

NON-ARCHIMEDEAN QUANTUM K-INVARIANTS

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ABSTRACT. We construct quantum K-invariants in non-archimedean analytic geometry. Our approach differs from the classical one in algebraic geometry via perfect obstruction theory. Instead, we build on our previous works on the foundation of derived non-archimedean geometry and Gromov compactness. We obtain a list of natural geometric relations of the stacks of stable maps, directly at the derived level, with respect to elementary operations on graphs, namely, products, cutting edges, forgetting tails and contracting edges. They imply immediately the corresponding properties of K-theoretic invariants. The derived approach produces highly intuitive statements and functorial proofs, without the need to manipulate perfect obstruction theories. The flexibility of our derived approach to quantum K-invariants allows us to impose not only simple incidence conditions for marked points, but also incidence conditions with multiplicities, which we discuss in the final section of the paper. This leads to a new set of enumerative invariants, which is not yet considered in the literature, to the best of our knowledge. For the proofs, we had to further develop the foundations of derived non-archimedean geometry in this paper: we study derived lci morphisms, relative analytification, and deformation to the normal bundle. Our motivations come from non-archimedean enumerative geometry and mirror symmetry.

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1. INTRODUCTION

Let X be a smooth projective complex variety. Its quantum K -invariants are the collection of linear maps

$$I_{g,n,\beta}^X: K_0(X)^{\otimes n} \longrightarrow K_0(\overline{M}_{g,n}),$$

for all $g, n \in \mathbb{N}$ and $\beta \in H_2(X)$, defined by

$$I_{g,n,\beta}^X(a_1 \otimes \cdots \otimes a_n) := \text{st}_* \left(\mathcal{O}_{\overline{M}_{g,n}(X,\beta)}^{\text{vir}} \otimes \text{ev}_1^* a_1 \otimes \cdots \otimes \text{ev}_n^* a_n \right),$$

where $\overline{M}_{g,n}(X, \beta)$ is the moduli stack of n -pointed genus g stable maps into X of class β , \mathcal{O}^{vir} is its virtual structure sheaf, st is the stabilization of domain, and ev_i are the evaluation maps (see [36]). Coupling with classes in $K_0(\overline{M}_{g,n})$, we obtain numerical invariants that are non-negative integers, which manifests (in a subtle way) the geometry of curves in X .

Quantum K -invariants were first studied by Givental [18] in the case $g = 0$ and X convex. In this case the moduli stack $\overline{M}_{g,n}(X, \beta)$ is smooth with expected dimension, the virtual structure sheaf is thus equal to the structure sheaf. The virtual structure sheaves in the general case are constructed by Lee [36], using the techniques of perfect obstruction theory and intrinsic normal cone developed by Behrend-Fantechi [8]. We refer to [37, 21, 26, 19, 53] for further developments of quantum K -theory in algebraic geometry, and to [20, 11, 46] for more specific examples and applications.

Motivated by the non-archimedean approach to mirror symmetry (see [34, 35, 44]), in this paper we construct quantum K -invariants in non-archimedean analytic geometry, more precisely, for proper smooth rigid analytic spaces over any discrete valuation field k of residue characteristic zero. Non-archimedean geometry comes into play naturally as we study degenerations of Calabi-Yau varieties in mirror symmetry. It has special features in comparison with classical approaches by symplectic geometry and complex geometry. In particular, we now have a rigorous foundation of non-archimedean SYZ (Strominger-Yau-Zaslow) fibration (see [44]), whose original archimedean version is yet far beyond reach. More importantly, the enumerative invariants responsible for instanton corrections in the SYZ conjecture [56] cannot be well-defined in the archimedean setting; while the non-archimedean enumerative invariants are more promising candidates, as already demonstrated in special cases of log Calabi-Yau varieties in [61, 62, 27, 23]. Via formal models

of non-archimedean analytic spaces, non-archimedean enumerative invariants are intimately related to logarithmic enumerative invariants of the special fiber (see the works of Abramovich, Chen, Gross, Siebert [13, 1, 22, 2]).

Our approach to the construction of non-archimedean quantum K-invariants differs from the classical one in algebraic geometry via perfect obstruction theory. Instead, we build upon our previous works on the foundations of derived non-archimedean geometry [49, 52, 50, 51]. The results of this paper are supposed to bring powerful techniques from derived geometry into the study of mirror symmetry quantum corrections via non-archimedean enumerative methods along the lines of [61, 62, 27, 23].

Now let us sketch the results of this paper. Recall that in order to construct any satisfactory theory of enumerative invariants, there are two main issues to be solved: compactness and transversality (see the excellent expository paper [47] on the two issues for various enumerative theories). The same applies to non-archimedean quantum K-invariants.

In this case, the issue of compactness is solved in [63] via non-archimedean Kähler structures and formal models of non-archimedean stable maps, referred to as the non-archimedean Gromov compactness theorem, which we will review in Section 7. We treat the issue of transversality in the current paper using derived geometry as explained below.

Fix X a smooth rigid k -analytic space.

Theorem 1.1 (see Theorems 4.13 and 4.15). *The moduli stack $\overline{M}_{g,n}(X)$ of n -pointed genus g stable maps into X admits a natural derived enhancement $\mathbb{R}\overline{M}_{g,n}(X)$ that is a derived k -analytic stack locally of finite presentation and derived lci.*

Now assuming X proper and endowed with a Kähler structure, combining with the non-archimedean Gromov compactness theorem (Theorem 7.5), we can immediately obtain the non-archimedean quantum K-invariants

$$\begin{aligned} I_{g,n,\beta}^X: K_0(X)^{\otimes n} &\longrightarrow K_0(\overline{M}_{g,n}) \\ a_1 \otimes \cdots \otimes a_n &\longmapsto \text{st}_*(\text{ev}_1^* a_1 \otimes \cdots \otimes \text{ev}_n^* a_n), \end{aligned}$$

where the maps st and ev_i all depart from the derived moduli stack $\mathbb{R}\overline{M}_{g,n}(X, \beta)$.

Next we prove that the non-archimedean quantum K-invariants satisfy all the expected properties analogous to the algebraic case. Instead of manipulating perfect obstruction theories as in [36, 7], our proofs of the properties of the invariants will follow directly from a list of natural and intuitive geometric relations between the derived moduli stacks, which we explain below.

In order to state the geometric relations, instead of working with n -pointed genus g stable maps, one has to work with a slight combinatorial refinement called (τ, β) -marked stable maps for an A-graph (τ, β) , introduced by Behrend-Manin [9] for the study of Gromov-Witten invariants. The A-graph (τ, β) imposes degeneration types on the domains of stable maps as well as more refined curve classes (see Section 4.3). We have the associated moduli stack of (τ, β) -marked stable maps, the corresponding derived enhancement, and thus the (more refined) non-archimedean quantum K-invariants $I_{\tau, \beta}^X$.

Theorem 1.2 (see Theorems 5.1, 5.2, 5.4, 5.7 and 5.19). *Let S be a rigid k -analytic space and X a rigid k -analytic space smooth over S . The derived moduli stack $\mathbb{R}\overline{M}(X/S, \tau, \beta)$ of (τ, β) -marked stable maps into X/S associated to an A-graph (τ, β) satisfies the following geometric relations with respect to elementary operations on A-graphs:*

(1) **Products:** *Let (τ_1, β_1) and (τ_2, β_2) be two A-graphs. We have a canonical equivalence*

$$\mathbb{R}\overline{M}(X/S, \tau_1 \sqcup \tau_2, \beta_1 \sqcup \beta_2) \xrightarrow{\sim} \mathbb{R}\overline{M}(X/S, \tau_1, \beta_1) \times_S \mathbb{R}\overline{M}(X/S, \tau_2, \beta_2),$$

where the projections are given by the natural forgetful maps.

(2) **Cutting edges:** *Let (σ, β) be an A-graph obtained from (τ, β) by cutting an edge e of τ . Let i, j be the two tails of σ created by the cut. We have a derived pullback diagram*

$$\begin{array}{ccc} \mathbb{R}\overline{M}(X/S, \tau, \beta) & \xrightarrow{c} & \mathbb{R}\overline{M}(X/S, \sigma, \beta) \\ \downarrow \text{ev}_e & & \downarrow \text{ev}_i \times \text{ev}_j \\ X & \xrightarrow{\Delta} & X \times_S X \end{array}$$

where Δ is the diagonal map, ev_e is evaluation at the section s_e corresponding to the edge e , and c is induced by cutting the domain curves at s_e .

(3) **Universal curve**¹: *Let (σ, β) be an A-graph obtained from (τ, β) by forgetting a tail t attached to a vertex w . Let $\overline{C}_w^{\text{pre}} \rightarrow \overline{M}_\sigma^{\text{pre}}$ be the universal curve corresponding*

¹We mention a slight discrepancy of terminology: our “universal curve” property corresponds to the “forgetting tails” property of [36, §3.4]; while our “forgetting tails” property corresponds to the “fundamental classes” property of loc. cit.. As the geometry becomes more transparent in our derived approach, we have adapted our terminology accordingly.

to w . We have a derived pullback diagram

$$\begin{array}{ccc} \mathbb{R}\overline{M}(X/S, \tau, \beta) & \longrightarrow & \mathbb{R}\overline{M}(X/S, \sigma, \beta) \\ \downarrow & & \downarrow \\ \overline{C}_w^{\text{pre}} & \longrightarrow & \overline{M}_\sigma^{\text{pre}}. \end{array}$$

(See Section 5.3 for the construction of the maps.)

(4) **Forgetting tails:** Context as above, we have a derived pullback diagram

$$\begin{array}{ccc} \mathbb{R}\overline{M}(X/S, \tau, \beta) & \longrightarrow & \overline{M}_\tau \times_{\overline{M}_\sigma} \mathbb{R}\overline{M}(X/S, \sigma, \beta) \\ \downarrow & & \downarrow \\ \overline{C}_w^{\text{pre}} & \longrightarrow & \overline{M}_\tau \times_{\overline{M}_\sigma} \overline{M}_\sigma^{\text{pre}}. \end{array}$$

(5) **Contracting edges:** Let (σ, β) be an A -graph obtained from (τ, β) by contracting an edge (possibly a loop) e . Let $(\tilde{\sigma}, \beta)$ be an A -graph obtained from (σ, β) by adding k tails. Let $(\tau_i^i, \beta_i^{i_j})$ denote the A -graphs obtained from (τ, β) by replacing e with a chain of l edges then attaching tails and curve classes in a compatible way (see Section 5.5 for details). We have a natural equivalence

$$\text{colim}_l \coprod_{i,j} \mathbb{R}\overline{M}(X/S, \tau_i^i, \beta_i^{i_j}) \xrightarrow{\sim} \overline{M}_\tau \times_{\overline{M}_\sigma} \mathbb{R}\overline{M}(X/S, \tilde{\sigma}, \beta).$$

We would like to point out that in the particular case where τ is a point, the statement (3) above asserts that the forgetful map

$$\mathbb{R}\overline{M}_{g,n+1}(X/S) \longrightarrow \mathbb{R}\overline{M}_{g,n}(X/S)$$

is equivalent to the universal curve $\mathbb{R}\overline{C}_{g,n}(X/S) \rightarrow \mathbb{R}\overline{M}_{g,n}(X/S)$. Note that such an intuitive statement in fact incorporates all the information about virtual counts with respect to forgetting a tail, which is classically expressed and proved in terms of pullback properties of perfect obstruction theories and intrinsic normal cones (see [36, Proposition 8], [7, Axiom IV]).

The construction of the forgetful map above is in fact nontrivial. Intuitively, we simply forget the last marked point then stabilize. We construct the stabilization étale locally, then we need to exhibit gluing data which are typically cumbersome to build in the derived setting. We prove a universal property (see Proposition 4.29), which solves the issue of gluing, as well as the independence of all the choices involved in the construction.

Next we state the properties of non-archimedean quantum K-invariants:

Theorem 1.3 (see Propositions 8.2, 8.6 to 8.8 and 8.11). *Let X be a proper smooth rigid k -analytic space equipped with a Kähler structure. Let (τ, β) be an A -graph and T_τ the set of tail vertices. The associated quantum K -invariants*

$$I_{\tau, \beta}^X: K_0(X)^{\otimes |T_\tau|} \longrightarrow K_0(\overline{M}_\tau)$$

satisfy the following properties:

(1) **Mapping to a point:** *Let $(\tau, 0)$ be any A -graph where β is 0. Let p_1, p_2 be the projections*

$$\begin{array}{ccc} \overline{M}_\tau \times X & \xrightarrow{p_2} & X \\ \downarrow p_1 & & \\ \overline{M}_\tau & & . \end{array}$$

For any $a_i \in K_0(X), i \in T_\tau$, we have

$$I_{\tau, 0}^X\left(\bigotimes_i a_i\right) = p_{1*} p_2^*\left(\left(\bigotimes_i a_i\right) \otimes \lambda_{-1}\left(\left(\mathbb{R}^1 \pi_* \mathcal{O}_{\overline{C}_\tau} \boxtimes \mathbb{T}_X^{\text{an}}\right)^\vee\right)\right),$$

where $\lambda_{-1}(F) := \sum_i (-1)^i \wedge^i F$.

(2) **Products:** *Let (τ_1, β_1) and (τ_2, β_2) be two A -graphs. For any $a_i \in K_0(X), i \in T_{\tau_1}$ and $b_j \in K_0(X), j \in T_{\tau_2}$, we have*

$$I_{\tau_1 \sqcup \tau_2, \beta_1 \sqcup \beta_2}^X\left(\left(\bigotimes_i a_i\right) \otimes \left(\bigotimes_j b_j\right)\right) = I_{\tau_1, \beta_1}^X\left(\bigotimes_i a_i\right) \boxtimes I_{\tau_2, \beta_2}^X\left(\bigotimes_j b_j\right).$$

(3) **Cutting edges:** *Let (σ, β) be an A -graph obtained from (τ, β) by cutting an edge e of τ . Let v, w be the two tails of σ created by the cut. Consider the following commutative diagram*

$$\begin{array}{ccc} \overline{M}_\tau & \xrightarrow{d} & \overline{M}_\sigma \\ \text{st}_\tau \uparrow & & \text{st}_\sigma \uparrow \\ \mathbb{R}\overline{M}(X, \tau, \beta) & \xrightarrow{c} & \mathbb{R}\overline{M}(X, \sigma, \beta) \\ \downarrow \text{ev}_e & & \downarrow \text{ev}_v \times \text{ev}_w \\ X & \xrightarrow{\Delta} & X \times_S X. \end{array}$$

For any $a_i \in K_0(X), i \in T_\tau$, we have

$$d_* I_{\tau, \beta}^X\left(\bigotimes_i a_i\right) = I_{\sigma, \beta}^X\left(\left(\bigotimes_i a_i\right) \otimes \Delta_* \mathcal{O}_X\right).$$

(4) **Forgetting tails:** *Let (σ, β) be an A -graph obtained from (τ, β) by forgetting a tail. Let $\Phi: \overline{M}_\tau \rightarrow \overline{M}_\sigma$ be the forgetful map from τ -marked stable curves to*

σ -marked stable curves. For any $a_i \in K_0(X), i \in T_\sigma$, we have

$$I_{\tau,\beta}^X\left(\left(\bigotimes_i a_i\right) \otimes \mathcal{O}_X\right) = \Phi^* I_{\sigma,\beta}^X\left(\bigotimes_i a_i\right).$$

(5) **Contracting edges:** Let (σ, β) be an A-graph obtained from (τ, β) by contracting an edge (possibly a loop) e . We follow the notations of Section 5.5. Let $\Phi: \overline{M}_\tau \rightarrow \overline{M}_\sigma$, $\Psi_l^i: \overline{M}_{\tau_l^i} \rightarrow \overline{M}_\tau$ and $\Omega: \overline{M}_{\tilde{\sigma}} \rightarrow \overline{M}_\sigma$ be the induced maps on the moduli stacks of stable curves. For any $a_v \in K_0(X), v \in T_{\tilde{\sigma}}$, we have

$$\Phi^* \Omega_* I_{\sigma,\beta}^X\left(\bigotimes_v a_v\right) = \sum_l (-1)^{l+1} \sum_{i,j} \Psi_{l^i}^i I_{\tau_l^i, \beta_l^{i,j}}^X\left(\bigotimes_v a_v\right).$$

The properties (2-5) in Theorem 1.3 follow readily from the corresponding properties in Theorem 1.2, with arguments given in Section 8. Nevertheless, the first property “mapping to a point” requires extra effort, whose proof is based on the construction of deformation to the normal bundle. We develop relative derived analytification in Section 6.1, which allows us to borrow the algebraic construction of deformation to the normal bundle from Gaitsgory-Rozenblyum [17]. Due to the lack of \mathbb{A}^1 -invariance of G-theory in analytic geometry (see [28, 29] for related discussions), we had to extend the construction in [17] over a projective line, and then apply the GAGA theorem for stacks [49] to rely on the \mathbb{A}^1 -invariance in algebraic geometry.

Another consequence of our relative derived analytification theory is a GAGA-type result for analytic vs algebraic formal moduli problems, which may deserve independent interest:

Theorem 1.4 (see Theorem 6.12). *Let $X \in \text{dAfd}_k$ be a derived k -affinoid space and let $A := \Gamma(\mathcal{O}_X^{\text{alg}})$. There is a canonical equivalence*

$$\text{FMP}_{/\text{Spec}(A)} \xrightarrow{\sim} \text{FMP}_{/X}^{\text{an}}.$$

The list of Kontsevich-Manin axioms as in [36, §4.3] holds also for the non-archimedean quantum K-invariants, which are immediate consequences of Theorem 1.3. We do not repeat those axioms here, as the properties above with respect to elementary operations on A-graphs are more fundamental properties of the invariants.

In the final section of this paper, we take advantage of the flexibility of our derived approach to introduce a generalized type of quantum K-invariants that allow not only simple incidence conditions for marked points, but also incidence conditions with multiplicities. They satisfy a list of properties parallel to Theorem 1.3. To the best of our knowledge, such invariants are not yet considered in the literature, even in algebraic geometry. We expect them to have intriguing relations with some K-theoretic version of logarithmic Gromov-Witten invariants (see [13, 1, 22]).

All of our constructions and statements in Sections 4, 5, 8 and 9 carry over almost verbatim to algebraic geometry, which provide alternative constructions and proofs of the algebraic quantum K-theory by Lee [36]. We remark that the derived approach towards Gromov-Witten theory in algebraic geometry dates back to Kontsevich [33] and Ciocan-Fontanine-Kapranov [14] using dg-manifolds, a precursor of derived algebraic geometry. The derived stack of stable maps are later studied by Toën [57, 59, 58], Schürg-Toën-Vezzosi [54] and Mann-Robalo [42, 43]. Nevertheless, the geometric relations as in Theorem 1.2 have never really been proved, even in algebraic geometry.

We end the introduction with a few remarks comparing our derived approach with the classical approach in algebraic geometry via perfect obstruction theory:

- (1) It is difficult to apply the classical approach in analytic geometry, because the classical approach relies on the global resolution of the perfect obstruction theory, which is an unnatural assumption in analytic geometry.
- (2) The derived approach allows us to work relatively over any base space S .
- (3) As we have seen in Theorem 1.2, the derived approach gives more intuitive and transparent proofs of the geometric properties of quantum K-invariants.
- (4) The derived approach completely separates the issue of transversality from the issue of compactness mentioned in the beginning, while in the classical approach, the compactness is involved in the solution of transversality, due to the global resolution assumption.

Notations and terminology. We refer to [52, 50, 51] for the foundations on derived non-archimedean analytic geometry.

We denote by \mathcal{S} the ∞ -category of spaces. We denote by \mathbf{A}_k^n the k -analytic n -dimensional affine space, by \mathbf{D}_k^n the k -analytic n -dimensional closed unit polydisk, and by \mathbf{P}_k^n the k -analytic n -dimensional projective space.

We use homological indexing convention, i.e. the differential in chain complexes lowers the degree by 1.

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2. DERIVED LCI MORPHISMS

We fix k a complete non-archimedean field with nontrivial valuation. In this section we study properties of derived lci morphisms in derived analytic geometry. We prove that derived lci morphisms have finite tor-amplitude. We introduce the notion of virtual dimension, and prove that a derived lci k -analytic space is underived if and only if its virtual dimension is equal to the dimension of its truncation.

Definition 2.1. A morphism $f: X \rightarrow Y$ of (underived) k -analytic stacks is called:

- (1) a *regular embedding* if it is a closed immersion and locally on Y the ideal defining X can be generated by a regular sequence;
- (2) *lci* if locally on X it can be factored as a regular embedding followed by a smooth morphism.

Definition 2.2. A morphism $f: X \rightarrow Y$ of derived k -analytic stacks is called *derived lci* if its analytic cotangent complex $\mathbb{L}_{X/Y}^{\text{an}}$ is perfect and in tor-amplitude $[1, -\infty)$.

Remark 2.3. Let X be a derived k -analytic stack. Recall from [51, Definition 7.5] that a coherent sheaf $\mathcal{F} \in \text{Coh}^+(X)$ is called a perfect complex if there exists a derived k -affinoid atlas $\{U_i\}$ of X such that $\Gamma(\mathcal{F}|_{U_i})$ belongs to the smallest full stable subcategory of $\Gamma(\mathcal{O}_{U_i}^{\text{alg}})\text{-Mod}$ closed under retracts and containing $\Gamma(\mathcal{O}_{U_i}^{\text{alg}})$. By [51, Lemma 7.6], $\mathcal{F} \in \text{Coh}^+(X)$ is perfect if and only if it has finite tor-amplitude. By [50, Corollary 5.40], $\mathbb{L}_{X/Y}^{\text{an}}$ belongs to $\text{Coh}^+(X)$. Hence $\mathbb{L}_{X/Y}^{\text{an}}$ is automatically perfect as soon as it has tor-amplitude $[1, -\infty)$.

Lemma 2.4. *A morphism $f: X \rightarrow Y$ of (underived) k -analytic stacks is lci as in Definition 2.1 if and only if it is derived lci as in Definition 2.2.*

Proof. The question being local on both X and Y , we may assume that X and Y are both affinoid. Write $X = \text{Sp}(B)$ and $Y = \text{Sp}(A)$, and factor f as

$$X \xrightarrow{j} Y \times \mathbf{D}_k^n \xrightarrow{p} Y,$$

where j is a closed immersion and p is the canonical projection. Using [50, Proposition 5.50] and the transitivity fiber sequence of analytic cotangent complex, we see that f is derived lci if and only if j is derived lci. We are therefore left to check that j is derived lci if and only if the ideal defining $X = \text{Sp}(B)$ inside

$Y \times \mathbf{D}_k^n = \mathrm{Sp}(A\langle T_1, \dots, T_n \rangle)$ can be generated by a regular sequence. Since j is a closed immersion, [50, Corollary 5.33] implies that there is a canonical equivalence

$$\Gamma(X; \mathbb{L}_{X/Y \times \mathbf{D}_k^n}^{\mathrm{an}}) \simeq \mathbb{L}_{B/A\langle T_1, \dots, T_n \rangle},$$

where the right hand side denotes the algebraic cotangent complex. As both $A\langle T_1, \dots, T_n \rangle$ and B are noetherian, the conclusion now follows from Avramov's second vanishing theorem [6, Theorem 1.2]. \square

Here are some basic properties of derived lci morphisms:

Lemma 2.5. (1) *If $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are derived lci morphisms, then so is the composition $g \circ f: X \rightarrow Z$.*

(2) *Let*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow f' & & \downarrow f \\ Y' & \longrightarrow & Y \end{array}$$

be a pullback square of derived k -analytic stacks. If f is derived lci, then so is f' .

(3) *Let $f: X \rightarrow Y$ be a smooth morphism and $s: Y \rightarrow X$ a section. Then s is derived lci.*

Proof. Statement (1) follows from the transitivity sequence of analytic cotangent complex [50, Proposition 5.10]. Statement (2) follows from [50, Proposition 5.12]. For statement (3), the transitivity sequence yields a fiber sequence

$$s^* \mathbb{L}_f^{\mathrm{an}} \longrightarrow \mathbb{L}_{\mathrm{id}_Y}^{\mathrm{an}} \longrightarrow \mathbb{L}_s^{\mathrm{an}}.$$

As $\mathbb{L}_{\mathrm{id}_Y}^{\mathrm{an}} \simeq 0$, we obtain $\mathbb{L}_s^{\mathrm{an}} \simeq s^* \mathbb{L}_f^{\mathrm{an}}[1]$. The conclusion now follows from [50, Proposition 5.50]. \square

2.1. The local structure of derived lci morphisms. Let X be a derived k -analytic stack, $\pi: E \rightarrow X$ a vector bundle, $s: X \rightarrow E$ a section, and $s_0: X \rightarrow E$ the zero section. Let $Z_X(s)$ denote the derived pullback

$$\begin{array}{ccc} Z_X(s) & \longrightarrow & X \\ \downarrow p & & \downarrow s \\ X & \xrightarrow{s_0} & E. \end{array}$$

Note that $t_0(Z(s))$ coincides with the usual zero locus of s .

Lemma 2.6. *The morphism $p: Z_X(s) \rightarrow X$ is derived lci and of finite tor-amplitude.*

Proof. Lemma 2.5(2) and (3) imply that $p: Z_S(s) \rightarrow X$ is derived lci. Now we prove that it has finite tor-amplitude. The question being local on X , we can assume $X = \mathrm{Sp}(A)$ to be affinoid. Up to shrinking X , we can assume that E is free of rank n over X . Then E is the analytification relative to A of a free rank n vector bundle E^{alg} on $\mathrm{Spec} A$ (see Section 6.1); moreover, s_0 and s are analytifications of algebraic sections s_0^{alg} and s^{alg} from $\mathrm{Spec} A$ to E^{alg} . Let $Z_X^{\mathrm{alg}}(s^{\mathrm{alg}})$ be the derived intersection of s_0^{alg} and s^{alg} . Then $Z_X(s)$ is the analytification relative to A of $Z_X^{\mathrm{alg}}(s^{\mathrm{alg}})$. By [50, Proposition 4.17], we are left to check that $Z_X^{\mathrm{alg}}(s^{\mathrm{alg}})$ has finite tor-amplitude. This follows from direct computation of $Z_X^{\mathrm{alg}}(s^{\mathrm{alg}})$ using the Koszul resolution of s_0^{alg} . \square

Lemma 2.7. *Let $f: X \rightarrow Y$ be a derived lci closed immersion of derived k -analytic stacks. Then locally on Y , there exists a vector bundle E on Y and a section $s: Y \rightarrow E$ such that X is equivalent to $Z_Y(s)$.*

Proof. This proof is inspired by [5, Proposition 2.1.10]. The question being local, we may assume that Y is affinoid. Since f is a closed immersion, then X is also affinoid. Set

$$A := \Gamma(Y; \mathcal{O}_Y^{\mathrm{alg}}) \quad , \quad B := \Gamma(X; \mathcal{O}_X^{\mathrm{alg}}).$$

Observe that

$$B \simeq \Gamma(Y; f_* \mathcal{O}_X^{\mathrm{alg}}).$$

As $f_* \mathcal{O}_X^{\mathrm{alg}}$ is a coherent sheaf on Y , the t-exact equivalence in [51, Theorem 3.4] implies that the canonical map $A \rightarrow B$ is a surjection on π_0 . Therefore, [50, Corollary 5.33] provides us with a canonical equivalence

$$\Gamma(X; \mathbb{L}_{X/Y}^{\mathrm{an}}) \simeq \mathbb{L}_{B/A}.$$

[51, Theorem 3.4] implies that $\mathbb{L}_{B/A}$ is perfect and in tor-amplitude $[1, 0]$. Let

$$E := \mathrm{Spec}(\mathrm{Sym}_B(\mathbb{L}_{B/A}[-1])),$$

and E^{an} the analytification of E relative to A . Then locally on A we can find elements $a_1, \dots, a_n \in \ker(\pi_0(A) \rightarrow \pi_0(B))$ whose differentials generate $\mathbb{L}_{B/A}[-1] \otimes_B \pi_0(B)$. Up to shrinking Y , we can lift these elements to a function

$$g = (g_1, \dots, g_n): Y \rightarrow \mathbf{A}_k^n.$$

Let Z be the derived fiber product

$$\begin{array}{ccc} Z & \longrightarrow & Y \\ \downarrow & & \downarrow g \\ \mathrm{Sp}(k) & \xrightarrow{0} & \mathbf{A}_k^n. \end{array}$$

Then f induces a canonical map $X \rightarrow Z$, which is a closed immersion and whose relative cotangent complex is zero, in other words $X \xrightarrow{\sim} Z$, completing the proof. \square

Proposition 2.8. *Let $f: X \rightarrow Y$ be a morphism of derived k -analytic stacks. The following are equivalent:*

- (1) *the morphism f is derived lci;*
- (2) *locally on X and Y we can factor f as*

$$X \xrightarrow{j} W \xrightarrow{p} Y,$$

where j is a closed immersion, p is smooth, and there exists a vector bundle E on W and a section s such that $X \simeq Z_W(s)$.

Proof. The implication (2) \Rightarrow (1) is the content of Lemma 2.6. For the converse, observe that locally on X and Y we can factor f as

$$X \xrightarrow{j} Y \times \mathbf{D}_k^n \xrightarrow{p} Y,$$

where j is a closed immersion and p is the canonical projection (see [50, Lemma 5.48]). As p is smooth and f is derived lci, the transitivity sequence for the analytic cotangent complex implies that j is derived lci. Thus we conclude from Lemma 2.7. \square

Corollary 2.9. *Let $f: X \rightarrow Y$ be a derived lci morphism of derived k -analytic stacks. Then f has finite tor-amplitude. In particular, if \mathcal{O}_Y is locally cohomologically bounded, then so is \mathcal{O}_X .*

Proof. This follows from Proposition 2.8 and Lemma 2.6. \square

2.2. Virtual dimension.

Definition 2.10. Let $X = (\mathcal{X}, \mathcal{O}_X)$ be a derived lci k -analytic space, $x \in \mathfrak{t}_0(X)$ a rigid point, and $\iota_x: \mathrm{Sp}(\kappa(x)) \rightarrow \mathfrak{t}_0(X) \hookrightarrow X$ the associated morphism. The *virtual dimension of X at x* , written $\mathrm{vdim}_x(X)$, is the Euler characteristic of $\iota_x^*(\mathbb{L}_X^{\mathrm{an}})$.

Lemma 2.11. *Let X be a derived k -analytic space such that $\mathfrak{t}_0(X)$ is connected. Let $\mathcal{F} \in \mathrm{Perf}(X)$ be a perfect complex on X . Then the Euler characteristic of $\iota_x^*(\mathcal{F})$ for any rigid point $x \in \mathfrak{t}_0(X)$ is independent of x .*

Proof. We may assume that $\mathfrak{t}_0(X) = \mathrm{Sp}(A)$ for some affinoid algebra A . Let $j: \mathfrak{t}_0(X) \hookrightarrow X$ denote the canonical inclusion. Then $j^*\mathcal{F}$ is a perfect complex on $\mathrm{Sp}(A)$. By [51, Theorem 3.4], $j^*\mathcal{F}$ is induced by a perfect complex M in $A\text{-Mod}$. Since A is discrete, [55, Tag 0658] shows that we can represent M by a bounded complex of projective A -modules:

$$\cdots \rightarrow 0 \rightarrow F^n \rightarrow F^{n-1} \rightarrow \cdots \rightarrow F^0 \rightarrow 0 \rightarrow \cdots .$$

Then for any rigid point $x \in \mathrm{Sp}(A)$, the perfect complex $\iota_x^* \mathcal{F}$ is quasi-isomorphic to the complex

$$\cdots \rightarrow 0 \rightarrow F^n \otimes_A \kappa(x) \rightarrow F^{n-1} \otimes_A \kappa(x) \rightarrow \cdots \rightarrow F^0 \otimes_A \kappa(x) \rightarrow 0 \rightarrow \cdots .$$

Since each F^i is a projective A -module and $\mathrm{Sp}(A)$ is connected, we have

$$\mathrm{rank}_A(F^i) = \mathrm{rank}_{\kappa(x)}(F^i \otimes_A \kappa(x)).$$

It follows that

$$\chi(\iota_x^* \mathcal{F}) = \sum_{i=0}^n (-1)^i \mathrm{rank}_{\kappa(x)}(F^i \otimes_A \kappa(x)) = \sum_{i=0}^n (-1)^i \mathrm{rank}_A(F^i),$$

and therefore the left hand side is independent of the rigid point x . \square

Corollary 2.12. *Let X be a derived lci k -analytic space such that $\mathfrak{t}_0(X)$ is connected. Then the virtual dimension of X at any rigid point $x \in \mathfrak{t}_0(X)$ is independent of x .*

Lemma 2.13. *Let X be a derived lci k -analytic space. Assume that $\mathrm{vdim}_x(X) = \dim_x(\mathfrak{t}_0(X))$ for any rigid point $x \in \mathfrak{t}_0(X)$. Then $\mathfrak{t}_0(X)$ is derived lci.*

Proof. The question being local on X , by [50, Lemma 5.48], we can assume that there exists a closed immersion

$$f: X \hookrightarrow \mathbf{D}_k^n.$$

By [50, Corollaries 5.26 and 5.37], $\mathbb{L}_{\mathbf{D}_k^n}^{\mathrm{an}}$ is free of rank n ; in particular, it is perfect and in tor-amplitude 0. Therefore, the transitivity fiber sequence

$$f^* \mathbb{L}_{\mathbf{D}_k^n}^{\mathrm{an}} \longrightarrow \mathbb{L}_X^{\mathrm{an}} \longrightarrow \mathbb{L}_{X/\mathbf{D}_k^n}^{\mathrm{an}}$$

shows that $\mathbb{L}_{X/\mathbf{D}_k^n}^{\mathrm{an}}$ is perfect and in tor-amplitude $[1, 0]$; in other words, f is derived lci. Let $g := \mathfrak{t}_0(f)$, we have a similar transitivity fiber sequence

$$g^* \mathbb{L}_{\mathbf{D}_k^n}^{\mathrm{an}} \longrightarrow \mathbb{L}_{\mathfrak{t}_0(X)}^{\mathrm{an}} \longrightarrow \mathbb{L}_{\mathfrak{t}_0(X)/\mathbf{D}_k^n}^{\mathrm{an}}.$$

Therefore, in order to prove that $\mathfrak{t}_0(X)$ is derived lci, it suffices to prove that g is derived lci.

Set $A := \Gamma(X; \mathcal{O}_X^{\mathrm{alg}})$ and $B := \pi_0(A)$. We have $\mathfrak{t}_0(X) \simeq \mathrm{Sp}(B)$. By [50, Corollary 5.33] and [51, Theorem 3.4], it is enough to check that the algebraic cotangent complex $\mathbb{L}_{B/k\langle T_1, \dots, T_n \rangle}$ is perfect and in tor-amplitude $[1, 0]$. By [50, Corollary 5.40], we only need to check that it is in tor-amplitude $[1, 0]$. This condition can be checked after base change to the local rings at the maximal ideals of B . For this, it is enough to check that for every rigid point $x \in \mathrm{Sp}(B)$, the local ring $B_{(x)}$ is cut out by a regular sequence. Let

$$I := \ker \left(k\langle T_1, \dots, T_n \rangle_{(x)} \rightarrow B_{(x)} \right).$$

Let \mathfrak{n} be the maximal ideal of $k\langle T_1, \dots, T_n \rangle_{(x)}$, and \mathfrak{m} the maximal ideal of $B_{(x)}$. Then $I/\mathfrak{m}I$ is a $\kappa(x)$ -vector space of finite dimension. Let m denote the dimension. Choose elements $g_1, \dots, g_m \in I$ mapping to a basis of $I/\mathfrak{m}I$. Notice that we can furthermore assume that $I \subset \mathfrak{m}^2$. Then the Nakayama lemma implies that

$$I = (g_1, \dots, g_m).$$

We claim that this is a regular sequence. Let

$$M := \mathbb{L}_{B_{(x)}/k\langle T_1, \dots, T_n \rangle_{(x)}}.$$

Then [50, Corollary 5.33 and Lemma 5.51] imply that

$$\pi_1(M \otimes_{B_{(x)}} \kappa(x)) \simeq I/\mathfrak{m}I \quad , \quad \pi_0(M \otimes_{B_{(x)}} \kappa(x)) \simeq \mathfrak{n}/\mathfrak{n}^2.$$

Using the long exact sequence associated to

$$j^* \mathbb{L}_{X/\mathbb{D}_k}^{\text{an}} \longrightarrow \mathbb{L}_{\mathfrak{t}_0(X)/\mathbb{D}_k}^{\text{an}} \longrightarrow \mathbb{L}_{\mathfrak{t}_0(X)/X}^{\text{an}}$$

we deduce that

$$\begin{aligned} \dim_{\kappa(x)} \left(\pi_1(\iota_x^* \mathbb{L}_{X/\mathbb{D}_k}^{\text{an}}) \right) &\geq \dim_{\kappa(x)}(I/\mathfrak{m}I), \\ \dim_{\kappa(x)} \left(\pi_0(\iota_x^* \mathbb{L}_{X/\mathbb{D}_k}^{\text{an}}) \right) &= \dim_{\kappa(x)}(\mathfrak{n}/\mathfrak{n}^2). \end{aligned}$$

Therefore

$$\begin{aligned} \dim_x(\mathfrak{t}_0(X)) = \text{vdim}_x(X) &= \dim_{\kappa(x)} \left(\pi_0(\iota_x^* \mathbb{L}_{X/\mathbb{D}_k}^{\text{an}}) \right) - \dim_{\kappa(x)} \left(\pi_1(\iota_x^* \mathbb{L}_{X/\mathbb{D}_k}^{\text{an}}) \right) \\ &\leq \dim_{\kappa(x)}(\mathfrak{n}/\mathfrak{n}^2) - \dim_{\kappa(x)}(I/\mathfrak{m}I) = n - m. \end{aligned}$$

But since $I = (g_1, \dots, g_m)$, we see that $\dim_x(\mathfrak{t}_0(X)) \geq n - m$. It follows that $\dim_x(\mathfrak{t}_0(X)) = n - m$, hence (g_1, \dots, g_m) is a regular sequence. \square

Proposition 2.14. *Let X be a derived lci derived k -analytic space, and $j: \mathfrak{t}_0(X) \rightarrow X$ the canonical closed immersion. The following are equivalent:*

- (1) j is an equivalence;
- (2) the relative analytic cotangent complex $\mathbb{L}_{\mathfrak{t}_0(X)/X}^{\text{an}}$ vanishes;
- (3) $\mathfrak{t}_0(X)$ is derived lci and the canonical map $\pi_1(j^* \mathbb{L}_X^{\text{an}}) \rightarrow \pi_1(\mathbb{L}_{\mathfrak{t}_0(X)}^{\text{an}})$ is an isomorphism;
- (4) $\text{vdim}_x(X) = \dim_x(\mathfrak{t}_0(X))$ for every rigid point $x \in \mathfrak{t}_0(X)$.

Proof. The equivalence (1) \Leftrightarrow (2) follows directly from [50, Corollary 5.35].

If (2) holds, then the fiber sequence

$$j^* \mathbb{L}_X^{\text{an}} \rightarrow \mathbb{L}_{\mathfrak{t}_0(X)}^{\text{an}} \rightarrow \mathbb{L}_{\mathfrak{t}_0(X)/X}^{\text{an}}$$

yields a canonical equivalence $j^* \mathbb{L}_X^{\text{an}} \simeq \mathbb{L}_{\mathfrak{t}_0(X)}^{\text{an}}$. In particular, $\mathfrak{t}_0(X)$ is derived lci and $\pi_1(j^* \mathbb{L}_X^{\text{an}}) \rightarrow \pi_1(\mathbb{L}_{\mathfrak{t}_0(X)}^{\text{an}})$ is an isomorphism. So (2) implies (3).

For the converse implication, the above fiber sequence yields for any $i \geq 0$ a long exact sequence

$$\pi_{i+1}(j^*\mathbb{L}_X^{\text{an}}) \rightarrow \pi_{i+1}(\mathbb{L}_{t_0(X)}^{\text{an}}) \rightarrow \pi_{i+1}(\mathbb{L}_{t_0(X)/X}^{\text{an}}) \rightarrow \pi_i(j^*\mathbb{L}_X^{\text{an}}) \rightarrow \pi_i(\mathbb{L}_{t_0(X)}^{\text{an}}).$$

If $i \geq 1$, the fact that both X and $t_0(X)$ are derived lci forces

$$\pi_{i+1}(\mathbb{L}_{t_0(X)/X}^{\text{an}}) = 0.$$

When $i = 0$, we have $\pi_1(j^*\mathbb{L}_X^{\text{an}}) \simeq \pi_1(\mathbb{L}_{t_0(X)}^{\text{an}})$ by hypothesis; while [50, Corollary 5.35] guarantees that

$$\pi_0(j^*\mathbb{L}_X^{\text{an}}) \simeq \pi_0(\mathbb{L}_{t_0(X)}^{\text{an}}).$$

Therefore we deduce

$$\pi_1(\mathbb{L}_{t_0(X)/X}^{\text{an}}) \simeq \pi_0(\mathbb{L}_{t_0(X)/X}^{\text{an}}) \simeq 0.$$

Hence (3) implies (2).

It follows from Lemma 2.4 that (1) implies (4). Now we will complete the proof by showing that (4) implies (3). First of all, $t_0(X)$ is derived lci by Lemma 2.13. In order to show that the map $\pi_1(j^*\mathbb{L}_X^{\text{an}}) \rightarrow \pi_1(\mathbb{L}_{t_0(X)}^{\text{an}})$ is an isomorphism, we can reason locally on X . We can thus assume that X is derived affinoid. In virtue of [52, Lemma 6.3], we can furthermore suppose that X admits a closed embedding $f: X \hookrightarrow \mathbf{D}_k^n$ for some n . Using [51, Theorem 3.4] and [50, Lemma 5.51], we can represent $j^*\mathbb{L}_X^{\text{an}}$ and $\mathbb{L}_{t_0(X)}^{\text{an}}$ respectively as two perfect complexes on $t_0(X)$ of the form

$$\begin{aligned} j^*\mathbb{L}_X^{\text{an}} &\simeq (0 \rightarrow E^{-1} \rightarrow j^*f^*\Omega_{\mathbf{D}_k^n}^{\text{an}} \rightarrow 0), \\ \mathbb{L}_{t_0(X)}^{\text{an}} &\simeq (0 \rightarrow F^{-1} \rightarrow j^*f^*\Omega_{\mathbf{D}_k^n}^{\text{an}} \rightarrow 0). \end{aligned}$$

Let

$$\begin{aligned} K_E &:= \ker(j^*f^*\Omega_{\mathbf{D}_k^n}^{\text{an}} \rightarrow \pi_0(j^*\mathbb{L}_X^{\text{an}})), \\ K_F &:= \ker(j^*f^*\Omega_{\mathbf{D}_k^n}^{\text{an}} \rightarrow \pi_0(\mathbb{L}_{t_0(X)}^{\text{an}})). \end{aligned}$$

We have a morphism of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_E & \longrightarrow & j^*f^*\Omega_{\mathbf{D}_k^n}^{\text{an}} & \longrightarrow & \pi_0(j^*\mathbb{L}_X^{\text{an}}) \longrightarrow 0 \\ & & \downarrow & & \parallel & & \downarrow \\ 0 & \longrightarrow & K_F & \longrightarrow & j^*f^*\Omega_{\mathbf{D}_k^n}^{\text{an}} & \longrightarrow & \pi_0(\mathbb{L}_{t_0(X)}^{\text{an}}) \longrightarrow 0. \end{array}$$

It follows from [50, Corollary 5.35] that the map $\pi_0(j^*\mathbb{L}_X^{\text{an}}) \rightarrow \pi_0(\mathbb{L}_{t_0(X)}^{\text{an}})$ is an isomorphism. We conclude that the canonical map $K_E \rightarrow K_F$ is also an isomorphism.

Thus, we can consider the morphism of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \pi_1(j^*\mathbb{L}_X^{\text{an}}) & \longrightarrow & E^{-1} & \longrightarrow & K_E \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \pi_1(\mathbb{L}_{t_0(X)}^{\text{an}}) & \longrightarrow & F^{-1} & \longrightarrow & K_F \longrightarrow 0. \end{array}$$

Using once again [50, Corollary 5.35] we deduce that the map $\pi_1(j^*\mathbb{L}_X^{\text{an}}) \rightarrow \pi_1(\mathbb{L}_{t_0(X)}^{\text{an}})$ is surjective. Hence the snake lemma implies that $E^{-1} \rightarrow F^{-1}$ is also surjective. Finally, since $\text{vdim}(X) = \dim(t_0(X))$ and $t_0(X)$ is derived lci, we conclude that $\text{rank}(E^{-1}) = \text{rank}(F^{-1})$. In particular, $E^{-1} \rightarrow F^{-1}$ is an isomorphism. It follows that $\pi_1(j^*\mathbb{L}_X^{\text{an}}) \rightarrow \pi_1(\mathbb{L}_{t_0(X)}^{\text{an}})$ is also an isomorphism. In particular, (3) holds, completing the proof. \square

3. THE STACK OF DERIVED k -ANALYTIC SPACES

In this section, we study the deformation theory of derived k -analytic spaces. We prove that the moduli functor of derived k -analytic spaces is cartesian, convergent and has a global analytic cotangent complex.

Let $(\text{dAfd}_k, \tau_{\text{ét}})$ be the ∞ -site of derived k -affinoid spaces equipped with the étale topology (see [52, §7]). Let

$$\text{St}(\text{dAfd}_k) := \text{Sh}(\text{dAfd}_k, \tau_{\text{ét}})^\wedge$$

denote the ∞ -category of stacks (i.e. hypercomplete sheaves) on this site. Consider the functor

$$\text{St}: \text{dAfd}_k^{\text{op}} \longrightarrow \text{Cat}_\infty$$

given by

$$X \mapsto \text{St}(\text{dAfd}_k)_{/X}.$$

The general theory of descent for ∞ -topoi (see [38, 6.1.3.9]) implies that the functor St satisfies descent for the étale topology. By Corollary 10.4, the functor St admits a subfunctor F which sends $X \in \text{dAfd}_k$ to the full subcategory $(\text{dAn}_k)_{/X}^{\text{flat}}$ of the overcategory $(\text{dAn}_k)_{/X}$ spanned by flat morphisms $Y \rightarrow X$. It follows from [52, Propositions 8.5 and 8.7] that F also satisfies étale descent.

Lemma 3.1. *Let $X \rightarrow S$, $X' \rightarrow S$ and $Y \rightarrow S$ be flat morphisms of derived k -analytic spaces. Let $i: X \hookrightarrow X'$, $j: X \hookrightarrow Y$ be closed immersions, and $Y' := Y \amalg_X X'$ the pushout (see [50, Theorem 6.5] for the construction). For any $T \in \text{dAfd}_k$ and any map $T \rightarrow S$, define*

$$X_T := X \times_S T, \quad X'_T := X' \times_S T, \quad Y_T := Y \times_S T, \quad Y'_T := Y' \times_S T.$$

Then the canonical map

$$X'_T \amalg_{X_T} Y_T \longrightarrow Y'_T$$

is an equivalence.

Proof. Notice that the squares

$$\begin{array}{ccc} X_T & \longrightarrow & X \\ \downarrow i_T & & \downarrow i \\ X'_T & \longrightarrow & X' \end{array} \quad , \quad \begin{array}{ccc} X_T & \longrightarrow & X \\ \downarrow j_T & & \downarrow j \\ Y_T & \longrightarrow & Y \end{array}$$

are derived pullback diagrams. In particular the morphisms i_T and j_T are closed immersions. So by [50, Theorem 6.5], the pushout $X'_T \amalg_{X_T} Y_T$ exists in the ∞ -category dAn_k of derived k -analytic spaces. Since $Y'_T \in \mathrm{dAn}_k$, by the universal property of pushout we obtain a canonical morphism

$$f: Y_T \amalg_{X_T} X'_T \longrightarrow Y'_T.$$

Now we show that it is an equivalence. By [50, (6.7)], Y' is flat over S . Using Corollary 10.4, we deduce that X_T , X'_T and Y_T are also flat over T , and therefore $X'_T \amalg_{X_T} Y_T$ is flat over T . Moreover, the same result implies that Y'_T is flat over T . It follows from [50, Lemma 5.47] that the canonical map f is strong. We are therefore left to check that f is an equivalence on the truncation.

Let R be an affinoid algebra, A , A' and B affinoid R -algebras, $A' \rightarrow A$ and $B \rightarrow A$ surjective morphisms, and $B' := A' \times_A B$. Note B' is an affinoid R -algebra by [50, Lemma 6.3]. Let $R \rightarrow R'$ be a morphism of affinoid algebras. Then it suffices to check that

$$\begin{array}{ccc} B' \widehat{\otimes}_R R' & \longrightarrow & B \widehat{\otimes}_R R' \\ \downarrow & & \downarrow \\ A' \widehat{\otimes}_R R' & \longrightarrow & A \widehat{\otimes}_R R' \end{array}$$

is pullback in CAlg_k , the ∞ -category of simplicial commutative k -algebras. By [10], we can choose formal models for the affinoid algebras preserving the flatness. Now the conclusion follows directly from the analogous algebraic statement. \square

Remark 3.2. The lemma should hold without the flatness assumption. One may prove it by reducing to the algebraic case via derived formal models (see Ant3nio [4]).

Lemma 3.3. *Let $f: X \rightarrow Y$ be a morphism of derived k -affinoid spaces. Let*

$$A := \Gamma(Y; \mathcal{O}_Y^{\mathrm{alg}}), \quad B := \Gamma(X; \mathcal{O}_X^{\mathrm{alg}}).$$

Then f is an equivalence if and only if the induced map $A \rightarrow B$ is an equivalence in CAlg_k .

Proof. The “only if” direction is obvious. Now assume that $A \rightarrow B$ is an equivalence. Then $\pi_0(A) \rightarrow \pi_0(B)$ is an isomorphism, which implies that $f_0 := t_0(f): t_0(X) \rightarrow t_0(Y)$ is an isomorphism. So it suffices to verify that f is strong. For this, we only have to check that the canonical map

$$f_0^*(\pi_i(\mathcal{O}_Y^{\text{alg}})) \longrightarrow \pi_i(\mathcal{O}_X^{\text{alg}})$$

is an equivalence for every i . As both sides are coherent sheaves on $t_0(X)$, the conclusion readily follows from the assumption that $A \rightarrow B$ is an equivalence and [51, Theorem 3.4]. \square

Proposition 3.4. *The functor $F: \text{dAfd}_k^{\text{op}} \rightarrow \text{Cat}_\infty$ is cartesian, in the sense that for any pushout diagram*

$$\begin{array}{ccc} X_{01} & \xleftarrow{j_0} & X_0 \\ \downarrow j_1 & & \downarrow \\ X_1 & \longrightarrow & X \end{array}$$

in dAfd_k where j_0 and j_1 are closed immersions, the canonical functor

$$\Psi: (\text{dAn}_k)_{/X}^{\text{flat}} \longrightarrow (\text{dAn}_k)_{/X_1}^{\text{flat}} \times_{(\text{dAn}_k)_{/X_{01}}^{\text{flat}}} (\text{dAn}_k)_{/X_0}^{\text{flat}}$$

is an equivalence.

Proof. We first construct a left adjoint

$$(3.5) \quad \Phi: (\text{dAn}_k)_{/X_1}^{\text{flat}} \times_{(\text{dAn}_k)_{/X_{01}}^{\text{flat}}} (\text{dAn}_k)_{/X_0}^{\text{flat}} \longrightarrow (\text{dAn}_k)_{/X}^{\text{flat}}$$

of Ψ as follows. Let

$$\text{Span} := \{ * \leftarrow * \rightarrow * \}$$

denote the span category. Consider the composition of the natural functors

$$(\text{dAn}_k)_{/X_1} \times_{(\text{dAn}_k)_{/X_{01}}} (\text{dAn}_k)_{/X_0} \xrightarrow{\phi} \text{Fun}(\text{Span}, \text{dAn}_k) \rightarrow \text{Fun}(\text{Span}, {}^{\text{R}}\mathcal{T}\text{op}(\mathcal{T}_{\text{an}}(k))).$$

We can informally describe the functor ϕ on objects as follows. An object in the fiber product on the left is the given of a diagram

$$(3.6) \quad \begin{array}{ccccc} Y_1 & \xleftarrow{i_0} & Y_{01} & \xrightarrow{i_1} & Y_1 \\ \downarrow & & \downarrow & & \downarrow \\ X_1 & \xleftarrow{j_1} & X_{01} & \xrightarrow{j_1} & X_0 \end{array}$$

where both squares are cartesian. Then ϕ sends this diagram to the top line $Y_1 \leftarrow Y_{01} \rightarrow Y_0$. Notice that i_0 and i_1 are again closed immersions. Therefore, it follows from [50, Theorem 6.5] that we can form the pushout Y in $\mathbf{R}\mathcal{T}\text{op}(\mathcal{T}_{\text{an}}(k))$ of

$$Y_1 \xleftarrow{i_1} Y_{01} \xrightarrow{i_0} Y_0;$$

moreover, the result is a derived k -analytic space. The universal property of pushout provides a canonical map $Y \rightarrow X$, and hence a functor

$$(\text{dAn}_k)_{/X_1} \times_{(\text{dAn}_k)_{/X_{01}}} (\text{dAn}_k)_{/X_0} \longrightarrow (\text{dAn}_k)_{/X},$$

which is left adjoint to the canonical functor

$$(\text{dAn}_k)_{/X} \longrightarrow (\text{dAn}_k)_{/X_1} \times_{(\text{dAn}_k)_{/X_{01}}} (\text{dAn}_k)_{/X_0}.$$

We claim that this functor restricts to the full subcategories spanned by flat morphisms. Indeed, the claim being local, we can assume that Y_0 , Y_1 and Y_{01} are all derived k -affinoid. In this case, the conclusion follows from [51, Theorem 3.4] and [41, 16.2.3.1(4)].

Next we claim that both the unit and the counit of this adjunction

$$u: \text{id} \longrightarrow \Psi \circ \Phi, \quad v: \Phi \circ \Psi \longrightarrow \text{id}$$

are equivalences. For v , this is a direct consequence of Lemmas 3.1. Now we show that u is also an equivalence. Consider an object in the fiber product

$$(\text{dAn}_k)_{/X_1}^{\text{flat}} \times_{(\text{dAn}_k)_{/X_{01}}^{\text{flat}}} (\text{dAn}_k)_{/X_0}^{\text{flat}}$$

which we depict as the diagram (3.6). Let Y be the pushout of that diagram. Then we have to check that the canonical morphisms

$$Y_0 \longrightarrow X_0 \times_X Y, \quad Y_1 \longrightarrow X_1 \times_X Y$$

are equivalences. The question being local, we can assume that Y (and therefore Y_0 , Y_1 and Y_{01}) are derived k -affinoid spaces. Write

$$\begin{aligned} A &:= \Gamma(X, \mathcal{O}_X^{\text{alg}}), & B &:= \Gamma(Y, \mathcal{O}_Y^{\text{alg}}), \\ A_i &:= \Gamma(X_i, \mathcal{O}_{X_i}^{\text{alg}}), & B_i &:= \Gamma(Y_i, \mathcal{O}_{Y_i}^{\text{alg}}). \end{aligned}$$

By Lemma 3.3, it is enough to check that the canonical map $Y_i \rightarrow X_i \times_X Y$ induces an equivalence in CAlg_k after passing to global sections. By [51, Proposition 4.2], we can compute the global sections of $X_i \times_X Y$ as the tensor product $A_i \otimes_A B$. We are therefore left to show that the canonical map

$$A_i \otimes_A B \longrightarrow B_i$$

is an equivalence. Combining diagram (6.7) in the proof of [50, Theorem 6.5] with [51, Theorem 3.4], we conclude that the diagram

$$\begin{array}{ccc} B & \longrightarrow & B_1 \\ \downarrow & & \downarrow \\ B_0 & \longrightarrow & B_{01} \end{array}$$

is a pullback diagram in $\mathrm{CAlg}_{A/}$, and hence in $A\text{-Mod}$. The conclusion now follows from [41, Proposition 16.2.2.1]. \square

Proposition 3.7. *The functor $F: \mathrm{dAfd}_k^{\mathrm{op}} \rightarrow \mathrm{Cat}_\infty$ is convergent, in the sense that for any $X \in \mathrm{dAfd}_k$ the canonical map*

$$F(X) \longrightarrow \lim_{n \in \mathbb{N}} F(t_{\leq n} X)$$

is an equivalence.

Proof. Let m be a non-negative integer. We say that a derived k -analytic space $X = (\mathcal{X}, \mathcal{O}_X)$ is m -truncated if $\pi_i(\mathcal{O}_X^{\mathrm{alg}}) \simeq 0$ for $i > m$. Let $\mathrm{dAn}_{/X}^{\leq m}$ denote the full subcategory of $\mathrm{dAn}_{/X}$ spanned by m -truncated derived k -analytic spaces. It follows from [52, Theorem 3.23] that the natural inclusion $\mathrm{dAn}_{/X}^{\leq m} \hookrightarrow \mathrm{dAn}_{/X}^{\leq m+1}$ admits a right adjoint $t_{\leq m}$. Observe that the natural morphism

$$(3.8) \quad \mathrm{dAn}_{/X} \longrightarrow \lim_{n \in \mathbb{N}^{\mathrm{op}}} \mathrm{dAn}_{/X}^{\leq n},$$

where the transition maps are given by the truncation functors, is an equivalence. Therefore, the vertical maps in the following commutative square

$$\begin{array}{ccc} \mathrm{dAn}_{/X} & \longrightarrow & \lim_{n \in \mathbb{N}^{\mathrm{op}}} \mathrm{dAn}_{/t_{\leq n} X} \\ \downarrow & & \downarrow \\ \lim_{m \in \mathbb{N}^{\mathrm{op}}} \mathrm{dAn}_{/X}^{\leq m} & \longrightarrow & \lim_{n \in \mathbb{N}^{\mathrm{op}}} \lim_{m \in \mathbb{N}^{\mathrm{op}}} \mathrm{dAn}_{/t_{\leq n} X}^{\leq m} \end{array}$$

are equivalences. Moreover, for every $n \geq m$ we have a canonical equivalence

$$\mathrm{dAn}_{/t_{\leq n} X}^{\leq m} \simeq \mathrm{dAn}_{/t_{\leq m} X}^{\leq m}.$$

It follows that the bottom horizontal morphism in the diagram above is also an equivalence, thus completing the proof. \square

Consider now the maximal ∞ -groupoid functor $(-)^{\simeq}: \mathrm{Cat}_\infty \rightarrow \mathcal{S}$. It is right adjoint to the inclusion $\mathcal{S} \hookrightarrow \mathrm{Cat}_\infty$, and in particular it commutes with limits. Let F^{\simeq} denote the composite functor

$$\mathrm{dAfd}_k^{\mathrm{op}} \xrightarrow{F} \mathrm{Cat}_\infty \xrightarrow{(-)^{\simeq}} \mathcal{S}.$$

We deduce that F^\simeq satisfies étale descent and that it is cartesian and convergent. To compute the analytic cotangent complex, we rely on the following lemma:

Lemma 3.9. *Let $G \in \text{St}(\text{dAfd}_k)$ and let $x: X \rightarrow G$ be a morphism with $X \in \text{dAfd}_k$. Let $\Omega_x G := X \times_G X$ and $\delta_x: X \rightarrow \Omega_x G$ the induced diagonal map. Assume that G is infinitesimally cartesian. Then G has an analytic cotangent complex at x if and only if $\Omega_x G$ has an analytic cotangent complex at δ_x . If this is the case, then there is a natural equivalence*

$$\mathbb{L}_{G,x}^{\text{an}} \simeq \mathbb{L}_{\Omega_x G, \delta_x}^{\text{an}}[-1].$$

Proof. For any $\mathcal{F} \in \text{Coh}^{\geq 1}(X)$, we have

$$\text{Der}_{\Omega_x G}^{\text{an}}(X; \mathcal{F}) \simeq \Omega \text{Der}_G^{\text{an}}(X; \mathcal{F}) \simeq \Omega \text{Der}_G^{\text{an}}(X; \mathcal{F}[-1][1]).$$

Since G is infinitesimally cartesian, we can use [50, Lemma 7.8] to rewrite this as

$$\text{Der}_G^{\text{an}}(X; \mathcal{F}[-1]) \simeq \text{Map}_{\text{Coh}^+(X)}(\mathbb{L}_{G,x}^{\text{an}}[1], \mathcal{F}).$$

Therefore, $\mathbb{L}_{\Omega_x G, \delta_x}^{\text{an}} \simeq \mathbb{L}_{G,x}^{\text{an}}[1]$. \square

Proposition 3.10. *Let $X \in \text{dAfd}_k$ and let $x: X \rightarrow F^\simeq$ be a point classifying a flat map $p: Y \rightarrow X$ from a derived k -analytic space Y . If p is proper, has finite coherent cohomological dimension and locally of finite presentation, then F^\simeq admits an analytic cotangent complex at x , given by the formula*

$$\mathbb{L}_{F^\simeq, x}^{\text{an}} \simeq p_+(\mathbb{L}_{Y/X}^{\text{an}})[-1],$$

where $p_+(\mathbb{L}_{Y/X}^{\text{an}}) := (p_*((\mathbb{L}_{Y/X}^{\text{an}})^\vee))^\vee$.

Proof. Consider the derived k -analytic stack

$$G := X \times_{F^\simeq} X$$

and let $\delta_x: X \rightarrow G$ be the diagonal morphism induced by x . Since F^\simeq is infinitesimally cartesian, Lemma 3.9 shows that F^\simeq has an analytic cotangent complex at x if and only if G has an analytic cotangent complex at δ_x ; moreover, in this case we have

$$\mathbb{L}_{F^\simeq, x}^{\text{an}} \simeq \mathbb{L}_{G, \delta_x}^{\text{an}}[1].$$

Fix $\mathcal{F} \in \text{Coh}^{\geq 0}(X)$. Unraveling the definitions, we have

$$\begin{aligned} \text{Der}_G^{\text{an}}(X; \mathcal{F}) &\simeq \text{Map}(X[\mathcal{F}], G) \times_{\text{Map}(X, G)} \{x\} \\ &\simeq \text{Map}_{/X[\mathcal{F}]}(Y[p^*\mathcal{F}], Y[p^*\mathcal{F}]) \times_{\text{Map}_{/X}(Y, Y)} \{\text{id}_Y\} \\ &\simeq \text{Map}_{/X}(Y[p^*\mathcal{F}], Y) \times_{\text{Map}_{/X}(Y, Y)} \{\text{id}_Y\} \\ &\simeq \text{Der}_{Y/X}^{\text{an}}(Y; p^*\mathcal{F}) \\ &\simeq \text{Map}_{\text{Coh}^+(Y)}(\mathbb{L}_{Y/X}^{\text{an}}, p^*\mathcal{F}). \end{aligned}$$

Since $p: Y \rightarrow X$ is locally of finite presentation, $\mathbb{L}_{Y/X}^{\text{an}}$ is perfect. Since p is also proper flat and has finite coherent cohomological dimension, [51, Proposition 7.11] allows us to rewrite

$$\text{Map}_{\text{Coh}^+(Y)}(\mathbb{L}_{Y/X}^{\text{an}}, p^* \mathcal{F}) \simeq \text{Map}_{\text{Coh}^+(X)}(p_+(\mathbb{L}_{Y/X}^{\text{an}}), \mathcal{F}),$$

completing the proof. \square

For application to the moduli stack of curves with marked points, we generalize the previous results concerning deformations of spaces to deformations of morphisms (varying source and target). Let $T \in \text{dAfd}_k$ and let $p: X \rightarrow T$, $q: Y \rightarrow T$ be two proper flat derived k -analytic spaces over T . Let $f: X \rightarrow Y$ be a morphism in dAn/T . Let

$$\text{Def}_f: \text{dAfd}_{T//T}^{\text{op}} \longrightarrow \mathcal{S}$$

be the moduli functor of deformations of f . In other words, for $T' \in \text{dAfd}_{T//T}$, $\text{Def}_f(T')$ is the space of commutative diagrams

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & & \\ & \searrow & \swarrow & & \\ & & X' & \xrightarrow{f'} & Y' \\ & \searrow p & \swarrow q & & \\ & & T & & \\ & & \searrow p' & & \swarrow q' \\ & & & & T' \end{array}$$

where p' and q' are proper and flat and the side squares are pullbacks.

Proposition 3.11. *The functor Def_f is infinitesimally cartesian and convergent. Moreover, let $x: T \rightarrow \text{Def}_f$ be the point corresponding to the trivial deformation. If p and q have finite coherent cohomological dimension, then Def_f admits an analytic cotangent complex at x , which fits in the following pushout diagram:*

$$\begin{array}{ccc} p_+ f^*(\mathbb{L}_{Y/T}^{\text{an}})[-1] & \longrightarrow & q_+(\mathbb{L}_{Y/T}^{\text{an}})[-1] \\ \downarrow & & \downarrow \\ p_+(\mathbb{L}_{X/T}^{\text{an}})[-1] & \longrightarrow & \mathbb{L}_{\text{Def}_f, x}^{\text{an}} \end{array}$$

Proof. Using Propositions 3.4 and 3.7, we deduce that Def_f is infinitesimally cartesian and convergent. Let G be the fiber product

$$G := T \times_{\text{Def}_f} T,$$

and $\delta_x: T \rightarrow G$ the diagonal morphism. Then Lemma 3.9 shows that Def_f has an analytic cotangent complex at x if and only if G has an analytic cotangent complex

at δ_x ; in this case they are related by the formula

$$\mathbb{L}_{\text{Def}_f, x}^{\text{an}} \simeq \mathbb{L}_{G, \delta_x}^{\text{an}}[-1].$$

Let us compute $\mathbb{L}_{G, \delta_x}^{\text{an}}$. Let $\mathcal{F} \in \text{Coh}^{\geq 0}(T)$. Unraveling the definitions, we have

$$\begin{aligned} \text{Der}_G^{\text{an}}(T; \mathcal{F}) &\simeq \text{Map}_{X/}(X[p^*\mathcal{F}], X) \times_{\text{Map}_{X/}(X[p^*\mathcal{F}], Y)} \text{Map}_{Y/}(Y[q^*\mathcal{F}], Y) \\ &\simeq \text{Map}_{\text{Coh}^+(X)}(\mathbb{L}_{X/T}^{\text{an}}, p^*\mathcal{F}) \times_{\text{Map}_{\text{Coh}^+(X)}(f^*\mathbb{L}_{Y/T}^{\text{an}}, p^*\mathcal{F})} \text{Map}_{\text{Coh}^+(X)}(\mathbb{L}_{Y/T}^{\text{an}}, q^*\mathcal{F}). \end{aligned}$$

Since p and q are proper, flat and have finite coherent cohomological dimension, we can rewrite the fiber product above as

$$\text{Map}_{\text{Coh}^+(X)}(p_+(\mathbb{L}_{X/T}^{\text{an}}), \mathcal{F}) \times_{\text{Map}_{\text{Coh}^+(X)}(p_+f^*(\mathbb{L}_{Y/T}^{\text{an}}), \mathcal{F})} \text{Map}_{\text{Coh}^+(X)}(q_+(\mathbb{L}_{Y/T}^{\text{an}}), \mathcal{F}).$$

The conclusion now follows from the Yoneda lemma. \square

4. THE DERIVED MODULI STACK OF NON-ARCHIMEDEAN STABLE MAPS

In this section, we introduce the derived moduli stacks of non-archimedean (pre)stable curves and (pre)stable maps. We prove that the derived moduli stack of stable curves is equivalent to the embedding of the classical moduli stack (see Proposition 4.5). This is not the case for stable maps. We prove that the derived moduli stack of stable maps is a derived k -analytic stack that is derived lci (see Theorems 4.13, 4.15).

4.1. The derived stack of (pre)stable curves.

Definition 4.1. Let $T \in \text{dAn}_k$ be a derived k -analytic space. An n -pointed genus g (pre)stable curve over T is a proper flat morphism $p: C \rightarrow T$ in dAn_k together with n distinct sections $s_i: T \rightarrow C$ such that every geometry fiber is an n -pointed genus g (pre)stable curve.

Remark 4.2. By definition, given $T \in \text{dAn}_k$, the ∞ -category of n -pointed genus g (pre)stable curves over T is a full subcategory of $\text{dAn}_{T \amalg n // T}$.

Observe that for any $T' \rightarrow T$ in dAn_k , the pullback of any n -pointed genus g (pre)stable curve over T is an n -pointed genus g (pre)stable curve over T' . Hence we obtain the derived stacks

$$\mathbb{R}\overline{M}_{g,n}^{\text{pre}}, \mathbb{R}\overline{M}_{g,n}: \text{dAfd}_k^{\text{op}} \longrightarrow \mathcal{S}$$

of n -pointed genus g prestable (resp. stable) curves. We denote

$$\overline{M}_{g,n}^{\text{pre}} := t_0(\mathbb{R}\overline{M}_{g,n}^{\text{pre}}) \xleftarrow{j} \mathbb{R}\overline{M}_{g,n}^{\text{pre}}, \quad \overline{M}_{g,n} := t_0(\mathbb{R}\overline{M}_{g,n}) \xleftarrow{i} \mathbb{R}\overline{M}_{g,n}.$$

Note that by the flatness condition, if T is underived, any (pre)stable curve over T is also underived. Hence $\overline{M}_{g,n}^{\text{pre}}$ and $\overline{M}_{g,n}$ parametrizes underived (pre)stable curves.

Lemma 4.3. *Let $T \in \mathrm{dAn}_k$ and let $[p: C \rightarrow T, (s_i)]$ be an n -pointed genus g prestable curve over T . Then p is derived lci, and smooth outside the nodes. Moreover, the sections s_i are derived lci.*

Proof. Since $p: C \rightarrow T$ is flat, the square

$$\begin{array}{ccc} t_0(C) & \xrightarrow{j_C} & C \\ \downarrow t_0(p) & & \downarrow p \\ t_0(T) & \xrightarrow{j_T} & T \end{array}$$

is a pullback square. So we have

$$j_C^* \mathbb{L}_{C/T}^{\mathrm{an}} \simeq \mathbb{L}_{t_0(C)/t_0(T)}^{\mathrm{an}}.$$

Since the map $t_0(p): t_0(C) \rightarrow t_0(T)$ is lci, it follows that $\mathbb{L}_{t_0(C)/t_0(T)}^{\mathrm{an}}$ is perfect and in tor-amplitude $[1, 0]$. Therefore, the same goes for $\mathbb{L}_{C/T}^{\mathrm{an}}$. Moreover, $t_0(p)$ is smooth outside the nodes; so by [50, Proposition 5.50], $\mathbb{L}_{t_0(C)/t_0(T)}^{\mathrm{an}}$ has tor-amplitude $[0, 0]$ outside the nodes. Thus, the same goes for $\mathbb{L}_{C/T}^{\mathrm{an}}$; and applying the same proposition again, we conclude that $p: C \rightarrow T$ is smooth outside the nodes.

Let $U \subset C$ be the open complementary of the nodes. For every $i = 1, \dots, n$, we see that $t_0(s_i): t_0(T) \rightarrow t_0(C)$ factors through $t_0(U)$, and hence s_i factor through U . In other words, the sections s_i are also sections of the smooth map $U \rightarrow T$. So it follows from Lemma 2.5(3) that they are derived lci. \square

Lemma 4.4. *The derived stack $\mathbb{R}\overline{M}_{g,n}^{\mathrm{pre}}$ is a derived k -analytic stack.*

Proof. Using Proposition 3.11, we deduce that $\mathbb{R}\overline{M}_{g,n}^{\mathrm{pre}}$ is infinitesimally cartesian, convergent and admits a global analytic cotangent complex. Moreover, its truncation is geometric, so [50, Theorem 7.1] implies that $\mathbb{R}\overline{M}_{g,n}^{\mathrm{pre}}$ is a derived k -analytic stack. \square

Proposition 4.5. *The morphisms $j: \overline{M}_{g,n}^{\mathrm{pre}} \rightarrow \mathbb{R}\overline{M}_{g,n}^{\mathrm{pre}}$ and $i: \overline{M}_{g,n} \rightarrow \mathbb{R}\overline{M}_{g,n}$ are equivalences.*

Proof. As $\overline{M}_{g,n}$ is an open substack in $\overline{M}_{g,n}^{\mathrm{pre}}$ and $\mathbb{R}\overline{M}_{g,n}$ is the corresponding open substack in $\mathbb{R}\overline{M}_{g,n}^{\mathrm{pre}}$, it is enough to prove that j is an equivalence. Since $\overline{M}_{g,n}^{\mathrm{pre}}$ is smooth, j is an equivalence if and only if $\mathbb{R}\overline{M}_{g,n}^{\mathrm{pre}}$ is smooth. Therefore, by [50, Proposition 5.50], it is enough to check that $\mathbb{L}_{\mathbb{R}\overline{M}_{g,n}^{\mathrm{pre}}}^{\mathrm{an}}$ is perfect and in tor-amplitude $[0, -1]$. By Remark 2.3, tor-amplitude $[0, -1]$ implies perfectness. Hence it is enough to prove that for every (underived) k -affinoid space $T \in \mathrm{Afd}_k$ and every map $x: T \rightarrow \mathbb{R}\overline{M}_{g,n}^{\mathrm{pre}}$, one has

$$(4.6) \quad \pi_i \left(x^* \mathbb{L}_{\overline{M}_{g,n}^{\mathrm{pre}}}^{\mathrm{an}} \right) \simeq 0 \text{ for every } i \geq 1.$$

Let $[p: C \rightarrow T, (s_i)]$ be the n -pointed genus g prestable curve classified by x . Let $\mathbb{T}_{C/T}^{\text{an}}(-\sum s_i)$ be given by the fiber sequence

$$(4.7) \quad \mathbb{T}_{C/T}^{\text{an}}(-\sum s_i) \longrightarrow \mathbb{T}_{C/T}^{\text{an}} \longrightarrow \bigoplus_{i=1}^n s_{i*} s_i^* \mathbb{T}_{C/T}^{\text{an}}.$$

By Proposition 3.11 we have

$$(4.8) \quad x^* \mathbb{L}_{\mathbb{R}\overline{M}_{g,n}^{\text{pre}}}^{\text{an}} \simeq \left(p_* \left(\mathbb{T}_{C/T}^{\text{an}}(-\sum s_i) \right) \right)^\vee[-1].$$

Since $p: C \rightarrow T$ is a curve, in order to check (4.6), it suffices to check that

$$\pi_{-2} \left(p_* \left(\mathbb{T}_{C/T}^{\text{an}}(-\sum s_i) \right) \right) = 0.$$

Applying p_* to the fiber sequence (4.7) and passing to the long exact sequence of homotopy groups, we are left to check the vanishings

$$\pi_{-2}(p_* \mathbb{T}_{C/T}^{\text{an}}) = 0 \quad \text{and} \quad \pi_{-2}(p_* s_{i*} s_i^* \mathbb{T}_{C/T}^{\text{an}}) = 0 \quad \text{for every } i = 1, \dots, n.$$

By Lemma 4.3, $p: C \rightarrow T$ is derived lci, hence $\mathbb{T}_{C/T}^{\text{an}}$ is in tor-amplitude $[0, -1]$. As T is underived, $s_i^* \mathbb{T}_{C/T}^{\text{an}}$ is in homological amplitude $[0, -1]$. Moreover, as T is affinoid, [51, Theorem 3.4] implies that the second vanishing holds. As for the first vanishing, using the hypercohomology spectral sequence, we are left to check that

$$\pi_0(p_*(\pi_{-2} \mathbb{T}_{C/T}^{\text{an}})) \simeq \pi_{-1}(p_*(\pi_{-1} \mathbb{T}_{C/T}^{\text{an}})) \simeq 0.$$

Since $p: C \rightarrow T$ is derived lci, $\pi_{-2} \mathbb{T}_{C/T}^{\text{an}} = 0$. Moreover, since $p: C \rightarrow T$ is smooth outside the nodes, $\pi_{-1}(\mathbb{T}_{C/T}^{\text{an}})$ is supported on finitely many sections of p . As T is affinoid, it follows that $\pi_{-1}(p_*(\pi_{-1} \mathbb{T}_{C/T}^{\text{an}})) \simeq 0$, completing the proof. \square

Corollary 4.9. *Let $\overline{C}_{g,n}^{\text{pre}} \rightarrow \overline{M}_{g,n}^{\text{pre}}$, $\overline{C}_{g,n} \rightarrow \overline{M}_{g,n}$, $\mathbb{R}\overline{C}_{g,n}^{\text{pre}} \rightarrow \mathbb{R}\overline{M}_{g,n}^{\text{pre}}$ and $\mathbb{R}\overline{C}_{g,n} \rightarrow \mathbb{R}\overline{M}_{g,n}$ be the universal curves. We have equivalences*

$$\begin{array}{ccc} \overline{C}_{g,n}^{\text{pre}} & \xrightarrow{\sim} & \mathbb{R}\overline{C}_{g,n}^{\text{pre}} \\ \downarrow & & \downarrow \\ \overline{M}_{g,n}^{\text{pre}} & \xrightarrow{\sim} & \mathbb{R}\overline{M}_{g,n}^{\text{pre}} \end{array} \quad \text{and} \quad \begin{array}{ccc} \overline{C}_{g,n} & \xrightarrow{\sim} & \mathbb{R}\overline{C}_{g,n} \\ \downarrow & & \downarrow \\ \overline{M}_{g,n} & \xrightarrow{\sim} & \mathbb{R}\overline{M}_{g,n} \end{array} .$$

Proof. By definition, the vertical maps are all flat. By Proposition 4.5, the stacks $\mathbb{R}\overline{M}_{g,n}^{\text{pre}}$ and $\mathbb{R}\overline{M}_{g,n}$ are underived. Hence the stacks $\mathbb{R}\overline{C}_{g,n}^{\text{pre}}$ and $\mathbb{R}\overline{C}_{g,n}$ are also underived, from which the conclusion follows. \square

4.2. The derived stack of (pre)stable maps. Fix S a rigid k -analytic space and X a rigid k -analytic space smooth over S .

Definition 4.10. Let $T \in \mathrm{dAn}/_S$ be a derived k -analytic space over S . An n -pointed genus g prestable map into X/S over T consists of an n -pointed genus g prestable curve $[C \rightarrow T, (s_i)]$ over T and an S -map $f: C \rightarrow X$. It is called *stable* if every geometric fiber $[C_t, (s_i(t)), f_t: C_t \rightarrow X]$ is a stable map, in the sense that its automorphism group is a finite analytic group.

Remark 4.11. An equivalent characterization for $[C_t, (s_i(t)), f_t: C_t \rightarrow X]$ to be stable is that every irreducible component of C_t of genus 0 (resp. 1) which maps to a point must have at least 3 (resp. 1) special points on its normalization, where by special point, we mean the preimage of either a marked point or a node by the normalization map. We refer to [33, 16] for the theory of stable maps in algebraic geometry.

Remark 4.12. By definition, given $T \in \mathrm{dAn}/_S$, the ∞ -category of n -pointed genus g (pre)stable maps into X/S over T is a full subcategory of $\mathrm{dAn}_{T^{\mathrm{un}}/T \times_S X}$.

Observe that for any $T' \rightarrow T$ in dAn_k , the pullback of any n -pointed genus g (pre)stable map into X/S over T is an n -pointed genus g (pre)stable map into X/S over T' . Hence we obtain the derived stacks

$$\mathbb{R}\overline{M}_{g,n}^{\mathrm{pre}}(X/S), \overline{M}_{g,n}(X/S): \mathrm{dAfd}_k^{\mathrm{op}} \longrightarrow \mathcal{S}$$

of n -pointed genus g prestable (resp. stable) maps into X/S . We denote

$$\overline{M}_{g,n}^{\mathrm{pre}}(X/S) := \mathrm{t}_0(\mathbb{R}\overline{M}_{g,n}^{\mathrm{pre}}(X/S)), \quad \overline{M}_{g,n}(X/S) := \mathrm{t}_0(\overline{M}_{g,n}(X/S)).$$

Note that by the flatness condition, if T is underived, any (pre)stable map into X/S over T is also underived. Hence $\overline{M}_{g,n}^{\mathrm{pre}}(X/S)$ and $\overline{M}_{g,n}(X/S)$ parametrize underived (pre)stable maps into X/S .

Theorem 4.13. *The derived stacks $\mathbb{R}\overline{M}_{g,n}^{\mathrm{pre}}(X/S)$ and $\overline{M}_{g,n}(X/S)$ are derived k -analytic stacks locally of finite presentation over S .*

Proof. By definition,

$$\mathbb{R}\overline{M}_{g,n}^{\mathrm{pre}}(X/S) \simeq \mathbf{Map}_{\mathbb{R}\overline{M}_{g,n}^{\mathrm{pre}} \times_S}(\mathbb{R}\overline{C}_{g,n}^{\mathrm{pre}} \times_S, X \times \mathbb{R}\overline{M}_{g,n}^{\mathrm{pre}}).$$

By Corollary 4.9, we have

$$\mathbf{Map}_{\mathbb{R}\overline{M}_{g,n}^{\mathrm{pre}} \times_S}(\mathbb{R}\overline{C}_{g,n}^{\mathrm{pre}} \times_S, X \times \mathbb{R}\overline{M}_{g,n}^{\mathrm{pre}}) \simeq \mathbf{Map}_{\overline{M}_{g,n}^{\mathrm{pre}} \times_S}(\overline{C}_{g,n}^{\mathrm{pre}} \times_S, X \times \overline{M}_{g,n}^{\mathrm{pre}}).$$

By [51, Theorem 1.1], the right hand side is a derived k -analytic stack locally of finite presentation over S . We conclude that $\mathbb{R}\overline{M}_{g,n}^{\mathrm{pre}}(X/S)$ is a derived k -analytic

stack locally of finite presentation over S . Since the stability condition is an open condition, the inclusion

$$\mathbb{R}\overline{M}_{g,n}(X/S) \hookrightarrow \mathbb{R}\overline{M}_{g,n}^{\text{pre}}(X/S)$$

is Zariski open. We deduce that $\mathbb{R}\overline{M}_{g,n}(X/S)$ is also a derived k -analytic stack locally of finite presentation over S . \square

Remark 4.14. If S is an algebraic variety over k and X is an algebraic variety over S , then by [25], we have

$$\mathbb{R}\overline{M}_{g,n}^{\text{pre}}(X/S)^{\text{an}} \xrightarrow{\sim} \mathbb{R}\overline{M}_{g,n}^{\text{pre}}(X^{\text{an}}/S^{\text{an}}),$$

and hence

$$\mathbb{R}\overline{M}_{g,n}(X/S)^{\text{an}} \xrightarrow{\sim} \mathbb{R}\overline{M}_{g,n}(X^{\text{an}}/S^{\text{an}}).$$

Theorem 4.15. *The derived k -analytic stacks $\mathbb{R}\overline{M}_{g,n}^{\text{pre}}(X/S)$ and $\mathbb{R}\overline{M}_{g,n}(X/S)$ are derived lci over S .*

Proof. Since $\mathbb{R}\overline{M}_{g,n}(X/S) \hookrightarrow \mathbb{R}\overline{M}_{g,n}^{\text{pre}}(X/S)$ is Zariski open, it suffices to prove that $\mathbb{R}\overline{M}_{g,n}^{\text{pre}}(X/S)$ is derived lci over S .

We start by computing explicitly the relative analytic cotangent complex of $\mathbb{R}\overline{M}_{g,n}^{\text{pre}}(X/S)$ over S . Let $T \in \text{dAn}/S$ and $x: T \rightarrow \mathbb{R}\overline{M}_{g,n}^{\text{pre}}(X/S)$. Recall $\overline{M}_{g,n}^{\text{pre}} \simeq \mathbb{R}\overline{M}_{g,n}^{\text{pre}}$ by Proposition 4.5. Let $y: T \rightarrow \overline{M}_{g,n}^{\text{pre}} \times S$ be the composition of x with the map $\mathbb{R}\overline{M}_{g,n}^{\text{pre}}(X/S) \rightarrow \overline{M}_{g,n}^{\text{pre}} \times S$ taking domain curves. Let

$$\begin{array}{ccc} C & \xrightarrow{f} & X \\ s_i \uparrow & & \downarrow p \\ & & T \end{array}$$

be the n -pointed genus g stable map classified by x . Then [51, Lemma 8.4] implies that we have a canonical equivalence

$$x^* \mathbb{L}_{\mathbb{R}\overline{M}_{g,n}^{\text{pre}}(X/S)/(\overline{M}_{g,n}^{\text{pre}} \times S)}^{\text{an}} \simeq p_+(f^* \mathbb{L}_{X/S}^{\text{an}}).$$

The transitivity fiber sequence for the analytic cotangent complex now yields

$$y^* \mathbb{L}_{\overline{M}_{g,n}^{\text{pre}} \times S}^{\text{an}} \longrightarrow x^* \mathbb{L}_{\mathbb{R}\overline{M}_{g,n}^{\text{pre}}(X/S)/S}^{\text{an}} \longrightarrow x^* \mathbb{L}_{\mathbb{R}\overline{M}_{g,n}^{\text{pre}}(X/S)/(\overline{M}_{g,n}^{\text{pre}} \times S)}^{\text{an}}.$$

By (4.8), we obtain

$$x^* \mathbb{T}_{\mathbb{R}\overline{M}_{g,n}^{\text{pre}}(X/S)/S}^{\text{an}} \simeq p_* \left(\text{cofib} \left(\mathbb{T}_{C/T}^{\text{an}}(-\sum s_i) \rightarrow f^* \mathbb{T}_{X/S}^{\text{an}} \right) \right).$$

At this point, in order to check that $\mathbb{R}\overline{M}_{g,n}^{\text{pre}}(X/S)$ is derived lci over S , it suffices to verify that for every $x: T \rightarrow \mathbb{R}\overline{M}_{g,n}^{\text{pre}}(X/S)$ with T underived over S , we have

$$\pi_j \left(x^* \mathbb{T}_{\mathbb{R}\overline{M}_{g,n}^{\text{pre}}(X/S)/S}^{\text{an}} \right) \simeq 0 \quad \text{for every } j \leq -2.$$

By Lemma 4.3, the map $p: C \rightarrow T$ and the sections $s_i: T \rightarrow C$ are all derived lci. Therefore, the fiber sequence (4.7) implies that $\mathbb{T}_{C/T}^{\text{an}}(-\sum s_i)$ is in homological amplitude $[0, -1]$. As $p: C \rightarrow T$ is a relative curve, we have

$$\pi_j \left(p_* \left(\mathbb{T}_{C/T}^{\text{an}}(-\sum s_i) \right) \right) \simeq 0 \quad \text{for every } j \leq -3.$$

Moreover, since X is smooth over S , $f^* \mathbb{T}_{X/S}^{\text{an}}$ is concentrated in degree 0, thus $\pi_j(p_* f^* \mathbb{T}_{X/S}^{\text{an}}) \simeq 0$ for every $j \leq -2$, whence the conclusion follows. \square

4.3. Stable maps associated to A-graphs. The stack of stable maps has a natural refinement indexed by combinatorial data which we describe below:

Definition 4.16. A *cycle* on a rigid k -analytic space X is a finite formal integral linear combination of closed irreducible reduced analytic subspaces. Two cycles Z and Z' on X are called *analytically equivalent* if there is a connected smooth rigid k -analytic curve T and a cycle V on $X \times T$ flat over T , such that

$$[V_t] - [V_{t'}] = [Z] - [Z']$$

for two rigid points t, t' on T . Let $A(X)$ denote the group of one-dimensional cycles on X modulo analytic equivalence.

For every class $\beta \in A(X)$, we denote by $\overline{M}_{g,n}(X/S, \beta)$ the connected component of $\overline{M}_{g,n}(X/S)$ consisting of stable maps such that the fundamental class of the domain curve maps to β , and we define $\mathbb{R}\overline{M}_{g,n}(X/S, \beta)$ similarly.

Definition 4.17 (see [9, Definition 1.5]). A *modular graph* consists of the following data:

- (1) A finite graph τ .
- (2) A subset T_τ of 1-valent vertices of τ called *tail vertices*, such that each edge e of τ contains at most one tail vertex; an edge containing a tail vertex is called a *tail*; we denote by V_τ the set of non-tail vertices of τ , and by E_τ the set of non-tail edges of τ . The tails are numbered, i.e. a bijection between T_τ and $\{1, \dots, |T_\tau|\}$ is fixed.
- (3) For each $v \in V_\tau$, a non-negative integer $g(v)$ called *genus*.

For every non-tail edge e , we choose a point p in the middle and will consider the two *half-edges* incident to p . Given $v \in V_\tau$, we denote by E_v the set of tails and half-edges connected to v ; we call the cardinality of E_v the *valence* of v , and denote it by $\text{val}(v)$. The *genus* $g(\tau)$ of the modular graph τ is the genus of the underlying graph plus the sum of all $g(v)$. An A-graph is a pair (τ, β) consisting of a modular graph τ and a map $\beta: V_\tau \rightarrow A(X)$.

Definition 4.18. Let τ be a modular graph and $T \in \mathrm{dAn}_k$ a derived k -analytic space. A τ -marked derived k -analytic space $[X_v, (s_{v,e})_{e \in E_\tau}]_{v \in V_\tau}$ over T is a collection of separated derived k -analytic spaces X_v over T equipped with sections $s_{v,e} : T \rightarrow X_v$ for every $e \in E_v$.

Note that τ -marked derived k -analytic spaces over T form naturally an ∞ -category dAn_T^τ , which is a full subcategory of the product $\prod_{v \in V_\tau} \mathrm{dAn}_{T \amalg \mathrm{val}(v)} // T$.

Construction 4.19. Let τ be a modular graph and let $[X_v, (s_{v,e})_{e \in E_\tau}]_{v \in V_\tau}$ be a τ -marked derived k -analytic space over T . Since each X_v is separated over T , the sections $s_{v,e} : T \rightarrow X_v$ are closed immersions. For each non-tail edge e of τ , let e_1 and e_2 be the two corresponding half-edges, and v_1, v_2 the two endpoints ($v_1 = v_2$ if e is a loop). Using [50, Theorem 6.5], we glue these sections and obtain a new derived k -analytic space over T . The universal property of the gluing induces a functor

$$\gamma_\tau : \mathrm{dAn}_T^\tau \longrightarrow \mathrm{dAn}_T.$$

We refer to γ_τ as the τ -gluing functor.

Definition 4.20. Given an n -pointed genus g prestable curve $[C, (s_i)]$ over an algebraically closed field, let Γ be the dual graph of C . For each vertex v of Γ , let $g(v)$ be the genus of the corresponding irreducible component C_v of C . For each marked point s_i , we add a vertex v_i to Γ , and an edge e_i connecting v_i and the vertex of Γ corresponding to the irreducible component of C containing s_i . Then we obtain a modular graph τ with $T_\tau = \{v_i\}$, which we call the modular graph associated to $[C, (s_i)]$.

Definition 4.21. Let τ be a modular graph and $T \in \mathrm{dAn}_k$ a derived k -analytic space. A τ -marked (pre)stable curve over T consists of a τ -marked derived k -analytic space $[C_v, (s_{v,e})_{e \in E_v}]_{v \in V_\tau}$ over T such that for every $v \in V_\tau$, $[C_v, (s_{v,e})_{e \in E_v}]$ is a $\mathrm{val}(v)$ -pointed genus $g(v)$ (pre)stable curve over T .

Definition 4.22. Let τ be a modular graph and $T \in \mathrm{dAn}_k$ a derived k -analytic space. Let $[C_v, (s_{v,e})]_{v \in V_\tau}$ be a τ -marked (pre)stable curve over T . Let C be its τ -gluing (see Construction 4.19). It inherits sections s_e for every non-tail edge e of τ and s_i for every tail edge i of τ . We call $[C, (s_i)_{i \in T_\tau}, (s_e)_{e \in E_\tau}]$ the *glued curve* associated to the τ -marked (pre)stable curve $[C_v, (s_{v,e})_{e \in E_v}]_{v \in V_\tau}$. Note that the arithmetic genus of C is equal to the genus of τ . Given any geometric point $t \in T$, consider the geometric fiber $[C_t, (s_i(t))_{i \in T_\tau}, (s_e(t))_{e \in E_\tau}]$. By construction, every $s_i(t)$ lies in the smooth locus of C_t , and every $s_e(t)$ is a node of C_t . We also remark that the modular graph associated to the pointed (pre)stable curve $[C_t, (s_i(t))_{i \in T_\tau}]$ is not τ in general, but it admits a contraction to τ .

Remark 4.23. Let $\overline{M}_\tau^{\text{pre}}$ and $\mathbb{R}\overline{M}_\tau^{\text{pre}}$ denote respectively the underived and the derived stack of τ -marked prestable curves. Let \overline{M}_τ and $\mathbb{R}\overline{M}_\tau$ denote respectively the underived and the derived stack of τ -marked stable curves. By definition, we have natural equivalences

$$\begin{aligned} \overline{M}_\tau^{\text{pre}} &\simeq \prod_{v \in V_\tau} \overline{M}_{g(v), \text{val}(v)}^{\text{pre}}, & \overline{M}_\tau &\simeq \prod_{v \in V_\tau} \overline{M}_{g(v), \text{val}(v)}, \\ \mathbb{R}\overline{M}_\tau^{\text{pre}} &\simeq \prod_{v \in V_\tau} \mathbb{R}\overline{M}_{g(v), \text{val}(v)}^{\text{pre}}, & \mathbb{R}\overline{M}_\tau &\simeq \prod_{v \in V_\tau} \mathbb{R}\overline{M}_{g(v), \text{val}(v)}. \end{aligned}$$

Let $\overline{C}_\tau^{\text{pre}}, \mathbb{R}\overline{C}_\tau^{\text{pre}}, \overline{C}_\tau$ and $\mathbb{R}\overline{C}_\tau$ be the universal glued curves over the moduli stacks $\overline{M}_\tau^{\text{pre}}, \mathbb{R}\overline{M}_\tau^{\text{pre}}, \overline{M}_\tau$ and $\mathbb{R}\overline{M}_\tau$ respectively. They are well-defined by Lemma 3.1. Then by Corollary 4.9, we obtain natural equivalences

$$\begin{array}{ccc} \overline{C}_\tau^{\text{pre}} & \xrightarrow{\sim} & \mathbb{R}\overline{C}_\tau^{\text{pre}} & & \overline{C}_\tau & \xrightarrow{\sim} & \mathbb{R}\overline{C}_\tau \\ \downarrow & & \downarrow & \text{and} & \downarrow & & \downarrow \\ \overline{M}_\tau^{\text{pre}} & \xrightarrow{\sim} & \mathbb{R}\overline{M}_\tau^{\text{pre}} & & \overline{M}_\tau & \xrightarrow{\sim} & \mathbb{R}\overline{M}_\tau \end{array}.$$

Definition 4.24. Let (τ, β) be an A-graph and let $T \in \text{dAn}/_S$ be a derived k -analytic space over S . A (τ, β) -marked prestable map into X/S over T consists of a τ -marked prestable curve $[C_v, (s_{v,e})_{e \in E_v}]_{v \in V_\tau}$ over T and an S -map f from the associated glued curve $[C, (s_i)_{i \in T_\tau}, (s_e)_{e \in E_\tau}]$ to X such that for every geometric point $t \in T$ and every $v \in V_\tau$, the composite map $C_{v,t} \rightarrow C_t \xrightarrow{f_t} X$ has class $\beta(v)$. It is called *stable* if every geometric fiber $[C_t, (s_i(t))_{i \in T_\tau}, f_t]$ is a stable map.

Note that (τ, β) -marked (pre)stable maps into X/S over T form naturally an ∞ -category, which is a full subcategory of the fiber product

$$\text{dAn}/_{T \times_S X} \times_{\text{dAn}/_T} \text{dAn}/_T^\tau,$$

where the functor $\text{dAn}/_T^\tau \rightarrow \text{dAn}/_T$ is the τ -gluing functor γ_τ of Construction 4.19.

Remark 4.25. Let $\overline{M}^{\text{pre}}(X/S, \tau, \beta)$ and $\mathbb{R}\overline{M}^{\text{pre}}(X/S, \tau, \beta)$ denote respectively the underived and the derived stack of (τ, β) -marked prestable maps into X/S . Let $\overline{M}(X/S, \tau, \beta)$ and $\mathbb{R}\overline{M}(X/S, \tau, \beta)$ denote respectively the underived and the derived stack of (τ, β) -marked stable maps into X/S . As in the proof Theorem 4.13, we have Zariski open embeddings

$$\begin{aligned} \overline{M}(X/S, \tau, \beta) &\hookrightarrow \overline{M}^{\text{pre}}(X/S, \tau, \beta) \simeq \mathbf{t}_0 \mathbf{Map}_{\overline{M}_\tau^{\text{pre}} \times_S}(\overline{C}_\tau^{\text{pre}} \times_S X \times \overline{M}_\tau^{\text{pre}}), \\ \mathbb{R}\overline{M}(X/S, \tau, \beta) &\hookrightarrow \mathbb{R}\overline{M}^{\text{pre}}(X/S, \tau, \beta) \simeq \mathbf{Map}_{\overline{M}_\tau^{\text{pre}} \times_S}(\overline{C}_\tau^{\text{pre}} \times_S X \times \overline{M}_\tau^{\text{pre}}). \end{aligned}$$

Hence $\overline{M}(X/S, \tau, \beta)$ is a rigid k -analytic stack over S , and $\mathbb{R}\overline{M}(X/S, \tau, \beta)$ is a derived k -analytic stack over S . Similar to Theorem 4.15, the derived k -analytic stack $\mathbb{R}\overline{M}(X/S, \tau, \beta)$ is moreover derived lci over S .

Let $\overline{C}^{\text{pre}}(X/S, \tau, \beta)$, $\mathbb{R}\overline{C}^{\text{pre}}(X/S, \tau, \beta)$, $\overline{C}(X/S, \tau, \beta)$ and $\mathbb{R}\overline{C}(X/S, \tau, \beta)$ be the universal glued curves over the moduli stacks $\overline{M}^{\text{pre}}(X/S, \tau, \beta)$, $\mathbb{R}\overline{M}^{\text{pre}}(X/S, \tau, \beta)$, $\overline{M}(X/S, \tau, \beta)$ and $\mathbb{R}\overline{M}(X/S, \tau, \beta)$ respectively; in other words, they are respectively the pullbacks of $\overline{C}_\tau^{\text{pre}}$, $\mathbb{R}\overline{C}_\tau^{\text{pre}}$, \overline{C}_τ and $\mathbb{R}\overline{C}_\tau$ along the maps taking domains.

4.4. The stabilization map. The goal of this subsection is to construct the stabilization of prestable maps in families

$$\text{stab}: \mathbb{R}\overline{M}^{\text{pre}}(X/S, \tau, \beta) \longrightarrow \mathbb{R}\overline{M}(X/S, \tau, \beta).$$

We begin by constructing the stabilizing contraction in the sense of the following definition:

Definition 4.26. Let $T \in \text{dAn}_S$ be a derived k -analytic space over S . Let $\mathbf{f} := ([C_v, (s_{v,e})_{e \in E_v}]_{v \in V_\tau}, f)$ and $\mathbf{f}' := ([C'_v, (s'_{v,e})_{e \in E_v}]_{v \in V_\tau}, f')$ be two τ -marked prestable maps into X/S over T . We say that a morphism $p: \mathbf{f} \rightarrow \mathbf{f}'$ is a *stabilizing contraction* if \mathbf{f}' is stable and for every other τ -marked stable map \mathbf{f}'' the morphism

$$\text{Map}_{/T}(\mathbf{f}', \mathbf{f}'') \longrightarrow \text{Map}_{/T}(\mathbf{f}, \mathbf{f}'')$$

is an equivalence. Here the mapping spaces are computed in the ∞ -category of τ -marked prestable maps into X/S over T (see Definition 4.24).

4.4.1. A criterion for stabilization. Before constructing the stabilizing contraction, we describe a criterion for checking its universal property.

Lemma 4.27. *Let $Y \in \text{dAn}_T$ be a derived k -analytic space over T , and $p: C \rightarrow C'$ a morphism in dAn_T . Assume the following:*

- (1) p is proper and has finite coherent cohomological dimension;
- (2) both C and C' are proper and flat over T ;
- (3) the canonical map

$$\mathcal{O}_{C'}^{\text{alg}} \longrightarrow p_* \mathcal{O}_C^{\text{alg}}$$

is an equivalence;

- (4) $Y \rightarrow T$ is flat, separated and locally of finite presentation.

Then $\mathbf{Map}_{/T}(C, Y)$ and $\mathbf{Map}_{/T}(C', Y)$ are derived k -analytic stacks locally of finite presentation over T , and the induced map

$$(4.28) \quad \pi: \mathbf{Map}_{/T}(C', Y) \longrightarrow \mathbf{Map}_{/T}(C, Y)$$

is étale. In particular the diagram

$$\begin{array}{ccc} \text{Map}_{/T}(C', Y) & \longrightarrow & \text{Map}_{/t_0(T)}(C'_0, t_0(Y)) \\ \downarrow & & \downarrow \\ \text{Map}_{/T}(C, Y) & \longrightarrow & \text{Map}_{/t_0(T)}(C_0, t_0(Y)) \end{array}$$

is a pullback square.

Proof. Let $U \in \mathrm{dAn}/T$, and assume that it is underived. Since C and Y are flat over T , we have

$$\begin{aligned} \mathbf{Map}_{/T}(C, Y)(U) &\simeq \mathrm{Map}_{/T}(C \times_T U, Y \times_T U) \\ &\simeq \mathrm{Map}_{/t_0(T)}(t_0(C) \times_{t_0(T)} U, t_0(Y) \times_T U) \\ &\simeq \mathbf{Map}_{/t_0(T)}(t_0(C), t_0(Y))(U). \end{aligned}$$

In other words, we have a canonical equivalence

$$t_0(\mathbf{Map}_{/T}(C, Y)) \simeq t_0(\mathbf{Map}_{/t_0(T)}(t_0(C), t_0(Y))).$$

Therefore, by [15, Proposition 5.3.3], the truncation of $\mathbf{Map}_{/T}(C, Y)$ is representable. Then it follows from [51, Proposition 8.5] that $\mathbf{Map}_{/T}(C, Y)$ and $\mathbf{Map}_{/T}(C', Y)$ are derived k -analytic stacks locally of finite presentation over T .

In order to prove that the map (4.28) is étale, it is enough to show that the relative analytic cotangent complex of is zero. Denote $F := \mathbf{Map}_{/T}(C, Y)$ and $F' := \mathbf{Map}_{/T}(C', Y)$. Then it is enough to check that for any (underived) k -affinoid space U over T and any $u: U \rightarrow F'$, the map

$$u^* \pi^* \mathbb{L}_{F/T}^{\mathrm{an}} \longrightarrow u^* \mathbb{L}_{F'/T}^{\mathrm{an}}$$

is an equivalence. Write $C_U := U \times_T C$ and $C'_U := U \times_T C'$. Let $f: C_U \rightarrow Y$ and $f': C'_U \rightarrow Y$ be the morphisms corresponding to $\pi \circ u$ and u , respectively. Let

$$q_U: C_U \rightarrow U, \quad q'_U: C'_U \rightarrow U, \quad p_U: C_U \rightarrow C'_U$$

be the induced morphisms. Then $f \simeq f' \circ p_U$ and $q_U \simeq q'_U \circ p_U$. Using [51, Lemma 8.4] we can rewrite

$$u^* \mathbb{T}_{F/T}^{\mathrm{an}} \simeq q_{U*}(f^*(\mathbb{T}_Y^{\mathrm{an}})), \quad u^* \pi^* \mathbb{T}_{F'/T}^{\mathrm{an}} \simeq q'_{U*}(f'^*(\mathbb{T}_Y^{\mathrm{an}})).$$

Since $p_U: C_U \rightarrow C'_U$ is proper, by assumption (3) and the projection formula ([51, Theorem 1.4]), we have

$$q_{U*}(f^*(\mathbb{T}_Y^{\mathrm{an}})) \simeq q'_{U*}(p_{U*}(p_U^*(f'^*(\mathbb{T}_Y^{\mathrm{an}})))) \simeq q'_{U*}(p_{U*} \mathcal{O}_{C_U}^{\mathrm{alg}} \otimes f'^*(\mathbb{T}_Y^{\mathrm{an}})) \simeq q'_{U*}(f'^*(\mathbb{T}_Y^{\mathrm{an}})),$$

completing the proof. \square

Proposition 4.29. *Let $T \in \mathrm{dAn}/S$ be a derived k -analytic space over S . Let $\mathbf{f} = ([C_v, (s_{v,e})_{e \in E_v}]_{v \in V_\tau}, f)$ and $\mathbf{f}' = ([C'_v, (s'_{v,e})_{e \in E_v}]_{v \in V_\tau}, f')$ be two τ -marked prestable maps into X/S over T and let $p: \mathbf{f} \rightarrow \mathbf{f}'$ be a morphism between them. Assume that:*

- (1) *the canonical map $\mathcal{O}_{C'_v}^{\mathrm{alg}} \rightarrow p_* \mathcal{O}_{C_v}^{\mathrm{alg}}$ is an equivalence;*
- (2) *the truncation $p_0: \mathbf{f}_0 \rightarrow \mathbf{f}'_0$ is a stabilizing contraction.*

Then p is a stabilizing contraction as well.

Proof. It follows from the definition that \mathbf{f}' is stable if and only if its truncation \mathbf{f}'_0 is stable. Let now \mathbf{f}'' be any other τ -marked stable map into X/S over T . Write $T_0 := t_0(T)$. By assumption (2), it is enough to prove that the diagram

$$(4.30) \quad \begin{array}{ccc} \mathrm{Map}_{/T}(\mathbf{f}', \mathbf{f}'') & \longrightarrow & \mathrm{Map}_{/T_0}(\mathbf{f}'_0, \mathbf{f}''_0) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{/T}(\mathbf{f}, \mathbf{f}'') & \longrightarrow & \mathrm{Map}_{/T_0}(\mathbf{f}_0, \mathbf{f}''_0) \end{array}$$

is a pullback square. Write

$$\begin{aligned} \mathbf{f} &= ([C_v, (s_{v,e})_{e \in E_v}]_{v \in V_\tau}, f), & \mathbf{C} &:= [C_v, (s_{v,e})_{e \in E_v}]_{v \in V_\tau}, \\ \mathbf{f}_0 &= ((C_v)_0, ((s_{v,e})_0)_{e \in E_v}]_{v \in V_\tau}, f_0, & \mathbf{C}_0 &:= [(C_v)_0, ((s_{v,e})_0)_{e \in E_v}]_{v \in V_\tau}. \end{aligned}$$

Let $[C, (s_i)_{i \in T_\tau}, (s_e)_{e \in E_\tau}]$ and $[C_0, ((s_i)_0)_{i \in T_\tau}, ((s_e)_0)_{e \in E_\tau}]$ be the associated glued curves. We introduce analogous notations for \mathbf{f}' and \mathbf{f}'' . By Definition 4.24, we have

$$\mathrm{Map}_{/T}(\mathbf{f}, \mathbf{f}'') \simeq \mathrm{Map}_{/T \times_S X}(C, C'') \times_{\mathrm{Map}_{/T}(C, C'')} \mathrm{Map}_{\mathrm{dAn}_T^\tau}(\mathbf{C}, \mathbf{C}'').$$

The other mapping spaces appearing in diagram (4.30) have analogous descriptions. We are therefore left to check that the diagrams

$$(4.31) \quad \begin{array}{ccc} \mathrm{Map}_{/T}(C', C'') & \longrightarrow & \mathrm{Map}_{/T_0}(C'_0, C''_0) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{/T}(C, C'') & \longrightarrow & \mathrm{Map}_{/T_0}(C_0, C''_0) \end{array},$$

$$(4.32) \quad \begin{array}{ccc} \mathrm{Map}_{/T \times_S X}(C', C'') & \longrightarrow & \mathrm{Map}_{/T_0 \times_S X}(C'_0, C''_0) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{/T \times_S X}(C, C'') & \longrightarrow & \mathrm{Map}_{/T_0 \times_S X}(C_0, C''_0) \end{array},$$

$$(4.33) \quad \begin{array}{ccc} \mathrm{Map}_{\mathrm{dAn}_T^\tau}(\mathbf{C}', \mathbf{C}'') & \longrightarrow & \mathrm{Map}_{\mathrm{dAn}_{T_0}^\tau}(\mathbf{C}'_0, \mathbf{C}''_0) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathrm{dAn}_T^\tau}(\mathbf{C}, \mathbf{C}'') & \longrightarrow & \mathrm{Map}_{\mathrm{dAn}_{T_0}^\tau}(\mathbf{C}_0, \mathbf{C}''_0) \end{array}$$

are pullback squares.

For diagram (4.31), the statement follows directly from Lemma 4.27. For diagram (4.32), we rewrite

$$\mathrm{Map}_{/T \times_S X}(C, C'') \simeq \mathrm{fib}\left(\mathrm{Map}_{/T}(C, C'') \rightarrow \mathrm{Map}_{/T}(C, T \times_S X)\right),$$

where the fiber is taken at the given map $f: C \rightarrow T \times_S X$. The other mapping spaces appearing in diagram (4.32) admit analogous descriptions. Applying Lemma 4.27 twice, we deduce that diagram (4.32) is also a pullback.

We are left to prove that diagram (4.33) is a pullback. Unraveling the definition of the ∞ -category $\mathrm{dAn}_{/T}^\tau$, we can rewrite

$$\mathrm{Map}_{\mathrm{dAn}_{/T}^\tau}(\mathbf{C}, \mathbf{C}'') \simeq \prod_{v \in V_\tau} \mathbf{Map}_{T^{\mathrm{II} \, \mathrm{val}(v)}/T}(C_v, C''_v).$$

Fix $v \in V_\tau$ and let $n := \mathrm{val}(v)$. It is then enough to prove that the diagram

$$(4.34) \quad \begin{array}{ccc} \mathrm{Map}_{T^{\mathrm{II}n}/T}(C'_v, C''_v) & \longrightarrow & \mathrm{Map}_{T_0^{\mathrm{II}n}/T_0}((C'_v)_0, (C''_v)_0) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{T^{\mathrm{II}n}/T}(C_v, C''_v) & \longrightarrow & \mathrm{Map}_{T_0^{\mathrm{II}n}/T_0}((C_v)_0, (C''_v)_0) \end{array}$$

is a pullback. Consider the diagram

$$\begin{array}{ccccc} \mathrm{Map}_{T^{\mathrm{II}n}/T}(C', C'') & \longrightarrow & \mathrm{Map}_{T^{\mathrm{II}n}/T}(C, C'') & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Map}_{/T}(C', C'') & \longrightarrow & \mathrm{Map}_{/T}(C, C'') & \longrightarrow & \mathrm{Map}_{/T}(T^{\mathrm{II}n}, C''), \end{array}$$

where the right vertical arrow selects the sections of C''_v . The outer and the right squares are pullbacks by definition. It follows that the left square is also a pullback. Similarly,

$$\begin{array}{ccc} \mathrm{Map}_{T_0^{\mathrm{II}n}/T_0}((C'_v)_0, (C''_v)_0) & \longrightarrow & \mathrm{Map}_{/T_0}((C'_v)_0, (C''_v)_0) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{T_0^{\mathrm{II}n}/T_0}((C_v)_0, (C''_v)_0) & \longrightarrow & \mathrm{Map}_{/T_0}((C_v)_0, (C''_v)_0) \end{array}$$

is a also pullback square. Therefore, in order to prove that diagram (4.34) is a pullback it is enough to prove that

$$\begin{array}{ccc} \mathrm{Map}_{/T}(C'_v, C''_v) & \longrightarrow & \mathrm{Map}_{/T_0}((C'_v)_0, (C''_v)_0) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{/T}(C_v, C''_v) & \longrightarrow & \mathrm{Map}_{/T_0}((C_v)_0, (C''_v)_0) \end{array}$$

is a pullback square. This follows from Lemma 4.27, completing the proof. \square

We remark that it is enough to check assumption (1) of Proposition 4.29 at the level of truncations, by the following lemma:

Lemma 4.35. *Let $T \in \mathrm{dAn}_k$, and let $p: C \rightarrow C'$ be a morphism in dAn/T . Assume that:*

- (1) $p: C \rightarrow C'$ is proper and has finite coherent cohomological dimension;
- (2) both C and C' are flat over T ;
- (3) let $p_0: C_0 \rightarrow C'_0$ denote the truncation of p , the canonical map

$$\mathcal{O}_{C'_0}^{\mathrm{alg}} \longrightarrow p_{0*} \mathcal{O}_{C_0}^{\mathrm{alg}}$$

is an equivalence.

Then the canonical map

$$\mathcal{O}_{C'}^{\mathrm{alg}} \longrightarrow p_* \mathcal{O}_C^{\mathrm{alg}}$$

is an equivalence as well.

Proof. Since both C and C' are flat over T , the diagram

$$\begin{array}{ccc} C_0 & \longrightarrow & C \\ \downarrow & & \downarrow \\ C'_0 & \xrightarrow{j} & C' \end{array}$$

is a pullback square. Since p is proper and has finite coherent cohomological dimension, we see that $p_* \mathcal{O}_C^{\mathrm{alg}}$ belongs to $\mathrm{Coh}^+(C')$. Therefore, the map $\alpha: \mathcal{O}_{C'}^{\mathrm{alg}} \rightarrow p_* \mathcal{O}_C^{\mathrm{alg}}$ is an equivalence if and only if $j^*(\alpha)$ is an equivalence. Since p is proper, by derived base change ([51, Theorem 1.5]), $j^*(\alpha)$ is canonically identified with the map $\mathcal{O}_{C'_0}^{\mathrm{alg}} \rightarrow p_{0*} \mathcal{O}_{C_0}^{\mathrm{alg}}$. The conclusion follows. \square

4.4.2. *Construction of stabilization.* Now let us construct the stabilizing contraction in Definition 4.26 for any τ -marked prestable map $\mathbf{f} := ([C_v, (s_{v,e})_{e \in E_v}]_{v \in V_\tau}, f)$ into X/S over T . We first reason étale locally on T . Let $t \in T$ be a geometric point and let $([(C_v)_t, (s_{v,e}(t))_{e \in E_v}]_{v \in V_\tau}, f_t)$ denote the associated stable map. Up to replacing T by a connected étale neighborhood of t , we can assume that for every geometric point $u = ([C_v]_u, (s_{v,e}(u))_{e \in E_v}]_{v \in V_\tau}, f_u) \in T_0 := t_0(T)$ and every $v \in V_\tau$, the following conditions are satisfied:

- (1) $(C_v)_u$ is a partial smoothing of $(C_v)_t$, in the sense that the deformation from $(C_v)_t$ to $(C_v)_u$ does not create any new nodes.
- (2) Let D be an irreducible component of $(C_v)_u$ specializing to a union E of irreducible components of $(C_v)_t$. If f_t is not constant on one component of E , then f_u is not constant on D .

To every irreducible component of $(C_v)_t$ on which f_t is not constant, we add three new marked points away from the nodes and the existing marked points. In this way we obtain a new τ -marked prestable map $([(C_v)_t, (s_{v,e}(t), p_{v,j}(t))]_{v \in V_\tau}, f_t)$. By [50, Proposition 5.50(3)], up to shrinking T , we can extend the marked points $p_{v,j}(t)$ to disjoint sections $p_{v,j}$ of $C_v \rightarrow T$, away from the nodes and the existing sections. Using the equivalence $\overline{M}_\tau^{\text{pre}} \simeq \mathbb{R}\overline{M}_\tau^{\text{pre}}$, we apply stabilization for τ -marked prestable curves (see [55, 0E8A]) to the family $[C_v, (s_{v,e}, p_{v,j})]_{v \in V_\tau}$, and obtain a τ -marked stable curve $[C'_v, (s'_{v,e}, p'_{v,j})]_{v \in V_\tau}$ together with stabilization maps $c_v: C_v \rightarrow C'_v$. This induces a stabilization map of glued curves $c: C \rightarrow C'$, and let $c_0: C_0 \rightarrow C'_0$ be its truncation. By construction, on every geometric fiber, the map c_0 contracts only irreducible components on which f_0 is constant. So the map $f_0: C_0 \rightarrow X$ factors as $C_0 \xrightarrow{c_0} C'_0 \xrightarrow{f'_0} X$. Moreover, since only rational components are contracted, the natural map

$$\mathcal{O}_{C'_0}^{\text{alg}} \longrightarrow c_{0*} \mathcal{O}_{C_0}^{\text{alg}}$$

is an equivalence. Using Lemma 4.35, we deduce that the natural map

$$(4.36) \quad \mathcal{O}_{C'}^{\text{alg}} \longrightarrow c_* \mathcal{O}_C^{\text{alg}}$$

is also an equivalence. Therefore Lemma 4.27 implies that the map $f'_0: C'_0 \rightarrow X$ can be lifted in a unique (up to a contractible space of choices) way to a map $f': C' \rightarrow X$ making the diagram

$$\begin{array}{ccc} C & \xrightarrow{f} & X \\ \downarrow c & \nearrow f' & \downarrow \\ C' & \longrightarrow & S \end{array}$$

commutative. Forgetting the new marked points, we obtain a τ -marked prestable map

$$\mathbf{f}' := ([C'_v, (s'_{v,e})_{e \in E_v}]_{v \in V_\tau}, f')$$

into X/S over T , together with a morphism

$$c: \mathbf{f} \longrightarrow \mathbf{f}'$$

We claim that c is a stabilizing contraction. Since the canonical map (4.36) is an equivalence, Proposition 4.29 implies that it is enough to show that the truncation $c_0: \mathbf{f}_0 \rightarrow \mathbf{f}'_0$ is a stabilizing contraction. This follows from the uniqueness statement in [55, Tag 08EA].

In order to glue this construction over T , it is enough to show that it is compatible with base change. In other words, we have to prove that if $c: \mathbf{f} \rightarrow \mathbf{f}'$ is the

stabilizing contraction over T constructed above and $T' \rightarrow T$ is any morphism, then the induced morphism

$$c \times_T T' : \mathbf{f} \times_T T' \longrightarrow \mathbf{f}' \times_T T'$$

is also a stabilizing contraction. Using once again Proposition 4.29, we are left to prove the same statement at the level of truncation, where it follows from [55, Tag 08EB]. At this point, we obtain the following proposition, the goal of Section 4.4.

Proposition 4.37. *There is a morphism of derived k -analytic stacks*

$$\text{stab} : \mathbb{R}\overline{M}^{\text{pre}}(X/S, \tau, \beta) \longrightarrow \mathbb{R}\overline{M}(X/S, \tau, \beta)$$

which sends every τ -marked prestable map \mathbf{f} over any $T \in \text{dAn}/S$ to the stabilization \mathbf{f}' of \mathbf{f} in the sense of Definition 4.26.

Proof. This follows from the construction of stabilizing contraction and its compatibility with base change. \square

5. GEOMETRY OF DERIVED STABLE MAPS

In this section, we prove a list of geometric relations between derived moduli stacks of k -analytic stable maps with respect to elementary operations on A -graphs, namely, products, cutting edges, universal curve, forgetting tails and contracting edges. These relations are very natural and intuitive, and help replace the study of compatibilities of perfect obstruction theories in the classical approach (cf. [7, 36]). They will give rise to the geometric properties of quantum K-invariants in Section 8.

We fix S a rigid k -analytic space, and X a rigid k -analytic space smooth over S . For any A -graph (τ, β) , let $\mathbb{R}\overline{M}(X/S, \tau, \beta)$ denote the derived stack of (τ, β) -marked stable maps into X/S as in Remark 4.25. We prove in the following a list of geometric properties of $\mathbb{R}\overline{M}(X/S, \tau, \beta)$ with respect to operations on (τ, β) .

5.1. Products. Let (τ_1, β_1) and (τ_2, β_2) be two A -graphs. Let $(\tau_1 \sqcup \tau_2, \beta_1 \sqcup \beta_2)$ be their disjoint union.

Theorem 5.1. *We have a canonical equivalence*

$$\mathbb{R}\overline{M}(X/S, \tau_1 \sqcup \tau_2, \beta_1 \sqcup \beta_2) \xrightarrow{\sim} \mathbb{R}\overline{M}(X/S, \tau_1, \beta_1) \times_S \mathbb{R}\overline{M}(X/S, \tau_2, \beta_2),$$

where the projections are given by the natural forgetful maps.

Proof. The forgetful maps

$$p_i : \overline{M}_{\tau_1 \sqcup \tau_2}^{\text{pre}} \longrightarrow \overline{M}_{\tau_i}^{\text{pre}}, \quad i = 1, 2$$

exhibit $\overline{M}_{\tau_1 \sqcup \tau_2}^{\text{pre}}$ as the product of $\overline{M}_{\tau_1}^{\text{pre}}$ and $\overline{M}_{\tau_2}^{\text{pre}}$. Let $p_i^* \overline{C}_{\tau_i}^{\text{pre}}$ be the pullback of the universal curve $\overline{C}_{\tau_i}^{\text{pre}} \rightarrow \overline{M}_{\tau_i}^{\text{pre}}$ along p_i . Then we have a canonical equivalence

$$\overline{C}_{\tau_1 \sqcup \tau_2}^{\text{pre}} \simeq p_1^* \overline{C}_{\tau_1}^{\text{pre}} \amalg p_2^* \overline{C}_{\tau_2}^{\text{pre}}.$$

Note that the disjoint union on the right is also a disjoint union of (representable) stacks. So $\mathbf{Map}_{\overline{M}_{\tau_1 \sqcup \tau_2}^{\text{pre}} \times S}(\overline{C}_{\tau_1 \sqcup \tau_2}^{\text{pre}} \times S, X \times \overline{M}_{\tau_1 \sqcup \tau_2}^{\text{pre}})$ is canonically equivalent to

$$\mathbf{Map}_{\overline{M}_{\tau_1 \sqcup \tau_2}^{\text{pre}} \times S}(p_1^* \overline{C}_{\tau_1}^{\text{pre}} \times S, X \times \overline{M}_{\tau_1 \sqcup \tau_2}^{\text{pre}}) \times_{\overline{M}_{\tau_1 \sqcup \tau_2}^{\text{pre}} \times S} \mathbf{Map}_{\overline{M}_{\tau_1 \sqcup \tau_2}^{\text{pre}} \times S}(p_2^* \overline{C}_{\tau_2}^{\text{pre}} \times S, X \times \overline{M}_{\tau_1 \sqcup \tau_2}^{\text{pre}}).$$

As $\overline{M}_{\tau_1 \sqcup \tau_2}^{\text{pre}} \simeq \overline{M}_{\tau_1}^{\text{pre}} \times \overline{M}_{\tau_2}^{\text{pre}}$, we have

$$\mathbf{Map}_{\overline{M}_{\tau_1 \sqcup \tau_2}^{\text{pre}} \times S}(p_1^* \overline{C}_{\tau_1}^{\text{pre}} \times S, X \times \overline{M}_{\tau_1 \sqcup \tau_2}^{\text{pre}}) \simeq \mathbf{Map}_{\overline{M}_{\tau_1}^{\text{pre}} \times S}(\overline{C}_{\tau_1}^{\text{pre}} \times S, X \times \overline{M}_{\tau_1}^{\text{pre}}) \times \overline{M}_{\tau_2}^{\text{pre}},$$

and similarly

$$\mathbf{Map}_{\overline{M}_{\tau_1 \sqcup \tau_2}^{\text{pre}} \times S}(p_2^* \overline{C}_{\tau_2}^{\text{pre}} \times S, X \times \overline{M}_{\tau_1 \sqcup \tau_2}^{\text{pre}}) \simeq \mathbf{Map}_{\overline{M}_{\tau_2}^{\text{pre}} \times S}(\overline{C}_{\tau_2}^{\text{pre}} \times S, X \times \overline{M}_{\tau_2}^{\text{pre}}) \times \overline{M}_{\tau_1}^{\text{pre}}.$$

Therefore, $\mathbf{Map}_{\overline{M}_{\tau_1 \sqcup \tau_2}^{\text{pre}} \times S}(\overline{C}_{\tau_1 \sqcup \tau_2}^{\text{pre}} \times S, X \times \overline{M}_{\tau_1 \sqcup \tau_2}^{\text{pre}})$ becomes canonically equivalent to

$$\mathbf{Map}_{\overline{M}_{\tau_1}^{\text{pre}} \times S}(\overline{C}_{\tau_1}^{\text{pre}} \times S, X \times \overline{M}_{\tau_1}^{\text{pre}}) \times \mathbf{Map}_{\overline{M}_{\tau_2}^{\text{pre}} \times S}(\overline{C}_{\tau_2}^{\text{pre}} \times S, X \times \overline{M}_{\tau_2}^{\text{pre}}).$$

Note that on the level of truncations, the inclusions

$$\overline{M}(X/S, \tau_1, \beta_1) \hookrightarrow \mathbf{t}_0 \mathbf{Map}_{\overline{M}_{\tau_1}^{\text{pre}} \times S}(\overline{C}_{\tau_1}^{\text{pre}} \times S, X \times \overline{M}_{\tau_1}^{\text{pre}}),$$

$$\overline{M}(X/S, \tau_2, \beta_2) \hookrightarrow \mathbf{t}_0 \mathbf{Map}_{\overline{M}_{\tau_2}^{\text{pre}} \times S}(\overline{C}_{\tau_2}^{\text{pre}} \times S, X \times \overline{M}_{\tau_2}^{\text{pre}}),$$

$$\overline{M}(X/S, \tau_1 \sqcup \tau_2, \beta_1 \sqcup \beta_2) \hookrightarrow \mathbf{t}_0 \mathbf{Map}_{\overline{M}_{\tau_1 \sqcup \tau_2}^{\text{pre}} \times S}(\overline{C}_{\tau_1 \sqcup \tau_2}^{\text{pre}} \times S, X \times \overline{M}_{\tau_1 \sqcup \tau_2}^{\text{pre}})$$

are all open, and we have a canonical isomorphism

$$\overline{M}(X/S, \tau_1 \sqcup \tau_2, \beta_1 \sqcup \beta_2) \simeq \overline{M}(X/S, \tau_1, \beta_1) \times \overline{M}(X/S, \tau_2, \beta_2).$$

Therefore, the conclusion follows from the equivalence of sites of [52, Lemma 7.16]. \square

5.2. Cutting edges. Let (σ, β) be an A-graph obtained from (τ, β) by cutting an edge e of τ . Let i, j be the two tails of σ created by the cut. The edge e gives a section s_e of the universal curve $\overline{C}_{\tau}^{\text{pre}} \rightarrow \overline{M}_{\tau}^{\text{pre}}$. Each geometric fiber of s_e gives the node corresponding to e .

Theorem 5.2. *We have a derived pullback diagram*

$$\begin{array}{ccc} \mathbb{R}\overline{M}(X/S, \tau, \beta) & \xrightarrow{c} & \mathbb{R}\overline{M}(X/S, \sigma, \beta) \\ \downarrow \text{ev}_e & & \downarrow \text{ev}_i \times \text{ev}_j \\ X & \xrightarrow{\Delta} & X \times_S X \end{array}$$

where Δ is the diagonal map, ev_e is evaluation at the section s_e , and c is induced by cutting the domain curves at s_e .

Proof of Theorem 5.2. In this case, we have naturally $\overline{M}_\tau^{\text{pre}} \simeq \overline{M}_\sigma^{\text{pre}}$, which we denote by $\overline{M}^{\text{pre}}$ for simplicity.

Let $s_i, s_j: \overline{M}^{\text{pre}} \rightarrow \overline{C}_\sigma^{\text{pre}}$ denote the two sections corresponding to the tails i and j . By Remark 4.23, we have a derived pushout square

$$\begin{array}{ccc} \overline{M}^{\text{pre}} \amalg \overline{M}^{\text{pre}} & \xrightarrow{s_i \sqcup s_j} & \overline{C}_\sigma^{\text{pre}} \\ \downarrow & & \downarrow \\ \overline{M}^{\text{pre}} & \xrightarrow{s_e} & \overline{C}_\tau^{\text{pre}} \end{array}$$

Then for any $T \in \text{dAn}/S$ and $T \rightarrow \overline{M}^{\text{pre}}$, by Lemma 3.1, the diagram above remains a pushout after base change to T . Therefore, we deduce that

$$(5.3) \quad \begin{array}{ccc} \mathbf{Map}_{\overline{M}^{\text{pre}} \times_S}(\overline{C}_\tau^{\text{pre}} \times S, X \times \overline{M}^{\text{pre}}) & \longrightarrow & \mathbf{Map}_{\overline{M}^{\text{pre}} \times_S}(\overline{C}_\sigma^{\text{pre}} \times S, X \times \overline{M}^{\text{pre}}) \\ \downarrow & & \downarrow \text{ev}_i \times \text{ev}_j \\ X & \xrightarrow{\Delta} & X \times_S X \end{array}$$

is a derived pullback square.

In order to complete the proof, it is enough to prove that the diagram

$$\begin{array}{ccc} \mathbb{R}\overline{M}(X/S, \tau, \beta) & \longrightarrow & \mathbb{R}\overline{M}(X/S, \sigma, \beta) \\ \downarrow & & \downarrow \\ \mathbf{Map}_{\overline{M}^{\text{pre}}}(\overline{C}_\tau^{\text{pre}}, X \times \overline{M}^{\text{pre}}) & \longrightarrow & \mathbf{Map}_{\overline{M}^{\text{pre}}}(\overline{C}_\sigma^{\text{pre}}, X \times \overline{M}^{\text{pre}}) \end{array}$$

is a derived pullback square as well. By Remark 4.25, the two vertical maps are Zariski open immersions, so it is enough to check that the diagram above is a pullback after passing to the truncations. At the truncated level, it follows directly from the definitions. \square

5.3. Universal curve. Let (σ, β) be an A-graph obtained from (τ, β) by forgetting a tail. By forgetting the marked point associated to the tail, we obtain a map of derived k -analytic stacks

$$\mathbb{R}\overline{M}^{\text{pre}}(X/S, \tau, \beta) \longrightarrow \mathbb{R}\overline{M}^{\text{pre}}(X/S, \sigma, \beta).$$

Let $\pi: \mathbb{R}\overline{M}(X/S, \tau, \beta) \rightarrow \mathbb{R}\overline{M}(X/S, \sigma, \beta)$ be the composition

$$\mathbb{R}\overline{M}(X/S, \tau, \beta) \rightarrow \mathbb{R}\overline{M}^{\text{pre}}(X/S, \tau, \beta) \rightarrow \mathbb{R}\overline{M}^{\text{pre}}(X/S, \sigma, \beta) \xrightarrow{\text{stab}} \mathbb{R}\overline{M}(X/S, \sigma, \beta),$$

where stab is the stabilization map in Proposition 4.37.

Assume that the forgotten tail was attached to the vertex w of σ . Let $\bar{C}_w^{\text{pre}} \rightarrow \bar{M}_\sigma^{\text{pre}}$ be the universal curve corresponding to the vertex w , in other words, \bar{C}_w^{pre} is defined by the following pullback diagram:

$$\begin{array}{ccc} \bar{C}_w^{\text{pre}} & \longrightarrow & \bar{M}_\sigma^{\text{pre}} \\ \downarrow & & \downarrow \\ \bar{C}_{g(w), \text{val}(w)}^{\text{pre}} & \longrightarrow & \bar{M}_{g(w), \text{val}(w)}^{\text{pre}}. \end{array}$$

For any derived k -analytic stack F and any map $f: F \rightarrow \bar{M}_\sigma^{\text{pre}}$, we have a fiber sequence

$$\text{Map}\left(F, F \times_{\bar{M}_{g(w), \text{val}(w)}^{\text{pre}}} \bar{C}_{g(w), \text{val}(w)}^{\text{pre}}\right) \longrightarrow \text{Map}(F, \bar{C}_w^{\text{pre}}) \longrightarrow \text{Map}(F, \bar{M}_\sigma^{\text{pre}}).$$

In other words, the space of maps $F \rightarrow \bar{C}_w^{\text{pre}}$ over $\bar{M}_\sigma^{\text{pre}}$ is equivalent to the space of sections of the prestable curve over F corresponding to the vertex w . Taking $F = \mathbb{R}\bar{M}(X/S, \tau, \beta)$ and the map f to be the composition

$$\mathbb{R}\bar{M}(X/S, \tau, \beta) \xrightarrow{\pi} \mathbb{R}\bar{M}(X/S, \sigma, \beta) \longrightarrow \bar{M}_\sigma^{\text{pre}},$$

and using the section associated to the forgotten tail in τ , we obtain a canonical map

$$\lambda: \mathbb{R}\bar{M}(X/S, \tau, \beta) \longrightarrow \bar{C}_w^{\text{pre}}.$$

We can now state the main result of this subsection:

Theorem 5.4. *The diagram*

$$\begin{array}{ccc} \mathbb{R}\bar{M}(X/S, \tau, \beta) & \xrightarrow{\pi} & \mathbb{R}\bar{M}(X/S, \sigma, \beta) \\ \downarrow \lambda & & \downarrow \\ \bar{C}_w^{\text{pre}} & \longrightarrow & \bar{M}_\sigma^{\text{pre}} \end{array}$$

is a derived pullback square.

Proof. The diagram commutes by construction. Write

$$\mathbb{R}\bar{C}_w(X/S, \sigma, \beta) := \bar{C}_w^{\text{pre}} \times_{\bar{M}_\sigma^{\text{pre}}} \mathbb{R}\bar{M}(X/S, \sigma, \beta).$$

We need to prove that the natural map

$$\mathbb{R}\bar{M}(X/S, \tau, \beta) \longrightarrow \mathbb{R}\bar{C}_w(X/S, \sigma, \beta)$$

is an equivalence. Let $T \in \text{dAn}/_S$ be any derived k -analytic space over S . We first show that the functor of ∞ -groupoids

$$(5.5) \quad \mathbb{R}\bar{M}(X/S, \tau, \beta)(T) \longrightarrow \mathbb{R}\bar{C}_w(X/S, \sigma, \beta)(T)$$

is fully faithful. Let $\mathbf{f} = ([C_v, (s_{v,e})]_{v \in V_\tau}, f)$ and $\tilde{\mathbf{f}} = ([\tilde{C}_v, (\tilde{s}_{v,e})]_{v \in V_\tau}, \tilde{f})$ be two τ -marked stable maps into X/S over T . Let $\mathbf{f}' = ([C'_v, (s'_{v,e})]_{v \in V_\sigma}, f')$ and $\tilde{\mathbf{f}}' = ([\tilde{C}'_v, (\tilde{s}'_{v,e})]_{v \in V_\sigma}, \tilde{f}')$ be their images via π . Let $s': T \rightarrow C'_v$ and $\tilde{s}': T \rightarrow \tilde{C}'_v$ be the images of the extra sections corresponding to the forgotten tail. So the pairs (\mathbf{f}', s') and $(\tilde{\mathbf{f}}', \tilde{s}')$ are the images of \mathbf{f} and $\tilde{\mathbf{f}}$ in $\mathbb{R}\bar{C}_w(X/S, \sigma, \beta)$ under the map (5.5). Given any $\alpha: (\mathbf{f}', s') \rightarrow (\tilde{\mathbf{f}}', \tilde{s}')$, denote by $\text{Map}_{/T}^\alpha(\mathbf{f}, \tilde{\mathbf{f}})$ the fiber of

$$\text{Map}_{/T}(\mathbf{f}, \tilde{\mathbf{f}}) \longrightarrow \text{Map}_{/T}((\mathbf{f}', s'), (\tilde{\mathbf{f}}', \tilde{s}'))$$

at α . In order to prove the full faithfulness of (5.5), it is enough to prove that for every equivalence $\alpha: (\mathbf{f}', s') \rightarrow (\tilde{\mathbf{f}}', \tilde{s}')$, the space $\text{Map}_{/T}^\alpha(\mathbf{f}, \tilde{\mathbf{f}})$ is contractible. This question is local on T , so we are free to replace T by an étale covering. Let U'_v be the complement in C'_v of all the sections and all the nodes, and let

$$U_v := C_v \times_{C'_v} U'_v.$$

Define similarly \tilde{U}'_v and \tilde{U}_v . Note that the equivalence α restricts to an equivalence $U'_v \simeq \tilde{U}'_v$. Moreover, the canonical morphisms $U_v \rightarrow U'_v$ and $\tilde{U}_v \rightarrow \tilde{U}'_v$ are also equivalences. It follows that α induces an equivalence $\beta: U_v \simeq \tilde{U}_v$. Up to replacing T by an étale covering, we can add extra sections $(p'_{v,j})$ to U'_v in such that $\mathbf{C}'^{\text{ex}} := [C'_v, (s'_{v,e}, p'_{v,j})]_{v \in V_\tau}$ is a stable pointed curve. Via the equivalence $U_v \simeq U'_v$, we lift the sections $p'_{v,j}$ of U'_v to sections $p_{v,j}$ of U_v . Write

$$\mathbf{C}^{\text{ex}} := [C_v, (s_{v,e}, p_{v,j})]_{v \in V_\tau}$$

and

$$\mathbf{f}^{\text{ex}} := (\mathbf{C}^{\text{ex}}, f), \quad \mathbf{f}'^{\text{ex}} := (\mathbf{C}'^{\text{ex}}, f').$$

Then the map $\mathbf{f} \rightarrow \mathbf{f}'$ induces a map $\mathbf{f}^{\text{ex}} \rightarrow \mathbf{f}'^{\text{ex}}$, and the underlying map $\mathbf{C}^{\text{ex}} \rightarrow \mathbf{C}'^{\text{ex}}$ is a stabilizing contraction of pointed curves (see Section 4.4.2). Using the equivalences α and β , we add compatible sections $\tilde{p}_{v,j}$ and $\tilde{p}'_{v,j}$ to \tilde{U}_v and \tilde{U}'_v . We define similar notations $\tilde{\mathbf{f}}^{\text{ex}}, \tilde{\mathbf{f}}'^{\text{ex}}, \tilde{\mathbf{C}}^{\text{ex}}, \tilde{\mathbf{C}}'^{\text{ex}}$. As above, the map $\tilde{\mathbf{f}} \rightarrow \tilde{\mathbf{f}}'$ induces a map $\tilde{\mathbf{f}}^{\text{ex}} \rightarrow \tilde{\mathbf{f}}'^{\text{ex}}$, and the underlying map $\tilde{\mathbf{C}}^{\text{ex}} \rightarrow \tilde{\mathbf{C}}'^{\text{ex}}$ is a stabilizing contraction of pointed curves. Write

$$\begin{aligned} \mathbf{f}_U &:= ([U_v]_{v \in V_\tau}, f), & \mathbf{f}_U^{\text{ex}} &:= ([U_v, (p_{v,j})]_{v \in V_\tau}, f), \\ \tilde{\mathbf{f}}_U &:= ([\tilde{U}_v]_{v \in V_\tau}, \tilde{f}), & \tilde{\mathbf{f}}_U^{\text{ex}} &:= ([\tilde{U}_v, (p_{v,j})]_{v \in V_\tau}, \tilde{f}). \end{aligned}$$

Let W_v denote the complement in C_v of the sections $p_{v,j}$, and define similarly \tilde{W}_v . Set

$$\begin{aligned} \mathbf{f}_W &:= \mathbf{f}_W^{\text{ex}} := ([W_v, (s_{v,e})]_{v \in V_\tau}, f), \\ \tilde{\mathbf{f}}_W &:= \tilde{\mathbf{f}}_W^{\text{ex}} := ([\tilde{W}_v, (\tilde{s}_{v,e})]_{v \in V_\tau}, \tilde{f}). \end{aligned}$$

We also introduce the notations $\mathbf{f}_{U \cap W}$, $\mathbf{f}_{U \cap W}^{\text{ex}}$, $\tilde{\mathbf{f}}_{U \cap W}$ and $\tilde{\mathbf{f}}_{U \cap W}^{\text{ex}}$ in the same way as above.

Observe that the diagrams

$$\begin{array}{ccc} \text{Map}_{/T}^{\alpha}(\mathbf{f}, \tilde{\mathbf{f}}) & \longrightarrow & \text{Map}_{/T}^{\alpha}(\mathbf{f}_W, \tilde{\mathbf{f}}_W) \\ \downarrow & & \downarrow \\ \text{Map}_{/T}^{\alpha}(\mathbf{f}_U, \tilde{\mathbf{f}}_U) & \longrightarrow & \text{Map}_{/T}^{\alpha}(\mathbf{f}_{U \cap W}, \tilde{\mathbf{f}}_{U \cap W}) \end{array}$$

and

$$\begin{array}{ccc} \text{Map}_{/T}^{\alpha}(\mathbf{f}^{\text{ex}}, \tilde{\mathbf{f}}^{\text{ex}}) & \longrightarrow & \text{Map}_{/T}^{\alpha}(\mathbf{f}_W^{\text{ex}}, \tilde{\mathbf{f}}_W^{\text{ex}}) \\ \downarrow & & \downarrow \\ \text{Map}_{/T}^{\alpha}(\mathbf{f}_U^{\text{ex}}, \tilde{\mathbf{f}}_U^{\text{ex}}) & \longrightarrow & \text{Map}_{/T}^{\alpha}(\mathbf{f}_{U \cap W}^{\text{ex}}, \tilde{\mathbf{f}}_{U \cap W}^{\text{ex}}) \end{array}$$

are pullback squares. Unraveling the definitions, we see that the natural maps

$$\begin{aligned} \text{Map}_{/T}^{\alpha}(\mathbf{f}_U^{\text{ex}}, \tilde{\mathbf{f}}_U^{\text{ex}}) &\longrightarrow \text{Map}_{/T}^{\alpha}(\mathbf{f}_U, \tilde{\mathbf{f}}_U) \\ \text{Map}_{/T}^{\alpha}(\mathbf{f}_W^{\text{ex}}, \tilde{\mathbf{f}}_W^{\text{ex}}) &\longrightarrow \text{Map}_{/T}^{\alpha}(\mathbf{f}_W, \tilde{\mathbf{f}}_W) \\ \text{Map}_{/T}^{\alpha}(\mathbf{f}_{U \cap W}^{\text{ex}}, \tilde{\mathbf{f}}_{U \cap W}^{\text{ex}}) &\longrightarrow \text{Map}_{/T}^{\alpha}(\mathbf{f}_{U \cap W}, \tilde{\mathbf{f}}_{U \cap W}) \end{aligned}$$

are equivalences. It follows that the induced map

$$\text{Map}_{/T}^{\alpha}(\mathbf{f}^{\text{ex}}, \tilde{\mathbf{f}}^{\text{ex}}) \longrightarrow \text{Map}_{/T}^{\alpha}(\mathbf{f}, \tilde{\mathbf{f}})$$

is an equivalence as well. By [38, Corollary 2.3.2.5], the forgetful map

$$\text{Map}_{/T}^{\alpha}(\mathbf{f}^{\text{ex}}, \tilde{\mathbf{f}}^{\text{ex}}) \longrightarrow \text{Map}_{/T}^{\alpha}(\mathbf{C}^{\text{ex}}, \tilde{\mathbf{C}}^{\text{ex}})$$

is an equivalence. As \mathbf{C}^{ex} and $\tilde{\mathbf{C}}^{\text{ex}}$ are stable pointed curves, it follows from [31, Corollary 2.6] that $\text{Map}_{/T}^{\alpha}(\mathbf{C}^{\text{ex}}, \tilde{\mathbf{C}}^{\text{ex}})$ is contractible. We deduce that $\text{Map}_{/T}^{\alpha}(\mathbf{f}, \tilde{\mathbf{f}})$ is contractible, and conclude the full faithfulness of the functor of ∞ -groupoids (5.5).

Now it remains to prove that the functor (5.5) is also essentially surjective. Since this functor is fully faithful, we can reason locally on T . At this point, using the equivalence $\overline{M}_{\tau} \simeq \mathbb{R}\overline{M}_{\tau}$, the same proof of [9, Proposition 4.5] applies. \square

Corollary 5.6. *The forgetful map*

$$\mathbb{R}\overline{M}_{g,n+1}(X/S) \longrightarrow \mathbb{R}\overline{M}_{g,n}(X/S)$$

canonically factors through the universal curve $\mathbb{R}\overline{C}_{g,n}(X/S)$, and the resulting morphism

$$\mathbb{R}\overline{M}_{g,n+1}(X/S) \longrightarrow \mathbb{R}\overline{C}_{g,n}(X/S)$$

is an equivalence.

Proof. By definition, $\mathbb{R}\overline{C}_{g,n}(X/S)$ fits in the pullback diagram

$$\begin{array}{ccc} \mathbb{R}\overline{C}_{g,n}(X/S) & \longrightarrow & \mathbb{R}\overline{M}_{g,n}(X/S) \\ \downarrow & & \downarrow \\ \overline{C}_{g,n}^{\text{pre}} & \longrightarrow & \overline{M}_{g,n}^{\text{pre}}. \end{array}$$

So the corollary follows directly from Theorem 5.4. \square

5.4. Forgetting tails. Let (σ, β) be an A-graph obtained from (τ, β) by forgetting a tail. Consider the commutative diagram

$$\begin{array}{ccc} \mathbb{R}\overline{M}(X/S, \tau, \beta) & \xrightarrow{\pi} & \mathbb{R}\overline{M}(X/S, \sigma, \beta) \\ \downarrow & & \downarrow \\ \overline{M}_\tau & \xrightarrow{\Phi} & \overline{M}_\sigma, \end{array}$$

where the upper map π is constructed in Section 5.3. This induces a map

$$\Psi: \mathbb{R}\overline{M}(X/S, \tau, \beta) \longrightarrow \overline{M}_\tau \times_{\overline{M}_\sigma} \mathbb{R}\overline{M}(X/S, \sigma, \beta).$$

Theorem 5.7. *The diagram*

$$\begin{array}{ccc} \mathbb{R}\overline{M}(X/S, \tau, \beta) & \xrightarrow{\Psi} & \overline{M}_\tau \times_{\overline{M}_\sigma} \mathbb{R}\overline{M}(X/S, \sigma, \beta) \\ \downarrow \lambda & & \downarrow \\ \overline{C}_w^{\text{pre}} & \xrightarrow{\rho} & \overline{M}_\tau \times_{\overline{M}_\sigma} \overline{M}_\sigma^{\text{pre}} \end{array}$$

is a derived pullback square. Moreover, the canonical map

$$\mathcal{O}_{\overline{M}_\tau \times_{\overline{M}_\sigma} \mathbb{R}\overline{M}(X/S, \sigma, \beta)}^{\text{alg}} \longrightarrow \Psi_* \left(\mathcal{O}_{\mathbb{R}\overline{M}(X/S, \tau, \beta)}^{\text{alg}} \right)$$

is an equivalence.

Proof. Consider the diagram

$$\begin{array}{ccccc} \mathbb{R}\overline{M}(X/S, \tau, \beta) & \xrightarrow{\Psi} & \overline{M}_\tau \times_{\overline{M}_\sigma} \mathbb{R}\overline{M}(X/S, \sigma, \beta) & \longrightarrow & \mathbb{R}\overline{M}(X/S, \sigma, \beta) \\ \downarrow \lambda & & \downarrow \mu & & \downarrow \\ \overline{C}_w^{\text{pre}} & \xrightarrow{\rho} & \overline{M}_\tau \times_{\overline{M}_\sigma} \overline{M}_\sigma^{\text{pre}} & \longrightarrow & \overline{M}_\sigma^{\text{pre}} \\ & & \downarrow & & \downarrow \\ & & \overline{M}_\tau & \xrightarrow{\Phi} & \overline{M}_\sigma. \end{array}$$

The right outer rectangle and the right lower square are pullbacks by definition. Hence the right upper square is also a pullback. The top outer rectangle is a pullback square in virtue of Theorem 5.4. It follows that the left upper square is a pullback, which shows the first statement in the proposition.

Now by derived base-change ([51, Theorem 6.8]), the canonical map

$$\mu^* \rho_* \left(\mathcal{O}_{\bar{C}_w^{\text{pre}}}^{\text{alg}} \right) \longrightarrow \Psi_* \lambda^* \left(\mathcal{O}_{\bar{C}_w^{\text{pre}}}^{\text{alg}} \right)$$

is an equivalence. Moreover, since ρ is a blow-down of rational curves (fiberwise over $\bar{M}_\sigma^{\text{pre}}$), we have

$$\mathcal{O}_{\bar{M}_\tau \times_{\bar{M}_\sigma} \bar{M}_\sigma^{\text{pre}}}^{\text{alg}} \xrightarrow{\sim} \rho_* \left(\mathcal{O}_{\bar{C}_w^{\text{pre}}}^{\text{alg}} \right).$$

Since pullbacks of structure sheaves are always structure sheaves, we conclude that the canonical map

$$\mathcal{O}_{\bar{M}_\tau \times_{\bar{M}_\sigma} \mathbb{R}\bar{M}(X/S, \sigma, \beta)}^{\text{alg}} \longrightarrow \Psi_* \left(\mathcal{O}_{\mathbb{R}\bar{M}(X/S, \tau, \beta)}^{\text{alg}} \right)$$

is an equivalence, completing the proof. \square

5.5. Contracting edges. Let σ be a modular graph obtained from τ by contracting an edge (possibly a loop) e , and $v_e \in V_\sigma$ the vertex of σ corresponding to the contracted edge e . Let $\tilde{\sigma}$ be a modular graph obtained from σ by adding k tails, and k_e the number of extra tails attached to v_e .

The main result here is Theorem 5.19. In order to rigorously construct the colimit diagram in (5.18), we start with the following:

Construction 5.8. For each integer $l \geq 1$, let τ_l denote the modular graph obtained from τ by replacing e with a chain γ_l of l edges and $l+1$ vertices such that all the $(l-1)$ new vertices have genus 0. Let $c_l: \tau_l \rightarrow \sigma$ be the map contracting γ_l . For every map $i: \{1, \dots, k_e\} \rightarrow V_{\tau_l}$, let τ_l^i be the graph obtained from τ_l by adding k extra tails in the unique way compatible with i , c_l and $\tilde{\sigma}$. Let $c_l^i: \tau_l^i \rightarrow \sigma$ be the composition of forgetting tails $\tau_l^i \rightarrow \tau_l$ with the contraction $c_l: \tau_l \rightarrow \sigma$. Let MG be the category of modular graphs whose morphisms are compositions of forgetting tails and contractions. Let \mathcal{D} be the full subcategory of MG/σ spanned by the morphisms $c_l^i: \tau_l^i \rightarrow \sigma$. Define

$$\Lambda := \Lambda_{\tau \rightarrow \sigma, \tilde{\sigma}} := \mathcal{D}/\tau \rightarrow \sigma.$$

An object in Λ consists of an integer $l \geq 1$, a function $i: \{1, \dots, k_e\} \rightarrow V_{\tau_l}$ and a map $a: \tau_l^i \rightarrow \tau$ in MG making the diagram

$$\begin{array}{ccc} \tau_l^i & & \\ a \downarrow & \searrow^{c_l^i} & \\ \tau & \longrightarrow & \sigma \end{array}$$

commutative. Note that the commutative diagram above can be extended in a unique way to

$$\begin{array}{ccc} \tau_l^i & \longrightarrow & \tilde{\sigma} \\ a \downarrow & & \downarrow \\ \tau & \longrightarrow & \sigma. \end{array}$$

Construction 5.9. We functorially attach to every object $(a: \tau_l^i \rightarrow \tau)$ in Λ the natural commutative diagram

$$(5.10) \quad \begin{array}{ccc} \overline{M}_{\tau_l^i}^{\text{pre}} & \longrightarrow & \overline{M}_{\tilde{\sigma}}^{\text{pre}} \\ \downarrow & & \downarrow \\ \overline{M}_{\tau}^{\text{pre}} & \longrightarrow & \overline{M}_{\sigma}^{\text{pre}}. \end{array}$$

Combining the universal properties of stabilization (see Definition 4.26) and of gluing along closed immersions, the diagram

$$(5.11) \quad \begin{array}{ccc} \overline{M}_{\tau}^{\text{pre}} & \longrightarrow & \overline{M}_{\sigma}^{\text{pre}} \\ \downarrow & & \downarrow \\ \overline{M}_{\tau} & \longrightarrow & \overline{M}_{\sigma} \end{array}$$

canonically commutes. Therefore we obtain a canonical map

$$(5.12) \quad \text{colim}_{(a: \tau_l^i \rightarrow \tau) \in \Lambda} \overline{M}_{\tau_l^i}^{\text{pre}} \longrightarrow \overline{M}_{\tau} \times_{\overline{M}_{\sigma}} \overline{M}_{\sigma}^{\text{pre}}.$$

Lemma 5.13. *The canonical map (5.12) is an equivalence in dAn_k .*

We will prove the following lemma first:

Lemma 5.14. *For any modular graph τ , the stabilization map*

$$\text{stab}: \overline{M}_{\tau}^{\text{pre}} \longrightarrow \overline{M}_{\tau}$$

is smooth.

Proof. Since both $\overline{M}_{\tau}^{\text{pre}}$ and \overline{M}_{τ} split as products as in Remark 4.23 and the stabilization map preserves the product, it is enough to prove the lemma in the case when τ has a single non-tail vertex. Fix $T \in \text{dAfd}_k$ and $x: T \rightarrow \overline{M}_{\tau}^{\text{pre}}$. Let $[p: C \rightarrow T, (s_i)]$ be the associated prestable pointed curve over T . Let $y := \text{stab} \circ x: T \rightarrow \overline{M}_{\tau}$, $[q: C' \rightarrow T, (s'_i)]$ the associated prestable pointed curve, and $c: C \rightarrow C'$ the stabilizing contraction. Then we have a fiber sequence

$$(5.15) \quad x^* \mathbb{T}_{\text{stab}}^{\text{an}} \longrightarrow x^* \mathbb{T}_{\overline{M}_{\tau}^{\text{pre}}}^{\text{an}} \longrightarrow y^* \mathbb{T}_{\overline{M}_{\tau}}^{\text{an}}.$$

To prove that the map stab is smooth, it is enough to show that $\pi_i(x^*\mathbb{T}_{\text{stab}}^{\text{an}}) \simeq 0$ for every $i < 0$. By (4.8), we have

$$x^*\mathbb{T}_{\overline{M}_\tau}^{\text{an}} \simeq p_*(\mathbb{T}_{C/T}^{\text{an}}(-\sum s_i))[1], \quad y^*\mathbb{T}_{\overline{M}_\tau}^{\text{an}} \simeq q_*(\mathbb{T}_{C'/T}^{\text{an}}(-\sum s'_i))[1].$$

By Lemma 4.35, we have a canonical equivalence $c_*(\mathcal{O}_C^{\text{alg}}) \simeq \mathcal{O}_{C'}^{\text{alg}}$. So by the projection formula, the unit transformation

$$\mathbb{T}_{C'/T}^{\text{an}} \longrightarrow c_*c^*\mathbb{T}_{C'/T}^{\text{an}}$$

is also an equivalence. In particular, the canonical map $\phi: \mathbb{T}_{C'/T}^{\text{an}} \rightarrow c^*\mathbb{T}_{C'/T}^{\text{an}}$ induces a morphism

$$\alpha: c_*\mathbb{T}_{C'/T}^{\text{an}} \longrightarrow \mathbb{T}_{C'/T}^{\text{an}}.$$

On the other hand, ϕ also induces a morphism

$$\beta: \bigoplus c_*s_{i*}s_i^*\mathbb{T}_{C'/T}^{\text{an}} \longrightarrow \bigoplus s'_{i*}s_i'^*\mathbb{T}_{C'/T}^{\text{an}}.$$

The morphisms α and β fit in the following commutative diagram

$$\begin{array}{ccc} c_*\mathbb{T}_{C'/T}^{\text{an}} & \longrightarrow & \bigoplus c_*s_{i*}s_i^*\mathbb{T}_{C'/T}^{\text{an}} \\ \downarrow \alpha & & \downarrow \beta \\ \mathbb{T}_{C'/T}^{\text{an}} & \longrightarrow & \bigoplus s'_{i*}s_i'^*\mathbb{T}_{C'/T}^{\text{an}}. \end{array}$$

We complete the diagram as follows:

$$\begin{array}{ccccc} \text{fib}(\gamma) & \longrightarrow & \text{fib}(\alpha) & \longrightarrow & \text{fib}(\beta) \\ \downarrow & & \downarrow & & \downarrow \\ c_*\mathbb{T}_{C'/T}^{\text{an}}(-\sum s_i) & \longrightarrow & c_*\mathbb{T}_{C'/T}^{\text{an}} & \longrightarrow & \bigoplus c_*s_{i*}s_i^*\mathbb{T}_{C'/T}^{\text{an}} \\ \downarrow \gamma & & \downarrow \alpha & & \downarrow \beta \\ \mathbb{T}_{C'/T}^{\text{an}}(-\sum s'_i) & \longrightarrow & \mathbb{T}_{C'/T}^{\text{an}} & \longrightarrow & \bigoplus s'_{i*}s_i'^*\mathbb{T}_{C'/T}^{\text{an}}. \end{array}$$

Comparing the fiber sequence (5.15) with the leftmost column of the above diagram, we obtain a canonical identification

$$q_*(\text{fib}(\gamma))[1] \simeq x^*\mathbb{T}_{\text{stab}}^{\text{an}}.$$

Therefore, in order to prove that $\pi_i(x^*\mathbb{T}_{\text{stab}}^{\text{an}}) \simeq 0$ for $i < 0$, it suffices to prove that

$$\pi_i(q_*(\text{fib}(\gamma))) = 0, \quad i \leq -2.$$

The fiber sequence (5.15) immediately implies that this vanishing holds for $i \leq -3$. We can therefore check that $\pi_{-2}(q_*(\text{fib}(\gamma))) = 0$ on the geometric points of T . In other words, we are left to prove the vanishing when T is the spectrum of an algebraically closed field. We now make the following observations:

- (1) The sheaf $\text{fib}(\alpha)$ is supported on finitely many points of C' . Indeed, applying c_* to the fiber sequence $\mathbb{T}_{C/C'}^{\text{an}} \rightarrow \mathbb{T}_C^{\text{an}} \rightarrow c^*\mathbb{T}_{C'}^{\text{an}}$, we obtain an equivalence $\text{fib}(\alpha) \simeq c_*\mathbb{T}_{C/C'}^{\text{an}}$, and the right hand side is supported on finitely many points of C' .
- (2) The sheaf $\text{fib}(\beta)$ is supported on finitely many points of C' .

These two observations imply that

$$\text{fib}(\gamma) \simeq \text{fib}(\text{fib}(\alpha) \rightarrow \text{fib}(\beta))$$

is also supported on finitely many points of C' . Hence, in order to obtain the vanishing $\pi_{-2}(q_*(\text{fib}(\gamma))) \simeq 0$, it suffices to show that $\pi_{-2}(\text{fib}(\gamma)) \simeq 0$. As $\pi_{-2}(c_*\mathbb{T}_C^{\text{an}}(-\sum s_i)) \simeq 0$, we are then left to prove that the natural map

$$\pi_{-1}(c_*\mathbb{T}_C^{\text{an}}(-\sum s_i)) \longrightarrow \pi_{-1}(\mathbb{T}_{C'}^{\text{an}}(-\sum s'_i)).$$

is surjective. We have

$$\begin{aligned} \pi_{-1}(c_*\mathbb{T}_C^{\text{an}}(-\sum s_i)) &\rightarrow \pi_0(c_*\pi_{-1}(\mathbb{T}_C^{\text{an}}(-\sum s_i))) \simeq c_*\left(\bigoplus_{\nu \text{ node of } C} i_{\nu*}\mathcal{O}_{\nu}\right) \\ &\rightarrow \bigoplus_{\nu' \text{ node of } C'} i_{\nu'*}\mathcal{O}_{\nu'} \simeq \pi_{-1}(\mathbb{T}_{C'}^{\text{an}}(-\sum s'_i)), \end{aligned}$$

where i_{ν} and $i_{\nu'}$ denote the inclusion of points. This completes the proof. \square

Proof of Lemma 5.13. The map $\overline{M}_{\sigma}^{\text{pre}} \rightarrow \overline{M}_{\sigma}$ factors as

$$\overline{M}_{\sigma}^{\text{pre}} \longrightarrow \overline{M}_{\sigma}^{\text{pre}} \longrightarrow \overline{M}_{\sigma},$$

where the first map is forgetting tails at the level of prestable curves, and the second map is stabilization. The first map is flat because forgetting each tail is isomorphic to the projection from the universal curve. The second map is flat by Lemma 5.14. So $\overline{M}_{\sigma}^{\text{pre}} \rightarrow \overline{M}_{\sigma}$ is flat, and therefore the fiber product $\overline{M}_{\tau} \times_{\overline{M}_{\sigma}} \overline{M}_{\sigma}^{\text{pre}}$ is underived. By the construction of gluing along closed immersions (see [50, (6.7)]), the colimit is also underived. Therefore, it is enough to prove the lemma at the underived level. Observe that the normal crossing substack

$$\overline{M}_{\tau} \times_{\overline{M}_{\sigma}} \overline{M}_{\sigma}^{\text{pre}} \subset \overline{M}_{\sigma}^{\text{pre}}$$

is stratified by the images of

$$\coprod_{(a: \tau_i^i \rightarrow \tau) \in \Lambda} \overline{M}_{\tau_i^i}^{\text{pre}},$$

and the stratum given by $\overline{M}_{\tau_i^i}^{\text{pre}}$ has codimension l . Note it is enough to prove the lemma étale locally on $\overline{M}_{\sigma}^{\text{pre}}$. After an étale base change, $\overline{M}_{\tau} \times_{\overline{M}_{\sigma}} \overline{M}_{\sigma}^{\text{pre}} \subset \overline{M}_{\sigma}^{\text{pre}}$ becomes a simple normal crossing divisor. At this point, the conclusion follows by induction from [50, Theorem 6.5]. \square

Next we upgrade Construction 5.8 to take into account curve classes.

Construction 5.16. Notation as in Construction 5.8. Given $\beta: V_\sigma \rightarrow A(X)$, we define a category Λ_β as follows. An object of Λ_β consists of an object $(a: \tau_l^i \rightarrow \tau)$ of Λ and a map $\beta_l^{ij}: V_{\tau_l^i} \rightarrow A(X)$ that adds up to β under the contraction $c_l^i: \tau_l^i \rightarrow \sigma$. A morphism in Λ_β is a morphism between the underlying objects in Λ that is compatible with the curve classes. Note we have a forgetful functor $\Lambda_\beta \rightarrow \Lambda$. (In the special case when σ has a single non-tail vertex, $g(\sigma) = 0$ and $k = 0$, the category Λ_β is equivalent to the one considered in [42, §4.2].)

Using the description via mapping stacks as in Remark 4.25, we can functorially attach to every object $(a: \tau_l^i \rightarrow \tau, \beta_l^{ij}) \in \Lambda_\beta$ a commutative diagram

$$(5.17) \quad \begin{array}{ccc} \mathbb{R}\overline{M}(X/S, \tau_l^i, \beta_l^{ij}) & \longrightarrow & \mathbb{R}\overline{M}(X/S, \tilde{\sigma}, \beta) \\ \downarrow & & \downarrow \\ \overline{M}_{\tau_l^i}^{\text{pre}} & \longrightarrow & \overline{M}_{\tilde{\sigma}}^{\text{pre}}. \end{array}$$

Combining with diagrams (5.10) and (5.11), we obtain a natural map

$$p_l^{ij}: \mathbb{R}\overline{M}(X/S, \tau_l^i, \beta_l^{ij}) \longrightarrow \overline{M}_\tau \times_{\overline{M}_\sigma} \mathbb{R}\overline{M}(X/S, \tilde{\sigma}, \beta).$$

By the functoriality of diagrams (5.17) and (5.10), we obtain a canonical map

$$(5.18) \quad \text{colim}_{(a: \tau_l^i \rightarrow \tau) \in \Lambda} \coprod_j \mathbb{R}\overline{M}(X/S, \tau_l^i, \beta_l^{ij}) \longrightarrow \overline{M}_\tau \times_{\overline{M}_\sigma} \mathbb{R}\overline{M}(X/S, \tilde{\sigma}, \beta).$$

Theorem 5.19. *The canonical map (5.18) is an equivalence in dAn_k . Moreover, we have*

$$\sum_l (-1)^{l+1} \sum_{i,j} p_{l*}^{ij} [\mathcal{O}_{\mathbb{R}\overline{M}(X/S, \tau_l^i, \beta_l^{ij})}] = [\mathcal{O}_{\overline{M}_\tau \times_{\overline{M}_\sigma} \mathbb{R}\overline{M}(X/S, \tilde{\sigma}, \beta)}]$$

in $K_0(\text{Coh}^b(\overline{M}_\tau \times_{\overline{M}_\sigma} \mathbb{R}\overline{M}(X/S, \tilde{\sigma}, \beta)))$.

Lemma 5.20. *For every $(a: \tau_l^i \rightarrow \tau) \in \Lambda$, the diagram*

$$\begin{array}{ccc} \coprod_j \mathbb{R}\overline{M}(X/S, \tau_l^i, \beta_l^{ij}) & \longrightarrow & \mathbb{R}\overline{M}(X/S, \tilde{\sigma}, \beta) \\ \downarrow & & \downarrow \\ \overline{M}_{\tau_l^i}^{\text{pre}} & \longrightarrow & \overline{M}_{\tilde{\sigma}}^{\text{pre}} \end{array}$$

is a pullback.

Proof. By Remark 4.23, we have an underived pullback square

$$\begin{array}{ccc} \overline{C}_{\tau_l^i}^{\text{pre}} & \longrightarrow & \overline{C}_{\tilde{\sigma}}^{\text{pre}} \\ \downarrow & & \downarrow \\ \overline{M}_{\tau_l^i}^{\text{pre}} & \longrightarrow & \overline{M}_{\tilde{\sigma}}^{\text{pre}}. \end{array}$$

It is also a derived pullback square as the vertical maps are flat. Hence we obtain a derived pullback square

$$\begin{array}{ccc} \mathbf{Map}_{\overline{M}_{\tau_l^i}^{\text{pre}} \times S}(\overline{C}_{\tau_l^i}^{\text{pre}} \times S, X \times \overline{M}_{\tau_l^i}^{\text{pre}}) & \longrightarrow & \mathbf{Map}_{\overline{M}_{\tilde{\sigma}}^{\text{pre}} \times S}(\overline{C}_{\tilde{\sigma}}^{\text{pre}} \times S, X \times \overline{M}_{\tilde{\sigma}}^{\text{pre}}) \\ \downarrow & & \downarrow \\ \overline{M}_{\tau_l^i}^{\text{pre}} & \longrightarrow & \overline{M}_{\tilde{\sigma}}^{\text{pre}}. \end{array}$$

Now it is enough to prove that the diagram

$$\begin{array}{ccc} \coprod_j \mathbb{R}\overline{M}(X/S, \tau_l^i, \beta_l^{ij}) & \longrightarrow & \mathbb{R}\overline{M}(X/S, \tilde{\sigma}, \beta) \\ \downarrow & & \downarrow \\ \mathbf{Map}_{\overline{M}_{\tau_l^i}^{\text{pre}} \times S}(\overline{C}_{\tau_l^i}^{\text{pre}} \times S, X \times \overline{M}_{\tau_l^i}^{\text{pre}}) & \longrightarrow & \mathbf{Map}_{\overline{M}_{\tilde{\sigma}}^{\text{pre}} \times S}(\overline{C}_{\tilde{\sigma}}^{\text{pre}} \times S, X \times \overline{M}_{\tilde{\sigma}}^{\text{pre}}) \end{array}$$

is a pullback square. By Remark 4.25, the two vertical maps are Zariski open immersions, so it is enough to check the pullback property after passing to the truncations. At the truncated level, it follows directly from the definitions. \square

Proof of Theorem 5.19. The first statement follows combining Lemmas 3.1, 5.13 and 5.20. For the second one, consider the diagram

$$\begin{array}{ccc} \text{colim}_{(a: \tau_l^i \rightarrow \tau) \in \Lambda} \coprod_j \mathbb{R}\overline{M}(X/S, \tau_l^i, \beta_l^{ij}) & \longrightarrow & \overline{M}_{\tau} \times_{\overline{M}_{\sigma}} \mathbb{R}\overline{M}(X/S, \tilde{\sigma}, \beta) \\ \downarrow d_l^i & & \downarrow d \\ \text{colim}_{(a: \tau_l^i \rightarrow \tau) \in \Lambda} \overline{M}_{\tau_l^i}^{\text{pre}} & \longrightarrow & \overline{M}_{\tau} \times_{\overline{M}_{\sigma}} \overline{M}_{\tilde{\sigma}}^{\text{pre}}, \end{array}$$

where $d: \mathbb{R}\overline{M}(X/S, \tilde{\sigma}, \beta) \rightarrow \overline{M}_{\tilde{\sigma}}^{\text{pre}}$ and $d_l^i: \coprod_j \mathbb{R}\overline{M}(X/S, \tau_l^i, \beta_l^{ij}) \rightarrow \overline{M}_{\tau_l^i}^{\text{pre}}$ are the maps induced by taking domain. It follows from [36, Lemma 3], that

$$\left[\mathcal{O}_{\overline{M}_{\tau} \times_{\overline{M}_{\sigma}} \overline{M}_{\tilde{\sigma}}^{\text{pre}}} \right] = \sum_l (-1)^{l+1} \sum_i q_l^i \left[\mathcal{O}_{\overline{M}_{\tau_l^i}^{\text{pre}}} \right]$$

in $K_0(\overline{M}_{\tilde{\sigma}}^{\text{pre}})$, where

$$q_l^i: \overline{M}_{\tau_l^i}^{\text{pre}} \longrightarrow \overline{M}_{\tau} \times_{\overline{M}_{\sigma}} \overline{M}_{\tilde{\sigma}}^{\text{pre}}.$$

Let

$$p_l^i: \coprod_j \mathbb{R}\overline{M}(X/S, \tau_l^i, \beta_l^{ij}) \longrightarrow \overline{M}_\tau \times_{\overline{M}_\sigma} \mathbb{R}\overline{M}(X/S, \tilde{\sigma}, \beta)$$

be induced from all p_l^{ij} . We have

$$\begin{aligned} [\mathcal{O}_{\overline{M}_\tau \times_{\overline{M}_\sigma} \mathbb{R}\overline{M}(X/S, \tilde{\sigma}, \beta)}] &= d^* [\mathcal{O}_{\overline{M}_\tau \times_{\overline{M}_\sigma} \overline{M}_\sigma^{\text{pre}}}] \\ &= \sum_l (-1)^{l+1} \sum_i d^* q_{l*}^i [\mathcal{O}_{\overline{M}_{\tau_l^i}^{\text{pre}}}] \\ &= \sum_l (-1)^{l+1} \sum_i p_{l*}^i d_l^{i*} [\mathcal{O}_{\overline{M}_{\tau_l^i}^{\text{pre}}}] \\ &= \sum_l (-1)^{l+1} \sum_{i,j} p_{l*}^{ij} [\mathcal{O}_{\mathbb{R}\overline{M}^{\text{pre}}(X/S, \tau_l^i, \beta_l^{ij})}], \end{aligned}$$

where the third equality follows from the derived base change theorem (see [51, Theorem 1.5]) and Lemma 5.20. \square

6. DEFORMATION TO THE NORMAL BUNDLE

In this section we construct the deformation to the normal bundle of the inclusion $t_0(X) \hookrightarrow X$ for any derived k -analytic space X . This is worked out in derived algebraic geometry by Gaitsgory-Rozenblyum in [17, §9.2]. Although it is possible to give a parallel treatment in the analytic setting, it is more straightforward to obtain this deformation via analytification relative to X of its algebraic counterpart. For this reason, we begin this section by discussing analytification relative to a derived k -analytic space.

6.1. Relative derived analytification. In [50, §3] we introduced and studied the derived analytification functor in derived non-archimedean geometry in the absolute case. Here we extend the construction to the relative setting, first over any derived k -affinoid space.

Let $\text{dAff}_k^{\text{aft}}$ be the ∞ -category of derived affine k -schemes almost of finite presentation, and $j: \text{dAff}_k^{\text{aft}} \hookrightarrow \text{dAff}_k$ the inclusion into the ∞ -category of all derived affine k -schemes. Equipping both sides with the étale topology, j becomes a continuous and cocontinuous morphism of sites (see [49, §2.4] for the terminology). We denote

$$\text{dSt}_k^{\text{laft}} := \text{Sh}(\text{dAff}_k^{\text{aft}}, \tau_{\text{ét}})^\wedge, \quad \text{dSt}_k := \text{Sh}(\text{dAff}_k, \tau_{\text{ét}})^\wedge,$$

the ∞ -categories of stacks (i.e. hypercomplete sheaves) on these sites. We refer to objects in $\text{dSt}_k^{\text{laft}}$ as *derived stacks locally almost of finite type*. Left Kan extension along j induces a fully faithful functor

$$j_s: \text{dSt}_k^{\text{laft}} \hookrightarrow \text{dSt}_k,$$

which has a right adjoint j^s .

Recall from Section 3 that

$$\mathrm{St}(\mathrm{dAfd}_k) := \mathrm{Sh}(\mathrm{dAfd}_k, \tau_{\acute{\mathrm{e}}\mathrm{t}})^\wedge.$$

The derived analytification functor (in the absolute case) introduced in [50, §3] induces a functor

$$(-)^{\mathrm{an}} : \mathrm{dAff}_k^{\mathrm{aft}} \longrightarrow \mathrm{St}(\mathrm{dAfd}_k).$$

Left Kan extension along $\mathrm{dAff}_k^{\mathrm{aft}} \hookrightarrow \mathrm{dSt}_k^{\mathrm{laft}}$ induces a functor

$$(-)^{\mathrm{an}, \mathrm{laft}} : \mathrm{dSt}_k^{\mathrm{laft}} \longrightarrow \mathrm{St}(\mathrm{dAfd}_k).$$

On the other hand, *right* Kan extension along $j : \mathrm{dAff}_k^{\mathrm{aft}} \hookrightarrow \mathrm{dAff}_k$ induces a functor

$$(-)^{\mathrm{an}} : \mathrm{dAff}_k \longrightarrow \mathrm{St}(\mathrm{dAfd}_k).$$

Example 6.1. Let $\mathrm{Spec}(A) \in \mathrm{dAff}_k$ be a derived affine k -scheme, not necessarily almost of finite type. Then

$$\mathrm{Spec}(A)^{\mathrm{an}} \xrightarrow{\sim} \lim_{B \rightarrow A} \mathrm{Spec}(B)^{\mathrm{an}},$$

where the limit ranges over all the morphisms $B \rightarrow A$ with B almost of finite presentation.

Lemma 6.2. *Let $U \rightarrow V$ be an étale covering in dAff_k and let U_\bullet be its Čech nerve. The canonical map*

$$|U_\bullet^{\mathrm{an}}| \longrightarrow V^{\mathrm{an}}$$

is an equivalence in $\mathrm{St}(\mathrm{dAfd}_k)$.

Proof. It follows from [39, Remark 2.2.8] that we can assume there exists $V' \in \mathrm{dAff}_k^{\mathrm{aft}}$ and an étale covering $U' \rightarrow V'$ such that

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ U' & \longrightarrow & V' \end{array}$$

is a pullback square. Let U'_\bullet be the Čech nerve of $U' \rightarrow V'$. Since

$$(-)^{\mathrm{an}} : \mathrm{dAff}_k^{\mathrm{aft}} \longrightarrow \mathrm{dAn}_k$$

is a morphism of sites, we see that

$$|(U'_\bullet)^{\mathrm{an}}| \longrightarrow (V')^{\mathrm{an}}$$

is an equivalence in $\mathrm{St}(\mathrm{dAfd}_k)$. Since colimits are stable under pullbacks in $\mathrm{St}(\mathrm{dAfd}_k)$, the diagram

$$\begin{array}{ccc} |U_{\bullet}^{\mathrm{an}}| & \longrightarrow & V \\ \downarrow & & \downarrow \\ |(U'_{\bullet})^{\mathrm{an}}| & \longrightarrow & (V')^{\mathrm{an}} \end{array}$$

is a pullback square. It follows that the top horizontal morphism is an equivalence, completing the proof. \square

The above lemma guarantees that left Kan extension along $(-)^{\mathrm{an}}: \mathrm{dAff}_k \rightarrow \mathrm{St}(\mathrm{dAfd}_k)$ induces a colimit preserving functor

$$(-)^{\mathrm{an}}: \mathrm{dSt}_k \longrightarrow \mathrm{St}(\mathrm{dAfd}_k).$$

The following lemma ensures that there is no ambiguity when analytifying a derived stack locally almost of finite type:

Lemma 6.3. *For any $F \in \mathrm{dSt}_k^{\mathrm{laft}}$, there is a canonical equivalence*

$$F^{\mathrm{an,laft}} \simeq (j_s(F))^{\mathrm{an}}.$$

Proof. As both $j_s: \mathrm{dSt}_k^{\mathrm{laft}} \rightarrow \mathrm{dSt}_k$ and $(-)^{\mathrm{an}}: \mathrm{dSt}_k \rightarrow \mathrm{St}(\mathrm{dAfd}_k)$ commute with colimits, it is enough to check that the two constructions agree when $F \in \mathrm{dAff}_k^{\mathrm{aft}}$. In this case, the result follows from the full faithfulness of the functors $\mathrm{dAff}_k^{\mathrm{aft}} \hookrightarrow \mathrm{dAff}_k$ and $\mathrm{dAff}_k^{\mathrm{aft}} \rightarrow \mathrm{dSt}_k^{\mathrm{laft}}$. \square

Fix now $X \in \mathrm{dAfd}_k$ a derived k -affinoid space. Let $A := \Gamma(\mathcal{O}_X^{\mathrm{alg}})$. For any $\mathrm{Spec}(B) \in \mathrm{dAff}_k^{\mathrm{afp}}$, we have

$$\begin{aligned} \mathrm{Map}_{\mathrm{dSt}_k}(\mathrm{Spec}(A), \mathrm{Spec}(B)) &\simeq \mathrm{Map}_{\mathrm{R}\mathcal{T}\mathrm{op}(\mathcal{T}_{\acute{e}t}(k))}(\mathrm{Spec}(A), \mathrm{Spec}(B)) \\ &\simeq \mathrm{Map}_{\mathrm{R}\mathcal{T}\mathrm{op}(\mathcal{T}_{\acute{e}t}(k))}(X^{\mathrm{alg}}, \mathrm{Spec}(B)) \\ &\simeq \mathrm{Map}_{\mathrm{dAn}_k}(X, \mathrm{Spec}(B)^{\mathrm{an}}) \end{aligned}$$

When $\mathrm{Spec}(B) \in \mathrm{dAff}_k$, we get

$$\begin{aligned} \mathrm{Map}_{\mathrm{dSt}_k}(\mathrm{Spec}(A), \mathrm{Spec}(B)) &\simeq \lim_{\mathrm{Spec}(B') \rightarrow \mathrm{Spec}(B)} \mathrm{Map}_{\mathrm{dSt}_k}(\mathrm{Spec}(A), \mathrm{Spec}(B')) \\ &\simeq \lim_{\mathrm{Spec}(B') \rightarrow \mathrm{Spec}(B)} \mathrm{Map}_{\mathrm{St}(\mathrm{dAfd}_k)}(X, \mathrm{Spec}(B')^{\mathrm{an}}) \\ &\simeq \mathrm{Map}_{\mathrm{St}(\mathrm{dAfd}_k)}(X, \mathrm{Spec}(B)^{\mathrm{an}}), \end{aligned}$$

where the colimits range over all the morphisms $\mathrm{Spec}(B') \rightarrow \mathrm{Spec}(B)$ with B' almost of finite presentation. In particular, taking $B = A$, we obtain a canonical morphism

$$\eta_X: X \rightarrow \mathrm{Spec}(A)^{\mathrm{an}}.$$

Pulling back along η_X gives a functor

$$\eta_X^*: \mathrm{dAnSt}/_{\mathrm{Spec}(A)^{\mathrm{an}} \times Y^{\mathrm{an}}} \longrightarrow \mathrm{dAnSt}/_{X \times Y^{\mathrm{an}}}.$$

At this point, we can give the following definition.

Definition 6.4. Let $X \in \mathrm{dAfd}_k$ be a derived k -affinoid space and let $A := \Gamma(\mathcal{O}_X^{\mathrm{alg}})$. The functor $(-)_X^{\mathrm{an}}$ of *analytification relative to X* is the composition

$$\mathrm{dSt}/_{\mathrm{Spec}(A)} \xrightarrow{(-)^{\mathrm{an}}} \mathrm{dAnSt}/_{\mathrm{Spec}(A)^{\mathrm{an}}} \xrightarrow{\eta_X^*} \mathrm{dAnSt}/_X,$$

Lemma 6.5. Let $X \in \mathrm{dAfd}_k$ be a derived k -affinoid space, $A := \Gamma(\mathcal{O}_X^{\mathrm{alg}})$ and $S := \mathrm{Spec}(A)$. The following hold:

- (1) The analytification of S relative to X coincides with X itself.
- (2) The analytification of \mathbb{A}_A^n relative to X coincides with \mathbf{A}_X^n .
- (3) The analytification of $t_{\leq n}S$ relative to X coincides with $t_{\leq n}X$.
- (4) The analytification of S_{red} relative to X coincides with X_{red} .

Proof. (1) follows directly from the definition. Let us prove (2). Since $(-)^{\mathrm{an}}: \mathrm{dSt}_k \rightarrow \mathrm{St}(\mathrm{dAfd}_k)$ commutes with finite limits, we have

$$(\mathbb{A}_A^n)^{\mathrm{an}} \simeq (\mathbb{A}_k^n)^{\mathrm{an}} \times S^{\mathrm{an}} \simeq \mathbf{A}_k^n \times S^{\mathrm{an}}.$$

Thus we deduce that

$$(\mathbb{A}_A^n)_X^{\mathrm{an}} \simeq \mathbf{A}_X^n.$$

For (3), we start by considering the composite functor

$$(\mathrm{dAff}_k^{\mathrm{aft}})_{S/} \rightarrow (\mathrm{dAff}_k^{\mathrm{aft}, \leq n})_{t_{\leq n}S/} \rightarrow (\mathrm{dAff}_k^{\mathrm{aft}})_{t_{\leq n}S/},$$

where $(\mathrm{dAff}_k^{\mathrm{aft}, \leq n})_{t_{\leq n}S/}$ denotes the full subcategory of $(\mathrm{dAff}_k^{\mathrm{aft}})_{t_{\leq n}S/}$ spanned by n -truncated derived affine schemes almost of finite type. The universal property of the truncation shows that the second functor is final.

Since the functor $t_{\leq n}(-)$ commutes with limits, the equivalence

$$S \simeq \lim_{S' \in (\mathrm{dAff}_k^{\mathrm{aft}})_{S/}} S',$$

implies

$$t_{\leq n}S \simeq \lim_{S' \in (\mathrm{dAff}_k^{\mathrm{aft}})_{S/}} t_{\leq n}S'.$$

For any n -truncated derived affine scheme almost of finite type $T \in \mathrm{dAff}_k^{\mathrm{aft}}$, [40, 7.2.4.26] shows that the canonical map

$$\mathrm{colim}_{S' \in (\mathrm{dAff}_k^{\mathrm{aft}})_{S/}} \mathrm{Map}_{\mathrm{dAff}_k}(t_{\leq n}S', T) \longrightarrow \mathrm{Map}_{\mathrm{dAff}_k}(t_{\leq n}S, T)$$

is an equivalence. This implies that $(\mathrm{dAff}_k^{\mathrm{aft}})_{S/} \rightarrow (\mathrm{dAff}_k^{\mathrm{aft}, \leq n})_{t_{\leq n}S/}$ is final as well.

Combining with Example 6.1, we deduce that

$$(\mathfrak{t}_{\leq n} S)^{\text{an}} \simeq \lim_{S' \in (\text{dAff}_k^{\text{aft}})_{S/}} (\mathfrak{t}_{\leq n} S')^{\text{an}}.$$

At this point, for any n -truncated k -affinoid space $U \in \text{dAfd}_k$, we have

$$\begin{aligned} \text{Map}_{\text{St}(\text{dAfd}_k)}(U, (\mathfrak{t}_{\leq n} S)^{\text{an}}) &\simeq \lim_{S' \in (\text{dAff}_k^{\text{aft}})_{S/}} \text{Map}_{\text{St}(\text{dAfd}_k)}(U, (\mathfrak{t}_{\leq n} S')^{\text{an}}) \\ &\simeq \lim_{S' \in (\text{dAff}_k^{\text{aft}})_{S/}} \text{Map}_{\text{St}(\text{dAfd}_k)}(U, \mathfrak{t}_{\leq n}(S'^{\text{an}})) \\ &\simeq \lim_{S' \in (\text{dAff}_k^{\text{aft}})_{S/}} \text{Map}_{\text{St}(\text{dAfd}_k)}(U, S'^{\text{an}}) \\ &\simeq \text{Map}_{\text{St}(\text{dAfd}_k)}(U, S^{\text{an}}). \end{aligned}$$

The second equivalence follows from the fact that B is almost of finite type and the flatness of the analytification (see [48, Corollary 5.15] and [50, Proposition 4.17]). The third equivalence follows from the universal property of the n -truncation. This implies that

$$\text{Map}_{\text{St}(\text{dAfd}_k)}(U, (\mathfrak{t}_{\leq n} S)_X^{\text{an}}) \simeq \text{Map}_{\text{St}(\text{dAfd}_k)}(U, X).$$

Therefore, by the universal property of the n -truncation again, we conclude that $(\mathfrak{t}_{\leq n} S)_X^{\text{an}}$ coincides with $\mathfrak{t}_{\leq n}(X)$.

Finally, (4) follows along the same lines of the proof of (3). \square

Proposition 6.6. *Let $X \in \text{dAfd}_k$ be a derived k -affinoid space, $A := \Gamma(\mathcal{O}_X^{\text{alg}})$, and Y a derived algebraic stack locally almost of finite type over $\text{Spec}(A)$. Then Y_X^{an} is a derived k -analytic stack.*

Proof. It follows by induction on the geometric level combining Lemma 6.5(2-3) as in the proof of [50, Proposition 3.7]. \square

We can further extend the definition of relative analytification over global (i.e. non-affinoid) bases as follows: Given $X \in \text{dAn}_k$ any derived k -analytic space, let $X_{\text{ét}}^{\text{afd}}$ denote the full subcategory of the étale site $X_{\text{ét}}$ spanned by étale morphisms $U \rightarrow X$ where U is derived k -affinoid. For every $U \in X_{\text{ét}}^{\text{afd}}$, write $A_U := \Gamma(\mathcal{O}_U^{\text{alg}})$. Then the derived analytification functors

$$(-)_U^{\text{an}} : \text{dSt}/_{\text{Spec}(A_U)} \longrightarrow \text{dAnSt}/_U$$

glue together to provide

$$(-)_X^{\text{an}} : \lim_{U \in X_{\text{ét}}^{\text{afd}}} \text{dSt}/_{\text{Spec}(A_U)} \longrightarrow \lim_{U \in X_{\text{ét}}^{\text{afd}}} \text{dAnSt}/_U \simeq \text{dAnSt}/_X,$$

where the last equivalence is a consequence of the descent theory for ∞ -topoi, see [38, 6.1.3.9].

6.2. Formal moduli problems in derived analytic geometry. As an application of the relative derived analytification introduced above, we prove here that all formal moduli problems over a given derived k -affinoid space are algebraic, see Theorem 6.12.

Definition 6.7. For any derived k -analytic space $X \in \mathrm{dAn}_k$, we define $X_{\mathrm{red}} := \mathfrak{t}_0(X)_{\mathrm{red}}$, called the reduced k -analytic space associated to X .

Definition 6.8. A morphism $Y \rightarrow X$ in dAn_k is called a *nil-isomorphism* if the induced morphism $Y_{\mathrm{red}} \rightarrow X_{\mathrm{red}}$ is an isomorphism. We denote by $\mathrm{Nil}_{/X}^{\mathrm{an}}$ the full subcategory of $(\mathrm{dAn}_k)_{/X}^{\mathrm{ft}}$ spanned by nil-isomorphisms $Y \rightarrow X$ of finite type.

Let X be a derived k -affinoid space. If $Y \in \mathrm{dAn}_k$ is a derived k -analytic space and $Y \rightarrow X$ is a nil-isomorphism, then Y is also a derived k -affinoid space. Moreover, we have the following basic result:

Lemma 6.9. *Let $Y \rightarrow X$ be a nil-isomorphism of derived k -affinoid spaces. Let $A := \Gamma(\mathcal{O}_X^{\mathrm{alg}})$ and $B := \Gamma(\mathcal{O}_Y^{\mathrm{alg}})$. Then:*

- (1) *The morphism $A \rightarrow B$ is a nil-isomorphism almost of finite type.*
- (2) *Let $\mathbb{L}_{B/A}^{\mathrm{an}} := \Gamma(\mathbb{L}_{Y/X}^{\mathrm{an}})$; the canonical morphism*

$$\mathbb{L}_{B/A} \longrightarrow \mathbb{L}_{B/A}^{\mathrm{an}}$$

is an equivalence.

- (3) *If $Y \rightarrow X$ is of finite type, then so is $A \rightarrow B$.*

Proof. It follows from [51, Theorem 3.4] that A_{red} and B_{red} are the global sections of X_{red} and Y_{red} , respectively. This implies that the morphism $A \rightarrow B$ is a nil-isomorphism. Let us show that it is also almost of finite type. Since both A and B are noetherian in the derived sense (see [40, 7.2.4.30]), the derived Hilbert basis theorem [40, 7.2.4.31] implies that $A \rightarrow B$ is almost of finite presentation if and only if $\pi_0(A) \rightarrow \pi_0(B)$ is of finite presentation. Since $A_{\mathrm{red}} \rightarrow B_{\mathrm{red}}$ is an isomorphism, we can find a map

$$\phi: \pi_0(A)[X_1, \dots, X_n] \longrightarrow \pi_0(B)$$

which becomes surjective after composing with $\pi_0(B) \rightarrow \pi_0(B)_{\mathrm{red}} = B_{\mathrm{red}}$. Since $\pi_0(B)$ is noetherian, its nilradical $\mathrm{Nil}_{\pi_0(B)}$ is generated by finitely many elements $b_1, \dots, b_m \in \pi_0(B)$. We now consider the extension of ϕ

$$\pi_0(A)[X_1, \dots, X_n, Y_1, \dots, Y_m] \longrightarrow \pi_0(A)$$

which sends Y_i to b_i . This map is surjective, and hence $\pi_0(B)$ is of finite type as $\pi_0(A)$ -algebra.

We now prove (2). Let $i: \mathrm{Spec}(B_{\mathrm{red}}) \rightarrow \mathrm{Spec}(B)$ be the natural inclusion. Since $A \rightarrow B$ is almost of finite type, [40, 7.4.3.18] implies that $\mathbb{L}_{B/A}$ is almost of finite presentation. Using [50, Corollary 5.40] and [51, Theorem 3.4], we deduce that $\mathbb{L}_{B/A}^{\mathrm{an}}$ is almost of finite presentation too. Therefore, by Nakayama's lemma, it is enough to check that $i^*\mathbb{L}_{B/A} \rightarrow i^*\mathbb{L}_{B/A}^{\mathrm{an}}$ is an equivalence. Consider the following diagram:

$$\begin{array}{ccccc} i^*\mathbb{L}_{B/A} & \longrightarrow & \mathbb{L}_{B_{\mathrm{red}}/A} & \longrightarrow & \mathbb{L}_{B_{\mathrm{red}}/B} \\ \downarrow & & \downarrow & & \downarrow \\ i^*\mathbb{L}_{B/A}^{\mathrm{an}} & \longrightarrow & \mathbb{L}_{B_{\mathrm{red}}/A}^{\mathrm{an}} & \longrightarrow & \mathbb{L}_{B_{\mathrm{red}}/B}^{\mathrm{an}}. \end{array}$$

Since $Y \rightarrow X$ is a nil-isomorphism, the map $Y_{\mathrm{red}} \simeq X_{\mathrm{red}} \rightarrow X$ is a closed immersion. Therefore, [50, Corollary 5.33] implies that both the middle and right vertical arrows are equivalences. As the horizontal lines are fiber sequences, it follows that the left vertical arrow is also an equivalence.

Finally, (3) follows immediately from (2). \square

Let $X \in \mathrm{dAfd}_k$ be a derived k -affinoid space and let $A := \Gamma(\mathcal{O}_X^{\mathrm{alg}})$. Denote by $\mathrm{Nil}/_{\mathrm{Spec}(A)}$ the ∞ -category of nil-isomorphisms of finite type $\mathrm{Spec}(B) \rightarrow \mathrm{Spec}(A)$. By the above lemma, taking global sections and applying Spec , we obtain a canonical functor

$$(-)_X^{\mathrm{alg}}: \mathrm{Nil}_X^{\mathrm{an}} \longrightarrow \mathrm{Nil}/_{\mathrm{Spec}(A)}.$$

The following result is a generalization of [48, Proposition 8.2].

Proposition 6.10. *Let $X \in \mathrm{dAfd}_k$ be a derived k -affinoid space and let $A := \Gamma(\mathcal{O}_X^{\mathrm{alg}})$. The functor $(-)_X^{\mathrm{alg}}: \mathrm{Nil}_X^{\mathrm{an}} \rightarrow \mathrm{Nil}/_{\mathrm{Spec}(A)}$ is an equivalence.*

Proof. Consider the relative analytification functor introduced in Section 6.1:

$$(-)_X^{\mathrm{an}}: (\mathrm{dSt}_k)_{/\mathrm{Spec}(A)}^{\mathrm{laft}} \longrightarrow \mathrm{dAnSt}/_X.$$

Let $\mathrm{Spec}(B) \rightarrow \mathrm{Spec}(A)$ be a nil-isomorphism of finite type. Set $Y := \mathrm{Spec}(B)_X^{\mathrm{an}}$. By the transitivity property for pullback squares ([38, 4.4.2.1]), given any $\mathrm{Spec}(B') \rightarrow \mathrm{Spec}(B)$, we have a canonical equivalence

$$\mathrm{Spec}(B')_Y^{\mathrm{an}} \simeq \mathrm{Spec}(B')_X^{\mathrm{an}}.$$

In particular, Lemma 6.5(4) implies that the functor $(-)_X^{\mathrm{an}}$ restricts to a functor

$$(-)_X^{\mathrm{an}}: \mathrm{Nil}/_{\mathrm{Spec}(A)} \longrightarrow \mathrm{Nil}_X^{\mathrm{an}}.$$

It is then enough to see that $(-)_X^{\mathrm{alg}}$ and $(-)_X^{\mathrm{an}}$ form an equivalence of ∞ -categories. First we observe that $(-)_X^{\mathrm{alg}}$ is left adjoint to $(-)_X^{\mathrm{an}}$. Moreover, $(-)_X^{\mathrm{alg}}$ is conservative:

indeed, let $Y' \rightarrow Y$ be a morphism in $\mathrm{Nil}/_X^{\mathrm{an}}$ such that the morphism $B \rightarrow B'$ obtained by passing to global sections is an equivalence. By Lemma 6.9, we have

$$\Gamma(Y', \mathbb{L}_{Y'/Y}^{\mathrm{an}}) \simeq \mathbb{L}_{B'/B}^{\mathrm{an}} \simeq 0.$$

Using [51, Theorem 3.4], we conclude that $\mathbb{L}_{Y'/Y}^{\mathrm{an}} \simeq 0$, and hence $Y' \rightarrow Y$ is étale. As $Y'_{\mathrm{red}} \rightarrow Y_{\mathrm{red}}$ is an isomorphism, we conclude that $Y' \rightarrow Y$ is an equivalence.

At this point, we are left to prove that for any $\mathrm{Spec}(B) \in \mathrm{Nil}/_{\mathrm{Spec}(A)}$ the unit morphism

$$\mathrm{Spec}(B) \longrightarrow (\mathrm{Spec}(B)_X^{\mathrm{an}})_X^{\mathrm{alg}}$$

is an equivalence. This follows from the compatibility of $(-)_X^{\mathrm{an}}$ and $(-)_X^{\mathrm{alg}}$ with the Postnikov tower decomposition, which are respectively guaranteed by [50, Proposition 4.17] and [52, Theorem 3.23]. \square

Definition 6.11. Let $X \in \mathrm{dAfd}_k$ be a derived k -affinoid space. An *analytic formal moduli problem over X* is a functor

$$F: (\mathrm{Nil}/_X^{\mathrm{an}})^{\mathrm{op}} \longrightarrow \mathcal{S}$$

satisfying the following conditions:

- (1) The space $F(X_{\mathrm{red}})$ is contractible;
- (2) Let

$$\begin{array}{ccc} S & \xrightarrow{f} & S' \\ \downarrow & & \downarrow \\ T & \longrightarrow & T' \end{array}$$

be a pushout square in $\mathrm{Nil}/_X$ where $f: S \rightarrow S'$ is a k -analytic square-zero extension. Then the canonical map

$$F(T') \longrightarrow F(T) \times_{F(S)} F(S')$$

is an equivalence.

We denote by $\mathrm{FMP}/_X^{\mathrm{an}}$ the ∞ -category of analytic formal moduli problems over X .

Theorem 6.12. *Let $X \in \mathrm{dAfd}_k$ be a derived k -affinoid space and let $A := \Gamma(\mathcal{O}_X^{\mathrm{alg}})$. There is a canonical equivalence*

$$\mathrm{FMP}/_{\mathrm{Spec}(A)} \xrightarrow{\sim} \mathrm{FMP}/_X^{\mathrm{an}}.$$

Proof. It follows from [17, Proposition 5.1.2.2(c)] and Proposition 6.10. \square

6.3. The deformation to the normal bundle in the algebraic setting. Here we review the deformation to the normal bundle in the algebraic setting following Gaitsgory-Rozenblyum [17]. Let

$$\mathrm{Bifurc}_{\mathrm{scaled}}^{\bullet} : \Delta^{\mathrm{op}} \longrightarrow \mathrm{dSt}_{/\mathbb{A}_k^1}$$

be as in [17, §9.2.2.6], which we will denote by \mathbf{B}^{\bullet} to simplify the notation. Recall that \mathbf{B}^n is obtained by gluing together $n+1$ copies of \mathbb{A}_k^1 along 0. Let $X \in \mathrm{dSt}_k^{\mathrm{laft}}$ be a derived algebraic stack locally almost of finite type over k , and $j: t_0(X) \hookrightarrow X$ the inclusion of its truncation. Consider the following fiber product in $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{dSt}_{/\mathbb{A}_k^1})$:

$$\begin{array}{ccc} \mathfrak{D}_X^{\bullet} & \longrightarrow & X \times \mathbb{A}_k^1 \\ \downarrow & & \downarrow \\ \mathrm{Map}_{/\mathbb{A}_k^1}(\mathbf{B}^{\bullet}, X_0 \times \mathbb{A}^1) & \longrightarrow & \mathrm{Map}_{/\mathbb{A}_k^1}(\mathbf{B}^{\bullet}, X \times \mathbb{A}_k^1), \end{array}$$

where $X \times \mathbb{A}_k^1$ is viewed as a constant simplicial object in $\mathrm{dSt}_{/\mathbb{A}_k^1}$. It is proven in [17, Lemma 9.2.3.4] that \mathfrak{D}_X^{\bullet} is a groupoid object in $\mathrm{dSt}_{/\mathbb{A}_k^1}$. In particular, Theorem 5.2.3.2 in loc. cit.² allows to associate to \mathfrak{D}_X^{\bullet} a formal moduli problem \mathfrak{D}_X under $X_0 \times \mathbb{A}_k^1$ and over $X \times \mathbb{A}_k^1$. Observe that this construction is functorial in X . The following proposition summarizes the main properties of \mathfrak{D}_X .

Proposition 6.13. *Let $X \in \mathrm{dSt}_k^{\mathrm{laft}}$ be a derived algebraic stack locally almost of finite type over k . The following hold:*

- (1) *The total space \mathfrak{D}_X is a derived algebraic stack locally almost of finite type over $X \times \mathbb{A}_k^1$. If X is a derived Deligne-Mumford stack (resp. a derived scheme, a derived affine scheme), then so is \mathfrak{D}_X .*
- (2) *There is a canonical equivalence*

$$t_0(\mathfrak{D}_X) \simeq t_0(X) \times \mathbb{A}_k^1.$$

- (3) *There is a canonical equivalence*

$$\mathfrak{D}_X \times_{\mathbb{A}_k^1} \mathbb{G}_m \simeq X \times \mathbb{G}_m.$$

- (4) *There is a canonical equivalence*

$$\mathfrak{D}_X \times_{\mathbb{A}_k^1} 0 \simeq \mathrm{Spec}_{t_0(X)} \left(\mathrm{Sym}_{\mathcal{O}_{t_0(X)}}(\mathbb{L}_{t_0(X)/X}[-1]) \right).$$

Moreover, given any morphism $f: X \rightarrow Y$, let $\mathfrak{f}: \mathfrak{D}_X \rightarrow \mathfrak{D}_Y$ denote its deformation. Then the fiber of \mathfrak{f} at $0 \in \mathbb{A}_k^1$

$$\mathfrak{f}_0: \mathfrak{D}_X \times_{\mathbb{A}_k^1} 0 \longrightarrow \mathfrak{D}_Y \times_{\mathbb{A}_k^1} 0$$

²See also [12, Theorem 3.9] and [45].

coincides with the morphism induced by the map

$$f^* \mathbb{L}_{t_0(Y)/Y} \longrightarrow \mathbb{L}_{t_0(X)/X}.$$

Proof. Property (2) follows from the canonical identification

$$t_0(\mathbf{Map}_{/\mathbb{A}_k^1}(\mathbf{B}^\bullet, X \times \mathbb{A}_k^1)) \simeq \mathbf{Hom}_{/\mathbb{A}_k^1}(\mathbf{B}^\bullet, t_0(X) \times \mathbb{A}_k^1),$$

which holds because \mathbf{B}^\bullet is flat over \mathbb{A}_k^1 . Property (4) follows from [17, Proposition 9.2.3.6].

Let us prove property (1). Proceeding as in §9.5.1 in loc. cit. and using Theorem 9.5.1.3 as main input, we construct a sequence

$$\mathfrak{X}^{(0)} \hookrightarrow \mathfrak{X}^{(1)} \hookrightarrow \mathfrak{X}^{(2)} \hookrightarrow \dots$$

of derived stacks enjoying the following properties:

- (i) We have $\mathfrak{X}^{(0)} = X_0 \times \mathbb{A}_k^1$, and $\mathfrak{X}^{(n+1)}$ is obtained as a square-zero extension of $\mathfrak{X}^{(n)}$ by an element in $\mathrm{Coh}^{\geq n}(\mathfrak{X}^{(n)})$. In particular, each $\mathfrak{X}^{(n)}$ is geometric, and the connectivity assumption guarantees that the colimit $\mathrm{colim}_n \mathfrak{X}^{(n)}$ is also geometric. If X is a Deligne-Mumford stack (resp. a derived scheme, a derived affine scheme), then the same goes for each $\mathfrak{X}^{(n)}$ and for the colimit.
- (ii) The above sequence is equipped with compatible maps $\mathfrak{X}^{(n)} \rightarrow \mathfrak{D}_X$, and the induced morphism

$$\mathrm{colim}_n \mathfrak{X}^{(n)} \longrightarrow \mathfrak{D}_X$$

is an equivalence. This is the content of Proposition 5.2.2 in loc. cit.

We are left to prove property (3). Observe that $\mathbf{B}^n \times_{\mathbb{A}_k^1} \mathbb{G}_m \simeq \mathbb{G}_m^{\sqcup n}$. Unraveling the definitions, we obtain

$$\mathfrak{D}_X^\bullet \times_{\mathbb{A}_k^1} \mathbb{G}_m \simeq \check{\mathcal{C}}(t_0(X) \times \mathbb{G}_m \rightarrow X \times \mathbb{G}_m).$$

Example 5.2.3.3 in loc. cit. shows that the formal moduli problem associated to this formal groupoid coincides with the formal completion of $t_0(X) \times \mathbb{G}_m$ inside $X \times \mathbb{G}_m$. This formal completion can be written as

$$(t_0(X) \times \mathbb{G}_m)_{\mathrm{dR}} \times_{(X \times \mathbb{G}_m)_{\mathrm{dR}}} (X \times \mathbb{G}_m).$$

As

$$(t_0(X) \times \mathbb{G}_m)_{\mathrm{dR}} \longrightarrow (X \times \mathbb{G}_m)_{\mathrm{dR}}$$

is an equivalence, the conclusion follows. \square

Remark 6.14. By Proposition 6.13(3) we can extend the deformation $\mathfrak{D}_X \rightarrow \mathbb{A}_k^1$ to a deformation

$$\tilde{\mathfrak{D}}_X \rightarrow \mathbb{P}_k^1,$$

which is trivial near infinity.

Remark 6.15. The construction of $\tilde{\mathfrak{D}}_X$ in Remark 6.14 induces a functor

$$\tilde{\mathfrak{D}}: \mathrm{dAff}_k^{\mathrm{aft}} \longrightarrow (\mathrm{dSt}_k)_{/\mathbb{P}_k^1}$$

sending every $X \in \mathrm{dAff}_k^{\mathrm{aft}}$ to $\tilde{\mathfrak{D}}_X$. Right Kan extension along $\mathrm{dAff}_k^{\mathrm{aft}} \hookrightarrow \mathrm{dAff}_k$ provides a functor

$$\tilde{\mathfrak{D}}: \mathrm{dAff}_k \longrightarrow (\mathrm{dSt}_k)_{/\mathbb{P}_k^1}.$$

Moreover, for every $X \in \mathrm{dAff}_k$, the deformation

$$\mathfrak{D}_X := \tilde{\mathfrak{D}}_X \times_{\mathbb{P}_k^1} \mathbb{A}_k^1$$

inherits all the properties of Proposition 6.13.

6.4. Analytifying the deformation. Let $X \in \mathrm{dAn}_k$ be a derived k -analytic space. As in Section 6.1, we denote by $X_{\acute{\mathrm{e}}\mathrm{t}}^{\mathrm{afd}}$ the full subcategory of the small étale site $X_{\acute{\mathrm{e}}\mathrm{t}}$ of X spanned by étale morphisms $U \rightarrow X$, where U is derived k -affinoid. For any $U \in X_{\acute{\mathrm{e}}\mathrm{t}}^{\mathrm{afd}}$, put $A_U := \Gamma(\mathcal{O}_U^{\mathrm{alg}})$ and $U^{\mathrm{alg}} := \mathrm{Spec}(A_U)$.

Lemma 6.16. *For any morphism $V \rightarrow U$ in $X_{\acute{\mathrm{e}}\mathrm{t}}^{\mathrm{afd}}$, the diagram*

$$\begin{array}{ccc} \tilde{\mathfrak{D}}_{V^{\mathrm{alg}}} & \longrightarrow & \tilde{\mathfrak{D}}_{U^{\mathrm{alg}}} \\ \downarrow & & \downarrow \\ V^{\mathrm{alg}} & \longrightarrow & U^{\mathrm{alg}} \end{array}$$

is a pullback square, where $\tilde{\mathfrak{D}}$ denotes the extended deformation introduced in Remark 6.15.

Proof. By construction, it suffices to prove the analogous statement for $\mathfrak{D}_{V^{\mathrm{alg}}}$ and $\mathfrak{D}_{U^{\mathrm{alg}}}$. Choose a cofiltered diagram $F: I \rightarrow (\mathrm{dAff}_k^{\mathrm{aft}})^{\Delta^1}$ whose limit $\lim_{i \in I} F(i)$ is equivalent to $V^{\mathrm{alg}} \rightarrow U^{\mathrm{alg}}$. We write

$$F(i): U_i \longrightarrow V_i.$$

Then, unraveling the definition, we find canonical equivalences

$$\mathfrak{D}_{U^{\mathrm{alg}}} \simeq \lim_{i \in I} \mathfrak{D}_{U_i}, \quad \mathfrak{D}_{V^{\mathrm{alg}}} \simeq \lim_{i \in I} \mathfrak{D}_{V_i}.$$

Applying Proposition 6.13(2) to each \mathfrak{D}_{U_i} and \mathfrak{D}_{V_i} , and passing to the limit, we see that

$$\mathrm{t}_0(\mathfrak{D}_{U^{\mathrm{alg}}}) \simeq \mathrm{t}_0(U^{\mathrm{alg}}) \times \mathbb{A}_k^1, \quad \mathrm{t}_0(\mathfrak{D}_{V^{\mathrm{alg}}}) \simeq \mathrm{t}_0(V^{\mathrm{alg}}) \times \mathbb{A}_k^1.$$

Therefore, the natural map

$$\mathfrak{D}_{V^{\mathrm{alg}}} \longrightarrow V^{\mathrm{alg}} \times_{U^{\mathrm{alg}}} \mathfrak{D}_{U^{\mathrm{alg}}}$$

induces an equivalence on the truncation. By Proposition 6.13(1), both $\mathfrak{D}_{V^{\text{alg}}}$ and $V^{\text{alg}} \times_{U^{\text{alg}}} \mathfrak{D}_{U^{\text{alg}}}$ are affine derived schemes, almost of finite presentation over V^{alg} . Write

$$\mathfrak{D}_{V^{\text{alg}}} \simeq \text{Spec}(R), \quad V^{\text{alg}} \times_{U^{\text{alg}}} \mathfrak{D}_{U^{\text{alg}}} \simeq \text{Spec}(R').$$

By [51, Theorem 1.3], A_V is noetherian in the sense of [40, Definition 7.2.4.30]. Since R and R' are A_V -algebras almost of finite presentation, Proposition 7.2.4.31 in loc. cit. implies that they are noetherian as well. Therefore applying the same result again, we see that the maps $R \rightarrow \pi_0(R)$ and $R' \rightarrow \pi_0(R)$ are almost of finite presentation. At this point, [24, Lemma 3.1.1] implies that the map $R' \rightarrow R$ is an equivalence if and only if the induced map of Čech nerves

$$\check{\mathcal{C}}(t_0(V^{\text{alg}}) \times \mathbb{A}_k^1 \rightarrow \mathfrak{D}_{V^{\text{alg}}}) \longrightarrow \check{\mathcal{C}}(t_0(V^{\text{alg}}) \times \mathbb{A}_k^1 \rightarrow V^{\text{alg}} \times_{U^{\text{alg}}} \mathfrak{D}_{V^{\text{alg}}})$$

is an equivalence of simplicial objects. We have

$$\check{\mathcal{C}}(t_0(V^{\text{alg}}) \times \mathbb{A}_k^1 \rightarrow \mathfrak{D}_{V^{\text{alg}}}) \simeq \lim_{i \in I} \check{\mathcal{C}}(t_0(V_i) \times \mathbb{A}_k^1 \rightarrow \mathfrak{D}_{V_i^{\text{alg}}}),$$

and similarly for the other Čech nerve. It follows from the construction of \mathfrak{D}_{V_i} (via [17, Theorem 5.2.3.2]) that we can identify this Čech nerve with

$$\mathfrak{D}_{V^{\text{alg}}}^\bullet := \lim_{i \in I} \mathfrak{D}_{V_i}^\bullet.$$

Similarly, we can identify the other Čech nerve with

$$V^{\text{alg}} \times_{U^{\text{alg}}} \mathfrak{D}_{U^{\text{alg}}}^\bullet := \lim_{i \in I} V_i \times_{U_i} \mathfrak{D}_{U_i}^\bullet.$$

We are thus reduced to prove that for every $[n] \in \mathbf{\Delta}$, the diagram

$$\begin{array}{ccc} \mathfrak{D}_{V^{\text{alg}}}^n & \longrightarrow & \mathfrak{D}_{U^{\text{alg}}}^n \\ \downarrow & & \downarrow \\ V^{\text{alg}} & \longrightarrow & U^{\text{alg}} \end{array}$$

is a pullback square. Unwinding the definitions, we see that it is enough to prove that the diagram

$$\begin{array}{ccc} \mathbf{Map}_{/\mathbb{A}_k^1}(B^n, t_0(V^{\text{alg}}) \times \mathbb{A}_k^1) & \longrightarrow & \mathbf{Map}_{/\mathbb{A}_k^1}(B^n, V^{\text{alg}} \times \mathbb{A}_k^1) \\ \downarrow & & \downarrow \\ \mathbf{Map}_{/\mathbb{A}_k^1}(B^n, t_0(U^{\text{alg}}) \times \mathbb{A}_k^1) & \longrightarrow & \mathbf{Map}_{/\mathbb{A}_k^1}(B^n, U^{\text{alg}} \times \mathbb{A}_k^1) \end{array}$$

is a pullback square. For this, it is enough to verify that

$$\begin{array}{ccc} t_0(V^{\text{alg}}) & \longrightarrow & V^{\text{alg}} \\ \downarrow & & \downarrow \\ t_0(U^{\text{alg}}) & \longrightarrow & U^{\text{alg}} \end{array}$$

is a pullback. Since $V \rightarrow U$ is étale, Lemma 10.2 implies that $A_U \rightarrow A_V$ is flat and therefore the previous square is indeed a pullback. The conclusion follows. \square

Thanks to the above lemma, we can interpret the assignment $U \mapsto \tilde{\mathfrak{D}}_{U^{\text{alg}}}$ as an element in the limit

$$\lim_{U \in X_{\text{ét}}^{\text{afd}}} \text{dSt}/U^{\text{alg}}.$$

We can therefore apply the relative analytification functor

$$(-)_X^{\text{an}}: \lim_{U \in X_{\text{ét}}^{\text{afd}}} \text{dSt}/U^{\text{alg}} \longrightarrow \text{dAnSt}/X,$$

and obtain an analytic deformation

$$\tilde{\mathfrak{D}}_X \longrightarrow X \times \mathbf{P}_k^1,$$

which enjoys the analytic version of all the properties given in Proposition 6.13.

7. NON-ARCHIMEDEAN GROMOV COMPACTNESS

In this section, we review the compactness results from [63]. Starting from this section until the end of the paper we will assume that the base field k has discrete valuation and residue characteristic zero.

In order to obtain a proper moduli stack of stable maps, we need to equip the ambient space X a Kähler structure. Let us recall the definition of Kähler structure in non-archimedean geometry following [63, §3].

Let X be a quasi-compact smooth rigid k -analytic space, \mathfrak{X} an snc formal model of X , and $S_{\mathfrak{X}}$ the associated Clemens polytope, i.e. the dual intersection complex of the special fiber \mathfrak{X}_s . Let $\{D_i\}_{i \in I_{\mathfrak{X}}}$ be the set of irreducible components of the reduction $(\mathfrak{X}_s)_{\text{red}}$ of \mathfrak{X}_s , and mult_i the multiplicity of D_i in \mathfrak{X}_s for every $i \in I_{\mathfrak{X}}$. For every non-empty subset $I \subset I_{\mathfrak{X}}$, let $D_I := \bigcap_{i \in I} D_i$. Let $N^1(D_I, \mathbb{R})$ denote the vector space of codimension-one cycles in D_I with real coefficients modulo numerical equivalence (with respect to proper curves in D_I).

Definition 7.1. A *simple function* φ on $S_{\mathfrak{X}}$ is a real valued function that is affine on every face of $S_{\mathfrak{X}}$. For $i \in I_{\mathfrak{X}}$, let $\varphi(i)$ denote the value of φ at the vertex i of $S_{\mathfrak{X}}$.

Definition 7.2. Let Δ^I , $I \subset I_{\mathfrak{X}}$ be a face of $S_{\mathfrak{X}}$. For any simple function φ on $S_{\mathfrak{X}}$, we define the *derivative* of φ with respect to Δ^I as

$$\partial_I \varphi = \sum_{i \in I_{\mathfrak{X}}} \text{mult}_i \cdot \varphi(i) \cdot [D_i]_{D_I} \in N^1(D_I, \mathbb{R}),$$

where $[D_i]$ denotes the divisor class of D_i . The function φ is said to be *linear* (resp. *convex*, *strictly convex*) along the open simplicial face $(\Delta^I)^\circ$ if $\partial_I \varphi$ is trivial (resp. nef, ample) in $N^1(D_I, \mathbb{R})$.

Let $\text{Lin}_{\mathfrak{X}}$ (resp. $\text{Conv}_{\mathfrak{X}}$, $\text{sConv}_{\mathfrak{X}}$) denote the sheaf of linear (resp. convex, strictly convex) functions on $S_{\mathfrak{X}}$.

Definition 7.3. A *virtual line bundle* L on the k -analytic space X with respect to the formal model \mathfrak{X} is a torsor over the sheaf $\text{Lin}_{\mathfrak{X}}$. A *strictly convex metrization* \widehat{L} of a virtual line bundle L is a global section of the sheaf $\text{sConv}_{\mathfrak{X}} \otimes_{\text{Lin}_{\mathfrak{X}}} L$, where $\text{Lin}_{\mathfrak{X}}$ acts on $\text{sConv}_{\mathfrak{X}}$ via additions $\psi \mapsto (\varphi \mapsto \varphi + \psi)$. Given \widehat{L} , we obtain a collection of numerical classes $\partial_i \varphi_i \in N^1(D_I, \mathbb{R})$ for every $i \in I_{\mathfrak{X}}$, which we call the *curvature* of \widehat{L} , and denote by $c(\widehat{L})$.

Definition 7.4. A *Kähler structure* \widehat{L} on X consists of a choice of an snc formal model \mathfrak{X} of X , a virtual line bundle L on X with respect to \mathfrak{X} equipped with a strictly convex metrization \widehat{L} .

Now we can state the Gromov compactness theorem in non-archimedean geometry, one main result of [63]:

Theorem 7.5. *Let X be a proper smooth k -analytic space equipped with a Kähler structure \widehat{L} . For any A -graph (τ, β) , the (underived) stack $\overline{M}(X, \tau, \beta)$ of (τ, β) -marked stable maps into X is a proper k -analytic stack.*

Proof. This follows from [63, Theorem 1.3] and the remark below. \square

Remark 7.6. For any closed 1-dimensional irreducible reduced analytic subspace $C \subset X$, we define $C \cdot \widehat{L}$ to be the degree of the pullback of \widehat{L} to the normalization \widetilde{C} of C (see [63, §5]). This extends to any 1-dimensional cycle Z of X by linearity, and factors through analytic equivalence, so $\beta \cdot \widehat{L}$ is well-defined for any $\beta \in A(X)$. Indeed, for any connected smooth k -affinoid curve T , $t, t' \in T$ two rigid points, and any closed 2-dimensional irreducible reduced analytic subspace V of $X \times T$ flat over T , we take a formal model \mathfrak{T} of T and a formal model \mathfrak{V} of V flat over \mathfrak{T} . Let $t_s, t'_s \in \mathfrak{T}_s$ be the specializations of t and t' respectively. Flatness implies that $[\mathfrak{V}_{s, t_s}] \cdot c(\widehat{L}) = [\mathfrak{V}_{s, t'_s}] \cdot c(\widehat{L})$. By [63, Propositions 4.3 and 5.7], we have $[\mathfrak{V}_{s, t_s}] \cdot c(\widehat{L}) = V_t \cdot \widehat{L}$ and $[\mathfrak{V}_{s, t'_s}] \cdot c(\widehat{L}) = V_{t'} \cdot \widehat{L}$, hence we obtain $V_t \cdot \widehat{L} = V_{t'} \cdot \widehat{L}$.

Let (τ, β) be an A -graph. Without loss of generality, we assume it to be connected. Let g be the genus of τ and n the number of tails of τ . Recall β is a map $V_\tau \rightarrow A(X)$; let $\beta \cdot \widehat{L} := \sum_{v \in V_\tau} \beta(v) \cdot \widehat{L}$. Let $\overline{M}_{g,n}(X, \beta \cdot \widehat{L})$ denote the stack of n -pointed genus g k -analytic stable maps into X whose degree with respect to \widehat{L} is at most $\beta \cdot \widehat{L}$ (as in [63]). We conclude that $\overline{M}(X, \tau, \beta)$ is a closed substack of $\overline{M}_{g,n}(X, \beta \cdot \widehat{L})$.

Lemma 7.7. *Let $\mathfrak{f}: \mathfrak{X} \rightarrow \mathfrak{Y}$ be a morphism of formal Deligne-Mumford stacks locally finitely presented over k° . Assume that \mathfrak{X} is proper. Let $f: X \rightarrow Y$ denote the induced map between the generic fibers. Then the higher direct image functor f_* sends $\text{Coh}^b(X)$ to $\text{Coh}^b(Y)$.*

Proof. The question being étale local on the target, we can assume that \mathfrak{Y} is affine. Let $|\mathfrak{X}_s|$ and $|\mathfrak{X}|$ denote the coarse moduli spaces. Then the map $\mathfrak{f}: \mathfrak{X} \rightarrow \mathfrak{Y}$ factors as $\mathfrak{X} \xrightarrow{p} |\mathfrak{X}| \xrightarrow{\tilde{f}} \mathfrak{Y}$, so $f: X \rightarrow Y$ factors as $X \xrightarrow{p} |\mathfrak{X}|_\eta \xrightarrow{\tilde{f}} Y$. By [3, Proposition 3.6], étale locally over $|\mathfrak{X}_s|$ the Deligne-Mumford stack \mathfrak{X}_s is isomorphic to the quotient of an affine scheme by a finite group. Using the equivalence between the étale sites of $|\mathfrak{X}_s| \simeq |\mathfrak{X}|_s$ and of $|\mathfrak{X}|$, we deduce that étale locally over $|\mathfrak{X}|_\eta$, the k -analytic Deligne-Mumford stack $X = \mathfrak{X}_\eta$ is isomorphic to the quotient of an affinoid space by a finite group. Since the base field is assumed to have characteristic zero, we deduce that the map $p: X \rightarrow |\mathfrak{X}|_\eta$ has cohomological dimension zero. By [15, §B.2], the generic fiber $|\mathfrak{X}|_\eta$ of the formal algebraic space $|\mathfrak{X}|$ is a k -analytic space. Since \mathfrak{X} is proper, so is $|\mathfrak{X}|_\eta$. Therefore, by Kiehl's proper mapping theorem [30], the higher direct image functor \tilde{f}_* sends $\text{Coh}^b(|\mathfrak{X}|_\eta)$ to $\text{Coh}^b(Y)$. As $f_* = \tilde{f}_* \circ p_*$, the conclusion follows. \square

Proposition 7.8. *Let X be a proper smooth k -analytic space equipped with a Kähler structure \widehat{L} . Let (τ, β) be an A -graph. Let st denote the composite map*

$$\text{st}: \overline{M}(X, \tau, \beta) \longrightarrow \overline{M}_\tau^{\text{pre}} \longrightarrow \overline{M}_\tau,$$

taking domain of stable map and then stabilizing the τ -marked prestable curve. Then the higher direct image functor st_ sends $\text{Coh}^b(\overline{M}(X, \tau, \beta))$ to $\text{Coh}^b(\overline{M}_\tau)$.*

Proof. Let g be the genus of τ , n the number of tails of τ , and $A := \beta \cdot \widehat{L}$. By Remark 7.6, it suffices to prove the proposition for $\overline{M}_{g,n}(X, A)$ instead of $\overline{M}(X, \tau, \beta)$. By [63, Theorem 8.9], the moduli stack $\overline{M}_{g,n}(\mathfrak{X}, A)$ of n -pointed genus g formal stable maps into \mathfrak{X} with degree bounded by A is a formal model of the k -analytic stack $\overline{M}_{g,n}(X, A)$. Moreover, the moduli stack $\overline{\mathfrak{M}}_{g,n}$ of n -pointed genus g formal stable curves over k° is a formal model of $\overline{M}_{g,n}$. Let

$$\text{st}: \overline{M}_{g,n}(\mathfrak{X}, A) \longrightarrow \overline{\mathfrak{M}}_{g,n}$$

be the map taking domain and then stabilizing. The generic fiber of \mathfrak{st} is canonically equivalent to st . Now the statement follows from [63, Theorem 1.3] and Lemma 7.7. \square

8. QUANTUM K-INVARIANTS

In this section, we introduce the quantum K-invariants in non-archimedean analytic geometry, and prove a list of natural geometric properties, namely, mapping to a point, products, cutting edges, forgetting tails and contracting edges. They are analogous to the axioms in algebraic quantum K-theory in the work of Lee [36], and to the Behrend-Manin axioms [9] for Gromov-Witten invariants. The proof of the mapping to a point property relies on the theory of deformation to the normal bundle that we studied in Section 6. The remaining properties are consequences of the corresponding geometric relations of the derived moduli stacks established in Section 5.

Fix X a proper smooth k -analytic space. For any A -graph (τ, β) , let $\mathbb{R}\overline{M}(X, \tau, \beta)$ denote the derived stack of (τ, β) -marked stable maps into X as in Remark 4.25. It is a proper derived k -analytic stack by Theorem 7.5. Let st denote the composite map

$$\text{st}: \mathbb{R}\overline{M}(X, \tau, \beta) \longrightarrow \overline{M}_\tau^{\text{pre}} \longrightarrow \overline{M}_\tau,$$

taking domain of stable map and then stabilizing the τ -marked prestable curve.

For every tail $i \in T_\tau$, we have a section s_i of the universal curve

$$\begin{array}{ccc} \mathbb{R}\overline{C}(X, \tau, \beta) & \xrightarrow{f} & X \\ s_i \uparrow \downarrow & & \\ \mathbb{R}\overline{M}(X, \tau, \beta) & & \end{array}$$

Composing with the universal map f , we obtain the evaluation map of the i -th marked point

$$\text{ev}_i: \mathbb{R}\overline{M}(X, \tau, \beta) \longrightarrow X.$$

Definition 8.1. The *quantum K-invariants* of X are the collection of linear maps

$$\begin{aligned} I_{\tau, \beta}^X: K_0(X)^{\otimes |T_\tau|} &\longrightarrow K_0(\overline{M}_\tau) \\ \bigotimes_{i \in T_\tau} a_i &\longmapsto \text{st}_* \left(\bigotimes_{i \in T_\tau} \text{ev}_i^*(a_i) \right), \end{aligned}$$

where (τ, β) is any A -graph. Proposition 7.8 ensures that the image of st_* lies in $K_0(\overline{M}_\tau)$.

8.1. Mapping to a point.

Proposition 8.2. *Let $(\tau, 0)$ be any A -graph where β is 0. Let p_1, p_2 be the projections*

$$\begin{array}{ccc} \overline{M}_\tau \times X & \xrightarrow{p_2} & X \\ \downarrow p_1 & & \\ \overline{M}_\tau & & . \end{array}$$

For any $a_i \in K_0(X), i \in T_\tau$, we have

$$I_{\tau,0}^X(\otimes_i a_i) = p_{1*} p_2^* \left((\otimes_i a_i) \otimes \lambda_{-1} \left((\mathbb{R}^1 \pi_* \mathcal{O}_{\overline{C}_\tau} \boxtimes \mathbb{T}_X^{\text{an}})^\vee \right) \right),$$

where $\lambda_{-1}(F) := \sum_i (-1)^i \wedge^i F$.

Let us introduce some notations before the proof. Consider the universal *underived* τ -marked stable map

$$\begin{array}{ccc} \overline{C}(X, \tau, 0) & \xrightarrow{q_2} & X \\ \downarrow p & \nearrow p_2 & \\ \overline{M}(X, \tau, 0) & & . \end{array}$$

Note $\beta = 0$ implies that every stable map above is constant, so the dashed arrow p_2 uniquely exists. Combining p_2 with $\text{st}: \overline{M}(X, \tau, 0) \rightarrow \overline{M}_\tau$, we obtain an isomorphism

$$\overline{M}(X, \tau, 0) \xrightarrow{\sim} \overline{M}_\tau \times X.$$

Similarly, we have $\overline{C}(X, \tau, 0) \simeq \overline{C}_\tau \times X$. Now we consider their derived enhancements:

$$(8.3) \quad \begin{array}{ccccccc} \overline{C}_\tau & \xleftarrow{q_1} & \overline{C}_\tau \times X & \xleftarrow{i} & \mathbb{R}\overline{C}(X, \tau, 0) & \xrightarrow{\quad} & X \\ \downarrow \pi & & \downarrow p & & \downarrow p_2 & & \downarrow \\ \overline{M}_\tau & \xleftarrow{p_1} & \overline{M}_\tau \times X & \xleftarrow{j} & \mathbb{R}\overline{M}(X, \tau, 0) & & . \end{array}$$

Note that contrary to the underived situation, in general, there is no map from $\mathbb{R}\overline{M}(X, \tau, 0)$ to X making the diagram commutative.

Lemma 8.4. *Let $j: \overline{M}(X, \tau, 0) \hookrightarrow \mathbb{R}\overline{M}(X, \tau, 0)$ be the canonical inclusion of the truncation. There is a canonical equivalence*

$$\mathbb{L}_j^{\text{an}}[-1] \simeq \left(\mathbb{R}^1 \pi_* \mathcal{O}_{\overline{C}_\tau} \boxtimes \mathbb{T}_X^{\text{an}} \right)^\vee[1].$$

Proof. Consider the fiber sequence

$$\mathbb{T}_j^{\text{an}} \longrightarrow \mathbb{T}_{\overline{M}(X, \tau, 0)}^{\text{an}} \longrightarrow j^* \mathbb{T}_{\mathbb{R}\overline{M}(X, \tau, 0)}^{\text{an}}.$$

As in Theorem 4.15, we have

$$j^* \mathbb{T}_{\mathbb{R}\overline{M}(X, \tau, 0)}^{\text{an}} \simeq p_* \left(\text{cofib} \left(\mathbb{T}_{\overline{C}(X, \tau, 0)/\overline{M}(X, \tau, 0)}^{\text{an}}(-\sum s_i) \rightarrow q_2^* \mathbb{T}_{X/S}^{\text{an}} \right) \right).$$

Since $\overline{C}(X, \tau, 0) \simeq \overline{C}_\tau \times X$ and $\overline{M}(X, \tau, 0) \simeq \overline{M}_\tau \times X$, we obtain

$$p_* \left(\mathbb{T}_{\overline{C}(X, \tau, 0)/\overline{M}(X, \tau, 0)}^{\text{an}}(-\sum s_i) \right) \simeq p_1^* \pi_* \mathbb{T}_{\overline{C}_\tau/\overline{M}_\tau}^{\text{an}}(-\sum s_i).$$

Moreover, the canonical map

$$p_1^* \pi_* \mathbb{T}_{\overline{C}_\tau/\overline{M}_\tau}^{\text{an}}(-\sum s_i) \longrightarrow p_* q_2^* \mathbb{T}_X^{\text{an}}$$

is nullhomotopic, and therefore

$$j^* \mathbb{T}_{\mathbb{R}\overline{M}(X, \tau, 0)}^{\text{an}} \simeq p_1^* \pi_* \mathbb{T}_{\overline{C}_\tau/\overline{M}_\tau}^{\text{an}}[1] \oplus p_* q_2^* \mathbb{T}_X^{\text{an}}.$$

On the other hand, using the equivalence $\overline{M}(X, \tau, 0) \simeq \overline{M}_\tau \times X$, we find:

$$\mathbb{T}_{\overline{M}(X, \tau, 0)}^{\text{an}} \simeq p_1^* \mathbb{T}_{\overline{M}_\tau}^{\text{an}} \oplus p_2^* \mathbb{T}_X^{\text{an}} \simeq p_1^* \pi_* \mathbb{T}_{\overline{C}_\tau/\overline{M}_\tau}^{\text{an}}[1] \oplus p_2^* \mathbb{T}_X^{\text{an}}.$$

Using the projection formula, we obtain:

$$\begin{aligned} \mathbb{T}_j^{\text{an}} &\simeq \text{fib} \left(p_2^* \mathbb{T}_X^{\text{an}} \rightarrow p_* q_2^* \mathbb{T}_X^{\text{an}} \right) \\ &\simeq \text{fib} \left(p_2^* \mathbb{T}_X^{\text{an}} \rightarrow p_1^* \pi_* \mathcal{O}_{\overline{C}_\tau} \otimes p_2^* \mathbb{T}_X^{\text{an}} \right) \\ &\simeq \text{fib} \left(p_2^* \mathbb{T}_X^{\text{an}} \rightarrow \pi_* \mathcal{O}_{\overline{C}_\tau} \boxtimes \mathbb{T}_X^{\text{an}} \right). \end{aligned}$$

The conclusion follows. \square

Proof of Proposition 8.2. Notation as in diagram (8.3), we observe that

$$\begin{array}{ccc} \overline{M}_\tau \times X & \xhookrightarrow{j} & \mathbb{R}\overline{M}(X, \tau, 0) \\ p_1 \downarrow & \swarrow \text{st} & \\ \overline{M}_\tau & & \end{array}$$

commutes. Let $\mathcal{F}_i \in \text{Perf}(X)$, $i \in T_\tau$ be representatives for the classes $a_i \in K_0(X)$. Lemma 8.4 implies the following equality in $K_0(\overline{M}_\tau \times X)$:

$$\lambda_{-1} \left((\mathbb{R}^1 \pi_* \mathcal{O}_{\overline{C}_\tau} \boxtimes \mathbb{T}_X^{\text{an}})^\vee \right) = \left[\text{Sym}_{\mathcal{O}_{\overline{M}_\tau \times X}} \left(\mathbb{L}_j^{\text{an}}[-1] \right) \right].$$

It is therefore enough to check that

$$\left[p_{1*} \left(p_2^* \left(\bigotimes_i \mathcal{F}_i \right) \otimes \text{Sym}_{\mathcal{O}_{\overline{M}_\tau \times X}} \left(\mathbb{L}_j^{\text{an}}[-1] \right) \right) \right] = \left[\text{st}_* \left(\bigotimes_i \text{ev}_i^* \mathcal{F}_i \right) \right]$$

in $K_0(\overline{M}_\tau)$. Let $\mathfrak{M}(X, \tau, 0)$ be the \mathbf{P}_k^1 -deformation to the normal bundle associated to $j: \overline{M}_\tau \times X \rightarrow \mathbb{R}\overline{M}(X, \tau, 0)$ we constructed in Section 6.4. The functoriality

of this construction allows to deform also the maps $\text{ev}_i: \mathbb{R}\overline{M}(X, \tau, 0) \rightarrow X$ and $\text{st}: \mathbb{R}\overline{M}(X, \tau, 0) \rightarrow \overline{M}_\tau$. Since the targets of these maps are underived, their associated deformations are constant. Thus we obtain maps

$$\widetilde{\text{ev}}_i: \mathfrak{M}(X, \tau, 0) \longrightarrow X \times \mathbf{P}_k^1, \quad \widetilde{\text{st}}: \mathfrak{M}(X, \tau, 0) \longrightarrow \overline{M}_\tau \times \mathbf{P}_k^1.$$

Proposition 6.13(3) implies that the fibers $\widetilde{\text{ev}}_i|_1$ and $\widetilde{\text{st}}|_1$ at $1 \in \mathbf{P}_k^1$ of $\widetilde{\text{ev}}_i$ and $\widetilde{\text{st}}$ coincide with the maps ev_i and st we started from. On the other hand, Proposition 6.13(4) shows that the fibers $\widetilde{\text{ev}}_i|_0$ and $\widetilde{\text{st}}|_0$ at $0 \in \mathbf{P}_k^1$ of $\widetilde{\text{ev}}_i$ and $\widetilde{\text{st}}$ fit in the following commutative diagram:

$$(8.5) \quad \begin{array}{ccc} & \text{Spec}_{\overline{M}_\tau \times X}(\text{Sym}_{\mathcal{O}_{\overline{M}_\tau \times X}}(\mathbb{L}_j^{\text{an}}[-1])) & \\ & \swarrow \widetilde{\text{st}}|_0 & \searrow \widetilde{\text{ev}}_i|_0 \\ \overline{M}_\tau & \xleftarrow{p_1} \overline{M}_\tau \times X \xrightarrow{p_2} & X, \end{array}$$

where the vertical map is the natural projection. Let $q: X \times \mathbf{P}_k^1 \rightarrow \mathbf{P}_k^1$ be the projection and consider the perfect complex

$$\mathcal{G} := \widetilde{\text{st}}_* \left(\bigotimes_i \widetilde{\text{ev}}_i^*(q^*(\mathcal{F}_i)) \right) \in \text{Perf}(\overline{M}_\tau \times \mathbf{P}_k^1).$$

Since $\overline{M}_\tau \times \mathbf{P}_k^1$ is algebraic and proper, the GAGA theorem ([49, Theorem 1.3]) implies that \mathcal{G} is an algebraic perfect complex. In particular, its restriction to $\overline{M}_\tau \times \mathbf{A}_k^1$ is also algebraic; and therefore the \mathbb{A}^1 -invariance for K -theory (see [60, Theorem 6.13]) implies that

$$[\mathcal{G}|_{\overline{M}_\tau \times 0}] = [\mathcal{G}|_{\overline{M}_\tau \times 1}]$$

in $K_0(\overline{M}_\tau)$. By proper base change ([51, Theorem 1.5]), we have

$$\mathcal{G}|_{\overline{M}_\tau \times 1} \simeq \text{st}_* \left(\bigotimes_i \text{ev}_i^*(\mathcal{F}_i) \right).$$

On the other hand, diagram (8.5) together with proper base change implies that

$$\begin{aligned} \mathcal{G}|_{\overline{M}_\tau \times 0} &\simeq p_{1*} r_* r^* p_2^* \left(\bigotimes_i \mathcal{F}_i \right) \\ &\simeq p_{1*} \left(p_2^* \left(\bigotimes_i \mathcal{F}_i \right) \otimes \text{Sym}_{\mathcal{O}_{\overline{M}_\tau \times X}}(\mathbb{L}_j^{\text{an}}[-1]) \right). \end{aligned}$$

Combining the isomorphisms and the equality above, we achieve the proof. \square

8.2. Products.

Proposition 8.6. *Let (τ_1, β_1) and (τ_2, β_2) be two A -graphs. Let $(\tau_1 \sqcup \tau_2, \beta_1 \sqcup \beta_2)$ be their disjoint union. For any $a_i \in K_0(X), i \in T_{\tau_1}$ and $b_j \in K_0(X), j \in T_{\tau_2}$, we have*

$$I_{\tau_1 \sqcup \tau_2, \beta_1 \sqcup \beta_2}^X \left(\left(\bigotimes_i a_i \right) \otimes \left(\bigotimes_j b_j \right) \right) = I_{\tau_1, \beta_1}^X \left(\bigotimes_i a_i \right) \boxtimes I_{\tau_2, \beta_2}^X \left(\bigotimes_j b_j \right).$$

Proof. For $i \in T_{\tau_1}$, $j \in T_{\tau_2}$, we denote the evaluation maps by

$$\begin{aligned} \text{ev}_i^1 &: \mathbb{R}\overline{M}(X, \tau_1, \beta_1) \longrightarrow X, \\ \text{ev}_j^2 &: \mathbb{R}\overline{M}(X, \tau_2, \beta_2) \longrightarrow X, \\ \text{ev}_i^{12}, \text{ev}_j^{12} &: \mathbb{R}\overline{M}(X, \tau_1 \sqcup \tau_2, \beta_1 \sqcup \beta_2) \longrightarrow X. \end{aligned}$$

By Theorem 5.1, we have the following commutative diagram

$$\begin{array}{ccc} \mathbb{R}\overline{M}(X, \tau_1, \beta_1) \times \mathbb{R}\overline{M}(X, \tau_2, \beta_2) & \xrightarrow{\sim} & \mathbb{R}\overline{M}(X, \tau_1 \sqcup \tau_2, \beta_1 \sqcup \beta_2) \xrightarrow{\text{ev}_i^{12}, \text{ev}_j^{12}} X \\ \downarrow \text{st}_1 \times \text{st}_2 & & \downarrow \text{st}_{12} \\ \overline{M}_{\tau_1} \times \overline{M}_{\tau_2} & \xrightarrow{\sim} & \overline{M}_{\tau_1 \sqcup \tau_2}. \end{array}$$

Let

$$\begin{aligned} p_1 &: \mathbb{R}\overline{M}(X, \tau_1 \sqcup \tau_2, \beta_1 \sqcup \beta_2) \longrightarrow \mathbb{R}\overline{M}(X, \tau_1, \beta_1), \\ p_2 &: \mathbb{R}\overline{M}(X, \tau_1 \sqcup \tau_2, \beta_1 \sqcup \beta_2) \longrightarrow \mathbb{R}\overline{M}(X, \tau_2, \beta_2) \end{aligned}$$

denote the projection maps. For any $i \in T_{\tau_1}$, $j \in T_{\tau_2}$, we have $\text{ev}_i^{12} = \text{ev}_i^1 \circ p_1$, $\text{ev}_j^{12} = \text{ev}_j^2 \circ p_2$. Hence we obtain

$$\begin{aligned} I_{\tau_1 \sqcup \tau_2, \beta_1 \sqcup \beta_2}^X \left((\otimes_i a_i) \otimes (\otimes_j b_j) \right) &= \text{st}_{12*} \left((\otimes_i \text{ev}_i^{12*} a_i) \otimes (\otimes_j \text{ev}_j^{12*} b_j) \right) \\ &= (\text{st}_1 \times \text{st}_2)_* \left((\otimes_i \text{ev}_i^{1*} a_i) \boxtimes (\otimes_j \text{ev}_j^{2*} b_j) \right) \\ &= \text{st}_{1*} \left(\otimes_i \text{ev}_i^{1*} a_i \right) \boxtimes \text{st}_{2*} \left(\otimes_j \text{ev}_j^{2*} b_j \right) \\ &= I_{\tau_1, \beta_1}^X \left(\otimes_i a_i \right) \boxtimes I_{\tau_2, \beta_2}^X \left(\otimes_j b_j \right), \end{aligned}$$

completing the proof. \square

8.3. Cutting edges.

Proposition 8.7. *Let (σ, β) be an A -graph obtained from (τ, β) by cutting an edge e of τ . Let v, w be the two tails of σ created by the cut. Consider the following commutative diagram*

$$\begin{array}{ccc} \overline{M}_{\tau} & \xrightarrow{d} & \overline{M}_{\sigma} \\ \text{st}_{\tau} \uparrow & & \text{st}_{\sigma} \uparrow \\ \mathbb{R}\overline{M}(X, \tau, \beta) & \xrightarrow{c} & \mathbb{R}\overline{M}(X, \sigma, \beta) \\ \downarrow \text{ev}_e & & \downarrow \text{ev}_v \times \text{ev}_w \\ X & \xrightarrow{\Delta} & X \times_S X. \end{array}$$

For any $a_i \in K_0(X), i \in T_\tau$, we have

$$d_* I_{\tau, \beta}^X \left(\bigotimes_i a_i \right) = I_{\sigma, \beta}^X \left(\left(\bigotimes_i a_i \right) \otimes \Delta_* \mathcal{O}_X \right).$$

Proof. For each $i \in T_\tau$, we denote the evaluation maps by

$$\text{ev}_i^\tau: \mathbb{R}\overline{M}(X, \tau, \beta) \longrightarrow X,$$

$$\text{ev}_i^\sigma: \mathbb{R}\overline{M}(X, \sigma, \beta) \longrightarrow X.$$

We have $\text{ev}_i^\tau = \text{ev}_i^\sigma \circ c$. By Theorem 5.2, the lower square of the commutative diagram in the statement is a derived pullback. Hence we obtain

$$\begin{aligned} d_* I_{\tau, \beta}^X \left(\bigotimes_i a_i \right) &= d_* \text{st}_{\tau*} \left(\bigotimes_i \text{ev}_i^{\tau*} a_i \right) \\ &= d_* \text{st}_{\tau*} \left(\left(\bigotimes_i \text{ev}_i^{\tau*} a_i \right) \otimes \text{ev}_e^* \mathcal{O}_X \right) \\ &= \text{st}_{\sigma*} c_* \left(c^* \left(\bigotimes_i \text{ev}_i^{\sigma*} a_i \right) \otimes \text{ev}_e^* \mathcal{O}_X \right) \\ &= \text{st}_{\sigma*} \left(\left(\bigotimes_i \text{ev}_i^{\sigma*} a_i \right) \otimes c_* \text{ev}_e^* \mathcal{O}_X \right) \\ &= \text{st}_{\sigma*} \left(\left(\bigotimes_i \text{ev}_i^{\sigma*} a_i \right) \otimes (\text{ev}_v \times \text{ev}_w)^* \Delta_* \mathcal{O}_X \right) \\ &= I_{\sigma, \beta}^X \left(\left(\bigotimes_i a_i \right) \otimes \Delta_* \mathcal{O}_X \right), \end{aligned}$$

where the fourth equality follows from the projection formula [51, Theorem 1.4], and the fifth equality follows from proper base change [51, Theorem 1.5]. \square

8.4. Forgetting tails.

Proposition 8.8. *Let (σ, β) be an A -graph obtained from (τ, β) by forgetting a tail. Let $\Phi: \overline{M}_\tau \rightarrow \overline{M}_\sigma$ be the forgetful map from τ -marked stable curves to σ -marked stable curves. For any $a_i \in K_0(X), i \in T_\sigma$, we have*

$$I_{\tau, \beta}^X \left(\left(\bigotimes_i a_i \right) \otimes \mathcal{O}_X \right) = \Phi^* I_{\sigma, \beta}^X \left(\bigotimes_i a_i \right).$$

Proof. Consider the commutative diagram as in the proof of Theorem 5.7:

$$\begin{array}{ccccc} & & \xrightarrow{\pi} & & \\ \mathbb{R}\overline{M}(X, \tau, \beta) & \xrightarrow{\Psi} & \overline{M}_\tau \times_{\overline{M}_\sigma} \mathbb{R}\overline{M}(X, \sigma, \beta) & \xrightarrow{\eta} & \mathbb{R}\overline{M}(X, \sigma, \beta) \\ \downarrow \lambda & \searrow & \downarrow \mu & & \downarrow \kappa \\ \overline{C}_w^{\text{pre}} & \xrightarrow{\rho} & \overline{M}_\tau \times_{\overline{M}_\sigma} \overline{M}_\sigma^{\text{pre}} & \xrightarrow{\quad} & \overline{M}_\sigma^{\text{pre}} \\ & \searrow \text{st}_\tau & \downarrow \xi & & \downarrow \theta \\ & & \overline{M}_\tau & \xrightarrow{\Phi} & \overline{M}_\sigma \end{array}$$

For each $i \in T_\sigma$, we denote the evaluation maps by

$$\begin{aligned} \text{ev}_i^\tau: \mathbb{R}\overline{M}(X, \tau, \beta) &\longrightarrow X, \\ \text{ev}_i^\sigma: \mathbb{R}\overline{M}(X, \sigma, \beta) &\longrightarrow X. \end{aligned}$$

We have $\text{ev}_i^\tau = \text{ev}_i^\sigma \circ \pi$. By Theorem 5.7, all the rectangles in the commutative diagram above are pullbacks; moreover, we have an equivalence

$$(8.9) \quad \mathcal{O}_{\overline{M}_\tau \times_{\overline{M}_\sigma} \mathbb{R}\overline{M}(X, \sigma, \beta)}^{\text{alg}} \xrightarrow{\sim} \Psi_* \left(\mathcal{O}_{\mathbb{R}\overline{M}(X, \tau, \beta)}^{\text{alg}} \right).$$

Let $t \in T_\tau$ denote the forgotten tail. We obtain

$$\begin{aligned} I_{\tau, \beta}^X \left(\left(\otimes_i a_i \right) \otimes \mathcal{O}_X \right) &= \text{st}_{\tau*} \left(\left(\otimes_i \text{ev}_i^{\tau*} a_i \right) \otimes \text{ev}_t^* \mathcal{O}_X \right) \\ &= \xi_* \mu_* \Psi_* \left(\Psi^* \eta^* \left(\otimes_i \text{ev}_i^{\sigma*} a_i \right) \otimes \mathcal{O}_{\mathbb{R}\overline{M}(X, \tau, \beta)}^{\text{alg}} \right) \\ &= \xi_* \mu_* \left(\eta^* \left(\otimes_i \text{ev}_i^{\sigma*} a_i \right) \otimes \Psi_* \left(\mathcal{O}_{\mathbb{R}\overline{M}(X, \tau, \beta)}^{\text{alg}} \right) \right) \\ &= \xi_* \mu_* \eta^* \left(\otimes_i \text{ev}_i^{\sigma*} a_i \right) \\ &= \Phi^* \theta_* \kappa_* \left(\otimes_i \text{ev}_i^{\sigma*} a_i \right) \\ &= \Phi^* \text{st}_{\sigma*} \left(\otimes_i \text{ev}_i^{\sigma*} a_i \right) \\ &= \Phi^* I_{\sigma, \beta}^X \left(\otimes_i a_i \right), \end{aligned}$$

where the third equality follows from the projection formula [51, Theorem 1.4], the fourth equality follows from (8.9), and the fifth equality follows from proper base change [51, Theorem 1.5]. \square

Remark 8.10. This property is called “fundamental class” in [36, §3.5, §4.3] because the name originates from [32, §2.2.3]. We chose to call it “forgetting tails” because it corresponds to the operation of forgetting a tail.

8.5. Contracting edges.

Proposition 8.11. *Let (σ, β) be an A -graph obtained from (τ, β) by contracting an edge (possibly a loop) e . We follow the notations of Section 5.5. Let $\Phi: \overline{M}_\tau \rightarrow \overline{M}_\sigma$, $\Psi_i^i: \overline{M}_{\tau_i^i} \rightarrow \overline{M}_\tau$ and $\Omega: \overline{M}_{\tilde{\sigma}} \rightarrow \overline{M}_\sigma$ be the induced maps on the moduli stacks of stable curves. For any $a_v \in K_0(X)$, $v \in T_\sigma^-$, we have*

$$\Phi^* \Omega_* I_{\sigma, \beta}^X \left(\otimes_v a_v \right) = \sum_l (-1)^{l+1} \sum_{i, j} \Psi_{l*}^i I_{\tau_i^i, \beta_l^{i, j}}^X \left(\otimes_v a_v \right).$$

Proof. For every l, i, j consider the following commutative diagram

$$\begin{array}{ccccc}
& & & & \xrightarrow{q_l^{ij}} \\
& & & & \searrow \\
\mathbb{R}\overline{M}(X, \tau_l^i, \beta_l^{ij}) & \xrightarrow{p_l^{ij}} & \mathbb{R}\overline{M}(X, \tilde{\sigma}, \beta) \times_{\overline{M}_\sigma} \overline{M}_\tau & \xrightarrow{\lambda} & \mathbb{R}\overline{M}(X, \tilde{\sigma}, \beta) \\
\downarrow \text{st}_l^{ij} & & \downarrow \mu & & \downarrow \text{st}_{\tilde{\sigma}} \\
\overline{M}_{\tau_l^i} & \xrightarrow{\Psi_l^i} & \overline{M}_\tau & \xrightarrow{\Phi} & \overline{M}_\sigma \\
& & & & \downarrow \Omega
\end{array}$$

For each $v \in T_{\tilde{\sigma}}$, we denote by v again the corresponding tail of τ_l^i , and we denote the evaluation maps by

$$\begin{aligned}
\text{ev}_v^{l,ij} &: \mathbb{R}\overline{M}(X, \tau_l^i, \beta_l^{ij}) \longrightarrow X, \\
\text{ev}_v^{\tilde{\sigma}} &: \mathbb{R}\overline{M}(X, \tilde{\sigma}, \beta) \longrightarrow X.
\end{aligned}$$

We have $\text{ev}_v^{l,ij} = \text{ev}_v^{\tilde{\sigma}} \circ q_l^{ij}$. By Theorem 5.19, we have

$$(8.12) \quad \sum_l (-1)^{l+1} \sum_{i,j} p_{l*}^{ij} \left[\mathcal{O}_{\mathbb{R}\overline{M}(X, \tau_l^i, \beta_l^{ij})} \right] = \left[\mathcal{O}_{\overline{M}_\tau \times_{\overline{M}_\sigma} \mathbb{R}\overline{M}(X, \tilde{\sigma}, \beta)} \right]$$

in $K_0(\text{Coh}^b(\overline{M}_\tau \times_{\overline{M}_\sigma} \mathbb{R}\overline{M}(X, \tilde{\sigma}, \beta)))$. We obtain

$$\begin{aligned}
\Phi^* \Omega_* I_{\tilde{\sigma}, \beta}^X (\otimes_v a_v) &= \Phi^* \Omega_* \text{st}_{\tilde{\sigma}*} (\otimes_v \text{ev}_v^{\tilde{\sigma}*} a_v) \\
&= \mu_* \lambda^* (\otimes_v \text{ev}_v^{\tilde{\sigma}*} a_v) \\
&= \mu_* \left(\lambda^* (\otimes_v \text{ev}_v^{\tilde{\sigma}*} a_v) \otimes \sum_l (-1)^{l+1} \sum_{i,j} p_{l*}^{ij} \left[\mathcal{O}_{\mathbb{R}\overline{M}(X, \tau_l^i, \beta_l^{ij})} \right] \right) \\
&= \sum_l (-1)^{l+1} \sum_{i,j} \mu_* \left(\lambda^* (\otimes_v \text{ev}_v^{\tilde{\sigma}*} a_v) \otimes p_{l*}^{ij} \left[\mathcal{O}_{\mathbb{R}\overline{M}(X, \tau_l^i, \beta_l^{ij})} \right] \right) \\
&= \sum_l (-1)^{l+1} \sum_{i,j} \mu_* p_{l*}^{ij} \left(p_l^{ij*} \lambda^* (\otimes_v \text{ev}_v^{\tilde{\sigma}*} a_v) \otimes \left[\mathcal{O}_{\mathbb{R}\overline{M}(X, \tau_l^i, \beta_l^{ij})} \right] \right) \\
&= \sum_l (-1)^{l+1} \sum_{i,j} \Psi_{l*}^i \text{st}_{l*}^{ij} q_l^{ij*} (\otimes_v \text{ev}_v^{\tilde{\sigma}*} a_v) \\
&= \sum_l (-1)^{l+1} \sum_{i,j} \Psi_{l*}^i \text{st}_{l*}^{ij} (\otimes_v \text{ev}_v^{l,ij*} a_v) \\
&= \sum_l (-1)^{l+1} \sum_{i,j} \Psi_{l*}^i I_{\tau_l^i, \beta_l^{ij}}^X (\otimes_v a_v),
\end{aligned}$$

where the second equality follows from proper base change [51, Theorem 1.5], the third equality follows from (8.12), and the fifth equality follows from the projection formula [51, Theorem 1.4]. \square

9. QUANTUM K-INVARIANTS WITH MULTIPLICITIES

The flexibility of our derived approach to quantum K-invariants allows us to impose not only simple incidence conditions for marked points as in the last section, but also incidence conditions with multiplicities. This leads to a new set of enumerative invariants, which is not yet considered in the literature, even in algebraic geometry, to the best of our knowledge.

Fix a proper smooth k -analytic space X and an A -graph (τ, β) . Let $i \in T_\tau$ be a tail vertex and m_i a positive integer. Let $s_i: \overline{M}_\tau^{\text{pre}} \rightarrow \overline{C}_\tau^{\text{pre}}$ be the i -th section of the universal curve, \mathcal{J}_i the ideal sheaf on $\overline{C}_\tau^{\text{pre}}$ cutting out the image of s_i , $(\overline{C}_\tau^{\text{pre}})_{(s_i^{m_i})}$ the closed substack given by the m_i -th power of the ideal \mathcal{J}_i , and $\mathbb{R}\overline{C}(X, \tau, \beta)_{(s_i^{m_i})}$ the pullback of $(\overline{C}_\tau^{\text{pre}})_{(s_i^{m_i})}$ along the map $\mathbb{R}\overline{M}(X, \tau, \beta) \rightarrow \overline{M}_\tau^{\text{pre}}$ taking domains. Let $X_i^{m_i}(\tau, \beta)$ denote the mapping stack

$$\mathbf{Map}_{\mathbb{R}\overline{M}(X, \tau, \beta)}\left(\mathbb{R}\overline{C}(X, \tau, \beta)_{(s_i^{m_i})}, X \times \mathbb{R}\overline{M}(X, \tau, \beta)\right).$$

Let $\text{ev}_i^{m_i}$ denote the composition of the natural map

$$\mathbb{R}\overline{M}(X, \tau, \beta) \longrightarrow \mathbf{Map}_{\mathbb{R}\overline{M}(X, \tau, \beta)}\left(\mathbb{R}\overline{C}(X, \tau, \beta), X \times \mathbb{R}\overline{M}(X, \tau, \beta)\right)$$

and the restriction map

$$\mathbf{Map}_{\mathbb{R}\overline{M}(X, \tau, \beta)}\left(\mathbb{R}\overline{C}(X, \tau, \beta), X \times \mathbb{R}\overline{M}(X, \tau, \beta)\right) \longrightarrow X_i^{m_i}(\tau, \beta).$$

We call

$$\text{ev}_i^{m_i}: \mathbb{R}\overline{M}(X, \tau, \beta) \longrightarrow X_i^{m_i}(\tau, \beta)$$

the evaluation map of the i -th marked point of order m_i .

Note that when $m_i = 1$, we have

$$(\overline{C}_\tau^{\text{pre}})_{(s_i^{m_i})} \simeq \overline{M}_\tau^{\text{pre}}, \quad \mathbb{R}\overline{C}(X, \tau, \beta)_{(s_i^{m_i})} \simeq \mathbb{R}\overline{M}(X, \tau, \beta)$$

and

$$X_i^{m_i}(\tau, \beta) \simeq X \times \mathbb{R}\overline{M}(X, \tau, \beta).$$

So we have $\text{ev}_i^1 \simeq \text{ev}_i \times \text{id}$.

Given any closed analytic subspace $Z_i \subset X$ such that the inclusion is lci, let

$$Z_i^{m_i}(\tau, \beta) := \mathbf{Map}_{\mathbb{R}\overline{M}(X, \tau, \beta)}\left(\mathbb{R}\overline{C}(X, \tau, \beta)_{(s_i^{m_i})}, Z_i \times \mathbb{R}\overline{M}(X, \tau, \beta)\right).$$

Lemma 9.1. *The diagram*

$$\begin{array}{ccc} t_0 Z_i^{m_i}(\tau, \beta) & \xrightarrow{t_0(\iota)} & t_0 X_i^{m_i}(\tau, \beta) \\ \downarrow & & \downarrow \\ Z_i^{m_i}(\tau, \beta) & \xrightarrow{\iota} & X_i^{m_i}(\tau, \beta) \end{array}$$

is a pullback square. Moreover, the morphisms $\iota: Z_i^{m_i}(\tau, \beta) \hookrightarrow X_i^{m_i}(\tau, \beta)$ and its truncation $t_0(\iota)$ are derived lci. Therefore $\iota_* \mathcal{O}_{Z_i^{m_i}(\tau, \beta)}$ is a perfect complex on $X_i^{m_i}(\tau, \beta)$.

Proof. Consider the diagram

$$\begin{array}{ccccc} t_0 Z_i^{m_i}(\tau, \beta) & \longrightarrow & t_0 X_i^{m_i}(\tau, \beta) & \longrightarrow & \overline{M}(X, \tau, \beta) \\ \downarrow & & \downarrow & & \downarrow \\ Z_i^{m_i}(\tau, \beta) & \longrightarrow & X_i^{m_i}(\tau, \beta) & \longrightarrow & \mathbb{R}\overline{M}(X, \tau, \beta). \end{array}$$

The maps

$$X_i^{m_i}(\tau, \beta) \longrightarrow \mathbb{R}\overline{M}(X, \tau, \beta)$$

and

$$Z_i^{m_i}(\tau, \beta) \longrightarrow \mathbb{R}\overline{M}(X, \tau, \beta)$$

are flat. This implies that the right and the outer squares in the above diagram are pullbacks. Therefore, so is the left square.

Let us now prove that the relative analytic tangent complex $\mathbb{T}_\iota^{\text{an}}$ is perfect and in tor-amplitude $[0, -1]$. Let $j: Z_i \times \mathbb{R}\overline{M}(X, \tau, \beta) \rightarrow X \times \mathbb{R}\overline{M}(X, \tau, \beta)$ be the closed immersion induced by $Z_i \subset X$. Consider the diagram

$$\begin{array}{ccc} \mathbb{R}\overline{C}(X, \tau, \beta)_{(s_i^{m_i})} \times_{\mathbb{R}\overline{M}(X, \tau, \beta)} Z_i^{m_i}(\tau, \beta) & \xrightarrow{e} & Z_i \times \mathbb{R}\overline{M}(X, \tau, \beta) \\ \downarrow q & & \\ Z_i^{m_i}(\tau, \beta) & & \end{array}$$

where e is the universal evaluation map and q is the projection. The construction in the proof of [51, Lemma 8.4] shows that

$$\mathbb{T}_\iota^{\text{an}} \simeq q_* e^* \mathbb{T}_j^{\text{an}}.$$

Since $Z_i \rightarrow X$ is lci, \mathbb{T}_j^{an} has tor-amplitude $[0, -1]$, and thus the same goes for $e^* \mathbb{T}_j^{\text{an}}$. The canonical map $(\overline{C}_\tau^{\text{pre}})_{(s_i^{m_i})} \rightarrow \overline{M}_\tau^{\text{pre}}$ is flat, so Corollary 10.4 implies that the map q is also flat. Moreover observe that q has relative dimension 0. These two properties of q imply that the functor q_* has cohomological dimension 0 and preserves the tor-amplitude. We conclude that $\mathbb{T}_\iota^{\text{an}}$ has tor-amplitude $[0, -1]$, and

therefore ι is derived lci. Combining Corollary 2.9 and [51, Proposition 7.8], we deduce that $\iota_*\mathcal{O}_{Z_i^{m_i}(\tau,\beta)}$ is a perfect complex on $X_i^{m_i}(\tau,\beta)$. \square

Definition 9.2. Given any A-graph (τ, β) , $\mathbf{m}_i = (m_i)_{i \in T_\tau}$ with $m_i \in \mathbb{N}_{>0}$, and $\mathbf{Z} = (Z_i)_{i \in T_\tau}$ with $Z_i \subset X$ closed lci, we define the associated quantum K-invariants of X with multiplicities

$$Q_{\tau,\beta,\mathbf{m}}^X(\mathbf{Z}) := \text{st}_* \left(\bigotimes_{i \in T_\tau} (\text{ev}_i^{m_i})^* \mathcal{O}_{Z_i^{m_i}(\tau,\beta)} \right) \in K_0(\overline{M}_\tau),$$

where the pullback $(\text{ev}_i^{m_i})^*$ is well-defined by Lemma 9.1, and Proposition 7.8 ensures that the image of st_* lies in $\text{Coh}^b(\overline{M}_\tau)$, thus gives an element in $K_0(\overline{M}_\tau)$.

The quantum K-invariants with multiplicities satisfy the following list of natural geometric properties exactly parallel to the previous section. As the proofs are also parallel to the previous section, we do not repeat them here.

9.1. Mapping to a point.

Proposition 9.3. *Let $(\tau, 0)$ be any A-graph where β is 0. Let p_1, p_2 be projections*

$$\begin{array}{ccc} \overline{M}_\tau \times X & \xrightarrow{p_2} & X \\ \downarrow p_1 & & \\ \overline{M}_\tau & & . \end{array}$$

Given any $\mathbf{m} = (m_i)_{i \in T_\tau}$ and $\mathbf{Z} = (Z_i)_{i \in T_\tau}$ as in Definition 9.2, for each $i \in T_\tau$, consider the truncations

$$t_0(\text{ev}_i^{m_i}): t_0\mathbb{R}\overline{M}(X, \tau, 0) \simeq \overline{M}_\tau \times X \longrightarrow t_0X_i^{m_i}(\tau, 0),$$

and

$$t_0(\iota): t_0Z_i^{m_i}(\tau, 0) \hookrightarrow t_0X_i^{m_i}(\tau, 0).$$

We have

$$Q_{\tau,\beta,\mathbf{m}}^X(\mathbf{Z}) = p_{1*} \left(\left(\bigotimes_{i \in T_\tau} t_0(\text{ev}_i^{m_i})^* \mathcal{O}_{t_0Z_i^{m_i}(\tau,0)} \right) \otimes p_2^* \lambda_{-1} \left((R^1\pi_* \mathcal{O}_{\overline{C}_\tau} \boxtimes \mathbb{T}_X^{\text{an}})^\vee \right) \right),$$

where $\lambda_{-1}(F) := \sum_i (-1)^i \wedge^i F$.

9.2. Products.

Proposition 9.4. *Let (τ_1, β_1) and (τ_2, β_2) be two A-graphs. Let $(\tau_1 \sqcup \tau_2, \beta_1 \sqcup \beta_2)$ be their disjoint union. Given any $\mathbf{m} = (m_i)_{i \in T_{\tau_1}}$, $\mathbf{Z} = (Z_i)_{i \in T_{\tau_1}}$, $\mathbf{n} = (n_j)_{j \in T_{\tau_2}}$, $\mathbf{W} = (W_j)_{j \in T_{\tau_2}}$ as in Definition 9.2, we have*

$$Q_{\tau_1 \sqcup \tau_2, \beta_1 \sqcup \beta_2, \mathbf{m} \sqcup \mathbf{n}}^X(\mathbf{Z} \sqcup \mathbf{W}) = Q_{\tau_1, \beta_1, \mathbf{m}}^X(\mathbf{Z}) \boxtimes Q_{\tau_2, \beta_2, \mathbf{n}}^X(\mathbf{W}).$$

9.3. Cutting edges.

Proposition 9.5. *Let (σ, β) be an A -graph obtained from (τ, β) by cutting an edge e of τ . Let v, w be the two tails of σ created by the cut. Consider the following commutative diagram*

$$\begin{array}{ccc} \overline{M}_\tau & \xrightarrow{d} & \overline{M}_\sigma \\ \text{st}_\tau \uparrow & & \text{st}_\sigma \uparrow \\ \mathbb{R}\overline{M}(X, \tau, \beta) & \xrightarrow{c} & \mathbb{R}\overline{M}(X, \sigma, \beta) \\ \downarrow \text{ev}_e & & \downarrow \text{ev}_v \times \text{ev}_w \\ X & \xrightarrow{\Delta} & X \times_S X. \end{array}$$

Given any $\mathbf{m} = (m_i)_{i \in T_\tau}$ and $\mathbf{Z} = (Z_i)_{i \in T_\tau}$ as in Definition 9.2, let

$$\mathbf{m}' := \mathbf{m} \sqcup (m_v = 1, m_w = 1).$$

We have

$$d_* Q_{\tau, \beta, \mathbf{m}}^X(\mathbf{Z}) = Q_{\sigma, \beta, \mathbf{m}'}^X(\mathbf{Z} \otimes \Delta_* \mathcal{O}_X),$$

where the right hand side means

$$\text{st}_* \left(\left(\bigotimes_{i \in T_\tau} (\text{ev}_i^{m_i})^* \mathcal{O}_{Z_i^{m_i}(\sigma, \beta)} \right) \otimes (\text{ev}_v \times \text{ev}_w)^* \Delta_* \mathcal{O}_X \right) \in K_0(\overline{M}_\sigma).$$

9.4. Forgetting tails.

Proposition 9.6. *Let (σ, β) be an A -graph obtained from (τ, β) by forgetting a tail t . Let $\Phi: \overline{M}_\tau \rightarrow \overline{M}_\sigma$ be the forgetful map from τ -marked stable curves to σ -marked stable curves. Given any $\mathbf{m} = (m_i)_{i \in T_\sigma}$ and $\mathbf{Z} = (Z_i)_{i \in T_\sigma}$ as in Definition 9.2, and any positive integer m_t , let*

$$\mathbf{m}' := \mathbf{m} \sqcup (m_t).$$

We have

$$Q_{\tau, \beta, \mathbf{m}'}^X(\mathbf{Z} \sqcup (X)) = \Phi^* Q_{\sigma, \beta, \mathbf{m}}^X(\mathbf{Z}).$$

9.5. Contracting edges.

Proposition 9.7. *Let (σ, β) be an A -graph obtained from (τ, β) by contracting an edge (possibly a loop) e . We follow the notations of Section 5.5. Let $\Phi: \overline{M}_\tau \rightarrow \overline{M}_\sigma$, $\Psi_l^i: \overline{M}_{\tau_l^i} \rightarrow \overline{M}_\tau$ and $\Omega: \overline{M}_{\tilde{\sigma}} \rightarrow \overline{M}_\sigma$ be the induced maps on the moduli stacks of stable curves. Given any $\mathbf{m} = (m_v)_{v \in T_{\tilde{\sigma}}}$ and $\mathbf{Z} = (Z_v)_{v \in T_{\tilde{\sigma}}}$ as in Definition 9.2, we have*

$$\Phi^* \Omega_* Q_{\sigma, \beta, \mathbf{m}}^X(\mathbf{Z}) = \sum_l (-1)^{l+1} \sum_{i, j} \Psi_{l^*}^i Q_{\tau_l^i, \beta_l^{i, j}, \mathbf{m}}^X(\mathbf{Z}).$$

10. APPENDIX: FLATNESS IN DERIVED k -ANALYTIC GEOMETRY

Definition 10.1. Let $f: X \rightarrow Y$ be a morphism of derived k -analytic spaces and let $\mathcal{F} \in \text{Coh}^+(X)$. We say that \mathcal{F} is *flat relative to Y* if \mathcal{F} is flat as $f^{-1}\mathcal{O}_Y^{\text{alg}}$ -module.

Lemma 10.2. *Let $f: X \rightarrow Y$ be a morphism of derived k -analytic spaces. Let $\mathcal{F} \in \text{Coh}^+(X)$. The following are equivalent:*

- (1) \mathcal{F} is flat relative to Y .
- (2) \mathcal{F} has tor-amplitude $[0, 0]$ relative to Y in the sense of [51, Definition 7.1].
- (3) For every $\mathcal{G} \in \text{Coh}^\heartsuit(Y)$, $\mathcal{F} \otimes_{\mathcal{O}_X} f^*(\mathcal{G})$ belongs to $\text{Coh}^\heartsuit(X)$.
- (4) Consider the pullback square

$$\begin{array}{ccc} X_0 & \xrightarrow{i} & X \\ \downarrow f_0 & & \downarrow f \\ t_0(Y) & \xrightarrow{j} & Y, \end{array}$$

where j is the inclusion of the truncation. Then $i^*(\mathcal{F})$ is flat relative to $t_0(Y)$.

Proof. We start by proving the equivalence (1) \Leftrightarrow (4). Write $X = (\mathcal{X}, \mathcal{O}_X)$ and $Y = (\mathcal{Y}, \mathcal{O}_Y)$. Since j is a closed immersion, [52, Propositions 3.17(iii) and 6.2(iv)] imply that

$$\begin{array}{ccc} f^{-1}\mathcal{O}_Y^{\text{alg}} & \longrightarrow & \mathcal{O}_X^{\text{alg}} \\ \downarrow & & \downarrow \\ f^{-1}(\pi_0(\mathcal{O}_Y^{\text{alg}})) & \longrightarrow & \mathcal{O}_{X_0}^{\text{alg}} \end{array}$$

is a pushout square in $\text{CAlg}_k(\mathcal{X})$. Then \mathcal{F} is flat as $f^{-1}\mathcal{O}_Y^{\text{alg}}$ -module if and only if the base change $\mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y^{\text{alg}}} f^{-1}(\pi_0(\mathcal{O}_Y^{\text{alg}}))$ is flat as $f^{-1}(\pi_0(\mathcal{O}_Y^{\text{alg}}))$ -module. The derived base change implies that

$$\mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y^{\text{alg}}} f^{-1}(\pi_0(\mathcal{O}_Y^{\text{alg}})) \simeq \mathcal{F} \otimes_{\mathcal{O}_X^{\text{alg}}} \mathcal{O}_{X_0}^{\text{alg}}.$$

Therefore (1) is equivalent to (4).

Next, we prove the equivalence (4) \Leftrightarrow (2). Since (4) is local on both X and Y , we may assume that X and Y are derived k -affinoid spaces. Write

$$A := \Gamma(X, \mathcal{O}_X^{\text{alg}}), \quad A_0 := \Gamma(X_0, \mathcal{O}_{X_0}^{\text{alg}}), \quad B := \Gamma(Y, \mathcal{O}_Y^{\text{alg}}), \quad M := \Gamma(X, \mathcal{F}).$$

Note that $i^*(\mathcal{F})$ is flat relative to $t_0(Y)$ if and only if $i^*(\mathcal{F}) \in \text{Coh}^\heartsuit(X_0) \simeq \text{Coh}^\heartsuit(t_0(X))$ and as a coherent sheaf on $t_0(X)$ it is flat relative to $t_0(Y)$ in the

underived sense. Using [51, Theorem 3.4], we conclude that $i^*(\mathcal{F})$ is flat relative to $t_0(Y)$ if and only if the module

$$\Gamma(X_0, i^*(\mathcal{F})) \simeq M \otimes_A A_0$$

is flat as $\pi_0(B)$ -module. By [51, Proposition 4.2], the diagram

$$\begin{array}{ccc} B & \xrightarrow{\phi} & A \\ \downarrow & & \downarrow \\ \pi_0(B) & \longrightarrow & A_0 \end{array}$$

is a pushout diagram in CAlg_k . This implies that

$$M \otimes_A A_0 \simeq M \otimes_B \pi_0(B).$$

If (4) is satisfied, then $M \otimes_B \pi_0(B)$ is flat as $\pi_0(B)$ -module. Hence, it has tor-amplitude $[0, 0]$ as $\pi_0(B)$ -module, which implies that M has tor-amplitude $[0, 0]$ as B -module, and hence M is flat as B -module. Thus (2) holds. Conversely, if (2) is satisfied, then M is flat as B -module, hence $M \otimes_A A_0$ is flat as $\pi_0(B)$ -module, so (4) holds.

We finally prove the equivalence (2) \Leftrightarrow (3). Once again, we can assume X and Y to be derived k -affinoid spaces. Keeping the same notations as above, let $\mathcal{G} \in \text{Coh}^\heartsuit(Y)$ and let $N := \Gamma(Y, \mathcal{G})$. By [51, Theorem 3.1], $\mathcal{F} \otimes_{\mathcal{O}_X} f^*(\mathcal{G})$ is discrete if and only if $M \otimes_A \phi^*(N)$ is discrete. On the other hand, $M \otimes_A \phi^*(N)$ is discrete if and only if

$$\phi_*(M \otimes_A \phi^*(N)) \simeq \phi_*(M) \otimes_B N$$

is discrete. This completes the proof. \square

Proposition 10.3. *Let*

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ \downarrow q & & \downarrow p \\ Y' & \xrightarrow{f} & Y \end{array}$$

be a pullback square of derived k -analytic spaces. Let $\mathcal{F} \in \text{Coh}^+(X)$. If \mathcal{F} is flat relative to Y , then $g^(\mathcal{F})$ is flat relative to Y' .*

Proof. Consider the extended diagram

$$\begin{array}{ccccc} X'_0 & \xrightarrow{i} & X' & \xrightarrow{g} & X \\ \downarrow q_0 & & \downarrow q & & \downarrow p \\ t_0(Y') & \xrightarrow{j} & Y' & \xrightarrow{f} & Y. \end{array}$$

By Lemma 10.2, it is enough to check that $i^*(\mathcal{F})$ is flat. We factor $t_0(Y') \rightarrow Y$ through $t_0(Y)$, which yields the following diagram

$$\begin{array}{ccccc} X'_0 & \xrightarrow{i'} & X_0 & \xrightarrow{i''} & X \\ \downarrow q_0 & & \downarrow p_0 & & \downarrow p \\ t_0(Y') & \longrightarrow & t_0(Y) & \longrightarrow & Y. \end{array}$$

Since \mathcal{F} is flat relative to Y , Lemma 10.2 shows that $(i'')^*(\mathcal{F})$ is flat relative to $t_0(Y)$. We are therefore left to prove the proposition assuming that X , Y and Y' are underived. Using the theory of formal models (see [10]), we see that $\pi_0(g^*(\mathcal{F}))$ is flat relative to Y' . Therefore, we are left to check that $g^*(\mathcal{F}) \in \text{Coh}^\heartsuit(X')$. Working locally, we can assume that Y and Y' are affinoid and that the map $Y' \rightarrow Y$ factors as

$$Y' \xleftarrow{i} \mathbf{D}_Y^n \xrightarrow{\pi} Y,$$

where i is a closed immersion and π is the canonical projection. We can therefore decompose the pullback square as

$$\begin{array}{ccccc} X' & \longrightarrow & \mathbf{D}_X^n & \xrightarrow{\pi_X} & X \\ \downarrow q & & \downarrow p' & & \downarrow p \\ Y' & \xleftarrow{i} & \mathbf{D}_Y^n & \xrightarrow{\pi_Y} & Y. \end{array}$$

As i is a closed immersion, [52, Proposition 3.17(iii) and 6.2(iv)] imply that it is enough to check that $\pi_X^*(\mathcal{F})$ is flat relative to \mathbf{D}_Y^n . For this, it is enough to check that $\pi_X^*(\mathcal{F})$ is discrete. By [51, Proposition 4.10], we see that π_X is flat, and therefore π_X^* is t -exact. The conclusion follows. \square

Corollary 10.4. *Let*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow q & & \downarrow p \\ Y' & \longrightarrow & Y \end{array}$$

be a pullback square of derived k -analytic stacks. If p is flat, then so is q .

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