

**ENUMERATION OF HOLOMORPHIC CYLINDERS
IN LOG CALABI-YAU SURFACES. II.
POSITIVITY, INTEGRALITY AND THE GLUING FORMULA**

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ABSTRACT. We prove three fundamental properties of counting holomorphic cylinders in log Calabi-Yau surfaces: positivity, integrality and the gluing formula. Positivity and integrality assert that the numbers of cylinders, defined via virtual techniques, are in fact nonnegative integers. The gluing formula roughly says that cylinders can be glued together to form longer cylinders, and the number of longer cylinders equals the product of the numbers of shorter cylinders. Our approach uses Berkovich geometry, tropical geometry, deformation theory and the ideas in the proof of associativity relations of Gromov-Witten invariants by Maxim Kontsevich. These three properties provide an evidence for a conjectural relation between counting cylinders and the broken lines of Gross-Hacking-Keel.

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1. INTRODUCTION

The motivation of this work comes from the study of mirror symmetry. Mirror symmetry is a conjectural duality between Calabi-Yau manifolds. A particular type of torus fibration, called *SYZ fibration*, plays an important role in the subject (see [49]). The enumerative geometry of an SYZ fibration associated to a Calabi-Yau manifold is intimately related to the construction of its mirror manifold. More precisely, we are interested in counting open holomorphic curves with boundaries on the torus fibers of SYZ fibrations. Remarkable progress was made in this direction, notably by Fukaya, Oh, Ohta, Ono and many others (see [18, 19]). However, there is a fundamental difficulty in the story: One observes that in general, the numbers from open curve counting are not invariants. They depend on various choices.

In order to overcome this difficulty, in [54], we explored a new approach using non-archimedean geometry. The counting of holomorphic cylinders was obtained for log Calabi-Yau surfaces. The key idea was to relate the counting of cylinders to the counting of particular types of rational curves. Since the approach was rather indirect, and went through the mysterious non-archimedean geometry, it is a priori unclear whether we have obtained the *right* invariants.

In this paper, we prove three fundamental properties of these invariants, indicating that these invariants really reflect open curve counting. The three properties are positivity, integrality and the gluing formula. Positivity and integrality assert that the numbers of cylinders, defined via virtual techniques, are in fact nonnegative integers. The gluing formula roughly says that cylinders can be glued together to form longer cylinders, and the number of longer cylinders equals the product of the numbers of shorter cylinders. In fact, we count cylinders with boundaries on fibers of (non-archimedean) SYZ torus fibrations. Given three torus fibers F_1, F_2, F_3 , we are interested in counting cylinders going from F_1 to F_2 , going from F_2 to F_3 , and going from F_1 to F_3 . The gluing formula claims that the number of cylinders going from F_1 to F_3 passing through F_2 is equal to the product of the number of cylinders going from F_1 to F_2 and the number of cylinders going from F_2 to F_3 .

Let us now give the precise statements of our theorems.

Let (Y, D) be a Looijenga pair, i.e. a connected smooth complex projective surface Y together with a singular nodal curve D representing the anti-canonical class $-K_Y$. Let $X := Y \setminus D$. Assume that the pair (Y, D) is positive¹, in the sense that the intersection matrix $(D_i \cdot D_j)$ of the components of D is not negative semi-definite,

¹This terminology was introduced in [24].

a particular case of which is when X is affine. Let $k := \mathbb{C}((t))$ be the field of formal Laurent series. Let $X_k := X \otimes_{\mathbb{C}} k$, and X_k^{an} the analytification in the sense of Berkovich (see [7]). We have a non-archimedean SYZ fibration $\tau: X_k^{\text{an}} \rightarrow B$, where B is a singular \mathbb{Z} -affine manifold homeomorphic to \mathbb{R}^2 .

A *spine* L in B is a triple $[\Gamma, (v_1, v_2), h: \Gamma \rightarrow B]$ where Γ is a chain of segments, (v_1, v_2) are the two endpoints of the chain, and h is a piecewise linear map. Here we omit some technical conditions on L (see Definition 3.7). Given a spine L in B and a curve class $\beta \in \text{NE}(Y)$, we constructed in [54] the number of holomorphic cylinders $N(L, \beta)$ associated to L and β .

Theorem 1.1 (Positivity and integrality, see Theorem 5.10). *The number $N(L, \beta)$ of holomorphic cylinders associated to the spine L and the curve class β is a nonnegative integer.*

In order to state the gluing formula, we start with two spines

$$\begin{aligned} L^1 &= [\Gamma^1, (v_1^1, v_2^1), h^1: \Gamma^1 \rightarrow B], \\ L^2 &= [\Gamma^2, (v_1^2, v_2^2), h^2: \Gamma^2 \rightarrow B]. \end{aligned}$$

Let e_2^1 denote the edge of Γ^1 connected to the vertex v_2^1 ; let e_1^2 denote the edge of Γ^2 connected to the vertex v_1^2 . Assume $h^1(v_2^1) = h^2(v_1^2)$ and $w_{v_2^1}(e_2^1) + w_{v_1^2}(e_1^2) = 0$, where $w(\cdot)$ denotes the weight vector, i.e. derivative along the edge. So we can glue L^1 and L^2 at the vertices v_2^1 and v_1^2 , and form a new spine L^3 (see Fig. 1). Let $\beta^3 \in \text{NE}(Y)$ be a curve class.

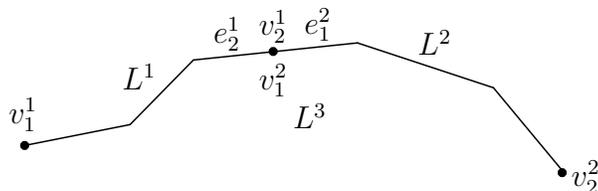


FIGURE 1. The spines L^1 , L^2 and L^3 .

Theorem 1.2 (Gluing formula). *We have the following gluing formula:*

$$\sum_{\beta^1 + \beta^2 = \beta^3} N(L^1, \beta^1) \cdot N(L^2, \beta^2) = N(L^3, \beta^3).$$

We use a combination of techniques from tropical geometry, Berkovich geometry, deformation theory and Gromov-Witten theory. The proof of the positivity and

integrality is based on three main ideas: First, the cylinder counts are invariant under deformations of the Looijenga pair (Y, D) ; Second, for generic (Y, D) in the deformation space, no bubbling can occur for our holomorphic cylinders in question; Third, by the deformation theory of rational curves in surfaces (see [32]), we deduce that the moduli space responsible for our counting is generically smooth and of expected dimension.

Now we sketch the proof of the gluing formula. We extend both L^1 and L^2 to infinity straight with respect to the \mathbb{Z} -affine structure, and glue them together to form a “double extended spine” L^d as in Construction 3.10 Fig. 3. Then we consider the moduli stack of 5-pointed rational non-archimedean analytic stable maps associated to L^d . The moduli space of 5-pointed rational curves has two special 0-strata corresponding to G and G' in Construction 3.14 Fig. 5. Then using a smoothness argument as in the proof of positivity and integrality, we prove that the counts of 5-pointed rational stable maps associated to G and G' are in fact equal. Finally we explain in Section 6 that the two counts give exactly the two sides of the gluing formula.

In order to make the idea above precise, we need to construct carefully various moduli spaces of tropical curves and the corresponding moduli spaces of analytic curves, and study their properties, which constitutes the bulk of Sections 3 and 4. We will rely on the work of Abramovich-Caporaso-Payne [2] and the author’s previous work [53]. Our proof of the gluing formula via different degenerations of domain curves can be seen as a refinement of the proof of the associativity relations of Gromov-Witten invariants (see [35, 21]). Comparing with the algebraic case, we need to take extra care of the properness of various moduli spaces in the proofs.

The results of this paper give us a deeper understanding of open curve counting of log Calabi-Yau surfaces. They provide evidence that our cylinder counts obtained via Berkovich geometry satisfy expected properties and reflect honest open curve enumeration. The counting of cylinders is intimately related to the combinatorial notion of broken line in the work of Gross-Hacking-Keel [24]². The positivity, integrality and the gluing formula proved in this paper suggest a precise conjectural relation with broken lines which we state below:

Let γ be a broken line in B for the canonical scattering diagram in the sense of [24, §2.3]. We follow the notations in loc. cit.. Let α be any negative number such

²The notion of broken line in general was developed by Gross, Hacking, Keel, Siebert and their coauthors in a series of papers [22, 11, 24, 29, 27]. It was also suggested by Abouzaid, Kontsevich and Soibelman in different occasions

that $(-\infty, \alpha]$ is a proper subset of the unique unbounded domain of linearity of γ . Then the restriction of γ to the interval $[\alpha, 0]$ gives rise to a spine in B , which we denote by L_γ . Let $c \cdot z^q$ denote the monomial attached to the last domain of linearity of γ . Write $q = \beta + \lambda$ for $\beta \in \text{NE}(Y)$ and $\lambda \in \Lambda_{\gamma(0)}$ (see [24, Construction 2.2 and Example 2.3]).

Conjecture 1.3. *The number $N(L_\gamma, \beta)$ of holomorphic cylinders associated to the spine L_γ and the curve class β is equal to the number c .*

Note that it is easy to verify the conjecture in simple cases, such as for toric (Y, D) . Moreover, we would like to mention that the number c is known to be a nonnegative integer via indirect combinatorial arguments (see [26]). If the conjecture above holds, then Theorem 1.1 gives a geometric understanding of this fact.

Related works. Besides the works already cited above, the following works are related to this paper in a broader context: the works of Kontsevich-Soibelman [37, 38], Auroux [3, 4], Gross-Siebert [28], Abouzaid [1] and Tu [51] on mirror symmetry, the works of Pandharipande-Solomon [43], Welschinger [52], Chan-Lau [12], Fang-Liu [17], Chan [13], Nishinou [42], Lin [39], Brini-Cavalieri-Ross [10] and Ross [48] on open curve enumeration, and the works of Chen-Satriano [14] and Ranganathan [45] on stable maps in logarithmic geometry. The technique of imposing tropical conditions on the moduli space of algebraic stable maps can also be achieved using logarithmic structures, see the works of Ranganathan-Santos-Parker-Wise [46, 47].

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2. REVIEW OF THE COUNTING OF CYLINDERS

In this section, we review the construction of the counting of holomorphic cylinders in log Calabi-Yau surfaces following [54].

We start with a Looijenga pair (Y, D) , i.e. a connected smooth complex projective surface Y together with a singular nodal curve³ D representing the anti-canonical class $-K_Y$. Let $X := Y \setminus D$. It is called a log Calabi-Yau surface.

Note that our initial data are all defined over the complex numbers. We will now use non-archimedean geometry as a tool in order to extract geometric invariants from X .

Let $k := \mathbb{C}((t))$ be the field of formal Laurent series. Let $X_k := X \otimes_{\mathbb{C}} k$, and let X_k^{an} be the analytification in the sense of Berkovich [7].

We have an essential skeleton $B \subset X_k^{\text{an}}$ and a canonical retraction map $\tau: X_k^{\text{an}} \rightarrow B$. As piecewise linear space, B is isomorphic to \mathbb{R}^2 . Outside the origin $O \in B$, the map τ is an affinoid torus fibration⁴ (see [54, Proposition 3.6], see also [41]). Our goal is to study the counting of holomorphic cylinders with boundaries on affinoid fiber tori.

Consider a holomorphic cylinder in X_k^{an} , i.e. a closed annulus C with two boundary points v_1, v_2 , and a holomorphic map $f: C \rightarrow X_k^{\text{an}}$. Let Γ be the skeleton of C , i.e. the path in C connecting v_1 and v_2 . Let $h := (\tau \circ f)|_{\Gamma}$. It is a piecewise linear map from Γ to B . We call $(\Gamma, (v_1, v_2), h)$ the spine associated to the holomorphic cylinder.

Now fix a spine L (in the sense of Definition 3.7) and a curve class $\beta \in \text{NE}(Y)$; we would like to count holomorphic cylinders in X_k^{an} whose associated spine is L and whose associated curve class is β . Unfortunately, the moduli space of such holomorphic cylinders is infinite dimensional. In order to obtain a finite number, our strategy in [54] is to restrict to cylinders whose boundaries satisfy an extra regularity condition: we require that when we make analytic continuation at the boundaries, our cylinders should extend straight to infinity in an appropriate sense. In this way, we are able to relate the counting of cylinders to the counting of particular types of rational curves.

The way to carry out the idea above is the following: First, using the \mathbb{Z} -affine structure on $B \setminus O$ induced by the non-archimedean SYZ fibration, we extend the spine L at both ends straight to an infinite spine \widehat{L} , which we call the associated extended spine (see Construction 3.9).

Put $\widetilde{B} := \mathbb{R} \times B$. The graph of the extended spine \widehat{L} gives an extended spine \widetilde{L} in \widetilde{B} . We also have an associated curve class $\widetilde{\beta}$ whose definition we omit here for

³A singular nodal curve is a singular curve with at worst ordinary double point singularities. It is not necessarily irreducible.

⁴Recall that a continuous map from a k -analytic space to a topological space is called an *affinoid torus fibration* if locally on the target, it is isomorphic to the pullback of the coordinate-wise valuation map $\text{val}^n: (\mathbb{G}_{m/k}^{\text{an}})^n \rightarrow \mathbb{R}^n$ along an open subset $U \subset \mathbb{R}^n$.

simplicity (see [54, §5]). Let \tilde{Y} be a toroidal compactification of $\mathbb{G}_{m/k} \times X_k$ which has two boundary divisors \tilde{D}_1, \tilde{D}_2 corresponding to the two infinite directions of \tilde{L} . Let $\tilde{\tau}: (\mathbb{G}_{m/k} \times X_k)^{\text{an}} \rightarrow \tilde{B}$ be the map induced by $\tau: X_k^{\text{an}} \rightarrow B$. Fix a point p in $(\mathbb{G}_{m/k} \times X_k)^{\text{an}}$ such that $\tilde{\tau}(p) \in \tilde{B}$ is the point in the image of \tilde{L} corresponding to an endpoint of L . Let \mathcal{M} be the stack of analytic stable maps into \tilde{Y}^{an} of class $\tilde{\beta}$ with three marked points, such that the first marked point maps to \tilde{D}_1^{an} , the second marked point maps to \tilde{D}_2^{an} , and the third marked point maps to p . For any stable map $[C, (s_1, s_2, s_3), f: C \rightarrow \tilde{Y}^{\text{an}}]$ in \mathcal{M} , let Γ be the path in C connecting s_1 and s_2 , and let $h := (\tilde{\tau} \circ f)|_{\Gamma \setminus \{s_1, s_2\}}$. We call $(\Gamma, (s_1, s_2), h)$ the extended spine in \tilde{B} associated to the stable map.

Let $\mathcal{M}(\tilde{L})$ be the substack of \mathcal{M} consisting of stable maps whose associated extended spine equals \tilde{L} . It is proved in [54, Theorem 5.4] that $\mathcal{M}(\tilde{L})$ is a proper analytic stack. Moreover, it possesses a virtual fundamental class of dimension zero. Finally, we define the number of holomorphic cylinders $N(L, \beta)$ associated to the spine L and the curve class β to be the degree of this virtual fundamental class. We have shown in loc. cit. that the number $N(L, \beta)$ is well-defined, i.e. it does not depend on the various choices we have made during the construction, in particular the choice of the point p .

Above is a sketch of the construction of the number $N(L, \beta)$. We remark that although $N(L, \beta)$ is supposed to represent the number of cylinders, its actual construction involves enumeration of rational curves and virtual fundamental classes. Therefore, the positivity and integrality in Theorem 1.1 and the gluing formula in Theorem 1.2 are not at all obvious. In order to prove these theorems, we will start by a series of constructions in tropical geometry. This is the subject of the next section.

3. TROPICAL CONSTRUCTIONS

In this section, we will make a series of constructions concerning tropical curves and tropical moduli spaces. Here is an overview:

Definitions 3.1, 3.2 and 3.5 set up the most basic notions regarding trees and \mathbb{Z} -affine maps in our context. Definition 3.7 introduces the notion of spine, the fundamental combinatorial object associated to holomorphic cylinders. As explained in Section 2, for the enumeration of holomorphic cylinders, we introduce the notion of extended spine in Definition 3.8, and the extension procedure in Construction 3.9. Construction 3.10 applies the extension procedure to the various spines in the statement of Theorem 1.2, and fix the notations for later use.

Constructions 3.12 and 3.13 concern subdivisions of the tropical bases, which will serve us for toric blowups in Section 4. Here is the idea behind Construction 3.13: In Section 4, it will be difficult to work directly with the tropical base B which is unbounded. So we approximate B by a bounded subset $B_f \subset B$. The conditions in Construction 3.13 ensure that the approximation is good enough, so that it captures enough information about tropical curves in B that are related to the spines L^i .

Constructions 3.14 and 3.15 define a path $\Delta \subset \overline{M}_{0,5}^{\text{trop}}$ over which we will deform analytic curves. Constructions 3.18 and 3.19 are based on Construction 3.10. They define the sets of tropical curves in B associated to the given spines. As explained above, we need to work with an approximation $B_f \subset B$ later in Section 4, so we make Constructions 3.21 and 3.22 to select out subsets of tropical curves in B_f associated to the given spines. The main property of these subsets is given in Proposition 3.23, which paves the way for the properness results in Section 4.

Definition 3.1. An *unbounded tree* $(\Gamma, V_\infty(\Gamma))$ consists of a finite tree⁵ Γ and a subset of 1-valent vertices $V_\infty(\Gamma)$ called *unbounded vertices*. Vertices that are not unbounded are called *bounded vertices*. We require that there is at least one bounded vertex.

Definition 3.2. An *unbounded \mathbb{Z} -affine tree* $(\Gamma, V_\infty(\Gamma))$ consists of an unbounded tree and

- (1) for every edge e with two bounded endpoints, a \mathbb{Z} -affine structure⁶ on e which is isomorphic to the interval $[0, \alpha] \subset \mathbb{R}$ for a positive real number α ,
- (2) for every edge e with one unbounded endpoint v_∞ , a \mathbb{Z} -affine structure on $e \setminus \{v_\infty\}$ which is isomorphic to the interval $[0, +\infty) \subset \mathbb{R}$.

We set $\Gamma^\circ := \Gamma \setminus V_\infty(\Gamma)$. We will simply say a *\mathbb{Z} -affine tree* when unbounded vertices are not present.

Remark 3.3. Definition 3.2 differs slightly from [54, Definition 4.2] in the sense that here we require that the tree is connected and contains at least one bounded vertex. The other cases are useless for our purpose.

Remark 3.4. Note that a one-dimensional \mathbb{Z} -affine structure is equivalent to a metric. So an unbounded \mathbb{Z} -affine tree without bounded 1-valent vertices is a rational tropical curve in the sense of [2, §4.1]. Our unbounded vertices correspond to the legs in loc. cit..

⁵Here “finite” means finitely many vertices. We allow 2-valent vertices.

⁶We refer to [54, §2] for the definition of \mathbb{Z} -affine structure.

Definition 3.5. Let $(\Gamma, V_\infty(\Gamma))$ be an unbounded \mathbb{Z} -affine tree. Let (Σ, Σ_0) be a polyhedral \mathbb{Z} -affine manifold with singularities, i.e., a polyhedral complex Σ equipped with a piecewise \mathbb{Z} -affine structure, an open subset $\Sigma_0 \subset \Sigma$ which is a manifold without boundary, and a \mathbb{Z} -affine structure on Σ_0 compatible with the piecewise \mathbb{Z} -affine structure on Σ (see [54, §2]). A \mathbb{Z} -affine map $h: \Gamma \setminus V_\infty(\Gamma) \rightarrow \Sigma$ is a continuous proper map such that every edge maps to a cell in Σ compatibly with the \mathbb{Z} -affine structures. A \mathbb{Z} -affine immersion is a \mathbb{Z} -affine map that does not contract any edge to a point⁷. Given a \mathbb{Z} -affine map $h: \Gamma \setminus V_\infty(\Gamma) \rightarrow \Sigma$, a bounded vertex $v \in V(\Gamma)$ mapping to Σ_0 , and an edge e connected to v , we denote by $w_v^0(e)$ the primitive integral tangent vector at v pointing to the direction of e , and by $w_v(e)$ the image of $w_v^0(e)$ in $T_{h(v)}\Sigma(\mathbb{Z})$, the integral lattice in the tangent space $T_{h(v)}\Sigma$. We call $w_v(e)$ the weight vector of the edge e at v . The \mathbb{Z} -affine map h is said to be *balanced* at the vertex v if $\sum_{e \ni v} w_v(e) = 0 \in T_{h(v)}\Sigma(\mathbb{Z})$, where the sum is taken over all edges of Γ connected to v .

Remark 3.6. Let B be as in Section 2, i.e. the base of the non-archimedean SYZ fibration $\tau: X_k^{\text{an}} \rightarrow B$. By construction it is homeomorphic to the dual intersection cone complex of D . Let $O \in B$ be the origin in B . The non-archimedean SYZ fibration induces a \mathbb{Z} -affine structure on $B \setminus O$ (see [54, Remark 3.7]). So the pair $(B, B \setminus O)$ is a polyhedral \mathbb{Z} -affine manifold with singularities.

Definition 3.7. A *spine* in the tropical base B consists of a \mathbb{Z} -affine tree Γ , two 1-valent vertices (v_1, v_2) of Γ , and a \mathbb{Z} -affine immersion $h: \Gamma \rightarrow B$ satisfying the following conditions:

- (1) The image of h does not contain the origin $O \in B$.
- (2) The vertices v_1 and v_2 are the only 1-valent vertices of Γ . In particular, Γ is homeomorphic to a closed interval.
- (3) There is no edge e of Γ such that $h(e)$ is contained in a ray starting from O .
- (4) For every 2-valent vertex v , the vector $-\sum_{e \ni v} w_v(e)$ is either zero or points towards the origin $O \in B$, where $w_v(e)$ denotes the weight vector defined in Definition 3.5 (see Fig. 2).

Definition 3.8. An *extended spine* in the tropical base B consists of an unbounded \mathbb{Z} -affine tree Γ , two unbounded vertices (v_1, v_2) , and a \mathbb{Z} -affine immersion $h: \Gamma^\circ \rightarrow B$ such that the Conditions (1)-(4) of Definition 3.7 hold.

⁷A \mathbb{Z} -affine immersion is not an injective map in general.

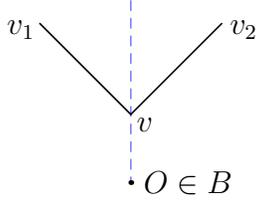


FIGURE 2.

Construction 3.9. Given any spine $L = [\Gamma, (v_1, v_2), h]$ in B , we can extend both its ends v_1 and v_2 straight with respect to the \mathbb{Z} -affine structure on $B \setminus O$. Since our Looijenga pair (Y, D) is assumed to be positive, by [23, Corollary 1.6], this extension gives an extended spine in B , which is called the extended spine associated to L , and which we denote by $\widehat{L} = [\widehat{\Gamma}, (u_1, u_2), \widehat{h}: \widehat{\Gamma}^\circ \rightarrow B]$. This is the only place in the paper where we use the positivity assumption.

Let B^1 denote the union of 1-dimensional cones in B viewed as a simplicial cone complex. For any $b \in B^1 \setminus O$, let \mathbf{e}_b denote the primitive integral vector at b in the direction of \overrightarrow{Ob} , and let D_b denote the irreducible component of D corresponding to the ray \overrightarrow{Ob} . Define

$$\delta := \sum_{x \in \widehat{h}^{-1}(B^1) \setminus \Gamma} |d_x \widehat{h} \wedge \mathbf{e}_{\widehat{h}(x)}| D_{\widehat{h}(x)} \in \text{NE}(Y),$$

where d_x denotes the derivative at x , and $|\cdot|$ denotes the lattice length of the wedge product. We call δ the curve class associated to the extension from L to \widehat{L} (see [54, §4]).

Construction 3.10. Fix a spine $L^0 = [\Gamma^0, (v_1^0, v_2^0), h^0: \Gamma^0 \rightarrow B]$ in B , which will play the role of an arbitrary spine in the paper. Let L^1, L^2 and L^3 be the spines in the statement of Theorem 1.2. For $i = 0, \dots, 3$, let

$$\widehat{L}^i = [\widehat{\Gamma}^i, (u_1^i, u_2^i), \widehat{h}^i: (\widehat{\Gamma}^i)^\circ \rightarrow B]$$

denote the extended spine in B associated to L^i .

Let us glue \widehat{L}^1 and \widehat{L}^2 along the vertices $v_2^1 \in \widehat{\Gamma}^1$ and $v_1^2 \in \widehat{\Gamma}^2$ to form an unbounded \mathbb{Z} -affine tree Γ^d with four unbounded vertices $u_1^1, u_2^1, u_1^2, u_2^2 \in \Gamma^d$. Let us denote the vertex $v_2^1 = v_1^2 \in \Gamma^d$ by v^d . The \mathbb{Z} -affine map $\widehat{h}^1: (\widehat{\Gamma}^1)^\circ \rightarrow B$ and $\widehat{h}^2: (\widehat{\Gamma}^2)^\circ \rightarrow B$ induce a \mathbb{Z} -affine map $h^d: (\Gamma^d)^\circ \rightarrow B$ (cf. Fig. 3). We denote

$$L^d := [\Gamma^d, (u_1^1, u_2^1, u_1^2, u_2^2, v^d), h^d].$$

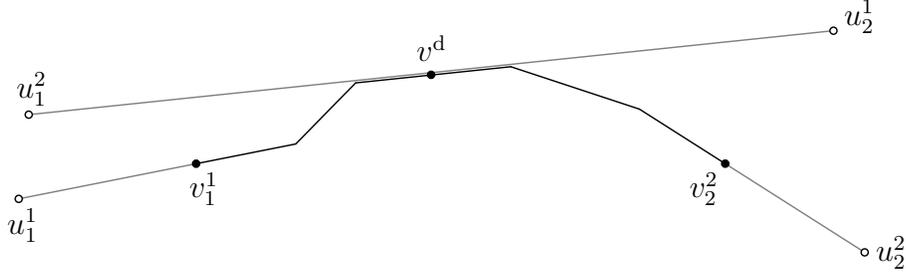


FIGURE 3.

Let $\widehat{\Gamma}^4$ be the path connecting u_1^2 to u_2^1 in Γ^d and let $\widehat{h}^4: (\widehat{\Gamma}^4)^\circ \rightarrow B$ the restriction of h^d . Denote $u_1^4 := u_1^2$, $u_2^4 := u_2^1$. We see that

$$\widehat{L}^4 = [\widehat{\Gamma}^4, (u_1^4, u_2^4), \widehat{h}^4: (\widehat{\Gamma}^4)^\circ \rightarrow B]$$

is an extended spine in B that is straight with respect to the \mathbb{Z} -affine structure on $B \setminus O$.

For $i = 0, \dots, 4$, let f_1^i, f_2^i denote the edges of $\widehat{\Gamma}^i$ connected to the unbounded vertices u_1^i, u_2^i respectively. Let u_1^i denote the other endpoint of the edge f_1^i , and u_2^i the other endpoint of the edge f_2^i . Let $w_1^i := w_{u_1^i}(f_1^i)$ and $w_2^i := w_{u_2^i}(f_2^i)$ denote the weight vectors.

Remark 3.11. Although the superscripts $0, \dots, 4$ in the construction above are not very enlightening, they are practical when we apply the same arguments uniformly to all these spines.

Construction 3.12. Let $\widetilde{B} := \mathbb{R} \times B$. We endow the first factor \mathbb{R} with the simplicial cone complex structure $\mathbb{R} = \mathbb{R}_{<0} \cup \{0\} \cup \mathbb{R}_{>0}$. Then \widetilde{B} obtains also the structure of a simplicial cone complex. Let \widetilde{B}' be a finite rational subdivision of \widetilde{B} into simplicial cones satisfying the following conditions:

- (1) Every simplicial cone in the subdivision is regular, i.e., the integer points in each simplicial cone can be generated by a subset of a basis of the underlying lattice. This condition is achievable by [33, Chapter I Theorem 11].
- (2) For every $i = 0, \dots, 4$, there are two rays \widetilde{r}_1^i and \widetilde{r}_2^i in the subdivision \widetilde{B}' pointing to the directions of the vectors $(1, w_1^i)$ and $(1, w_2^i)$ respectively.

Construction 3.13. Let $C_B := \mathbb{R}_{\geq 0} \times B$ and we embed B into C_B by the map $x \mapsto (1, x)$. Let C'_B be a finite rational subdivision of C_B into simplicial cones satisfying the following conditions:

- (1) Every simplicial cone in the subdivision is regular.

- (2) Let B' denote the subdivision of B induced by C'_B . Let B_f denote the union of the bounded faces of B' . The subscript f in the notation is short for “finite”. Let ∂B_f denote the boundary of B_f viewed as a subset of B and let B_f° denote the interior. We require that the origin O and the images of the spines L^0, L^1, L^2 are all contained in the interior B_f° .
- (3) For every $i = 0, \dots, 4$, for every unbounded 2-dimensional cell ρ in the subdivision B' containing the ray $\widehat{h}^i(f_1^i \setminus u_1^i)$ (resp. $\widehat{h}^i(f_2^i \setminus u_2^i)$) except a finite part, we require that ρ has two infinite 1-dimensional boundary faces pointing to the same direction.
- (4) Let B'_∞ denote the conical subdivision of B induced by the directions⁸ of the infinite 1-dimensional cells in B' . We require that the projection $\widetilde{B} \rightarrow B$ induces a map of simplicial cone complexes $\widetilde{B}' \rightarrow B'_\infty$.

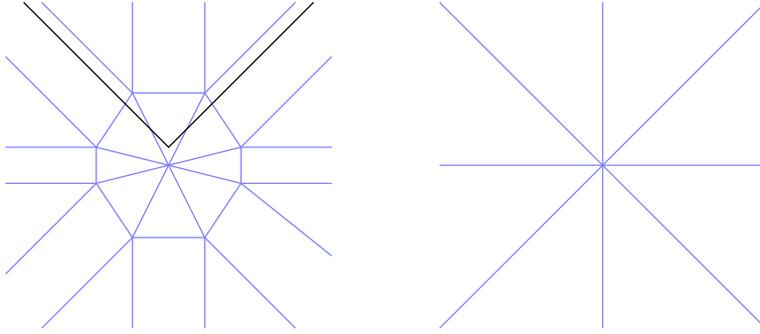


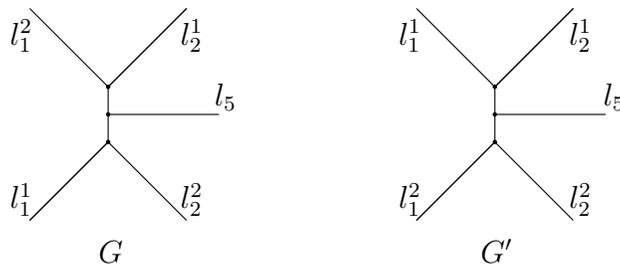
FIGURE 4. An example of the subdivision B' and the associated B'_∞ .

Construction 3.14 (see Fig. 5). We define two graphs with legs:

- The graph G consists of three vertices v_1, v_2, v_3 , an edge connecting v_1, v_3 , an edge connecting v_2, v_3 , two legs l_1^2, l_2^1 attached to the vertex v_1 , two legs l_1^1, l_2^2 attached to the vertex v_2 , and one leg l_5 attached to the vertex v_3 .
- The graph G' consists of three vertices v_1, v_2, v_3 , an edge connecting v_1, v_3 , an edge connecting v_2, v_3 , two legs l_1^1, l_2^1 attached to the vertex v_1 , two legs l_1^2, l_2^2 attached to the vertex v_2 , and one leg l_5 attached to the vertex v_3 .

Note that the only difference between G and G' is the labeling of the legs.

⁸Each infinite 1-dimensional cell in B' has a direction vector which gives a ray in B starting from O .


 FIGURE 5. The graphs G and G' .

Construction 3.15. Following Abramovich-Caporaso-Payne [2], let $\overline{M}_{g,n}^{\text{trop}}$ denote the coarse moduli space of extended tropical curves of genus g with n legs. For the extended tropical curves parametrized by $\overline{M}_{0,5}^{\text{trop}}$, we label the five legs by $l_1^1, l_2^1, l_1^2, l_2^2, l_5$ respectively. For any $r \in (0, +\infty]$, let G_r (resp. G'_r) denote the extended tropical curve such that

- (1) the underlying graph is G (resp. G'),
- (2) the lengths of both vertical edges (as shown in Fig. 5) are equal to r .

Note that for $r = 0$, we have the degenerate case $G_0 = G'_0$. Now varying r from 0 to $+\infty$, G_r and G'_r give two paths in $\overline{M}_{0,5}^{\text{trop}}$. We denote the union of the two paths by $\Delta \subset \overline{M}_{0,5}^{\text{trop}}$.

Definition 3.16. A *pointed extended tropical cylinder* in the tropical base B consists of an unbounded \mathbb{Z} -affine tree Γ with exactly two unbounded vertices (v_1, v_2) , a marked point $v_3 \in \Gamma$, and a \mathbb{Z} -affine immersion $h: \Gamma^\circ \rightarrow B$ satisfying the following condition: For every bounded vertex v of Γ such that $h(v) \neq O$, the \mathbb{Z} -affine map h is balanced at v .

Definition 3.17. A *pointed extended tropical double cylinder* in the tropical base B consists of an unbounded \mathbb{Z} -affine tree Γ with exactly four unbounded vertices v_1, v_2, v_3, v_4 , a marked point $v_5 \in \Gamma$, and a \mathbb{Z} -affine immersion $h: \Gamma^\circ \rightarrow B$ satisfying the same condition as in Definition 3.16.

Construction 3.18. For $i = 0, \dots, 4$, let T^i be the set of pointed extended tropical cylinders $(\Gamma, (v_1, v_2, v_3), h)$ satisfying the following conditions:

- (1) If $i = 0$, then we have $h(v_3) = h^0(v_1^0)$.
- (2) If $i = 1, \dots, 4$, then we have $h(v_3) = h^d(v^d)$. (Recall that $h^d(v^d) = h^1(v_2^1) = h^2(v_1^2)$.)

- (3) Let Γ^s denote the path in Γ connecting the unbounded vertices v_1 and v_2 . Then $(\Gamma^s, (v_1, v_2, v_3), h|_{\Gamma^s})$ is equal to $(\widehat{\Gamma}^i, (u_1^i, u_2^i, v_1^i), \widehat{h}^i)$, where $v_1^3 := v_2^1 = v_1^2 \in \widehat{\Gamma}^3$ and $v_1^4 := v^d \in \widehat{\Gamma}^4$.

Construction 3.19. Let T^d be the set of pointed extended tropical double cylinders $(\Gamma, (v_1, v_2, v_3, v_4, v_5), h)$ satisfying the following conditions:

- (1) We have $h(v_5) = h^d(v^d) \in B$.
- (2) Let Γ^s denote the convex hull of the vertices v_1, \dots, v_4 in Γ . Then $(\Gamma^s, (v_1, \dots, v_5), h|_{\Gamma^s})$ is equal to $(\Gamma^d, (u_1^1, u_2^1, u_1^2, u_2^2, v^d), h^d)$.

Definition 3.20. A *pointed rational tropical curve in B_f with n boundary points* consists of a \mathbb{Z} -affine tree Γ , $(n+1)$ marked points $v_1, \dots, v_{n+1} \in \Gamma$, and a \mathbb{Z} -affine immersion $h: \Gamma \rightarrow B_f$ satisfying the following conditions:

- (1) For $i = 1, \dots, n$, $h(v_i)$ lies in the boundary ∂B_f .
- (2) For any vertex v of Γ , if $h(v)$ lies in $B_f^\circ \setminus O$, then the \mathbb{Z} -affine map h is balanced at v .
- (3) For any vertex v of Γ , if $h(v)$ lies in the relative interior of a 1-dimensional cell σ of ∂B_f , let ρ denote the unbounded 2-dimensional cell in B' containing σ . Since the subdivision B' of B is induced by the simplicial conical subdivision C'_B of C_B , the two other boundary faces of ρ are two rays pointing to the same direction. Let w_ρ denote this direction. Consider the quotient $T_{h(v)}B/w_\rho$, where $T_{h(v)}B$ denotes the tangent space of B at $h(v)$. We require that $\sum_{e \ni v} w_v(e)$ is zero in $T_{h(v)}B/w_\rho$, where the sum is taken over all edges of Γ connected to v .
- (4) If $h^{-1}(\partial B_f)$ is a finite set, then $h^{-1}(\partial B_f) = \bigcup_{i=1}^n v_i$.

A pointed rational tropical curve in B_f with n boundary points is called *simple* if there is no balanced 2-valent vertex. Note that any pointed rational tropical curve in B_f with n boundary points can be made simple by removing all the balanced 2-valent vertices and gluing the corresponding edges.

Construction 3.21. Let $M_{n+1}(B_f)$ denote the space of simple pointed rational tropical curves in B_f with n boundary points. It has a natural Hausdorff topological structure given by deformations of tropical curves (cf. [53, Theorem 6.1]).

For $i = 0, \dots, 4$, let $M_{2+1}^i(B_f) \subset M_{2+1}(B_f)$ denote the subset consisting of pointed rational tropical curves in B_f with 2 boundary points $(\Gamma, (v_1, v_2, v_3), h: \Gamma \rightarrow B_f)$ such that $h(v_3) = h^0(v_1^0)$ for $i = 0$, and $h(v_3) = h^d(v^d)$ for $i = 1, \dots, 4$.

Let $M_{4+1}^d(B_f) \subset M_{4+1}(B_f)$ denote the subset consisting of pointed rational tropical curves in B_f with 4 boundary points $(\Gamma, (v_1, \dots, v_5), h: \Gamma \rightarrow B_f)$ satisfying the following conditions:

- (1) We have $h(v_5) = h^d(v^d) \in B_f$.
- (2) Let P_{13} denote the path in Γ connecting v_1, v_3 , and P_{24} the path connecting v_2, v_4 . Then the intersection $P_{13} \cap P_{24}$ is nonempty. Moreover, the vertex v_5 lies in the intersection.

Construction 3.22. In the context of Construction 3.13, for every unbounded cell ρ in the subdivision B' , we consider the linear projection from ρ to $\rho \cap \partial B_f$. This gives a natural retraction map from B to B_f . Under the retraction map, pointed extended tropical cylinders and pointed extended tropical double cylinders induce simple pointed rational tropical curves in B_f . So the sets T^i , for $i = 0, \dots, 4$, and T^d induce subsets of $M(B_f)$, which we denote by T_f^i and T_f^d respectively. By construction, T_f^i is a subset of $M_{2+1}^i(B_f)$, and T_f^d is a subset of $M_{4+1}^d(B_f)$.

Proposition 3.23. *For $i = 0, \dots, 4$, the subset $T_f^i \subset M_{2+1}^i(B_f)$ is a union of connected components of $M_{2+1}^i(B_f)$. Similarly, the subset $T_f^d \subset M_{4+1}^d(B_f)$ is a union of connected components of $M_{4+1}^d(B_f)$.*

Lemma 3.24. *Let Γ be a \mathbb{Z} -affine tree which is not a point, and r a vertex of Γ called the root. Let $h: \Gamma \rightarrow B$ be a \mathbb{Z} -affine immersion satisfying the following conditions:*

- (1) *The image $h(r) \neq O \in B$.*
- (2) *For any vertex v of Γ except the root, if $h(v)$ lies in $B \setminus O$, then the \mathbb{Z} -affine map h is balanced at v .*

Then there exists a ray R in B starting from $O \in B$ with rational slope such that the image $h(\Gamma)$ lies in R . In particular, the image $h(r)$ lies in R .

Proof. Let $V(\Gamma)$ denote the set of vertices of Γ . We call 1-valent vertices in $V(\Gamma) \setminus \{r\}$ leaves of Γ . By Condition (2), if v is a leaf of Γ , then $h(v) = O \in B$. Therefore, for every edge e of Γ containing a leaf, the image $h(e)$ is contained in a ray R_e in B starting from $O \in B$. Since h is a \mathbb{Z} -affine immersion, the ray R_e has rational slope. Let v be any vertex of Γ except r such that $h(v) \neq O$. Let e_0, \dots, e_n be the edges connected to v . If we know that for $i = 1, \dots, n$, the image $h(e_i)$ is contained in a ray R_{e_i} in B starting from O , then the rays R_{e_1}, \dots, R_{e_n} are all equal; moreover, by Condition (2), the image $h(e_0)$ is also contained in the same ray. Therefore, reasoning from the leaves to the root, we conclude that for all edge e of Γ , the image $h(e)$ is contained in a same ray R starting from $O \in B$ with rational slope. \square

Proof of Proposition 3.23. The conditions in Constructions 3.18 and 3.19 are all given by equalities, therefore, the subsets $T_f^i \subset M_{2+1}^i(B_f)$, $T_f^d \subset M_{4+1}^d(B_f)$ are closed. Now let us prove that they are open.

Let $K = [\Gamma, (v_1, v_2, v_3), h: \Gamma \rightarrow B_f] \in T_f^i$, and let Γ^s denote the path connecting v_1 and v_2 in Γ . We claim that the image $h(\Gamma^s) \subset B_f$ is fixed during small deformation of K inside $M_{2+1}^i(B_f)$ (see Fig. 6).

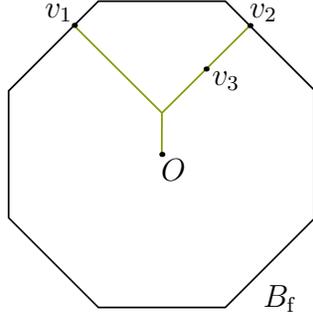


FIGURE 6. An example of a tropical curve in T_f^i having one bending vertex on the spine.

Let v be a 2-valent vertex of Γ^s and let e_1, e_2 denote the two edges connected to v . We call v a bending vertex of Γ^s if we have $w_v(e_1) + w_v(e_2) \neq 0$ for the weight vectors. When we deform K in $M_{2+1}^i(B_f)$, by the \mathbb{Z} -affine property, the slopes of the edges are fixed. Note that the image $h(v_3) \in B_f$ is always fixed by Construction 3.21. Therefore, if Γ^s contains no bending vertex, the image $h(\Gamma^s) \subset B_f$ must be fixed during small deformation of K .

Now assume that v is a bending vertex of Γ^s . Let Γ_v be the closure of a connected component of $\Gamma \setminus \Gamma^s$ such that Γ_v contains v . Applying Lemma 3.24 to Γ_v , v and $h|_{\Gamma_v}$, we see that $h(v)$ lies in a ray R_v in B starting from $O \in B$ with rational slope. Therefore, during any small deformation of K inside $M_{2+1}^i(B_f)$, the image $h(v)$ must stay on the ray R_v . Since the image $h(v_3)$ and the slopes of the edges of Γ^s are all fixed during deformation, we conclude that the image $h(\Gamma^s) \subset B_f$ is fixed during small deformation of K . So the claim holds.

By Construction 3.13 Condition (3), the preimage $h^{-1}(\partial B_f)$ in Γ is exactly $\{v_1, v_2\}$. Therefore, by Definition 3.20 Conditions (1) and (4), the preimage $h^{-1}(\partial B_f)$ stays $\{v_1, v_2\}$ during small deformation of K inside $M_{2+1}^i(B_f)$. We conclude that $[\Gamma^s, (v_1, v_2, v_3), h|_{\Gamma^s}]$ is fixed during small deformation of K inside $M_{2+1}^i(B_f)$. In other words, the subset $T_f^i \subset M_{2+1}^i(B_f)$ is open, hence it is a union of connected components.

Next let us show that the subset $T_f^d \subset M_{4+1}^d(B_f)$ is open using the same idea. Let $K = [\Gamma, (v_1, \dots, v_5), h: \Gamma \rightarrow B_f] \in T_f^d$, and let Γ^s denote the convex hull of v_1, \dots, v_4 in Γ . Let P_{13} be the path connecting v_1, v_3 in Γ , and P_{24} the path connecting v_2, v_4 in Γ . We have $P_{13} \cap P_{24} = v_5$. By Construction 3.21 Condition (2), during deformation of K inside $M_{4+1}^d(B_f)$, the intersection $P_{13} \cap P_{24}$ stays nonempty. Therefore, by the balancing condition, the 4-valent vertex of Γ^s cannot deform into two 3-valent vertices. So the intersection $P_{13} \cap P_{24}$ is always a point during deformation, which is always the vertex v_5 .

Now let P_{14} be the path connecting v_1, v_4 in Γ , and P_{23} the path connecting v_2, v_3 in Γ . The two paths intersect at v_5 . By Construction 3.21, the image $h(v_5) \in B_f$ is fixed during deformation of K inside $M_{4+1}^d(B_f)$. Applying our reasoning for T_f^i in the previous paragraphs to the paths P_{14} and P_{23} , we deduce that $[\Gamma^s, (v_1, \dots, v_5), h|_{\Gamma^s}]$ is fixed during small deformation of K inside $M_{4+1}^d(B_f)$. We conclude that the subset $T_f^d \subset M_{4+1}^d(B_f)$ is open, hence it is a union of connected components. \square

4. ANALYTIC CONSTRUCTIONS

In this section, we apply the tropical constructions in Section 3 to the constructions of analytic moduli spaces. The main results are the properness statements in Propositions 4.12 and 4.15.

We begin by replacing our target spaces by toric blowups for two technical reasons: first, we want our curves to touch only the smooth locus of the boundary divisor; the second reason is related to the approximation $B_f \subset B$ explained in the beginning of Section 3. The toric blowups cause a bit of notation complications regarding curve classes, which are dealt with in Constructions 4.1 - 4.3. First time readers are advised to skip these technical considerations.

We endow the projective line \mathbb{P}_k^1 with the standard toric structure. The subdivision \tilde{B}' in Construction 3.12 induces a toric blowup⁹ $\tilde{Y} \rightarrow \mathbb{P}_k^1 \times Y_k$. The subdivision B'_∞ in Construction 3.13 induces a toric blowup $Y' \rightarrow Y_k$. Let $\partial\tilde{Y}$ and $\partial Y'$ denote the toroidal boundaries of \tilde{Y} and Y' respectively. By Construction 3.13 Condition (4), the map of simplicial cone complexes $\tilde{B}' \rightarrow B'_\infty$ induces a map of k -varieties $p_{Y,2}: \tilde{Y} \rightarrow Y'$.

For $i = 0, \dots, 4$, let m_1^i be the lattice length of the weight vector w_1^i defined before Construction 3.12. Let \tilde{r}_1^i be as in Construction 3.12(2). Let r_1^i be the image of \tilde{r}_1^i under the map $\tilde{B}' \rightarrow B'_\infty$. Let \tilde{D}_1^i be the divisor in \tilde{Y} corresponding to the ray \tilde{r}_1^i .

⁹We refer to [33, Chapter II] for the relationship between subdivisions and blowups.

Let D_1^i be the divisor in Y' corresponding to the ray r_1^i . Similarly, we define m_2^i , \widetilde{D}_2^i and D_2^i .

Now, let $i \in \{0, \dots, 4\}$, and let β be an element in $\text{NE}(Y)$, the cone of curves of Y .

Construction 4.1. Let $s(\widehat{L}^i, \beta)$ denote the subset of $\text{NE}(Y')$ consisting of elements $\gamma \in \text{NE}(Y')$ satisfying the following conditions:

- (1) The element γ admits a representative which is a closed rational curve C contained in $(Y' \setminus \partial Y') \cup (D_1^i \cup D_2^i)$ that intersects the divisor D_1^i at exactly one point with multiplicity m_1^i , and the divisor D_2^i at exactly one point with multiplicity m_2^i .
- (2) Under the composite morphism $\text{NE}(Y') \rightarrow \text{NE}(Y_k) \xrightarrow{\sim} \text{NE}(Y)$, the image of γ equals $\beta \in \text{NE}(Y)$.

Construction 4.2. Let $s^{\text{p}}(\widehat{L}^i, \beta)$ denote the subset of $\text{NE}(\widetilde{Y})$ consisting of elements $\gamma \in \text{NE}(\widetilde{Y})$ satisfying the following conditions:

- (1) The element γ admits a representative which is a closed rational curve C contained in $(\widetilde{Y} \setminus \partial \widetilde{Y}) \cup \widetilde{D}_1^i \cup \widetilde{D}_2^i$ that intersects the divisor \widetilde{D}_1^i transversally at exactly one point, and the divisor \widetilde{D}_2^i also transversally at exactly one point.
- (2) Under the projection $\text{NE}(\widetilde{Y}) \rightarrow \text{NE}(\mathbb{P}_k^1)$, the image of γ is the fundamental class of \mathbb{P}_k^1 .
- (3) Under the composite morphism $\text{NE}(\widetilde{Y}) \rightarrow \text{NE}(Y_k) \xrightarrow{\sim} \text{NE}(Y)$, the image of γ equals $\beta \in \text{NE}(Y)$.

The superscript p in the notation is short for “parametrized”.

Construction 4.3. Let $s^{\text{d}}(\beta)$ denote the subset of $\text{NE}(Y')$ consisting of elements $\gamma \in \text{NE}(Y')$ satisfying the following conditions:

- (1) The element γ admits a representative which is a closed rational curve C contained in $(Y' \setminus \partial Y') \cup (D_1^1 \cup D_2^1 \cup D_1^2 \cup D_2^2)$ that intersects the divisor D_1^1 (resp. D_2^1, D_1^2, D_2^2) at exactly one point with multiplicity m_1^1 (resp. m_2^1, m_1^2, m_2^2).
- (2) Under the composite morphism $\text{NE}(Y') \rightarrow \text{NE}(Y_k) \xrightarrow{\sim} \text{NE}(Y)$, the image of γ equals $\beta \in \text{NE}(Y)$.

Remark 4.4. Each of the subsets we defined in Constructions 4.1 - 4.3 contains at most one element. To see this for $s(\widehat{L}^i, \beta)$, note that $N^1(Y')$ is a direct sum of $N^1(Y_k)$ with classes of exceptional curves, where $N^1(-)$ denotes divisors modulo numerical equivalence. So the conditions in Construction 4.1 determine the curve class uniquely. The same reasoning applies to Constructions 4.2 and 4.3 too.

Recall that a stable map into a variety is a map from a pointed proper nodal curve to the variety having only finitely many automorphisms. An equivalent characterization of having finitely many automorphisms is that every irreducible component of the domain nodal curve of genus 0 (resp. 1) which maps to a point must have at least 3 (resp. 1) special points on its normalization, where by special point, we mean the preimage of either a marked point or a node by the normalization map. We refer to [36, 20] for the theory of stable maps in algebraic geometry, and to [55] for the theory in non-archimedean analytic geometry.

Construction 4.5. Let $\mathcal{M}(\widehat{L}^i, \beta)$ denote the stack of 3-pointed rational stable maps into Y' satisfying the following conditions:

- (1) The first marked point maps to D_1^i with multiplicity greater than or equal to m_1^i ; the second marked point maps to D_2^i with multiplicity greater than or equal to m_2^i . (By “multiplicity greater than or equal to”, we also allow the irreducible component containing the marked point to lie completely inside the divisor. So this condition is a Zariski closed condition.)
- (2) The image of the fundamental class of the domain curve belongs to $s(\widehat{L}^i, \beta)$.

Construction 4.6. Let $\mathcal{M}^p(\widehat{L}^i, \beta)$ denote the stack of 3-pointed rational stable maps into \widetilde{Y} satisfying the following conditions:

- (1) The first marked point maps to \widetilde{D}_1^i ; the second marked point maps to \widetilde{D}_2^i .
- (2) The image of the fundamental class of the domain curve belongs to $s^p(\widehat{L}^i, \beta)$.

The above parametrized version of moduli spaces is used in the construction of the counts $N(L, \beta)$ reviewed in Section 2, it is also essential for Proposition 6.6.

Construction 4.7. Let $\mathcal{M}^d(\beta)$ denote the stack of 5-pointed rational stable maps into Y' satisfying the following conditions:

- (1) The first four marked points map respectively to the divisors $D_1^1, D_2^1, D_1^2, D_2^2$ with multiplicities greater than or equal to $m_1^1, m_2^1, m_1^2, m_2^2$ respectively.
- (2) The image of the fundamental class of the domain curve belongs to $s^d(\beta)$.

Remark 4.8. Let $\mathcal{M}(\widehat{L}^i, \beta)^{\text{an}}$ denote the analytification of the algebraic stack $\mathcal{M}(\widehat{L}^i, \beta)$ (see [44, §6.1]). By [55, Theorem 8.7], the k -analytic stack $\mathcal{M}(\widehat{L}^i, \beta)^{\text{an}}$ parametrizes 3-pointed rational analytic stable maps into the analytification $(Y')^{\text{an}}$ satisfying the conditions of Construction 4.5. Similar remarks apply also to the stacks in Constructions 4.6 and 4.7.

The subdivision B' of B gives rise to an snc log-model (\mathcal{X}, H) of the k -variety X_k (see [54, §2]). Note that the generic fiber of \mathcal{X} is isomorphic to the k -variety Y' . In other words, \mathcal{X} is an snc model of Y' . Up to a finite ground field extension, we can assume furthermore that \mathcal{X} is a strictly semi-stable model of Y' (see [33]). By construction, the Clemens polytope associated to \mathcal{X} is isomorphic to B_f (i.e. the finite part of B') as a simplicial complex. Let

$$\tau_f: (Y')^{\text{an}} \rightarrow B_f$$

denote the corresponding retraction map (see [55, Proposition 2.7]). Note that the toroidal boundary of $(Y')^{\text{an}}$ maps onto ∂B_f .

The following lemma relates analytic stable maps in $(Y')^{\text{an}}$ to tropical curves in B_f .

Lemma 4.9. *Let $[C, (s_1, s_2, s_3), f: C \rightarrow (Y')^{\text{an}}]$ be an analytic stable map in $\mathcal{M}(\widehat{L}^i, \beta)^{\text{an}}$. The composite map $C \xrightarrow{f} (Y')^{\text{an}} \xrightarrow{\tau_f} B_f$ factors as $C \xrightarrow{c} \Gamma \xrightarrow{h} B_f$, where $C \xrightarrow{c} \Gamma$ contracts all paths in C that are contracted by $\tau_f \circ f$. Let $v_i := c(s_i)$ for $i = 1, 2, 3$. Then $[\Gamma, (v_1, v_2, v_3), h]$ is a pointed rational tropical curve in B_f with 2 boundary points in the sense of Definition 3.20.*

Proof. Let \mathfrak{X} be the completion of \mathcal{X} along its special fiber. By [55, Theorem 1.5], up to passing to a finite ground field extension, we can find a strictly semi-stable formal model \mathfrak{C} of C and a morphism of formal schemes $\mathfrak{f}: \mathfrak{C} \rightarrow \mathfrak{X}$ such that $\mathfrak{f}_\eta \simeq f$. Let $S_{\mathfrak{C}}$ be the Clemens polytope of \mathfrak{C} and $\tau_{\mathfrak{C}}: \mathfrak{C} \rightarrow S_{\mathfrak{C}}$ the retraction map. By the functoriality of Clemens polytope (see [55, §2]), we obtain a commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{f} & (Y')^{\text{an}} \\ \downarrow \tau_{\mathfrak{C}} & & \downarrow \tau_f \\ S_{\mathfrak{C}} & \xrightarrow{S_f} & B_f, \end{array}$$

where S_f is affine on every edge of $S_{\mathfrak{C}}$. The map $S_f: S_{\mathfrak{C}} \rightarrow B_f$ factors as $S_{\mathfrak{C}} \xrightarrow{c'} \overline{S}_{\mathfrak{C}} \xrightarrow{\overline{S}_f} B_f$ where c' contracts all edges of $S_{\mathfrak{C}}$ that are contracted by S_f , so that \overline{S}_f is an immersion. Thus the map $\Gamma \xrightarrow{h} B_f$ is isomorphic to $\overline{S}_{\mathfrak{C}} \xrightarrow{\overline{S}_f} B_f$. This shows that Γ is a finite \mathbb{Z} -affine tree, and $h: \Gamma \rightarrow B_f$ is a \mathbb{Z} -affine immersion.

Now let us verify Conditions (1)-(4) in Definition 3.20. Since $f(s_1), f(s_2) \in (\partial Y')^{\text{an}}$ and $\tau_f((\partial Y')^{\text{an}}) = \partial B_f \subset B_f$, Condition (1) is satisfied. Next, note that any point $p \in B_f^\circ \setminus O$ has an open neighborhood V in B_f such that the map $\tau_f^{-1}(V) \rightarrow V$ is a trivial affinoid torus fibration. So Condition (2) is satisfied (see [5, Theorem 6.14]). For Condition (3), consider the extension of the valuation map $\text{val}: \mathbb{G}_{m/k}^{\text{an}} \rightarrow \mathbb{R}$ to the

map $\overline{\text{val}}: (\mathbb{A}_k^1)^{\text{an}} \rightarrow \mathbb{R} \cup \{+\infty\}$. Let $\phi: \mathbb{R} \cup \{+\infty\} \rightarrow (-\infty, 0]$ be the map contracting $[0, +\infty]$ to the point 0. Let $\psi := \phi \circ \overline{\text{val}}: (\mathbb{A}_k^1)^{\text{an}} \rightarrow (-\infty, 0]$ be the composition. For any point p in the relative interior of a 1-dimensional cell of ∂B , there exists a small open neighborhood V of p in B_f such that the map $\tau_f^{-1}(V) \rightarrow V$ is isomorphic to the map $(\psi, \text{val}): (\mathbb{A}_k^1)^{\text{an}} \times \mathbb{G}_{m/k}^{\text{an}} \rightarrow (-\infty, 0] \times \mathbb{R}$ localized over an open subset of $(-\infty, 0] \times \mathbb{R}$. Therefore, by projecting $\tau_f^{-1}(V)$ to the factor $\mathbb{G}_{m/k}^{\text{an}}$, [5, Theorem 6.14] implies Condition (3) too. Finally, if $h^{-1}(\partial B_f)$ is a finite set, then $f(C)$ does not contain any component of $(\partial Y')^{\text{an}}$. So by Constructions 4.1 and 4.5, $f(C)$ cannot have extra intersections with $(\partial Y')^{\text{an}}$ except at s_1 and s_2 . Thus the following lemma implies Condition (4). \square

Remark 4.10. We mention a generalization [53, Theorem 4.1] and a variant [45, Theorem 3.4.2] of Lemma 4.9.

Lemma 4.11. *In the context of Lemma 4.9, let v be a vertex of Γ and U a small closed connected neighborhood of v in Γ . Assume that $h(v) \in \partial B_f$, $h(U \setminus v) \cap \partial B_f = \emptyset$, and $O \notin h(U)$. Then there is a point $x \in C$ such that $c(x) = v$ and $f(x) \in (\partial Y')^{\text{an}}$.*

Proof. Suppose to the contrary that such a point x does not exist. Since $h(U \setminus v) \cap \partial B_f = \emptyset$, then $f(c^{-1}(U)) \cap (\partial Y')^{\text{an}} = \emptyset$. Using formal models as in the proof of Lemma 4.9, we see that the composite map $c^{-1}(U) \xrightarrow{f} X_k^{\text{an}} \xrightarrow{\tau} B$ factors as $c^{-1}(U) \xrightarrow{c'} \Delta \xrightarrow{h'} B$ where $c^{-1}(U) \xrightarrow{c'} \Delta$ contracts all paths in $c^{-1}(U)$ that are contracted by $(\tau \circ f)$. Then the composite map $\Delta \xrightarrow{h'} B \rightarrow B_f$ factors as $\Delta \xrightarrow{r} U \xrightarrow{h} B_f$, where r contracts a subtree of Δ to the point $v \in U$. Note that we have a natural embedding $U \subset \Delta$ and $r: \Delta \rightarrow U$ is a retraction map. Since h' is balanced at v , there is at least one edge e of Δ connected to v such that $r(e) = v$. Let Δ_e be the closure of the connected component of $\Delta \setminus v$ containing e . Let v' be a 1-valent vertex of Δ_e other than v . Since $O \notin h(U)$, we have $h'(v') \neq O$. Then the \mathbb{Z} -affine map h' must be balanced at the vertex v' , which contradicts the fact that v' is 1-valent. So the proof is complete. \square

By Lemma 4.9, we obtain a tropicalization map

$$\tau_{\mathcal{M}}^i: \mathcal{M}(\widehat{L}^i, \beta)^{\text{an}} \rightarrow M_{2+1}(B_f).$$

Similarly, we have a tropicalization map for the stack $\mathcal{M}^{\text{d}}(\beta)$

$$\tau_{\mathcal{M}^{\text{d}}}: \mathcal{M}^{\text{d}}(\beta)^{\text{an}} \rightarrow M_{4+1}(B_f).$$

A point in the stack $\mathcal{M}^{\text{p}}(\widehat{L}^i, \beta)^{\text{an}}$ corresponds to a 3-pointed rational analytic stable map into $\widetilde{Y}^{\text{an}}$. If we compose this stable map with the map $p_{Y'_2}^{\text{an}}: \widetilde{Y}^{\text{an}} \rightarrow (Y')^{\text{an}}$, we

obtain a map from a nodal curve to $(Y')^{\text{an}}$. Its tropicalization is a pointed rational tropical curve in B_f with 2 boundary points in the sense of Definition 3.20. So we obtain a tropicalization map

$$\tau_{\mathcal{M}^p}^i: \mathcal{M}^p(\widehat{L}^i, \beta)^{\text{an}} \rightarrow M_{2+1}(B_f).$$

Let us denote

$$\begin{aligned} \mathcal{M}(\widehat{L}^i, \beta, T) &:= (\tau_{\mathcal{M}}^i)^{-1}(T_f^i), \\ \mathcal{M}^p(\widehat{L}^i, \beta, T) &:= (\tau_{\mathcal{M}^p}^i)^{-1}(T_f^i), \\ \mathcal{M}^d(\beta, T) &:= (\tau_{\mathcal{M}^d})^{-1}(T_f^d), \end{aligned}$$

where the preimages are understood merely as functors of points (without extra geometric structures a priori).

Let

$$\begin{aligned} W^0 &:= \tau_f^{-1}(h^0(v_1^0)) \subset (Y')^{\text{an}}, \\ W^p &:= \mathbb{G}_{m/k}^{\text{an}} \times W^0 \subset \widetilde{Y}^{\text{an}}, \\ W^d &:= \tau_f^{-1}(h^d(v^d)) \subset (Y')^{\text{an}}. \end{aligned}$$

Proposition 4.12. *Consider the evaluation map of the third marked point for the stack $\mathcal{M}(\widehat{L}^0, \beta)^{\text{an}}$*

$$\text{ev}_3: \mathcal{M}(\widehat{L}^0, \beta)^{\text{an}} \rightarrow (Y')^{\text{an}}.$$

Let $\mathcal{M}(\widehat{L}^0, \beta)_W := \text{ev}_3^{-1}W^0$ and

$$\mathcal{M}(\widehat{L}^0, \beta, T)_W := \mathcal{M}(\widehat{L}^0, \beta)_W \cap \mathcal{M}(\widehat{L}^0, \beta, T).$$

Then $\mathcal{M}(\widehat{L}^0, \beta, T)_W$ is a k -analytic stack, and the restriction

$$\text{ev}_3|_{\mathcal{M}(\widehat{L}^0, \beta, T)_W}: \mathcal{M}(\widehat{L}^0, \beta, T)_W \rightarrow W^0$$

is a proper map. Similarly, consider the evaluation map of the third marked point for the stack $\mathcal{M}^p(\widehat{L}^0, \beta)^{\text{an}}$

$$\text{ev}_3: \mathcal{M}^p(\widehat{L}^0, \beta)^{\text{an}} \rightarrow \widetilde{Y}^{\text{an}}.$$

Let $\mathcal{M}^p(\widehat{L}^0, \beta)_W := \text{ev}_3^{-1}W^p$ and

$$\mathcal{M}^p(\widehat{L}^0, \beta, T)_W := \mathcal{M}^p(\widehat{L}^0, \beta)_W \cap \mathcal{M}^p(\widehat{L}^0, \beta, T).$$

Then $\mathcal{M}^p(\widehat{L}^0, \beta, T)_W$ is a k -analytic stack, and the restriction

$$\text{ev}_3|_{\mathcal{M}^p(\widehat{L}^0, \beta, T)_W}: \mathcal{M}^p(\widehat{L}^0, \beta, T)_W \rightarrow W^p$$

is a proper map.

Proof. Let us prove the first statement. The same argument applies to the second statement. Consider the tropicalization map

$$\tau_{\mathcal{M}}^0: \mathcal{M}(\widehat{L}^0, \beta)^{\text{an}} \rightarrow M_{2+1}(B_f).$$

Let $[C, (p_1, p_2, p_3), f: C \rightarrow (Y')^{\text{an}}]$ be the analytic stable map corresponding to a point in $\mathcal{M}(\widehat{L}^0, \beta)_W$. Let $[\Gamma, (v_1, v_2, v_3), h: \Gamma \rightarrow B_f]$ be the associated rational tropical curve in B_f with two boundary points. Since $\mathcal{M}(\widehat{L}^0, \beta)_W = \text{ev}_3^{-1}(W^0)$ and $W^0 = \tau_f^{-1}(h^0(v_1^0))$, we have $\tau_f(f(p_3)) = h^0(v_1^0)$. So $h(v_3) = h^0(v_1^0)$. We deduce that

$$\tau_{\mathcal{M}}^0(\mathcal{M}(\widehat{L}^0, \beta)_W) \subset M_{2+1}^0(B_f),$$

where $M_{2+1}^0(B_f)$ is defined in Construction 3.21.

By Proposition 3.23, the subset $T_f^0 \subset M_{2+1}^0(B_f)$ is a union of connected components. The tropicalization map $\tau_{\mathcal{M}}^0$ is continuous by [53, Theorem 8.1]. Therefore, the substack $\mathcal{M}(\widehat{L}^0, \beta, T)_W \subset \mathcal{M}(\widehat{L}^0, \beta)_W$ is a union of connected components; in particular, it is a k -analytic stack. Since the algebraic stack $\mathcal{M}(\widehat{L}^0, \beta)$ is proper over k , by [44, Proposition 6.4], its analytification $\mathcal{M}(\widehat{L}^0, \beta)^{\text{an}}$ is a proper k -analytic stack. So the map

$$\text{ev}_3: \mathcal{M}(\widehat{L}^0, \beta)^{\text{an}} \rightarrow (Y')^{\text{an}}$$

is proper. By base change, the map

$$\text{ev}_3|_{\mathcal{M}(\widehat{L}^0, \beta)_W}: \mathcal{M}(\widehat{L}^0, \beta)_W \rightarrow W^0$$

is proper. Since the substack $\mathcal{M}(\widehat{L}^0, \beta, T)_W \subset \mathcal{M}(\widehat{L}^0, \beta)_W$ is a union of connected components, we deduce that the map

$$\text{ev}_3|_{\mathcal{M}(\widehat{L}^0, \beta, T)_W}: \mathcal{M}(\widehat{L}^0, \beta, T)_W \rightarrow W^0$$

is proper, completing the proof. \square

Lemma 4.13. *Any analytic stable map $[C, (\text{marked points}), f: C \rightarrow (Y')^{\text{an}}]$ in $\mathcal{M}(\widehat{L}^0, \beta, T)$ (resp. $\mathcal{M}^{\text{d}}(\beta, T)$) satisfies the following properties:*

- (1) *There is no irreducible component of C mapping into the boundary $(\partial Y')^{\text{an}}$.*
- (2) *The preimage $f^{-1}((\partial Y')^{\text{an}})$ contains exactly the first two (resp. four) marked points on C .*
- (3) *Let C' be an irreducible component of C that contains at most two special points; then $f(C')$ does not meet the boundary $(\partial Y')^{\text{an}}$ and $(\tau \circ f)(C') = O \in B$.*
- (4) *Let $C_{\text{m}} \subset C$ be the union of irreducible components containing the marked points. Let $\text{st}(C)$ be the stabilization of the pointed domain curve (obtained by iterated contractions of non-stable components). We have $C_{\text{m}} = \text{st}(C)$.*

Proof. Let us prove the lemma in the case of $\mathcal{M}^d(\beta, T)$. The proof in the case of $\mathcal{M}(\widehat{L}^0, \beta, T)$ is similar and simpler.

Let $Z = (\Gamma, (v_1, \dots, v_5), h: \Gamma \rightarrow B_f)$ denote the pointed rational tropical curve in B_f with four boundary points associated to the stable map. Let C_b denote the union of the irreducible components of C mapping into the boundary $(\partial Y')^{\text{an}}$. Let C'_b be a connected component of C_b . Since Z belongs to T_f^d , the intersection $h(\Gamma) \cap \partial B_f$ consists of finitely many points in ∂B_f . So $(\tau_f \circ f)(C'_b)$ is a point in ∂B_f , which we denote by b . Since $\tau_f^{-1}(b)$ is an affinoid domain in $(Y')^{\text{an}}$, the image $f(C'_b) \subset (Y')^{\text{an}}$ must be a point in $(\partial Y')^{\text{an}}$.

Let C_i be the union of the irreducible components of C not contained in C_b . We claim that $C'_b \cap C_i$ contains exactly one point. To see this, let $v \in \{v_1, \dots, v_4\}$ be the vertex of Γ corresponding to C'_b ; let $x \in C'_b \cap C_i$ be an intersection point; and let K be the irreducible component of C_i containing x . If $(\tau_f \circ f)(K) \subset \partial B_f$, using the fact that $h(\Gamma) \cap \partial B_f$ is a finite set, and that the fibers of $\tau_f: (Y')^{\text{an}} \rightarrow B_f$ over ∂B_f are affinoid, we see that $f(K)$ is a point in $(Y')^{\text{an}}$. Since $K \subset C_i$, $f(K)$ is a point in $(Y' \setminus \partial Y')^{\text{an}}$. Then C'_b and K cannot intersect. Therefore, $(\tau_f \circ f)(K)$ cannot be contained in ∂B_f . So the intersection point $x = C'_b \cap K$ gives rise to an edge of Γ connected to v . Since Z belongs to T_f^d , v is a 1-valent vertex. Since each intersection point of $C'_b \cap C_i$ gives rise to a different edge of Γ connected to v , we conclude that $C'_b \cap C_i$ contains exactly one point.

Moreover, $Z \in T_f^d$ implies that v_1, \dots, v_4 are four different points on Γ . So C'_b contains at most one marked point. Let C_1 be the union of the irreducible components of C not contained in C'_b . We have $C_1 = (C_b \setminus C'_b) \cup C_i$. So

$$C'_b \cap C_1 = C'_b \cap ((C_b \setminus C'_b) \cup C_i) = C'_b \cap C_i$$

contains exactly one point. Recall that C'_b has arithmetic genus zero and contains at most one marked point. So there exists an irreducible component E of C'_b which contains at most two special points. As f is constant on E , we get a contradiction to the stability condition of the stable map f . Therefore, C_b must be empty. This proves Property (1).

Using Property (1) and the conditions on the intersections numbers in Construction 4.1 (resp. Construction 4.3), we obtain Property (2).

Now let C' be the irreducible component of C in Property (3). In this case, C is reducible, so C' contains at least one node of C . By Property (2), $f(C')$ can meet the boundary $(\partial Y')^{\text{an}}$ in at most one point. If $f(C')$ meets $(\partial Y')^{\text{an}}$ at exactly one

point, then we claim that $(\tau_f \circ f)(C')$ must contain the intersection between B_f and a rational ray starting from O in B .

In order to see the claim, note that the retraction map $\tau: X_k^{\text{an}} \rightarrow B$ extends to a retraction map $\bar{\tau}: (Y'_k)^{\text{an}} \rightarrow \bar{B}'$, where \bar{B}' is the closure of B' inside $(Y'_k)^{\text{an}}$ (see [30, Theorem 4.13]). The composite map $C \xrightarrow{f} (Y')^{\text{an}} \xrightarrow{\bar{\tau}} \bar{B}'$ factors as $C \xrightarrow{\bar{c}} \bar{\Gamma} \xrightarrow{\bar{h}} \bar{B}'$ where $C \xrightarrow{\bar{c}} \bar{\Gamma}$ contracts all paths in C that are contracted by $\bar{\tau} \circ f$. If $f(C')$ meets $(\partial Y')^{\text{an}}$ at exactly one point, then there is exactly one point r of $\bar{\Gamma}$ such that $\bar{h}(r) \in \partial \bar{B}'$. Let \bar{R} denote the ray connecting O and $\bar{h}(r)$ in \bar{B}' . Applying the same argument as in the proof of Lemma 3.24 to $\bar{h}: \bar{\Gamma} \rightarrow \bar{B}'$ with root r , we see that the image $\bar{h}(\bar{\Gamma})$ is equal to the ray \bar{R} . Then the image $(\tau_f \circ f)(C') \subset B_f$ contains the intersection $\bar{R} \cap B_f$. So the claim holds.

The claim contradicts the fact that the pointed rational tropical curve Z belongs to T_f^{d} , because the image of any tropical curve in T_f^{d} cannot contain such a ray. Therefore, $f(C')$ must lie in the interior $(Y' \setminus \partial Y')^{\text{an}} \simeq X_k^{\text{an}}$. In this case, by the balancing condition, $(\tau \circ f)(C')$ is a point in B . By the stability condition, $f|_{C'}$ is nontrivial. Moreover, since $\tau: X_k^{\text{an}} \rightarrow B$ is an affinoid torus fibration outside $O \in B$, the only possibility for the image $(\tau \circ f)(C')$ is the origin $O \in B$. This proves Property (3).

Let C_m and $\text{st}(C)$ be as in Property (4). Let us first prove that C_m is connected. Assume on the contrary that C_m is not connected. Let $C_n \subset C$ be the union of irreducible components of C not contained in C_m . Let C'_n be a connected component of C_n that meets at least two connected components of C_m .

We claim that $(\tau_f \circ f)(C'_n)$ must be the point $h^{\text{d}}(v^{\text{d}}) \in B_f$. To see this, say that C'_n meets two connected components C_m^1 and C_m^2 of C_m . Let Γ_m^1, Γ_m^2 and Γ'_n be the connected subtrees of Γ associated to C_m^1, C_m^2 and C'_n respectively. The restrictions of h to these subtrees are balanced away from $\partial B_f \cup O$. Note that each of Γ_m^1 and Γ_m^2 contains some marked points. Since Z belongs to T_f^{d} , it follows from the shape of Z and the balancing condition that Γ'_n must be the point v_5 . Hence we obtain $(\tau_f \circ f)(C'_n) = h(\Gamma'_n) = h(v_5) = h^{\text{d}}(v^{\text{d}})$.

Since $\tau_f^{-1}(h^{\text{d}}(v^{\text{d}}))$ is affinoid, the image $f(C'_n)$ is a point in X_k^{an} . Then by the stability condition, the intersection $C'_n \cap C_m$ has at least three points. Since C is rational, we see that C_m must have at least three connected components. Using again the fact that Z belongs to T_f^{d} , we see that only the following situation is possible: C_m has exactly three connected components, one containing the last marked point, one containing two among the first four marked points, and one containing the remaining two marked points. We denote the three connected components by C_m^1, C_m^2, C_m^3

respectively. Since C is rational, C'_n intersects each of C_m^1 , C_m^2 and C_m^3 exactly once. Since C_m^1 has only one marked point, by Property (3), C_m^1 must contain two nodes of C . So there exists a connected component C''_n of C_n different from C'_n that meets C_m^1 . By the same reasoning as for C'_n , the intersection $C''_n \cap C_m$ has at least three points. However, since C is rational and $C_m \cup C'_n$ is connected, we see that $C''_n \cap (C_m \cup C'_n)$ can contain at most one point. So we have reached a contradiction. Therefore, C_m is connected.

By Property (3), every irreducible component in C_m contains at least three special points. Combining with the fact that C_m is connected, we see that C_m is a stable pointed curve. Note that for any pointed rational nodal curve $C' \supset C_m$ whose marked points are all contained in C_m , there exists at least one irreducible component of C' not contained in C_m which contains at most two nodes. Therefore, after iterated contractions of non-stable components of C , we will end up with C_m . In other words, we have $C_m = \text{st}(C)$, completing the proof of Property (4). \square

Construction 4.14. Let $\overline{\mathcal{M}}_{0,5}$ denote the moduli space of 5-pointed rational algebraic stable curves over k . Let $\tau_{\overline{\mathcal{M}}_{0,5}}: \overline{\mathcal{M}}_{0,5}^{\text{an}} \rightarrow \overline{\mathcal{M}}_{0,5}^{\text{trop}}$ denote the tropicalization map sending pointed analytic stable curves to the associated extended tropical curves (see [2]). Let $\Delta \subset \overline{\mathcal{M}}_{0,5}^{\text{trop}}$ be as in Construction 3.15. We set $\tilde{\Delta} := \tau_{\overline{\mathcal{M}}_{0,5}}^{-1}(\Delta) \subset \overline{\mathcal{M}}_{0,5}^{\text{an}}$.

Proposition 4.15. *Let*

$$\Phi = (\text{st}, \text{ev}_5): \mathcal{M}^d(\beta)^{\text{an}} \rightarrow \overline{\mathcal{M}}_{0,5}^{\text{an}} \times (Y')^{\text{an}}$$

be the map given by stabilization of domain curves and the evaluation map of the fifth marked point. Let

$$\mathcal{M}^d(\beta)_W := \Phi^{-1}(\tilde{\Delta} \times W^d),$$

$$\mathcal{M}^d(\beta, T)_W := \mathcal{M}^d(\beta)_W \cap \mathcal{M}^d(\beta, T).$$

Then $\mathcal{M}^d(\beta, T)_W$ is a k -analytic stack, and the restriction

$$\Phi|_{\mathcal{M}^d(\beta, T)_W}: \mathcal{M}^d(\beta, T)_W \rightarrow \tilde{\Delta} \times W^d$$

is a proper map.

Proof. Consider the tropicalization map

$$\tau_{\mathcal{M}^d}: \mathcal{M}^d(\beta)^{\text{an}} \rightarrow M_{4+1}(B_f).$$

Let $[C, (p_1, \dots, p_5), f: C \rightarrow (Y')^{\text{an}}]$ be the analytic stable map corresponding to a point in $\mathcal{M}^d(\beta)_W$. Let $[\Gamma, (v_1, \dots, v_5), h: \Gamma \rightarrow B_f]$ be the associated pointed rational

tropical curve in B_f with four boundary points. Since $\mathcal{M}^d(\beta)_W = \Phi^{-1}(\tilde{\Delta} \times W^d)$ and $W^d = \tau_f^{-1}(h^d(v^d))$, we have $\tau_f(f(p_5)) = h^d(v^d)$. So $h(v_5) = h^d(v^d)$.

Let $\text{st}(C)$ denote the stabilization of the 5-pointed curve $(C, (p_1, \dots, p_5))$. Since $\mathcal{M}^d(\beta)_W = \Phi^{-1}(\tilde{\Delta} \times W^d)$ and $\tilde{\Delta} = \tau_{\overline{\mathcal{M}}_{0,5}}^{-1}(\Delta)$, we have $\tau_{\overline{\mathcal{M}}_{0,5}}(\text{st}(C)) \in \Delta$. Let Γ^s be the convex hull of v_1, \dots, v_4 in Γ . By Lemma 4.13(4), Γ^s is a contraction (in terms of graph) of $\tau_{\overline{\mathcal{M}}_{0,5}}(\text{st}(C))$. So by the definition of Δ , the path connecting v_1, v_3 and the path connecting v_2, v_4 in Γ^s intersect, and the vertex v_5 lies in the intersection. Combining with the equality $h(v_5) = h^d(v^d)$ shown in the last paragraph, we deduce that

$$\tau_{\mathcal{M}^d}(\mathcal{M}^d(\beta)_W) \subset \mathcal{M}_{4+1}^d(B_f),$$

where $\mathcal{M}_{4+1}^d(B_f)$ is defined in Construction 3.21.

By Proposition 3.23, the subset $T_f^d \subset \mathcal{M}_{4+1}^d(B_f)$ is a union of connected components. The tropicalization map $\tau_{\mathcal{M}^d}$ is continuous by [53, Theorem 8.1]. Therefore, the substack $\mathcal{M}^d(\beta, T)_W \subset \mathcal{M}^d(\beta)_W$ is a union of connected components; in particular, it is a k -analytic stack. Since the algebraic stack $\mathcal{M}^d(\beta)$ is proper over k , by [44, Proposition 6.4], its analytification $\mathcal{M}^d(\beta)^{\text{an}}$ is a proper k -analytic stack. So the map

$$\Phi: \mathcal{M}^d(\beta)^{\text{an}} \rightarrow \overline{\mathcal{M}}_{0,5}^{\text{an}} \times (Y')^{\text{an}}$$

is proper. By base change, the map

$$\Phi|_{\mathcal{M}^d(\beta)_W}: \mathcal{M}^d(\beta)_W \rightarrow \tilde{\Delta} \times W^d$$

is proper. Since the substack $\mathcal{M}^d(\beta, T)_W \subset \mathcal{M}^d(\beta)_W$ is a union of connected components, we deduce that the map

$$\Phi|_{\mathcal{M}^d(\beta, T)_W}: \mathcal{M}^d(\beta, T)_W \rightarrow \tilde{\Delta} \times W^d$$

is also proper, completing the proof. \square

5. SMOOTHNESS OF MODULI SPACES

In Section 4, we have shown the properness of various evaluation maps. In this section, we study the smoothness of moduli spaces. The main result is Proposition 5.9, which shows that after a small deformation of the target log Calabi-Yau surface, the evaluation maps become generically finite étale. As a consequence, we obtain the positivity and integrality theorem.

Proposition 5.1. *Let $\mathcal{M}(\widehat{L}^0, \beta)_0$ denote the substack of $\mathcal{M}(\widehat{L}^0, \beta)$ consisting of stable maps $[C, (s_1, s_2, s_3), f: C \rightarrow Y']$ satisfying the following conditions:*

- (1) *The domain curve C is irreducible.*

(2) The preimage $f^{-1}(\partial Y')$ is a Cartier divisor supported at the two marked points s_1 and s_2 .

Then there exists a nonempty Zariski open subset $V \subset Y'$ over which the evaluation map

$$\mathrm{ev}_3: \mathcal{M}(\widehat{L}^0, \beta)_0 \rightarrow Y'$$

is smooth.

Similarly, we define the substack $\mathcal{M}^p(\widehat{L}^0, \beta)_0 \subset \mathcal{M}^p(\widehat{L}^0, \beta)$. Then there exists a nonempty Zariski open subset $V \subset \widetilde{Y}$ over which the evaluation map

$$\mathrm{ev}_3: \mathcal{M}^p(\widehat{L}^0, \beta)_0 \rightarrow \widetilde{Y}$$

is smooth.

Proof. Let us prove the statement for the stack $\mathcal{M}(\widehat{L}^0, \beta)$. The same arguments work for the stack $\mathcal{M}^p(\widehat{L}^0, \beta)$. Let x be a closed point in $\mathcal{M}(\widehat{L}^0, \beta)_0$ and let $[C, (s_1, s_2, s_3), f: C \rightarrow Y']$ be the corresponding stable map. Let E denote the vector bundle $f^*T_{Y'}(-\log \partial Y')$. The derivative dev_3 of the evaluation map ev_3 at x is given by the map

$$H^0(C, E) \longrightarrow f^*(T_{Y'})_{s_3}.$$

By Condition (2), the image of s_3 does not lie in the boundary $\partial Y'$. So we have

$$f^*(T_{Y'})_{s_3} \simeq E_{s_3}.$$

Assume that the derivative dev_3 is surjective. Then the vector bundle E is globally generated at the point s_3 . Since C is a projective line, the vector bundle E is semi-positive, i.e. it is isomorphic to a direct sum of line bundles $\mathcal{O}(i)$ with $i \geq 0$. Therefore, the first cohomology $H^1(C, E)$ is zero. So by the dimension estimate in [32, Proposition 5.3], the moduli space $\mathcal{M}(\widehat{L}^0, \beta)_0$ is smooth at x . By the surjectivity of the derivative dev_3 again, we deduce that the evaluation map ev_3 is smooth at x . Now let \mathcal{M}' be the set of closed points in $\mathcal{M}(\widehat{L}^0, \beta)_0$ where the derivative dev_3 is not surjective. By [31, Proposition III.10.6], the Zariski closure of the image $\mathrm{ev}_3(\mathcal{M}')$ has dimension less than or equal to one. So we can take V to be the complement, completing the proof of the proposition. \square

Lemma 5.2. *Let C be a nodal curve of arithmetic genus 0. Let E be a vector bundle on C such that $c_1(E) \cdot C' = 0$ for every irreducible component C' of C . Then E is globally generated at a point $p \in C$ if and only if E is a trivial vector bundle.*

Proof. Note that C is a tree of projective lines. Let $C' \subset C$ be an irreducible component of C and let $p \in C'$ be a point. Assume that E is globally generated at p . Then $E|_{C'}$ is a direct sum of line bundles $\mathcal{O}(i)$ with $i \geq 0$. Since $c_1(E) \cdot C' = 0$, the vector bundle $E|_{C'}$ is trivial. Thus $H^0(C', E|_{C'}) \rightarrow E_q$ is an isomorphism for all $q \in C'$. So the sections in $H^0(C, E)$ which generate E_p also generate E_q for all $q \in C'$, in particular for any nodal point of C contained in C' . Now we can propagate the argument over the full tree and deduce that E is a trivial vector bundle. The proof of the other direction is obvious. \square

Proposition 5.3. *Let $\mathcal{M}^d(\beta)_0$ denote the substack of $\mathcal{M}^d(\beta)$ consisting of stable maps $[C, (s_1, \dots, s_5), f: C \rightarrow Y']$ satisfying the following conditions:*

- (1) *The domain curve $[C, (s_1, \dots, s_5)]$ is a stable 5-pointed curve.*
- (2) *The preimage $f^{-1}(\partial Y')$ is a Cartier divisor supported at the first four marked points.*

Let $\mathcal{M}_{\text{sm}}^d(\beta)_0 \subset \mathcal{M}^d(\beta)_0$ denote the open substack consisting of stable maps such that $f^*T_{Y'}(-\log \partial Y')$ is a trivial vector bundle on C . The following hold:

- (1) *The map*

$$\Phi := (\text{st}, \text{ev}_5): \mathcal{M}^d(\beta)_0 \longrightarrow \overline{\mathcal{M}}_{0,5} \times Y'$$

is smooth over $\mathcal{M}_{\text{sm}}^d(\beta)_0$.

- (2) *Given any closed point $\mu \in \overline{\mathcal{M}}_{0,5}$, let*

$$\Phi_\mu: \mathcal{M}^d(\beta)_{0,\mu} \rightarrow Y'$$

denote the restriction of the map Φ to the fibers over μ . There exists a Zariski dense open subset $V_\mu \subset Y'$ such that

$$\Phi_\mu^{-1}(V_\mu) \subset \mathcal{M}_{\text{sm}}^d(\beta)_0.$$

Proof. Fix any closed point $\mu \in \overline{\mathcal{M}}_{0,5}$. Let $x \in \mathcal{M}^d(\beta)_0$ be a closed point over μ . Let $[C, (s_1, \dots, s_5), f: C \rightarrow Y']$ be the corresponding stable map. Let E denote the vector bundle $f^*T_{Y'}(-\log \partial Y')$. The derivative $d\Phi_\mu$ at the point x is given by the map

$$H^0(C, E) \rightarrow f^*(T_{Y'})_{s_5} \simeq E_{s_5}.$$

Assume that x lies in $\mathcal{M}_{\text{sm}}^d(\beta)_0$. In this case, E is a trivial vector bundle. Hence the derivative $d\Phi_\mu$ is surjective at x . Moreover, the first cohomology $H^1(C, E)$ vanishes.

By [34, Chapter II Theorem 1.7], the dimension of $\mathcal{M}^d(\beta)_0$ at the point x is at least

$$h^0(C, E) - h^1(C, E) + \dim \overline{\mathcal{M}}_{0,5} = h^0(C, E) + \dim \overline{\mathcal{M}}_{0,5}.$$

Note that the dimension of the Zariski tangent space of $\mathcal{M}^d(\beta)_{0,\mu}$ at x is at most $h^0(C, E)$. Moreover, the dimension of the Zariski tangent space of $\mathcal{M}^d(\beta)_0$ at x is at most the dimension of the Zariski tangent space of $\mathcal{M}^d(\beta)_{0,\mu}$ at x plus the dimension of the Zariski tangent space of $\overline{\mathcal{M}}_{0,5}$ at m . Therefore, the dimension of the Zariski tangent space of $\mathcal{M}^d(\beta)_0$ at x equals the dimension of the space $\mathcal{M}^d(\beta)_0$ itself at x . So $\mathcal{M}^d(\beta)_0$ is smooth at x . Furthermore, the surjectivity of the derivative $d\Phi_\mu$ at x implies the surjectivity of the derivative $d\Phi$ at x . Thus the map Φ is smooth at x . This shows the first statement in the proposition.

Next we drop the assumption that x lies in $\mathcal{M}_{\text{sm}}^d(\beta)_0$. If the derivative $d\Phi_\mu$ is surjective at x , then the vector bundle E is globally generated at the point s_5 ; hence by Lemma 5.2, E is trivial. In other words, $\mathcal{M}_{\text{sm}}^d(\beta)_{0,\mu} \subset \mathcal{M}^d(\beta)_{0,\mu}$ is exactly the locus where the derivative $d\Phi_\mu$ is surjective. Hence by [31, Proposition III.10.6], the Zariski closure of the image $\Phi_\mu(\mathcal{M}^d(\beta)_{0,\mu} \setminus \mathcal{M}_{\text{sm}}^d(\beta)_{0,\mu})$ has dimension less than the dimension of Y' . This shows the second statement in the proposition. \square

Proposition 5.4. *Given any Looijenga pair (Y, D) , there is a deformation \mathcal{Y} of Y preserving D such that \mathcal{Y} does not contain any proper curve disjoint from D .*

Proof. Denote

$$D^\perp := \{ \alpha \in \text{Pic}(Y) \mid \alpha \cdot [D'] = 0 \text{ for every irreducible component } D' \text{ of } D \}.$$

Let $\text{Pic}^0(D) \subset \text{Pic}(D)$ denote the subset consisting of line bundles on D of degree 0 on every irreducible component of D . Fix a cyclic ordering of the irreducible components of D . By [25, Lemma 2.1(1)], it induces an identification $\text{Pic}^0(D) = \mathbb{G}_{\text{m}/\mathbb{C}}$.

Let \mathcal{Y} be a deformation of Y preserving D . Restriction of line bundles gives a homomorphism

$$\tilde{\phi}_{\mathcal{Y}}: \text{Pic}(\mathcal{Y}) \rightarrow \text{Pic}(D).$$

Since $\text{Pic}(Y) \simeq H^2(Y, \mathbb{Z})$, the lattice $\text{Pic}(\mathcal{Y})$ can be canonically identified with $\text{Pic}(Y)$, together with the intersection form and the classes of the irreducible components of D . So $\tilde{\phi}_{\mathcal{Y}}$ induces a homomorphism

$$\phi_{\mathcal{Y}}: D^\perp \rightarrow \text{Pic}^0(D) = \mathbb{G}_{\text{m}/\mathbb{C}}.$$

Let $\text{Def}(Y, D)$ denote the versal deformation space of Y preserving D . The construction above gives a local period mapping

$$\phi: \text{Def}(Y, D) \rightarrow \text{Hom}(D^\perp, \mathbb{G}_{\text{m}/\mathbb{C}}),$$

which is a local analytic isomorphism (see [25, Proposition 4.1] and [40, II.2.5]).

For each $\alpha \in D^\perp \setminus 0$, let $T_\alpha \subset \text{Hom}(D^\perp, \mathbb{G}_{m/\mathbb{C}})$ be the subtorus consisting of $\psi \in \text{Hom}(D^\perp, \mathbb{G}_{m/\mathbb{C}})$ such that $\psi(\alpha) = 1$. Let \mathcal{Y} be a deformation of Y in the preimage

$$\phi^{-1}\left(\text{Hom}(D^\perp, \mathbb{G}_{m/\mathbb{C}}) \setminus \left(\bigcup_{\alpha \in D^\perp \setminus 0} T_\alpha\right)\right).$$

By construction, there is no nontrivial line bundle α on \mathcal{Y} in D^\perp whose restriction to D is trivial. In particular, there are no proper curves in \mathcal{Y} disjoint from D . \square

By [54, Theorem 5.4], the moduli stack responsible for the enumeration of holomorphic cylinders is a union of connected components of the moduli stack of stable maps (with certain incidence conditions). Moreover, the virtual fundamental class we use is the restriction of the Gromov-Witten virtual fundamental class. Therefore, the deformation invariance of Gromov-Witten invariants implies that the enumeration of holomorphic cylinders for a Looijenga pair (Y, D) is invariant under deformations of Y preserving D . Thus, in virtue of Proposition 5.4, we can assume from now on the following:

Assumption 5.5. *We assume that for the Looijenga pair (Y, D) , the projective surface Y does not contain any proper curve disjoint from D .*

The main point of the following two lemmas is that under the assumption above, the analytic stable maps with desired tropicalizations will have stable domains.

Lemma 5.6. *Let $\mathcal{M}(\widehat{L}^0, \beta)_0$ and $\mathcal{M}^d(\beta)_0$ be as in Propositions 5.1 and 5.3 respectively. We have $\mathcal{M}(\widehat{L}^0, \beta, T) \subset \mathcal{M}(\widehat{L}^0, \beta)_0^{\text{an}}$ and $\mathcal{M}^d(\beta, T) \subset \mathcal{M}^d(\beta)_0^{\text{an}}$.*

Proof. Under Assumption 5.5, Lemma 4.13(3) implies that the pointed domain curve of any analytic stable map in $\mathcal{M}(\widehat{L}^0, \beta, T)$ or $\mathcal{M}^d(\beta, T)$ is stable. Therefore, combining with Lemma 4.13(2), we obtain the two inclusions in the statement of the lemma. \square

Similarly, we have the following:

Lemma 5.7. *Let $p_{Y,1}: \widetilde{Y} \rightarrow \mathbb{P}_k^1$ denote the projection map induced by the map of simplicial cone complexes $\widetilde{B}' \rightarrow \mathbb{R} = \mathbb{R}_{<0} \cup \{0\} \cup \mathbb{R}_{>0}$. Consider the composite map*

$$(p_{Y,1}^{\text{an}} \circ \text{ev}_3): \mathcal{M}^p(\widehat{L}^0, \beta, T) \rightarrow (\mathbb{P}_k^1)^{\text{an}}.$$

Let $\mathcal{M}^p(\widehat{L}^0, \beta, T)_1 := (p_{Y,1}^{\text{an}} \circ \text{ev}_3)^{-1}(\mathbb{G}_{m/k}^{\text{an}})$. Let $\mathcal{M}^p(\widehat{L}^0, \beta)_0$ be as in Proposition 5.1. Then we have $\mathcal{M}^p(\widehat{L}^0, \beta, T)_1 \subset \mathcal{M}^p(\widehat{L}^0, \beta)_0^{\text{an}}$.

Proof. Under Assumption 5.5, [23, Lemma 5.9(1.2)] implies that there exists an ample effective divisor A on Y' with support equal to $\partial Y'$. As \tilde{Y} is a toric blowup of $\mathbb{P}_k^1 \times Y'$, we deduce that there also exists an ample effective divisor \tilde{A} on \tilde{Y} with support equal to $\partial \tilde{Y}$.

Let $\tilde{\tau}: (\tilde{Y} \setminus \partial \tilde{Y})^{\text{an}} \rightarrow \tilde{B}$ denote the retraction map. Let $[C, (s_1, s_2, s_3), f: C \rightarrow \tilde{Y}^{\text{an}}]$ be a stable map in $\mathcal{M}^{\text{p}}(\hat{L}^0, \beta, T)_1$. By definition, there is an irreducible component C_0 of C with three marked points $\{\bar{s}_1, \bar{s}_2, \bar{s}_3\}$ such that the restriction of $(\tilde{\tau} \circ f)$ to the path $(\bar{s}_1, \bar{s}_2) \subset C_0$ (excluding the endpoints) is equal to

$$\tilde{h}^0 := (\iota, \hat{h}^0): (\hat{\Gamma}^0)^\circ \rightarrow \mathbb{R} \times B = \tilde{B},$$

where $\iota: (\hat{\Gamma}^0)^\circ \rightarrow \mathbb{R}$ is the homeomorphism preserving the \mathbb{Z} -affine structures. Therefore, by Construction 4.2(1), for any $\gamma \in \text{NE}(\tilde{Y})$, we have

$$\gamma \cdot \tilde{A} \leq [f(C_0)] \cdot \tilde{A}.$$

Since $[f(C)] \cdot \tilde{A} = \gamma \cdot \tilde{A}$ for some $\gamma \in \text{NE}(\tilde{Y})$ and \tilde{A} is ample, for any irreducible component E of C other than C_0 , $f|_E$ is necessarily constant. This implies that the pointed domain curve $[C, (s_1, s_2, s_3)]$ is stable. Thus composing f with the projection $p_{Y,2}: \tilde{Y} \rightarrow Y'$, we obtain a stable map in $\mathcal{M}(\hat{L}^0, \beta, T)$. Now we conclude from Lemma 5.6. \square

Lemma 5.8. *The k -analytic stacks $\mathcal{M}(\hat{L}^0, \beta, T)$, $\mathcal{M}^{\text{p}}(\hat{L}^0, \beta, T)_1$ and $\mathcal{M}^{\text{d}}(\beta, T)$ are k -analytic spaces.*

Proof. The stack $\mathcal{M}(\hat{L}^0, \beta)$ is an algebraic Deligne-Mumford stack over k , so it is a quotient of an étale groupoid in k -schemes of finite type. Then its analytification $\mathcal{M}(\hat{L}^0, \beta)^{\text{an}}$ is a quotient of an étale groupoid in k -analytic spaces, so is the analytic domain $\mathcal{M}(\hat{L}^0, \beta, T)$ in $\mathcal{M}(\hat{L}^0, \beta)^{\text{an}}$. Note that a stable pointed rational curve does not have non-trivial automorphisms. So by Lemma 5.6, the points in $\mathcal{M}(\hat{L}^0, \beta, T)$ do not have non-trivial automorphisms. Then we can write $\mathcal{M}(\hat{L}^0, \beta, T)$ as a quotient of an étale equivalence relation in k -analytic spaces. Since the k -analytic stack $\mathcal{M}(\hat{L}^0, \beta, T)$ is separated, it follows from [15, Theorem 1.2.2] that it is isomorphic to a k -analytic space. Using Lemma 5.7 and Lemma 5.6, the same argument shows that the k -analytic stacks $\mathcal{M}^{\text{p}}(\hat{L}^0, \beta, T)_1$ and $\mathcal{M}^{\text{d}}(\beta, T)$ are also k -analytic spaces. \square

We recall that a map of k -analytic spaces $\phi: X' \rightarrow X$ is said to be *finite* if there is an affinoid G -covering $\{V_i\}$ of X such that every $\phi^{-1}(V_i) \rightarrow V_i$ is a finite map of k -affinoid spaces, i.e. the corresponding homomorphism of k -affinoid algebras is finite (see [8, §1.3] for the G -topology).

We refer to [8, 16] for the notion of étale map of k -analytic spaces. For a finite étale map of k -analytic spaces $\phi: X' \rightarrow X$, we have an affinoid G -covering $\{V_i\}$ of X such that every $\phi^{-1}(V_i) \rightarrow V_i$ is a finite étale map of k -affinoid spaces, i.e. the corresponding homomorphism of k -affinoid algebras is finite étale. Therefore, $\phi_*\mathcal{O}_{X'}$ is a finite G -locally free \mathcal{O}_X -module. We define the *degree* of the finite étale map ϕ to be the rank of $\phi_*\mathcal{O}_{X'}$ viewed as a finite G -locally free \mathcal{O}_X -module.

By Lemma 5.8, all the moduli stacks involved in the following proposition and the following theorem are k -analytic spaces.

Proposition 5.9. *There exists three Zariski dense open subsets $V^0 \subset W^0$, $V^p \subset W^p$ and $V^d \subset \tilde{\Delta} \times W^d$, such that the three maps*

$$\begin{aligned} \text{ev}_3: \mathcal{M}(\widehat{L}^0, \beta, T)_W &\rightarrow W^0 \\ \text{ev}_3: \mathcal{M}^p(\widehat{L}^0, \beta, T)_W &\rightarrow W^p \\ \Phi = (\text{st}, \text{ev}_5): \mathcal{M}^d(\beta, T)_W &\rightarrow \tilde{\Delta} \times W^d \end{aligned}$$

considered in Section 4 are finite étale over V^0 , V^p and V^d respectively. Moreover, we can require that the projection $V^d \rightarrow \tilde{\Delta}$ is surjective. We will denote respectively by $\mathcal{M}(\widehat{L}^0, \beta, T)_V$, $\mathcal{M}^p(\widehat{L}^0, \beta, T)_V$ and $\mathcal{M}^d(\beta, T)_V$ the preimages of V^0, V^p and V^d under the three maps considered above.

Proof. Let us prove for the first map. Proposition 5.1 and Lemma 5.6 imply that there exists a Zariski dense open subset $V^0 \subset W^0$ such that the first map is smooth over V^0 . Using the notations in the proof of Proposition 5.1, the dimension of the space $\mathcal{M}(\widehat{L}^0, \beta, T)_V$ is equal to the dimension of $H^0(C, E)$. Using the Riemann-Roch formula, we compute that

$$\dim(H^0(C, E)) = \chi(C, E) = \text{rank } E + \text{deg } E = 2 + 0 = 2.$$

So the dimension of $\mathcal{M}(\widehat{L}^0, \beta, T)_V$ equals the dimension of W^0 . Thus the map $\text{ev}_3: \mathcal{M}(\widehat{L}^0, \beta, T)_W \rightarrow W^0$ is étale over V^0 . Moreover, the map $\text{ev}_3: \mathcal{M}(\widehat{L}^0, \beta, T)_W \rightarrow W^0$ is proper by Proposition 4.12. So it is finite over V^0 by [7, Corollary 3.3.8]. To conclude, the map $\text{ev}_3: \mathcal{M}(\widehat{L}^0, \beta, T)_W \rightarrow W^0$ is finite étale over V^0 .

For the second map, the same reasoning applies, using Proposition 5.1, Lemma 5.7 and Proposition 4.12.

Now we consider the third map. By Proposition 4.15, the map

$$\Phi: \mathcal{M}^d(\beta, T)_W \rightarrow \tilde{\Delta} \times W^d$$

is proper. Let $\mathcal{M}_{\text{sm}}^{\text{d}}(\beta)_0 \subset \mathcal{M}^{\text{d}}(\beta)_0 \subset \mathcal{M}^{\text{d}}(\beta)$ be as in Proposition 5.3. By Lemma 5.6, we have $\mathcal{M}^{\text{d}}(\beta, T) \subset \mathcal{M}^{\text{d}}(\beta)_0^{\text{an}}$. Let

$$V^{\text{d}} := \tilde{\Delta} \times W^{\text{d}} \setminus \Phi\left(\mathcal{M}^{\text{d}}(\beta, T)_W \setminus \mathcal{M}_{\text{sm}}^{\text{d}}(\beta)_0^{\text{an}}\right).$$

The properness of Φ implies that $V^{\text{d}} \subset \tilde{\Delta} \times W^{\text{d}}$ is Zariski open. Proposition 5.3 shows that Φ is smooth over V^{d} and that the projection $V^{\text{d}} \rightarrow \tilde{\Delta}$ is surjective. Using the notations in the proof of Proposition 5.3, the dimension of $\mathcal{M}^{\text{d}}(\beta, T)_V$ is equal to $h^0(C, E) + \dim \overline{\mathcal{M}}_{0,5}$. Using the Riemann-Roch formula again, we see that the dimension of $\mathcal{M}^{\text{d}}(\beta, T)_V$ is equal to the dimension of $\tilde{\Delta} \times W^{\text{d}}$. Combining with smoothness, we conclude that Φ is étale over V^{d} . Finally, combining with properness, we conclude that Φ is finite étale over V^{d} . \square

Theorem 5.10. *Let β' denote the curve class associated to the extension from L^0 to \widehat{L}^0 (see Construction 3.9). Let $\beta^0 := \beta - \beta'$. The finite étale maps*

$$\begin{aligned} \text{ev}_3: \mathcal{M}(\widehat{L}^0, \beta, T)_V &\rightarrow V^0, \text{ and} \\ \text{ev}_3: \mathcal{M}^{\text{p}}(\widehat{L}^0, \beta, T)_V &\rightarrow V^{\text{p}} \end{aligned}$$

have the same degree. The degree is equal to the number $N(L^0, \beta^0)$ of holomorphic cylinders associated to the spine L^0 and the curve class β^0 . As a consequence, the number $N(L^0, \beta^0)$ is a nonnegative integer.

Remark 5.11. Here β is the curve class of the rational stable maps, while $\beta^0 := \beta - \beta'$ is the curve class of the holomorphic cylinders, because β' represents the sum of curve classes of the two auxiliary disks glued to our cylinders in order to make them into rational curves. The notion of curve class of a curve with boundary is defined via specialization as in [54, Definition 5.10].

Proof of Theorem 5.10. Let $\mathcal{M}(\widehat{L}^0, \beta)_0$ and $\mathcal{M}^{\text{p}}(\widehat{L}^0, \beta)_0$ be as in Proposition 5.1. Recall that we have two projections $p_{Y,1}: \widetilde{Y} \rightarrow \mathbb{P}_k^1$ and $p_{Y,2}: \widetilde{Y} \rightarrow Y'$. Composing the evaluation map of the third marked point with the first projection, we get a natural map

$$p_{Y,1} \circ \text{ev}_3: \mathcal{M}^{\text{p}}(\widehat{L}^0, \beta)_0 \rightarrow \mathbb{P}_k^1.$$

Composing the stable maps with the second projection, we get a natural map

$$\pi_2: \mathcal{M}^{\text{p}}(\widehat{L}^0, \beta)_0 \rightarrow \mathcal{M}(\widehat{L}^0, \beta)_0.$$

Combining the two, we obtain a natural map

$$\Psi := (p_{Y,1} \circ \text{ev}_3, \pi_2): \mathcal{M}^{\text{p}}(\widehat{L}^0, \beta)_0 \rightarrow \mathbb{P}_k^1 \times \mathcal{M}(\widehat{L}^0, \beta)_0.$$

By construction, we see that Ψ is an isomorphism over $\mathbb{G}_{m/k} \times \mathcal{M}(\widehat{L}^0, \beta)_0$. Combining with Lemmas 5.6 and 5.7, we deduce that the finite étale maps

$$\begin{aligned} \text{ev}_3: \mathcal{M}(\widehat{L}^0, \beta, T)_V &\rightarrow V^0, \text{ and} \\ \text{ev}_3: \mathcal{M}^p(\widehat{L}^0, \beta, T)_V &\rightarrow V^p \end{aligned}$$

have the same degree. The construction of the number $N(L^0, \beta^0)$ in [54], as reviewed in Section 2, involves a virtual fundamental class. Here in our situation, it follows from étaleness that the virtual fundamental class is equal to the fundamental class (see [6, Example after Proposition 5.3]). Therefore, the number $N(L^0, \beta^0)$ is equal to the degree of the finite étale maps above. As a consequence, the number $N(L^0, \beta^0)$ is a nonnegative integer. \square

Remark 5.12. Recall that we have the extended spine

$$\widehat{L}^0 = (\widehat{\Gamma}^0, (u_1^0, u_2^0), \widehat{h}^0: (\widehat{\Gamma}^0)^\circ \rightarrow B)$$

associated to the spine $L^0 = (\Gamma^0, (v_1^0, v_2^0), h^0: \Gamma^0 \rightarrow B)$. Let $r \in \widehat{\Gamma}^0$ be any point in the interior of an edge of $\widehat{\Gamma}^0$, or any bounded vertex where the \mathbb{Z} -affine map \widehat{h}^0 is balanced. Fix a point $v \in V^0 \subset X_k^{\text{an}}$. Let v' be a general point in the fiber of $\tau: X_k^{\text{an}} \rightarrow B$ over $\widehat{h}^0(r) \in B$. Let $\mathcal{M}(\widehat{L}^0, \beta, T)_v$ and $\mathcal{M}(\widehat{L}^0, \beta, T)_{v'}$ denote respectively the fibers of the evaluation map

$$\text{ev}_3: \mathcal{M}(\widehat{L}^0, \beta, T) \rightarrow (Y')^{\text{an}}$$

over the points v and v' in $(Y')^{\text{an}}$. The symmetry property of [54, Theorem 6.3] implies that

$$\deg \mathcal{M}(\widehat{L}^0, \beta, T)_v = \deg \mathcal{M}(\widehat{L}^0, \beta, T)_{v'}$$

when r is the vertex $v_2^0 \in \widehat{\Gamma}^0$. In fact, the same proof as that of [54, Theorem 6.3] shows that the equality above always holds.

6. THE GLUING FORMULA

With the preparations in the previous sections, we will prove the gluing formula in this section.

Consider the moduli space $\overline{\mathcal{M}}_{0,5}$ of 5-pointed rational stable curves over k . We label the 5 marked points by $s_1^1, s_2^1, s_1^2, s_2^2$ and s_5 . The graph G (resp. G') and the labeling of its legs in Construction 3.14 give rise to a 0-dimensional stratum in $\overline{\mathcal{M}}_{0,5}$ which we denote by m (resp. m').

Let V^d be as in Proposition 5.9 and consider the projection $V^d \rightarrow \tilde{\Delta}$. Let w be a point in the fiber over m and let w' be a point in the fiber over m' . Let \bar{w} and \bar{w}' be respectively the images of w and w' under the projection $V^d \rightarrow W^d$.

Consider the map

$$\Phi = (\text{st}, \text{ev}_5): \mathcal{M}^d(\beta, T)_V \rightarrow V^d.$$

Let $\mathcal{M}^d(\beta, T)_w$ and $\mathcal{M}^d(\beta, T)_{w'}$ be the two fibers over the points w and w' respectively.

Proposition 6.1. *Let $[C, (s_1^1, s_2^1, s_1^2, s_2^2, s_5), f: C \rightarrow (Y')^{\text{an}}]$ be a stable map in $\mathcal{M}^d(\beta, T)_w$. Let C^5 be the irreducible component of C containing the marked point s_5 . Then we have $f(C^5) = \bar{w}$. The same applies to w' .*

Proof. Since $w \in V^d$ lies in the fiber over $m \in \overline{\mathcal{M}}_{0,5}$, the irreducible component C^5 does not contain the other marked points $s_1^1, s_2^1, s_1^2, s_2^2$. By Lemma 4.13(2), the image $f(C^5)$ lies in the interior $(Y' \setminus \partial Y')^{\text{an}} \simeq X_k^{\text{an}}$. The map $f|_{C^5}: C^5 \rightarrow X_k^{\text{an}}$ gives rise to a tropical curve $h: \Gamma \rightarrow B$, where Γ is a \mathbb{Z} -affine tree and h is a \mathbb{Z} -affine immersion that is balanced everywhere outside of $h^{-1}(O)$.

We claim that Γ must be a point. Suppose the contrary. Then since h is an immersion, the image $h(\Gamma)$ is not a point. So by Lemma 3.24, the image $h(\Gamma)$ is contained in a ray R in B starting from $O \in B$ with rational slope. Let b be the point in $h(\Gamma)$ that is farthest from O , and let v be a vertex in Γ that maps to b . Then h cannot be balanced at v , which is a contradiction.

Since $f(s_5) = \bar{w}$, the claim implies that the image $h(\Gamma)$ is equal to the point $\tau(\bar{w}) \in B$. Since the preimage $\tau^{-1}(\tau(\bar{w}))$ is affinoid, it cannot contain any proper curve. So we must have $f(C^5) = \bar{w}$, completing the proof. The same argument applies to w' . \square

Consider the evaluation map of the third marked point

$$\text{ev}_3: \mathcal{M}(\widehat{L}^1, \beta, T) \rightarrow (Y')^{\text{an}}.$$

Let $\mathcal{M}(\widehat{L}^1, \beta, T)_{\bar{w}'}$ be the fiber over the point $\bar{w}' \in W^d$. Similarly, we define $\mathcal{M}(\widehat{L}^2, \beta, T)_{\bar{w}'}$, $\mathcal{M}(\widehat{L}^3, \beta, T)_{\bar{w}'}$ and $\mathcal{M}(\widehat{L}^4, \beta, T)_{\bar{w}'}$.

By the construction of the subset $V^d \subset \tilde{\Delta} \times W^d$ in Proposition 5.9, the space $\mathcal{M}(\widehat{L}^2, \beta, T)_{\bar{w}'}$ (resp. $\mathcal{M}(\widehat{L}^3, \beta, T)_{\bar{w}'}$, $\mathcal{M}(\widehat{L}^4, \beta, T)_{\bar{w}'}$) is finite étale over the point \bar{w}' (resp. \bar{w}, \bar{w}).

For $i = 1, \dots, 3$, let $(\beta^i)'$ denote the curve class associated to the extension of the spine from L^i to \widehat{L}^i (see [54, Definition 4.10]).

Proposition 6.2. *We have the following equalities:*

$$\begin{aligned} \deg \mathcal{M}(\widehat{L}^1, \beta, T)_{\overline{w'}} &= N(L^1, \beta - (\beta^1)'), \\ \deg \mathcal{M}(\widehat{L}^2, \beta, T)_{\overline{w'}} &= N(L^2, \beta - (\beta^2)'), \\ \deg \mathcal{M}(\widehat{L}^3, \beta, T)_{\overline{w}} &= N(L^3, \beta - (\beta^3)'). \end{aligned}$$

Proof. Let v be a general point in the fiber of the map $\tau: X_k^{\text{an}} \rightarrow B$ over $h^1(v_1^1) \in B$. Let $\mathcal{M}(\widehat{L}^1, \beta, T)_v$ (resp. $\mathcal{M}(\widehat{L}^3, \beta, T)_v$) be the fiber of the evaluation map

$$\text{ev}_3: \mathcal{M}(\widehat{L}^1, \beta, T) \rightarrow (Y')^{\text{an}} \quad (\text{resp. } \text{ev}_3: \mathcal{M}(\widehat{L}^3, \beta, T) \rightarrow (Y')^{\text{an}})$$

over the point v . From Theorem 5.10, replacing L^0 with L^1, L^2, L^3 respectively, we deduce the following equalities

$$\begin{aligned} \deg \mathcal{M}(\widehat{L}^1, \beta, T)_v &= N(L^1, \beta - (\beta^1)'), \\ \deg \mathcal{M}(\widehat{L}^2, \beta, T)_{\overline{w'}} &= N(L^2, \beta - (\beta^2)'), \\ \deg \mathcal{M}(\widehat{L}^3, \beta, T)_v &= N(L^3, \beta - (\beta^3)'). \end{aligned}$$

By the symmetry property in [54, §6], as we recalled in Remark 5.12, we have

$$\begin{aligned} \deg \mathcal{M}(\widehat{L}^1, \beta, T)_v &= \deg \mathcal{M}(\widehat{L}^1, \beta, T)_{\overline{w'}}, \\ \deg \mathcal{M}(\widehat{L}^3, \beta, T)_v &= \deg \mathcal{M}(\widehat{L}^3, \beta, T)_{\overline{w}}. \end{aligned}$$

We conclude the proof by combining all the equalities above. \square

Lemmas 6.3, 6.4 and Proposition 6.5 will serve the proof of Proposition 6.6.

Lemma 6.3. *Let $\mathbb{G}_{\mathfrak{m}/k}^n := \text{Spec } k[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$. Let $\sigma: (\mathbb{G}_{\mathfrak{m}/k}^n)^{\text{an}} \rightarrow \mathbb{R}^n$ be the continuous map given by taking coordinate-wise valuations. Let $\Omega \subset \mathbb{R}^n$ be a rational convex polyhedron, i.e., it is given by finitely many linear inequalities with rational coefficients and is compact. Assume that Ω has nonempty interior. Let T_Ω denote the affinoid algebra of the affinoid space $\sigma^{-1}(\Omega)$. Let ϕ be an automorphism of $\sigma^{-1}(\Omega)$ preserving the map σ . Then for any $\gamma \in \mathbb{Z}^n$, we have $\phi^*(X^\gamma) = c_\gamma \cdot X^\gamma(1 + f_\gamma)$, where c_γ is an element in k of norm 1, and f_γ is an element in T_Ω of spectral norm less than 1.*

Proof. Write $\phi^*(X^\gamma) = \sum_{\nu \in \mathbb{Z}^n} c_\nu X^\nu$ with $c_\nu \in k$. Since ϕ preserves the map σ , $\phi^*(X^\gamma)$ has no zeros in $\sigma^{-1}(\Omega)$. We claim that there exists $\nu_0 \in \mathbb{Z}^n$ such that $|(c_{\nu_0} X^{\nu_0})(x)| > |(c_\nu X^\nu)(x)|$ for all $\nu \neq \nu_0$ and all rigid points (i.e. points associated to maximal ideals) $x \in \sigma^{-1}(\Omega)$. Otherwise, there is a rigid point $x \in \sigma^{-1}(\Omega)$, $\nu_1, \nu_2 \in \mathbb{Z}^n$ such that $|(c_{\nu_1} X^{\nu_1})(x)| = |(c_{\nu_2} X^{\nu_2})(x)| \geq |(c_\nu X^\nu)(x)|$ for all $\nu \in \mathbb{Z}^n$. Fixing the first

$(n - 1)$ coordinates of x and varying the last one, by [9, Lemma 9.7.1/1], we see that $\phi^*(X^\gamma)$ is not invertible, which is a contradiction. So we have justified our claim. Using again the assumption that ϕ preserves the map σ , we see that $\nu_0 = \gamma$ and $|c_\gamma| = 1$, completing the proof. \square

Lemma 6.4. *Let $\mathbb{A}_k^n := \text{Spec } k[X_1, \dots, X_n]$. Let $\bar{\sigma}: (\mathbb{A}_k^n)^{\text{an}} \rightarrow (-\infty, +\infty]^n$ be the continuous map given by taking coordinate-wise valuations. Let $\bar{\Omega} \subset (-\infty, +\infty]^n$ be a rational convex polyhedron, i.e., it is given by finitely many linear inequalities with rational coefficients and is compact. Assume that $\bar{\Omega}$ has nonempty interior. Let $T_{\bar{\Omega}}$ be the affinoid algebra of the affinoid space $\bar{\sigma}^{-1}(\bar{\Omega})$. Let $f_1, \dots, f_n \in T_{\bar{\Omega}}$ be n elements of spectral norm less than 1. Let φ be the endomorphism of $T_{\bar{\Omega}}$ given by $X_i \mapsto X_i(1 + f_i)$. Then φ is an automorphism. It induces an automorphism of $\bar{\sigma}^{-1}(\bar{\Omega})$ preserving the map $\bar{\sigma}$.*

Proof. Decompose $\varphi = \text{id} + B$ as an operator on the Banach space $T_{\bar{\Omega}}$. Since f_i has spectral norm less than 1, we see that B has operator norm less than 1. Therefore, the operator φ has an inverse $\varphi^{-1} = \sum_{j=0}^{\infty} (-1)^j B^j$. Since φ preserves the multiplication on $T_{\bar{\Omega}}$, so does φ^{-1} . We conclude that φ is an automorphism of the affinoid algebra $T_{\bar{\Omega}}$ and it induces an automorphism of the affinoid space $\bar{\sigma}^{-1}(\bar{\Omega})$ preserving the map $\bar{\sigma}$. \square

Proposition 6.5. *Let $n \in \mathbb{Z}_{\geq 0}$, $r_1, \dots, r_n, r'_1, \dots, r'_n, p, q, s, t \in \sqrt{|k|} \cup \{+\infty\}$, (the notation $a \in \sqrt{|k|}$ means that some positive integer power of a lies in $|k|$). Assume $0 < r'_i < r_i < +\infty$, $0 \leq p < s < t < q \leq +\infty$. Let $R := \prod_{i=1}^n [r'_i, r_i] \subset \mathbb{R}^n$. Let X be a k -analytic space and $\rho: X \rightarrow R \times [p, q]$ a continuous map. Let $\pi: (\text{Spec } k[T_1^{\pm 1}, \dots, T_n^{\pm 1}, S^{\pm 1}])^{\text{an}} \rightarrow (0, +\infty)^{n+1}$ be the continuous map given by taking coordinate-wise norms. Let $\bar{\pi}: ((\mathbb{P}_k^1)^{\text{an}})^{n+1} \rightarrow [0, +\infty]^{n+1}$ be the continuous extension of π . Assume that $\rho^{-1}(R \times [p, t])$ and $\rho^{-1}(R \times [s, q])$ are affinoid subdomains of X , and we have isomorphisms*

$$\begin{array}{ccc}
 \rho^{-1}(R \times [p, t]) & \xrightarrow{\alpha \sim} & \bar{\pi}^{-1}(R \times [p, t]) \\
 \searrow \rho & & \swarrow \bar{\pi} \\
 & R \times [p, t] &
 \end{array}$$

and

$$\begin{array}{ccc}
 \rho^{-1}(R \times [s, q]) & \xrightarrow[\sim]{\beta} & \bar{\pi}^{-1}(R \times [s, q]) \\
 \searrow \rho & & \swarrow \bar{\pi} \\
 & R \times [s, q] &
 \end{array}$$

Then we have an isomorphism

$$\begin{array}{ccc}
 X & \xrightarrow{\sim} & \bar{\pi}^{-1}(R \times [p, q]) \\
 \searrow \rho & & \swarrow \bar{\pi} \\
 & R \times [p, q] &
 \end{array}$$

Proof. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be the affinoid algebras of the affinoid spaces $\rho^{-1}(R \times [s, t]), \rho^{-1}(R \times [p, t]), \rho^{-1}(R \times [s, q])$ respectively. Let $\|\cdot\|_{\mathcal{A}}, \|\cdot\|_{\mathcal{B}}, \|\cdot\|_{\mathcal{C}}$ denote the spectral norms on $\mathcal{A}, \mathcal{B}, \mathcal{C}$ respectively. Assume that the isomorphisms α and β are given by the following isomorphisms of affinoid algebras:

$$\begin{aligned}
 k\langle r_1^{-1}T_1, r_1' T_1^{-1}, \dots, r_n^{-1}T_n, r_n' T_n^{-1}, t^{-1}S, pS^{-1} \rangle &\xrightarrow{\phi} \mathcal{B}, \quad T_i \mapsto f_i, S \mapsto f, \\
 k\langle r_1^{-1}T_1, r_1' T_1^{-1}, \dots, r_n^{-1}T_n, r_n' T_n^{-1}, sS', q^{-1}(S')^{-1} \rangle &\xrightarrow{\psi} \mathcal{C}, \quad T_i \mapsto g_i, S' \mapsto g.
 \end{aligned}$$

In the case $p = 0$, we remove the term pS^{-1} from the expression above. In the case $q = +\infty$, we remove the term $q^{-1}(S')^{-1}$ from the expression above. Consider the isomorphism

$$\mathcal{B}\langle sf^{-1} \rangle \rightarrow \mathcal{C}\langle t^{-1}g^{-1} \rangle, \quad f_i \mapsto g_i, f \mapsto g^{-1}.$$

Up to multiplying g_1, \dots, g_n and g by elements in k of norm 1, it follows from Lemma 6.3 that we can assume that $\|f_i - g_i\|_{\mathcal{A}} < \|f_i\|_{\mathcal{A}}$ and $\|f - g^{-1}\|_{\mathcal{A}} < \|f\|_{\mathcal{A}}$. Now we proceed as in the proof of [50, Lemma 3.2]. Let \mathcal{B}_+ and \mathcal{C}_+ be the subspaces of \mathcal{B} and \mathcal{C} consisting of elements of the form $\sum_{\nu \in \mathbb{Z}^n, j \geq 0} \lambda_{\nu, j} f_1^{\nu_1} \cdots f_n^{\nu_n} f^j$ and $\sum_{\nu \in \mathbb{Z}^n, j \geq 0} \mu_{\nu, j} g_1^{\nu_1} \cdots g_n^{\nu_n} g^j$ respectively. It is shown in the proof of [50, Lemma 3.2] that each element $a \in \mathcal{A}$ can be decomposed as $a = b + c$ with $b \in \mathcal{B}_+, c \in \mathcal{C}_+$, and $\|b\|_{\mathcal{B}}, \|c\|_{\mathcal{C}} \leq \|a\|_{\mathcal{A}}$. So we have $f_i - g_i = b_i + c_i$ and $f - g^{-1} = b + c$, where $b_i, b \in \mathcal{B}_+, c_i, c \in \mathcal{C}_+, \|b_i\|_{\mathcal{B}}, \|c_i\|_{\mathcal{C}} \leq \|f_i - g_i\|_{\mathcal{A}}$ and $\|b\|_{\mathcal{B}}, \|c\|_{\mathcal{C}} \leq \|f - g^{-1}\|_{\mathcal{A}}$. By Lemma 6.4, the homomorphisms

$$\begin{aligned}
 k\langle r_1^{-1}T_1, r_1' T_1^{-1}, \dots, r_n^{-1}T_n, r_n' T_n^{-1}, t^{-1}S, pS^{-1} \rangle &\xrightarrow{\phi} \mathcal{B}, \quad T_i \mapsto f_i - b_i, S \mapsto f - b, \\
 k\langle r_1^{-1}T_1, r_1' T_1^{-1}, \dots, r_n^{-1}T_n, r_n' T_n^{-1}, sS', q^{-1}(S')^{-1} \rangle &\xrightarrow{\psi} \mathcal{C}, \\
 T_i \mapsto g_i + c_i, S' \mapsto (g^{-1} + c)^{-1} &= g(1 + gc)^{-1}
 \end{aligned}$$

are isomorphisms. So we deduce the isomorphism claimed in the proposition. \square

Proposition 6.6. *Let $\beta^4 = (\beta^1)' + (\beta^2)' - (\beta^3)'$. We have*

$$\deg \mathcal{M}(\widehat{L}^4, \beta, T)_{\bar{w}} = \begin{cases} 1 & \text{for } \beta = \beta^4, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For any rigid point $w^4 \in \widetilde{Y}^{\text{an}}$, let $\mathcal{M}^{\text{P}}(\widehat{L}^4, \beta, T)_{w^4}$ be the fiber of the evaluation map

$$\text{ev}_3: \mathcal{M}^{\text{P}}(\widehat{L}^4, \beta, T) \rightarrow \widetilde{Y}^{\text{an}}$$

over w^4 . By Proposition 5.9 and Theorem 5.10, replacing \widehat{L}^0 with \widehat{L}^4 , there exists a rigid point $w^4 \in \mathbb{G}_{\text{m}/k}^{\text{an}} \times W^{\text{d}} \subset \widetilde{Y}^{\text{an}}$ such that $\mathcal{M}^{\text{P}}(\widehat{L}^4, \beta, T)_{w^4}$ is finite étale over w^4 and its degree equals the degree of $\mathcal{M}(\widehat{L}^4, \beta, T)_{\bar{w}}$.

Recall the notation

$$\widehat{L}^4 = [\widehat{\Gamma}^4, (u_1^4, u_2^4), \widehat{h}^4: (\widehat{\Gamma}^4)^\circ \rightarrow B].$$

Let \widetilde{B} and \widetilde{B}' be as in Construction 3.12. For each d -dimensional simplicial cone in \widetilde{B}' , we compactify it as $[0, +\infty)^d \subset [0, +\infty]^d$. The gluing of all such compactifications gives a natural compactification of \widetilde{B}' , which we denote by \mathbf{B} . Note that \mathbf{B} is isomorphic to the closure of \widetilde{B}' inside $\widetilde{Y}^{\text{an}}$ (see [30, Remark 4.12]). The retraction map $\tilde{\tau}: (\widetilde{Y} \setminus \partial\widetilde{Y})^{\text{an}} \rightarrow \widetilde{B}'$ extends to a retraction map $\sigma: \widetilde{Y}^{\text{an}} \rightarrow \mathbf{B}$ (see [30, Theorem 4.13]).

Let

$$h := (\iota, \widehat{h}^4): (\widehat{\Gamma}^4)^\circ \rightarrow \mathbb{R} \times B = \widetilde{B},$$

where the map $\iota: (\widehat{\Gamma}^4)^\circ \rightarrow \mathbb{R}$ is the homeomorphism preserving the \mathbb{Z} -affine structures. Let x_0 be a point in $\widehat{\Gamma}^4$ and let γ be a nonzero tangent vector at x_0 . Let δ and ϵ be two small rational tangent vectors at $h(x_0) \in \widetilde{B}$, so that together with $h(\gamma)$, they generate the tangent space of \widetilde{B} at $h(x_0)$.

Let

$$U := \left\{ x + a\delta_x + b\epsilon_x \mid x \in (\widehat{\Gamma}^4)^\circ, a, b \in [0, 1] \right\} \subset \widetilde{B},$$

where δ_x and ϵ_x denote respectively the parallel transport of δ and ϵ along $h((\widehat{\Gamma}^4)^\circ)$ to the point $h(x)$. We choose δ and ϵ sufficiently small, so that U is disjoint from $\mathbb{R} \times \{O\} \subset \widetilde{B}$.

Let \bar{U} be the closure of U in \mathbf{B} and let $\widetilde{U} := \sigma^{-1}(\bar{U}) \subset \widetilde{Y}^{\text{an}}$. Note that the retraction map $\sigma: \widetilde{Y}^{\text{an}} \rightarrow \mathbf{B}$, outside $[-\infty, +\infty] \times \{O\} \subset \mathbf{B}$, locally on the target, is modeled on the coordinate-wise valuation map $(\mathbb{P}_k^1)^{\text{an}} \rightarrow [-\infty, +\infty]$. Therefore, by Proposition 6.5, the analytic domain $\widetilde{U} \subset \widetilde{Y}^{\text{an}}$ is isomorphic to the product of a

projective line $(\mathbb{P}_k^1)^{\text{an}}$ with a 2-dimensional polyannulus. By construction, for any stable map in $\mathcal{M}^{\text{P}}(\widehat{L}^4, \beta, T)_{w^4}$, its image lies in \widetilde{U} , and its degree over $(\mathbb{P}_k^1)^{\text{an}}$ is equal to 1. So its curve class lies in $s^{\text{P}}(\widehat{L}^4, \beta^4)$. Moreover, the product structure of \widetilde{U} implies that when $\beta = \beta^4$, the degree of $\mathcal{M}^{\text{P}}(\widehat{L}^4, \beta, T)_{w^4}$ over w^4 is 1. Finally we conclude by Theorem 5.10. \square

Let $\overline{\mathcal{M}}_{0,3}(\bar{w})$ denote the stack of 3-pointed rational stable maps into the point \bar{w} . The stack $\overline{\mathcal{M}}_{0,3}(\bar{w})$ is just a point. Similarly, we denote $\overline{\mathcal{M}}_{0,3}(\bar{w}')$ for the point \bar{w}' . Let $\beta^{\text{d}} := \beta^3 + (\beta^1)' + (\beta^2)'$. It follows from Proposition 6.1 that we have two isomorphisms

$$(6.1) \quad \mathcal{M}^{\text{d}}(\beta^{\text{d}}, T)_{w'} \simeq \coprod_{\widehat{\beta}^1 + \widehat{\beta}^2 = \beta^{\text{d}}} \mathcal{M}(\widehat{L}^1, \widehat{\beta}^1, T)_{\bar{w}'} \times \overline{\mathcal{M}}_{0,3}(\bar{w}') \times \mathcal{M}(\widehat{L}^2, \widehat{\beta}^2, T)_{\bar{w}'},$$

$$(6.2) \quad \mathcal{M}^{\text{d}}(\beta^{\text{d}}, T)_w \simeq \coprod_{\widehat{\beta}^3 + \widehat{\beta}^4 = \beta^{\text{d}}} \mathcal{M}(\widehat{L}^3, \widehat{\beta}^3, T)_{\bar{w}} \times \overline{\mathcal{M}}_{0,3}(\bar{w}) \times \mathcal{M}(\widehat{L}^4, \widehat{\beta}^4, T)_{\bar{w}}.$$

By Propositions 6.2 and 6.6, we deduce from (6.1) and (6.2) the following equalities:

$$\begin{aligned} \deg \mathcal{M}^{\text{d}}(\beta^{\text{d}}, T)_{w'} &= \sum_{\widehat{\beta}^1 + \widehat{\beta}^2 = \beta^{\text{d}}} \deg \mathcal{M}(\widehat{L}^1, \widehat{\beta}^1, T)_{\bar{w}'} \cdot \deg \mathcal{M}(\widehat{L}^2, \widehat{\beta}^2, T)_{\bar{w}'} \\ &= \sum_{\widehat{\beta}^1 + \widehat{\beta}^2 = \beta^{\text{d}}} N(L^1, \widehat{\beta}^1 - (\beta^1)') \cdot N(L^2, \widehat{\beta}^2 - (\beta^2)') \\ (6.3) \quad &= \sum_{\beta^1 + \beta^2 = \beta^3} N(L^1, \beta^1) \cdot N(L^2, \beta^2), \end{aligned}$$

$$\begin{aligned} \deg \mathcal{M}^{\text{d}}(\beta^{\text{d}}, T)_w &= \sum_{\widehat{\beta}^3 + \widehat{\beta}^4 = \beta^{\text{d}}} \deg \mathcal{M}(\widehat{L}^3, \widehat{\beta}^3, T)_{\bar{w}} \cdot \deg \mathcal{M}(\widehat{L}^4, \widehat{\beta}^4, T)_{\bar{w}} \\ &= \deg \mathcal{M}(\widehat{L}^3, \beta^3 + (\beta^3)', T)_{\bar{w}} \\ (6.4) \quad &= N(L^3, \beta^3). \end{aligned}$$

By Proposition 5.9 and the connectedness of V^{d} , we have $\deg \mathcal{M}^{\text{d}}(\beta^{\text{d}}, T)_w = \deg \mathcal{M}^{\text{d}}(\beta^{\text{d}}, T)_{w'}$. Combining (6.3) and (6.4), we conclude the proof of Theorem 1.2.

REFERENCES

- [1] Mohammed Abouzaid. Family floer cohomology and mirror symmetry. *arXiv preprint arXiv:1404.2659*, 2014.
- [2] Dan Abramovich, Lucia Caporaso, and Sam Payne. The tropicalization of the moduli space of curves. *Ann. Sci. Éc. Norm. Supér. (4)*, 48(4):765–809, 2015.
- [3] Denis Auroux. Mirror symmetry and T -duality in the complement of an anticanonical divisor. *J. Gökova Geom. Topol. GGT*, 1:51–91, 2007.

- [4] Denis Auroux. Special Lagrangian fibrations, wall-crossing, and mirror symmetry. In *Surveys in differential geometry. Vol. XIII. Geometry, analysis, and algebraic geometry: forty years of the Journal of Differential Geometry*, volume 13 of *Surv. Differ. Geom.*, pages 1–47. Int. Press, Somerville, MA, 2009.
- [5] Matthew Baker, Sam Payne, and Joseph Rabinoff. Nonarchimedean geometry, tropicalization, and metrics on curves. *arXiv preprint arXiv:1104.0320*, 2011.
- [6] K. Behrend and B. Fantechi. The intrinsic normal cone. *Invent. Math.*, 128(1):45–88, 1997.
- [7] Vladimir G. Berkovich. *Spectral theory and analytic geometry over non-Archimedean fields*, volume 33 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1990.
- [8] Vladimir G. Berkovich. Étale cohomology for non-Archimedean analytic spaces. *Inst. Hautes Études Sci. Publ. Math.*, (78):5–161 (1994), 1993.
- [9] S. Bosch, U. Güntzer, and R. Remmert. *Non-Archimedean analysis*, volume 261 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1984. A systematic approach to rigid analytic geometry.
- [10] Andrea Brini, Renzo Cavalieri, and Dustin Ross. Crepant resolutions and open strings. *arXiv preprint arXiv:1309.4438*, 2013.
- [11] Michael Carl, Max Pumperla, and Bernd Siebert. A tropical view of landau-ginzburg models. Preprint, 2010.
- [12] Kwokwai Chan and Siu-Cheong Lau. Open Gromov-Witten invariants and superpotentials for semi-Fano toric surfaces. *Int. Math. Res. Not. IMRN*, (14):3759–3789, 2014.
- [13] Kwokwai Chan, Siu-Cheong Lau, Naichung Conan Leung, and Hsian-Hua Tseng. Open Gromov-Witten invariants, mirror maps, and Seidel representations for toric manifolds. *arXiv preprint arXiv:1209.6119*, 2012.
- [14] Qile Chen and Matthew Satriano. Chow quotients of toric varieties as moduli of stable log maps. *Algebra Number Theory*, 7(9):2313–2329, 2013.
- [15] Brian Conrad and Michael Temkin. Non-Archimedean analytification of algebraic spaces. *J. Algebraic Geom.*, 18(4):731–788, 2009.
- [16] Antoine Ducros. Families of Berkovich spaces. *arXiv preprint arXiv:1107.4259*, 2011.
- [17] Bohan Fang and Chiu-Chu Melissa Liu. Open Gromov-Witten invariants of toric Calabi-Yau 3-folds. *Comm. Math. Phys.*, 323(1):285–328, 2013.
- [18] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono. *Lagrangian intersection Floer theory: anomaly and obstruction*, volume 46 of *AMS/IP Studies in Advanced Mathematics*. American Mathematical Society, Providence, RI; International Press, Somerville, MA, 2009.
- [19] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono. Lagrangian Floer theory on compact toric manifolds: survey. In *Surveys in differential geometry. Vol. XVII*, volume 17 of *Surv. Differ. Geom.*, pages 229–298. Int. Press, Boston, MA, 2012.
- [20] W. Fulton and R. Pandharipande. Notes on stable maps and quantum cohomology. In *Algebraic geometry—Santa Cruz 1995*, volume 62 of *Proc. Sympos. Pure Math.*, pages 45–96. Amer. Math. Soc., Providence, RI, 1997.
- [21] Andreas Gathmann and Hannah Markwig. Kontsevich’s formula and the WDVV equations in tropical geometry. *Adv. Math.*, 217(2):537–560, 2008.

- [22] Mark Gross. Mirror symmetry for \mathbb{P}^2 and tropical geometry. *Adv. Math.*, 224(1):169–245, 2010.
- [23] Mark Gross, Paul Hacking, and Sean Keel. Mirror symmetry for log Calabi-Yau surfaces I. *arXiv preprint arXiv:1106.4977v1*, 2011.
- [24] Mark Gross, Paul Hacking, and Sean Keel. Mirror symmetry for log Calabi-Yau surfaces I. *Publications mathématiques de l’IHÉS*, pages 1–104, 2015.
- [25] Mark Gross, Paul Hacking, and Sean Keel. Moduli of surfaces with an anti-canonical cycle. *Compos. Math.*, 151(2):265–291, 2015.
- [26] Mark Gross, Paul Hacking, Sean Keel, and Maxim Kontsevich. Canonical bases for cluster algebras. *arXiv preprint arXiv:1411.1394*, 2014.
- [27] Mark Gross, Paul Hacking, Sean Keel, and Bernd Siebert. Theta functions on varieties with effective anti-canonical class. In progress, 2015.
- [28] Mark Gross and Bernd Siebert. From real affine geometry to complex geometry. *Ann. of Math. (2)*, 174(3):1301–1428, 2011.
- [29] Mark Gross and Bernd Siebert. Theta functions and mirror symmetry. *arXiv preprint arXiv:1204.1991*, 2012.
- [30] Walter Gubler, Joseph Rabinoff, and Annette Werner. Skeletons and tropicalizations. *arXiv preprint arXiv:1404.7044*, 2014.
- [31] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.
- [32] Seán Keel and James McKernan. Rational curves on quasi-projective surfaces. *Mem. Amer. Math. Soc.*, 140(669):viii+153, 1999.
- [33] G. Kempf, Finn Faye Knudsen, D. Mumford, and B. Saint-Donat. *Toroidal embeddings. I*. Lecture Notes in Mathematics, Vol. 339. Springer-Verlag, Berlin, 1973.
- [34] János Kollár. *Rational curves on algebraic varieties*, volume 32 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 1996.
- [35] M. Kontsevich and Yu. Manin. Gromov-Witten classes, quantum cohomology, and enumerative geometry. *Comm. Math. Phys.*, 164(3):525–562, 1994.
- [36] Maxim Kontsevich. Enumeration of rational curves via torus actions. In *The moduli space of curves (Texel Island, 1994)*, volume 129 of *Progr. Math.*, pages 335–368. Birkhäuser Boston, Boston, MA, 1995.
- [37] Maxim Kontsevich and Yan Soibelman. Homological mirror symmetry and torus fibrations. In *Symplectic geometry and mirror symmetry (Seoul, 2000)*, pages 203–263. World Sci. Publ., River Edge, NJ, 2001.
- [38] Maxim Kontsevich and Yan Soibelman. Affine structures and non-Archimedean analytic spaces. In *The unity of mathematics*, volume 244 of *Progr. Math.*, pages 321–385. Birkhäuser Boston, Boston, MA, 2006.
- [39] Yu-Shen Lin. Open Gromov-Witten invariants on elliptic K3 surfaces and wall-crossing. *arXiv preprint arXiv:1412.4455*, 2014.
- [40] Eduard Looijenga. Rational surfaces with an anticanonical cycle. *Ann. of Math. (2)*, 114(2):267–322, 1981.

- [41] Johannes Nicaise, Chenyang Xu, and Tony Yue Yu. The non-archimedean SYZ fibration. *Compos. Math.*, 155(5):953–972, 2019.
- [42] Takeo Nishinou. Disk counting on toric varieties via tropical curves. *Amer. J. Math.*, 134(6):1423–1472, 2012.
- [43] R. Pandharipande, J. Solomon, and J. Walcher. Disk enumeration on the quintic 3-fold. *J. Amer. Math. Soc.*, 21(4):1169–1209, 2008.
- [44] Mauro Porta and Tony Yue Yu. Higher analytic stacks and GAGA theorems. *Advances in Mathematics*, 302:351–409, 2016.
- [45] Dhruv Ranganathan. Skeletons of stable maps I: Rational curves in toric varieties. *arXiv preprint arXiv:1506.03754*, 2015.
- [46] Dhruv Ranganathan, Keli Santos-Parker, and Jonathan Wise. Moduli of stable maps in genus one and logarithmic geometry I. *arXiv preprint arXiv:1708.02359*, 2017.
- [47] Dhruv Ranganathan, Keli Santos-Parker, and Jonathan Wise. Moduli of stable maps in genus one and logarithmic geometry II. *arXiv preprint arXiv:1709.00490*, 2017.
- [48] Dustin Ross. Localization and gluing of orbifold amplitudes: the Gromov-Witten orbifold vertex. *Trans. Amer. Math. Soc.*, 366(3):1587–1620, 2014.
- [49] Andrew Strominger, Shing-Tung Yau, and Eric Zaslow. Mirror symmetry is T -duality. *Nuclear Phys. B*, 479(1-2):243–259, 1996.
- [50] M. Temkin. On local properties of non-Archimedean analytic spaces. *Math. Ann.*, 318(3):585–607, 2000.
- [51] Junwu Tu. On the reconstruction problem in mirror symmetry. *Adv. Math.*, 256:449–478, 2014.
- [52] Jean-Yves Welschinger. Open Gromov-Witten invariants in dimension six. *Math. Ann.*, 356(3):1163–1182, 2013.
- [53] Tony Yue Yu. Tropicalization of the moduli space of stable maps. *Mathematische Zeitschrift*, 281(3):1035–1059, 2015.
- [54] Tony Yue Yu. Enumeration of holomorphic cylinders in log Calabi–Yau surfaces. I. *Math. Ann.*, 366(3-4):1649–1675, 2016.
- [55] Tony Yue Yu. Gromov compactness in non-archimedean analytic geometry. *J. Reine Angew. Math.*, 741:179–210, 2018.

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