lines $L_\theta = \{z \in \mathbb{H} \,|\, \Re(z) = \theta\}$ oriented from the point at infinity to the point θ on the real line. Upon choosing an arbitrary base point $x \in L_\beta$ let $x(s)$ denote the point on L_β at an arc-length distance s from x in the orientation direction. One considers the homology class $\{x, x(s)\}_G \in H_1(X_G, \mathbb{R})$ determined by the geodesic arc between x and $x(s)$. The limiting modular symbol is defined as the limit (when it exists)

$$(4.1) \qquad \{\{\star, \theta\}\}_G := \lim_{s \to \infty} \frac{1}{s}\{x, x(s)\}_G \in H_1(X_G, \mathbb{R}).$$

It was shown in [23] that the limit (4.1) exists on a full measure set and can be computed by an ergodic average. More generally, it was shown in [26] that there is a multifractal decomposition of the real line by level sets of the Lyapunov exponent of the shift of the continued fraction expansion plus an exceptional set where the limit does not exist. On the level sets of the Lyapunov exponent the limit is again given by an average of modular symbols associated to the successive terms of the continued

fraction expansion. More precisely, as in the previous section, let $T : [0,1] \to [0,1]$ denote the shift map of the continued fraction expansion,

$$T(x) = \frac{1}{x} - \left[\frac{1}{x}\right],$$

extended to a map $T : [0,1] \times \mathbb{P} \to [0,1] \times \mathbb{P}$ with $\mathbb{P} = \Gamma/G$. The Lyapunov exponent of the shift map is given by the limit (when it exists)

$$(4.2) \qquad \lambda(x) = \lim_{n\to\infty} \frac{1}{n} \log |(T^n)'(x)| = 2 \lim_{n\to\infty} \frac{1}{n} \log q_n(x),$$

where $q_n(x)$ are the successive denominators of the continued fraction expansion of $x \in [0,1]$. There is a decomposition $[0,1] = \cup_\lambda \mathcal{L}_\lambda \cup \mathcal{L}'$ where $\mathcal{L}_\lambda = \{x \in [0,1] \,|\, \lambda(x) = \lambda\}$ and $\mathcal{L}'$ the set on which the limit (4.2) does not exist. For all $\theta \in \mathcal{L}_\lambda$ the limiting modular symbol (4.1) is then given by
(4.3)

$$\{\{\star,\theta\}\}_G = \lim_{n\to\infty} \frac{1}{\lambda n} \sum_{k=1}^{n} \{g_k^{-1}(\theta) \cdot 0, g_k^{-1}(\theta) \cdot \infty\}_G = \lim_{n\to\infty} \frac{1}{\lambda n} \sum_{k=1}^{n} \{\frac{p_{k-1}(\theta)}{q_{k-1}(\theta)}, \frac{p_k(\theta)}{q_k(\theta)}\}_G.$$

The results of [23] and [26] show that the limiting modular symbol (4.3) vanishes almost everywhere, with respect to the Hausdorff measure of $\mathcal{L}_\lambda$. However, non-vanishing values of the limiting modular symbols are obtained, for example, for all the quadratic irrationalities.

In the case of quadratic irrationalities, one obtains two equivalent descriptions of the limiting modular symbol, one that corresponds to integration on the closed geodesic $C_\theta = \Gamma_\theta \backslash S_\theta$ with S_θ the infinite geodesic with endpoints the Galois conjugate pair θ, θ' and the other in terms of averaged integration on the modular symbols associated to the (eventually periodic) continued fraction expansion. We obtain the identification of homology classes in $H_1(X_G, \mathbb{R})$

$$(4.4) \qquad \{\{\star,\theta\}\}_G = \frac{\sum_{k=1}^{\ell} \{\frac{p_{k-1}(\theta)}{q_{k-1}(\theta)}, \frac{p_k(\theta)}{q_k(\theta)}\}_G}{\lambda(\theta)\ell} = \frac{\{0, g \cdot 0\}_G}{\ell(g)} \in H_1(X_G, \mathbb{R}).$$

In the first expression ℓ is the minimal positive integer for which $T^\ell(\theta) = \theta$ and the limit defining the Lyapunov exponent $\lambda(\theta)$ exists for quadratic irrationalities. In the second expression $g \in \Gamma$ is the hyperbolic generator of Γ_θ with fixed points θ, θ', with eigenvalues $\Lambda_g^\pm$ and $\{0, g \cdot 0\}_G$ denotes the homology class in $H_1(X_G, \mathbb{R})$ of the closed geodesic C_θ and $\ell(g) = \log \Lambda_g^- = 2 \log \epsilon$ is the length of C_θ.

A more complete analysis of the values of the limiting modular symbols was then carried out in [14], where it was shown that, in fact, the limiting modular symbols is non-vanishing on a multifractal stratification of Cantor sets of positive Hausdorff dimension. We will recall more precisely this result in §4.3 below.

The construction recalled above of limiting modular symbols determine non-trivial real homology classes in the quotient X_G associated to geodesics in $\mathbb{H}$ with endpoints in one of the multifractal level sets of [14]. These homology classes pair with 1-forms on X_G, and in particular with weight 2 cusp forms for the finite index subgroup G.

Let $\mathcal{M}_{G,k}$ the $\mathbb{C}$-vector space of modular forms of weight k for the finite index subgroup $G \subset \mathrm{GL}_2(\mathbb{Z})$ and let $\mathcal{S}_{G,k}$ be the subspace of cusp forms. Let $X_G = \mathrm{GL}_2(\mathbb{Z})\backslash(\mathbb{H}^{\pm} \times \mathbb{P})$ be the associated modular curve. We denote by

$$(4.5) \qquad \langle \cdot, \cdot \rangle : \mathcal{S}_{G,2} \times H_1(X_G, \mathbb{R}) \to \mathbb{C}$$

the perfect pairing between cusp forms of weight two and modular symbols., which we equivalently write as integration

$$(4.6) \qquad \langle \psi, \{\alpha, \beta\}_G \rangle = \int_{\{\alpha,\beta\}_G} \psi(z)\, dz.$$

4.3. Boundary values.

We consider now a linear map, constructed using cusp forms and limiting modular symbols, that assigns to an observable of the bulk GL_2-system a boundary value.

Let $\mathcal{L} \subset [0,1]$ denote the subset of points such that the Lyapunov exponent (4.2) of the shift of the continued fraction exists. The set $\mathcal{L}$ is stratified by level sets $\mathcal{L}_\lambda = \{x \in [0,1] \,|\, \lambda(x) = \lambda\}$, with the Lyapunov spectrum given by the Hausdorff dimension function $\delta(\lambda) = \dim_H(\mathcal{L}_\lambda)$. Recall also that, for a continuous function ϕ on a T-invariant subset $E \subset [0,1]$, the Birkhoff spectrum (see [11]) is the function

$$(4.7) \qquad f_E(\alpha) := \dim_H \mathcal{L}_{\phi,E,\alpha},$$

where $\mathcal{L}_{\phi,E,\alpha}$ are the level sets of the Birkhoff average by

$$(4.8) \qquad \mathcal{L}_{\phi,E,\alpha} := \{x \in E \,|\, \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi \circ T^k(x) = \alpha\}.$$

In particular, $\mathcal{L}_\lambda = \mathcal{L}_{\phi,\lambda}$ for $\phi(x) = \log|T'(x)|$. Lyapunov and Birkhoff spectra for the shift of the continued fraction expansion are analyzed in [29], [12].

In a similar way, one can consider the multifractal spectrum associated to the level sets of the limiting modular symbol, as analyzed in [14]. Let $f_1, \ldots, f_g$ be a basis of the complex vectors space $\mathcal{S}_{G,2}$ of cusp forms of weight 2 for the finite index subgroup $G \subset \mathrm{GL}_2(\mathbb{Z})$. Let $\Re(f_i)$, $\Im(f_i)$ be the corresponding basis as a real $2g$-dimensional vector space. Under the pairing (4.5), (4.6), which identifies $\mathcal{S}_{G,2}$ with the dual of $H_1(X_G, \mathbb{R})$, we can define as in [14] the level sets E_α, for $\alpha \in \mathbb{R}^{2g}$, of the limiting modular symbol as

$$(4.9) \qquad E_\alpha := \{(x,s) \in [0,1] \times \mathbb{P} \,|\, (\langle f_i, \{\{\star, x\}\}_G \rangle = \alpha \in \mathbb{R}^{2g}\}.$$

Equivalently, we write this as

$$E_\alpha = \{(x,s) \in [0,1] \times \mathbb{P} \,|\, (\lim_{n\to\infty} \frac{1}{\lambda(x)n} \int_{\{g_k^{-1}(x)\cdot 0, g_k^{-1}(x)\cdot\infty\}_G} f_i(z)\, dz)_{i=1\ldots,g} = \alpha \in \mathbb{R}^{2g}\}.$$

For $(x,s) \in E_\alpha$ we have

$$\lim_{n\to\infty} \frac{1}{\lambda(x)n} \{g_k^{-1}(x)\cdot 0, g_k^{-1}(x)\cdot\infty\}_G = h_\alpha \in H_1(X_G, \mathbb{R}),$$

where the homology class h_α is uniquely determined by the property that $\langle f_i, h_\alpha \rangle = \int_{h_\alpha} f_i(z)dz = \alpha_i$.

The main result of [14] shows that for a given finite index subgroup $G \subset \mathrm{GL}_2(\mathbb{Z})$ with X_G of genus $g \geq 1$, there is a strictly convex and differentiable function $\beta_G : \mathbb{R}^{2g} \to \mathbb{R}$ such that, for all $\alpha \in \nabla \beta_G(\mathbb{R}^{2g}) \subset \mathbb{R}^{2g}$

$$(4.10) \qquad\qquad \dim_H(E_\alpha) = \hat{\beta}_G(\alpha),$$

where $\hat{\beta}_G(\alpha) = \inf_{v \in \mathbb{R}^{2g}}(\beta_G(v) - \langle \alpha, v \rangle)$ is the Legendre transform, and for all $(x, s) \in E_\alpha$

$$(4.11) \qquad\qquad \lim_{n \to \infty} \frac{1}{\lambda(x)n}\{g_k^{-1}(x) \cdot g \cdot 0, g_k^{-1}(x) \cdot g \cdot \infty\}_G = h_\alpha(x, s)$$

with g a representative of the class $s \in \mathbb{P} = \mathrm{GL}_2(\mathbb{Z})/G$.

We let $\mathcal{A}_{\partial,G,\mathcal{P},h_\alpha} = B_{\partial,E_\alpha} \rtimes \mathrm{Red}$ be the subalgebra of $\mathcal{A}_{\partial,G,\mathcal{P}} = B_\partial \rtimes \mathrm{Red}$ associated to the invariant set E_α, as in Corollary 3.10. As an immediate consequence of the results (4.10), (4.11) of [14] we have the following.

Lemma 4.1. *The choice of a cusp form $\psi \in \mathcal{S}_{G,2}$ determines a bounded linear operator $\mathcal{I}_{\psi,\alpha}$ from $\mathcal{A}_{\mathrm{GL}_2(\mathbb{Z}),G,\mathcal{P}}$ to B_{∂,E_α} with for $(x,s) \in E_\alpha$*

$$
\begin{aligned}
\mathcal{I}_{\psi,\alpha}(f)(g, \rho, x, s) &= \int_{\{\star, x\}_G} f(g, \rho, z, s)\, \psi(z)\, dz \\
(4.12) \qquad &= \lim_{n \to \infty} \frac{1}{\lambda(x)\, n} \sum_{k=1}^n \int_{\{g_k^{-1}(x) \cdot 0, g_k^{-1}(x) \cdot \infty\}_G} f(g, \rho, z, s)\, \psi(z)\, dz \\
&= \int_{h_\alpha(x)} f(g, \rho, z, s)\, \psi(z)\, dz.
\end{aligned}
$$

Here we pair the form $\omega(z) = f(z)\psi(z)dz$ with the limiting modular symbol $h_\alpha(x, s)$ of (4.11). We use the notation $\omega(z) = \omega_{\rho,s}(z)$ to highlight the dependence on the variables (ρ, s) that come from choosing an element f in the arithmetic algebra $\mathcal{A}^{ar}_{\mathrm{GL}_2(\mathbb{Z}),G,\mathcal{P}}$.

Note that $\mathcal{I}_{\psi,\alpha}$ is only a linear operator and *not* an algebra homomorphism. We obtain a subalgebra of $\mathcal{A}_{\partial,G,\mathcal{P}}$ as follows.

Definition 4.2. *Let $\mathcal{A}_{\mathcal{I},G,\mathcal{P}}$ denote the subalgebra of $\mathcal{A}_{\partial,G,\mathcal{P}} = B_\partial \rtimes \mathrm{Red}$ generated by all the images $\mathcal{I}_{\psi,\alpha}(f)$ for $f \in \mathcal{A}_{\mathrm{GL}_2(\mathbb{Z}),G,\mathcal{P}}$, for $\psi \in \mathcal{S}_{G,2}$, and for $\alpha \in \nabla \beta_G(\mathbb{R}^{2g})$, and by the S_k, S_k^* with the relations as in Definition 3.7. The arithmetic algebra $\mathcal{A}^{ar}_{\mathcal{I},G,\mathcal{P}}$ is obtained in the same way as the algebra generated by the images $\mathcal{I}_{\psi,\alpha}(f)$ with f in the arithmetic algebra $\mathcal{A}^{ar}_{\mathrm{GL}_2(\mathbb{Z}),G,\mathcal{P}}$ of §2.2, for all $\psi \in \mathcal{S}_{G,2}$ and $\alpha \in \nabla \beta_G(\mathbb{R}^{2g})$, and by the S_k, S_k^* as above.*

4.4. **Evaluation of ground states on boundary values.** When we evaluate zero-temperature KMS states on the elements $\mathcal{I}_{\psi,\alpha}(f)$, for an element $f \in \mathcal{A}_{\mathrm{GL}_2(\mathbb{Z}),G,\mathcal{P}}^{ar}$, we obtain the pairing of a cusp form with a limiting modular symbol,

$$(4.13) \qquad \varphi_{\infty,\rho,x,s}(\mathcal{I}_{\psi,\alpha}(f)) = \langle \omega_{\rho,s}, h_\alpha(x) \rangle,$$

where $\omega_{\rho,s}(z) = f(1,\rho,z,s)\psi(z)dz$ is a cusp form in $\mathcal{S}_{G,2}$ for all (ρ,s). Since elements $f \in \mathcal{A}_{\mathrm{GL}_2(\mathbb{Z}),G,\mathcal{P}}^{ar}$ depend on the variable $\rho \in M_2(\hat{\mathbb{Z}})$ through some finite projection $\pi_N(\rho) \in \mathbb{Z}/N\mathbb{Z}$, we can write $\omega_{\rho,s}(z)$ as a finite collection $\{\omega_{i,s}(z)\}_{i \in \mathbb{Z}/N\mathbb{Z}}$.

To illustrate the properties of the values of ground states on arithmetic boundary elements, we consider here the particular case where $G = \Gamma_0(N)$ and a state $\varphi_{\infty,\rho,x,s}$ with $s \in \mathbb{P}$. We also choose f and ψ so that the resulting $\omega_{i,s}$ are cusp forms for $\Gamma_0(N)$ that are Hecke eigenforms for all the Hecke operators $T(m)$.

Recall (see [30]) that the Hecke operators $T(m)$ given by

$$T_m = \sum_{\gamma \,:\, \det(\gamma)=m} \Gamma_0(N)\gamma\Gamma_0(N)$$

satisfying $T_n T_m = T_m T_n$ for $(m,n) = 1$ and $T_{p^n} T_p = T_{p^{n+1}} + p T_{p^{n-1}} R_p$ with R_λ the scaling operator that acts on a modular form of weight $2k$ as multiplication by λ^{-2k}. The Hecke operators T_m and the scaling operators R_λ generate a commutative algebra, and the action of T_m on a modular form of weight $2k$ is given by

$$T_m f(z) = n^{2k-1} \sum_{a \geq 1, ad=n, 0 \leq b < d} d^{-2k} f\left(\frac{az+b}{d}\right).$$

Proposition 4.3. *Let $G = \Gamma_0(N)$ and let ω_{i,s_0}, with $i = 0,\ldots,N-1$ and $s_0 = \Gamma_0(N)g_0 \in \mathbb{P}$, be Hecke eigencuspforms of weight 2. For $s \in \mathbb{P}$ with $s = \Gamma_0(N)g_0\gamma$, with $\gamma \in \mathrm{GL}_2(\mathbb{Z})$, let $\omega_{i,s} = \omega_{i,s_0\gamma} := \omega_{i,s_0} \circ \gamma^{-1}$. Consider the pairing*

$$\xi_\omega(s) = \langle \omega_{\rho,s}, h_\alpha(x) \rangle = \int_{h_\alpha(x)} \omega_{i,s}$$

with the limiting modular symbol $h_\alpha(x)$, as in (4.12) and (4.13). For $(m,N) = 1$ we have the relations

$$a_{i,m}\,\xi_\omega(s) = \sum_{M \in \bar{A}_m} u_m\,\xi_\omega(s\,M)$$

where $a_{i,m}$ are the Hecke eigenvalues and $A_m = \{M \in M_2(\mathbb{Z}) : \det(M) = m\}$, with $\bar{A}_m = A_m/\{\pm 1\}$ and $\sum_M u_M M \in \mathbb{Z}\bar{A}_m$ is the Manin–Heilbronn lift of the Hecke operator T_m.

Proof. The condition that $\omega_{i,s\gamma} = \omega_{i,s}\circ\gamma^{-1}$ implies that $\langle \omega_{i,s}, h_\alpha(x) \rangle = \langle \omega_{i,s_0}, h_\alpha(x,s) \rangle$. The following facts are known from [21], [27]. Let

$$A_{m,N} = \{M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) : \det(M) = m, \ N|c\}.$$

Let R be a set of representatives for the classes $\Gamma_0(N)\backslash A_{m,N}$ The Hecke operators act on the modular symbols by $T_m\{\alpha,\beta\} = \sum_{\lambda \in R}\{\lambda\alpha, \lambda\beta\}$. For $(m,N) = 1$ there

is a bijection between the cosets $\Gamma_0(N)\backslash A_{m,N}$ and $A_m/\mathrm{SL}_2(\mathbb{Z})$. For $s \in \mathbb{P}$ consider the assignment $\xi_\omega : s \mapsto \xi_\omega(s) = \langle \omega_{i,s}, h_\alpha(x,s) \rangle$, where $\omega_{i,s}$ is a Hecke eigencuspform. It is shown in [21], [27] that there is a lift Θ_m of the action of the Hecke operators $T_m \circ \xi = \xi \circ \Theta_m$ (the Manin–Heilbronn lift), which is given by $\Theta_m = \sum_{\gamma \in A_m/\mathrm{SL}_2(\mathbb{Z})} \Upsilon_\gamma$, where Υ_γ is a formal chain of level m connecting ∞ to 0 and of class γ. This means that $\Upsilon_\gamma = \sum_{k=0}^{n-1} \gamma_k$ in $\mathbb{Z}A_m$, for some $n \in \mathbb{N}$ where

$$\gamma_k = \begin{pmatrix} u_k & u_{k+1} \\ v_k & v_{k+1} \end{pmatrix}$$

with $u_0/v_0 = \infty$ and $u_n/v_n = 0$ and where γ_k agrees with γ in $A_m/\mathrm{SL}_2(\mathbb{Z})$. An argument in [27] based on the continued fraction expansion and modular symbols shows that it is always possible to construct such formal chains with $\gamma_k = \gamma g_k$ with $g_k \in \mathrm{SL}_2(\mathbb{Z})$ and that the resulting Θ_m is indeed a lift of the Hecke operators. (The length n of the chain of the Manin–Heilbronn lift is also computed, see §3.2 of [27].) Using the notation of Theorem 4 of [27], we write the Manin–Heilbronn lift as $\Theta_m = \sum_{M \in \bar{A}_m} u_M M$ as an element of $\mathbb{Z}A_m$. Each element $M \in \bar{A}_m$ maps $s \mapsto s\,M$ in $\mathcal{P}$, hence one obtains a map $\Theta_m : \mathbb{Z}\mathcal{P} \to \mathbb{Z}\mathcal{P}$. In particular, as shown in Theorem 2 of [27], for $s = \Gamma_0(N)g$ in $\mathbb{P}$ one has $\Theta_m(s) = \sum_{\gamma \in R} \sum_{k=0}^{n-1} \phi(g\gamma\gamma_k)$ where $\phi : A_m \to \Gamma_0(N)\backslash\mathrm{SL}_2(\mathbb{Z})$ is the map that assigns

$$A_m \ni \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \Gamma_0(N) \begin{pmatrix} w & t \\ u & v \end{pmatrix} \in \Gamma_0(N)\backslash\mathrm{SL}_2(\mathbb{Z})$$

with $(c : d) = (u : v)$ in $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z}) \simeq \mathbb{P} = \Gamma_0(N)\backslash\mathrm{SL}_2(\mathbb{Z})$. Thus, as in Theorem 2 of [27] we then have $\Theta_m(s) = \sum_{\gamma \in R} \sum_{k=0}^{n-1} \phi(g\gamma)g_k = \sum_{\gamma \in R} \phi(g\gamma)\gamma^{-1}$, seen here as an element in $\mathbb{Z}\mathcal{P}$. Thus, in the pairing of limiting modular symbols and boundary elements we find

$$T_m \xi_\omega(s) = T_m \int_{h_\alpha(x,s)} \omega_{i,s_0} = \int_{h_\alpha(x,s)} T_m \omega_{i,s_0} = a_{i,m} \int_{h_\alpha(x,s)} \omega_{i,s_0},$$

where $a_{i,m}$ are the Hecke eigenvalues of the eigenform ω_{i,s_0}, with $a_{i,1} = 1$. On the other hand, using the Manin–Heilbronn lift we have

$$T_m \xi_\omega(s) = \xi_\omega(\Theta_m(s)) = \sum_{\gamma \in R} \int_{h_\alpha(x,s_\gamma)} \omega_{i,s_0}$$

with $s_\gamma \in \mathcal{P}$ given by $s_\gamma = \phi(g\gamma)\gamma^{-1}$ as above. We write the latter expression in the form

$$\sum_{M \in \bar{A}_m} u_M \int_{h_\alpha(x,s\,M)} \omega_{i,s_0}$$

for consistency with the notation of Theorem 4 of [27]. □

Proposition 4.4. *Under the same hypothesis as Proposition 4.3, let $L_{\omega_{i,s_0}}(\sigma) = \sum_m a_{i,m} m^{-\sigma}$ be the L-series associated to the cusp form $\omega_{i,s_0} = \sum_m a_{i,m} q^m$. For x*

a quadratic irrationality the evaluation (4.13) *of KMS$_\infty$ states satisfies*

$$(4.14) \qquad \langle \omega_{i,s}, h(x) \rangle = \frac{1}{\lambda(x)n} \sum_{k=1}^n \langle \omega_{i,s_k}, \{0, \infty\} \rangle = \frac{1}{\lambda(x)n} \sum_{k=1}^n L_{\omega_{i,s_k}}(1),$$

where $s_k = \Gamma_0(N)gg_k^{-1}(x)$ for $s = \Gamma_0(N)g$.

Proof. The special value $L_{\omega_{i,s_0}}(1)$ of the L-function gives the pairing with the modular symbol $\langle \omega_{i,s_0}, \{0, \infty\} \rangle$. Similarly, for $s \in \mathbb{P}$ with $s = \Gamma_0(N)g$, the special value gives

$$L_{\omega_{i,s}}(1) = \langle \omega_{i,s}, \{0, \infty\} \rangle = \langle \omega_{i,s_0}, \{g \cdot 0, g \cdot \infty\} \rangle.$$

In the case of a quadratic irrationality the limiting modular symbol satisfies

$$h(x) = \frac{1}{\lambda(x)n} \sum_{k=1}^n \{g_k^{-1}(x) \cdot 0, g_k^{-1}(x) \cdot \infty\}_G,$$

where n is the length of the period of the continued fraction expansion of x and $\lambda(x)$ is the Lyapunov exponent. For $s_k = \Gamma_0(N)gg_k^{-1}(x)$ with $s = \Gamma_0(N)g$, we have $\langle \omega_{i,s_k}, \{0, \infty\} \rangle = \langle \omega_{i,s}, \{g_k^{-1}(x) \cdot 0, g_k^{-1}(x) \cdot \infty\} \rangle$ hence one obtains (4.14). $\qquad \square$

As shown in Theorem 3.3 of [21], the special value $L_{\omega_{i,s_0}}(1)$ satisfies

$$\Big(\sum_{d|m} d - a_{i,m}\Big) L_{\omega_{i,s_0}}(1) = \sum_{d|m, \, b \bmod d} \int_{\{0, b/d\}_G} \omega_{i,s_0},$$

since one has

$$\int_0^\infty T_m \omega_{i,s_0} = a_{i,m} \int_0^\infty \omega_{i,s_0} = \sum_{d|m} \sum_{b=0}^{d-1} \int_{\{b/d, 0\}_G + \{0, \infty\}_G} \omega_{i,s_0}.$$

For a normalized Hecke eigencuspform $f = \sum_n a_n q^n$, let $L_f(s) = \sum_n a_n n^{-s}$ be the associated L-function and $\Lambda_f(s) = (2\pi)^{-s} \Gamma(s) L_f(s)$ the completed L-function, the Mellin transform $\Lambda_f(s) = \int_0^\infty f(iz) z^{s-1} dz$.

The relation between special values of L-functions and periods of Hecke eigenforms generalizes for higher weights, and it was shown in [22] that ratios of these special values of the same parity are algebraic (in the field generated over $\mathbb{Q}$ by the Hecke eigenvalues). For a normalized Hecke eigencuspform $f = \sum_n a_n q^n$ of weight k the coefficients of the period polynomial $r_f(z)$ are expressible in terms of special values of the L-function,

$$r_f(z) = -i \sum_{j=0}^{k-2} \binom{k-2}{j} (iz)^j \Lambda_f(j+1).$$

Manin's Periods Theorem shows that, for $\mathbb{K}_f$ the field of algebraic numbers generated over $\mathbb{Q}$ by the Fourier coefficients, there are $\omega_\pm(f) \in \mathbb{R}$ such that for all $1 \le s \le k-1$ with s even $\Lambda_f(s)/\omega_+(f) \in \mathbb{K}_f$, respectively $\Lambda_f(s)/\omega_-(f) \in \mathbb{K}_f$ for s odd.

Shokurov gave a geometric argument based on Kuga varieties and a higher weight generalization of modular symbols, [31]. It is expected that the limiting modular

symbols of [23], as well as the quantum statistical mechanics of the GL_2-system and its boundary described here, will generalize to the case of Kuga varieties, with the relations between periods of Hecke eigencuspforms described in [22] arising in the evaluation of zero-temperature KMS states of these systems.

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