Some Remarks Concerning the Real and Imaginary Parts of the Characteristic Roots of a Finite Matrix

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Some theorems are obtained on the existence of certain determinantal equations whose roots are separately the real or imaginary parts of the characteristic roots of a given matrix with simple elementary divisors. When the elementary divisors are not simple, similar, but somewhat less precise, results are obtained.

THE purpose of this paper is to generalize some previously known theorems on the real parts of the characteristic roots of a finite real matrix (cf. footnote references 2 and 5), to refine their proofs by presenting them from a unified point of view, and to show that the same methods yield similar results concerning the imaginary parts of the roots. Also, analogous results for complex matrices are indicated.

In the sequel we use the following notation: If X is a matrix, $\bar{X}$ is the conjugate complex of X, X' is the transpose of X, and $X^*=\bar{X}'$.

THEOREM 1

Given any (not necessarily real) matrix A of order n with simple elementary divisors and with characteristic roots, $\alpha_1, \cdots, \alpha_n$. Then it is always possible to find a positive definite Hermitian matrix G (of order n) such that the roots $\lambda_1, \cdots, \lambda_n$ of the equation $\det(\sigma GA+\bar{\sigma}A^*G-2\lambda G)=0$ are the real parts of $\sigma\alpha_1, \cdots, \sigma\alpha_n$, where σ is any complex number and $\bar{\sigma}$ is its conjugate complex. Moreover, if A is real, G may be chosen so as to be real also. In any case, G is independent of σ.

Proof

Since the matrix A is similar to a diagonal matrix, we have

$$S^{-1}AS=\text{diagonal } (\alpha_1,\cdots,\alpha_n)$$

for some suitably chosen nonsingular matrix S. Hence

$$S^{-1}\sigma AS=\text{diagonal } (\sigma\alpha_1,\cdots,\sigma\alpha_n),$$
$$S^*\bar{\sigma}A^*S^{*-1}=\text{diagonal } (\bar{\sigma}\bar{\alpha}_1,\cdots,\bar{\sigma}\bar{\alpha}_n).$$

Hence $\frac{1}{2}(S^{-1}\sigma AS+S^*\bar{\sigma}A^*S^{*-1})$ has as characteristic roots the real parts of the $\sigma\alpha_i$. This latter matrix is equal to

$$\tfrac{1}{2}S^*(\sigma S^{*-1}S^{-1}ASS^*+\bar{\sigma}A^*)S^{*-1},$$

which is similar to $\frac{1}{2}(\sigma GAG^{-1}+\bar{\sigma}A^*)$, if we put $SS^*=G^{-1}$. But the statement that $\frac{1}{2}(\sigma GAG^{-1}+\bar{\sigma}A^*)$ has the real parts of the $\sigma\alpha_i$ as its characteristic roots is equivalent to the statement to be proved about the roots of the above-mentioned determinantal equation in λ. The fact that $G=S^{*-1}S^{-1}$ is positive definite, Hermitian, and independent of σ is obvious. To show that G can be chosen to be real when A is real, we observe that S has as its columns the characteristic vectors of A. These characteristic vectors can be chosen real for real characteristic roots and conjugate complex for a pair of conjugate complex roots. It follows that $\bar{S}=SE$ where E is a permutation matrix which is a direct sum of a unit matrix and matrices of the form

$$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

Hence E is symmetric. We then have $S^*=ES'$. Hence $SS^*=SES'=\bar{S}S'$ and $(SS^*)'=S\bar{S}'=SS^*$. Since $G^{-1}=SS^*$ is Hermitian and symmetric, it must be real.[1]

Corollary 1.1

Given any real matrix A with simple elementary divisors, it is always possible to find a positive definite symmetric matrix G (of the same order as A) such that the roots $\lambda_1, \cdots, \lambda_n$ of the equation $\det(B-2\lambda G)=0$, where $B=GA+A'G$, are the real parts of the characteristic roots of A.

This is deduced immediately from Theorem 1 by taking $\sigma=1$.

This result was first stated and proved by Lewis[2] with the unnecessary restriction that A be nonsingular. The proof there given was based on concepts of tensor analysis instead of the purely matrix methods of the present paper.

Corollary 1.2

The positive definite symmetric matrix G of Corollary 1.1 may be chosen so as not only to satisfy the conditions of that corollary but so that simultaneously the roots of the equation $\det(C-2\lambda G)=0$, where $C=GA-A'G$, are the

[1] An alternative proof would have been to replace every pair of complex conjugate elements in diagonal $(\alpha_1,\cdots,\alpha_n)$ by the real matrix

$$\begin{pmatrix}R\alpha & I\alpha\\ -I\alpha & R\alpha\end{pmatrix}.$$

[2] D. C. Lewis, Am. J. Math. 73, 48 (1951).

imaginary parts, multiplied by $\sqrt{-1}$, of the characteristic roots of A.

This is deduced from Theorem 1 by taking $\sigma=\sqrt{-1}$, with emphasis on the fact that the G of Theorem 1 is independent of σ.

Corollary 1.3

The matrix B of Corollary 1.1 has the same signature as the real parts of the characteristic roots of A.

This follows from the fact that the signature of the roots of $\det(B-2\lambda G)=0$ is the same as the signature of the characteristic roots of BG^{-1}, which latter matrix is similar to $G^{-\frac{1}{2}}BG^{-\frac{1}{2}}$, which has the same signature as B.

Remark

Corollary 1.1 (and hence Theorem 1) ceases to be true if the hypothesis about the simple elementary divisors is omitted. As an example take

$$A=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}.$$

If

$$G=\begin{pmatrix}a & b\\ b & c\end{pmatrix},$$

which is the most general symmetric matrix of order two, we find that

$$B=GA+A'G=\begin{pmatrix}2a & a+2b\\ a+2b & 2b+2c\end{pmatrix}.$$

We now find easily enough that $\det(GA+A'G-2\lambda G)=0$ has a root $\lambda=1$ if and only if

$$\begin{vmatrix}0 & a\\ a & 2b\end{vmatrix}=0, \quad \text{or} \quad a=0.$$

Hence G cannot be positive definite, as required by the Corollary.

We are, however, able to state the following theorem in which A is not required to have simple elementary divisors.

THEOREM 2

Given any (not necessarily real) matrix A of order n with characteristic roots, $\alpha_1, \cdots, \alpha_n$. Given also a positive number ϵ. Then it is always possible to find a positive definite Hermitian matrix G (of order n) such that the roots $\lambda_1, \cdots, \lambda_n$ of the equation $\det(\sigma GA+\bar{\sigma}A^*G-2\lambda G)=0$ are real and differ from the real parts of $\sigma\alpha_1, \cdots, \sigma\alpha_n$ by not more than $|\sigma|\epsilon$, where σ is any complex number. Moreover, if A is real, G may be chosen so as to be real also. In any case, G, though dependent on A and ϵ, is independent of σ.

The proof is a modification of the proof of Theorem 1. Let S be a matrix which transforms A into a modified Jordan canonical form J_ϵ:

$$S^{-1}AS=J_\epsilon,$$

where J_ϵ consists of a diagonal containing the characteristic roots of A and a superdiagonal containing 0's and ϵ's. All other elements of J_ϵ are 0. As in Theorem 1, if we set $G=(SS^*)^{-1}$ we find that the matrix $\frac{1}{2}(G\sigma AG^{-1}+\bar{\sigma}A^*)$ is similar to a Hermitian matrix K whose elements are all 0, except for the diagonal, which contains the real parts of the numbers $\sigma\alpha_1, \cdots, \sigma\alpha_n$, and for some of the elements in the super- and subdiagonal which are equal to $\sigma\epsilon/2$ and $\bar{\sigma}\epsilon/2$, respectively. Since K splits up into blocks with equal elements on the diagonal, it follows that any characteristic root of K must differ from some one of the real parts of the numbers $\sigma\alpha_1, \cdots, \sigma\alpha_n$ by a quantity x which satisfies the equation

$$\begin{vmatrix} x & \sigma\epsilon/2 & 0 & 0 & \cdots \\ \bar{\sigma}\epsilon/2 & x & \sigma\epsilon/2 & 0 & \cdots \\ 0 & \bar{\sigma}\epsilon/2 & x & \sigma\epsilon/2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{vmatrix}=0,$$

where the determinant on the left contains a suitable number of rows. It can be proved that $|x|\leqq|\sigma|\epsilon$. This follows from the Gersgorin-Brauer theorem.[3]

Furthermore, if A is real, S can be chosen so that SS^* is real. This follows in a way analogous to the corresponding part of the proof of Theorem 1, only here the matrix is built up from the so-called "principal vectors" which correspond to the characteristic roots. These have been studied by Wielandt.[4]

Corollary 2.1

Given any real matrix A and a positive number ϵ, it is always possible to find a positive definite symmetric matrix G such that the roots $\lambda_1, \cdots, \lambda_n$ of the equation $\det(B-2\lambda G)=0$, where $B=GA+A'G$, differ from the real parts of the characteristic roots of A by not more than ϵ.

This follows from Theorem 2 by taking $\sigma=1$.

Corollary 2.2

The positive definite symmetric matrix G of Corollary 2.1 may be chosen so as not only to satisfy the conditions of that corollary but so that simultaneously the roots of the equation $\det(C-2\lambda G)=0$, where $C=GA-A'G$, are pure imaginary and differ from the imaginary parts, multiplied by $\sqrt{-1}$, of the characteristic roots of A by quantities which in absolute value do not exceed ϵ.

This is deduced from Theorem 2 by taking $\sigma=\sqrt{-1}$, with emphasis on the fact that the G of Theorem 2 is independent of σ.

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[3] O. Taussky, Am. Math. Monthly **56**, 672 (1949).

[4] R. Zurmühl, *Matrizen* (Springer–Verlag, Berlin, 1950), pp. 211–226.

Corollary 2.3

If the real parts of the characteristic roots of A are all different from zero, the matrix B of Corollary 2.1 has the same signature as the real parts of the characteristic roots of A, provided that $\epsilon <$ minimum of the absolute value of the real parts of the characteristic roots of A.

This follows from the fact that the signature of the roots of $\det(B-2\lambda G)=0$ is the same as the signature of the characteristic roots of BG^{-1}, which latter matrix is similar to $G^{-\frac{1}{2}}BG^{-\frac{1}{2}}$, which has the same signature as B.

This theorem was first discovered by Bass.[5] If some of the real parts of the characteristic roots are zero then it is not always possible to find a positive definite G such that $AG+GA'$ has the same signature as the real parts of the roots of A.

In all six corollaries we have restricted attention to real matrices A. But, of course, they can all be modified to hold for complex matrices A, if we are willing to accept G as a Hermitian matrix instead of insisting that it be real and symmetric.

[5] R. W. Bass, (to be published).