

The critical two-point function for long-range percolation on the hierarchical lattice

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Abstract. We prove up-to-constants bounds on the two-point function (i.e., point-to-point connection probabilities) for critical long-range percolation on the d -dimensional hierarchical lattice. More precisely, we prove that if we connect each pair of points x and y by an edge with probability $1 - \exp(-\beta\|x - y\|^{-d-\alpha})$, where $0 < \alpha < d$ is fixed and $\beta \geq 0$ is a parameter, then the critical two-point function satisfies

$$\mathbb{P}_{\beta_c}(x \leftrightarrow y) \asymp \|x - y\|^{-d+\alpha}$$

for every pair of distinct points x and y . We deduce in particular that the model has mean-field critical behaviour when $\alpha < d/3$ and does *not* have mean-field critical behaviour when $\alpha > d/3$.

1 Introduction

In this paper we study critical long-range percolation on the *hierarchical lattice*. The hierarchical lattice \mathbb{H}_L^d is in some ways similar to the usual Euclidean lattice \mathbb{Z}^d but has additional symmetries and an exact recursive nesting structure that often makes hierarchical models of statistical mechanics much easier to understand than their Euclidean counterparts. First introduced by Dyson [17] to study the Ising model in 1969, there is now an extensive literature studying statistical mechanics on hierarchical lattices, with notable works studying the critical behaviour of the Ising model [7, 23], the φ^4 model [5, 31], and self-avoiding walk [8, 10, 11]. Hierarchical models have been particularly popular when e.g. analyzing spin systems via the renormalization group, where the exact recursive structure is extremely helpful [5, 20]. We refer the reader to e.g. [5, 38] for further background on hierarchical models, and to [5, Section 4.2] in particular for a detailed overview of the literature.

While several papers have been written about Bernoulli percolation on hierarchical lattices [13, 14, 21, 32], we think it is fair to say that hierarchical models have received rather less attention within percolation theory than within other parts of mathematical physics, and one goal of this paper is to attract more interest to hierarchical models within the percolation community more broadly. To this end, let us note by way of advertisement that not only do hierarchical models offer a much more tractable alternative to low-dimensional Euclidean models, they are also arguably *more realistic* than Euclidean models as descriptions of certain real-world phenomena in epidemiology and the social sciences [19, 36]. Let us also remark that existing analyses of other hierarchical models at criticality are mostly based on block-renormalization of spins and therefore do not apply to percolation, which is not rigorously known to have any spin-system representations.

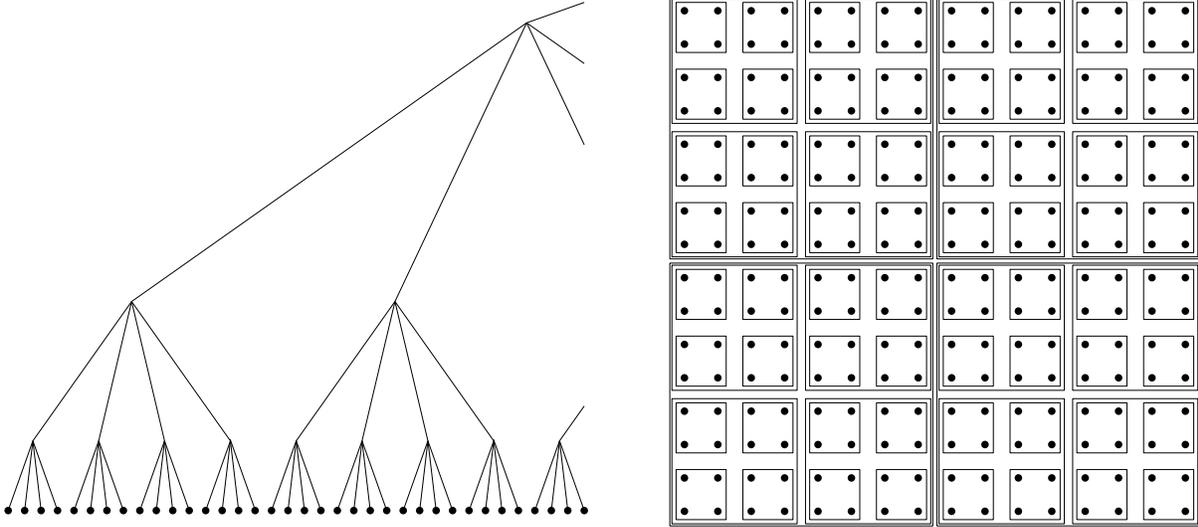


Figure 1: Two graphical representations of the hierarchical lattice \mathbb{H}_2^2 , which can be identified with \mathbb{H}_4^1 by a bijection that transforms distances by a power. In the picture on the left, only the leaves of the tree represent vertices of \mathbb{H}_4^1 . In the picture on the right, the distance between two points is equal to the side-length of the smallest dyadic box containing both points.

1.1 The model

Let us now define the hierarchical lattice. Let $d \geq 1$, $L \geq 2$, and let $\mathbb{T}_L^d = (\mathbb{Z}/L\mathbb{Z})^d$ be the discrete d -dimensional torus of side length L . The **hierarchical lattice** \mathbb{H}_L^d is defined to be the countable abelian group $\bigoplus_{i=1}^{\infty} \mathbb{T}_L^d = \{x = (x_1, x_2, \dots) \in (\mathbb{T}_L^d)^{\mathbb{N}} : x_i = 0 \text{ for all but finitely many } i \geq 0\}$ equipped with the group-invariant ultrametric defined by

$$\|y - x\| = \begin{cases} 0 & x = y \\ L^{h(x,y)} & x \neq y \end{cases} \quad \text{where } h(x, y) = \max\{i \geq 1 : x_i \neq y_i\}.$$

We will also use the ‘Japanese bracket’ notation $\langle x \rangle := 1 \vee \|x\|$ to avoid dividing by zero. Note that the hierarchical lattice \mathbb{H}_L^d is indeed d -dimensional in the sense that if $B(0, r)$ denotes the ball of radius r around the origin then

$$\frac{r^d}{L^d} \leq |B(0, r)| = |B(0, L^{\lfloor \log_L r \rfloor})| = \left| \bigoplus_{i=1}^{\lfloor \log_L r \rfloor} \mathbb{T}_L^d \right| = L^{d \lfloor \log_L r \rfloor} \leq r^d$$

for every $r \geq 1$. As mentioned above, hierarchical lattices have much more symmetry than their Euclidean counterparts, and this symmetry can often be very useful when studying statistical-

mechanics models on them. Indeed, the hierarchical lattice \mathbb{H}_L^d is **distance-transitive**, meaning that if $w, x, y, z \in \mathbb{H}_L^d$ are such that $\|w - x\| = \|y - z\|$ then there exists an isometry γ of \mathbb{H}_L^d such that $\gamma(w) = y$ and $\gamma(x) = z$.

A function $J : \mathbb{H}_L^d \rightarrow [0, \infty)$ is said to be **symmetric** if $J(x) = J(-x)$ for every $x \in \mathbb{H}_L^d$ and is said to be **integrable** if $\sum_{x \in \mathbb{H}_L^d} J(x) < \infty$. We say that J is **radially symmetric** if $J(x)$ can be expressed as a function of $\|x\|$, i.e., if $J(x) = J(y)$ for every $x, y \in \mathbb{H}_L^d$ such that $\|x\| = \|y\|$. Equivalently, J is radially symmetric if it is invariant under isometries of \mathbb{H}_L^d . For our purposes, a particularly interesting choice of integrable, radially symmetric J is given by $J(x) = \langle x \rangle^{-d-\alpha}$, which is integrable for $\alpha > 0$. Given a symmetric, integrable function $J : \mathbb{H}_L^d \rightarrow [0, \infty)$ and $\beta \geq 0$, **long-range percolation** on \mathbb{H}_L^d is defined to be the random graph with vertex set \mathbb{H}_L^d in which each pair $\{x, y\}$ is included as an edge of the graph independently at random with inclusion probability $1 - e^{-\beta J(x-y)}$. (The precise form of this function is not very important; the important thing is that it belongs to $(0, 1)$ when $J(x-y)$ is positive and satisfies $1 - e^{-\beta J(x-y)} \sim \beta J(x-y)$ when $J(x-y)$ is small.) We write $\mathbb{P}_\beta = \mathbb{P}_{\beta, J}$ and $\mathbb{E}_\beta = \mathbb{E}_{\beta, J}$ for probabilities and expectations taken with respect to the law of the resulting random graph. The integrability assumption on J ensures that this graph is locally finite (i.e., has finite vertex degrees) almost surely. The connected components of the resulting random graph are known as **clusters** and the **critical probability** $\beta_c = \beta_c(d, L, J)$ is defined by

$$\beta_c = \inf\{\beta \geq 0 : \text{there exists an infinite cluster with positive probability}\}.$$

Elementary path-counting arguments yield that $\beta_c \geq 1/\sum_x J(x) > 0$ when J is integrable.

For J of the form $J(x) = \langle x \rangle^{-d-\alpha}$ with $\alpha > 0$, it has been shown independently by Koval, Meester, and Trapman [32] and Dawson and Gorostiza [13] that $\beta_c < \infty$ (i.e., that the phase transition is non-trivial) if and only if $\alpha < d$. Koval, Meester, and Trapman [32] have also shown under the same assumptions that the phase transition is *continuous*, meaning that there is no infinite cluster at β_c almost surely. This is a hierarchical version of a theorem of Berger [6], which establishes a similar result for long-range percolation on \mathbb{Z}^d with $\alpha < d$. (In contrast, long-range percolation on \mathbb{Z} with $\alpha = 1$ is known to undergo a *discontinuous* phase transition by a theorem of Aizenman and Newman [3]; see also [15] for a new proof of this result.) Both theorems are proven by showing that the set $\{\beta > 0 : \text{there is an infinite cluster at } \beta\}$ is open, and do not yield any quantitative control of the models at criticality.

In our recent work [30] we established new, quantitative versions of both continuity theorems, yielding power-law upper bounds on the distribution of the cluster of the origin in critical long-range percolation on both \mathbb{Z}^d and \mathbb{H}_L^d . The power-law bounds proven in [30] are not expected to be sharp, and it remained open to compute the exact critical exponents describing these models.

1.2 Our results

The goal of this paper is to establish a sharp quantitative understanding of critical long-range percolation on the hierarchical lattice by proving up-to-constants estimates on the two-point function (i.e., on the point-to-point connection probabilities $\mathbb{P}_\beta(x \leftrightarrow y)$). Before stating our main result, let us briefly introduce some further relevant definitions. For each $n \geq 0$ we write $\Lambda_n = \bigoplus_{i=1}^n \mathbb{T}_L^d = \{y \in \mathbb{H}_L^d : \langle y \rangle \leq L^n\}$ and write $\Lambda_n(x) = \Lambda_n + x = \{y \in \mathbb{H}_L^d : \langle y - x \rangle \leq L^n\}$ for

each $n \geq 0$ and $x \in \mathbb{H}_L^d$. Given $x, y \in \mathbb{H}_L^d$ and $A \subseteq \mathbb{H}_L^d$ we write $\{x \overset{A}{\longleftrightarrow} y\}$ for the event that x and y are connected by a path in A .

Theorem 1.1. *Let $J : \mathbb{H}_L^d \rightarrow [0, \infty)$ be a symmetric, integrable function, let $0 < \alpha < d$, and suppose that there exist positive constants c and C such that $c\|x\|^{-d-\alpha} \leq J(x) \leq C\|x\|^{-d-\alpha}$ for every $x \in \mathbb{H}_L^d \setminus \{0\}$. Then there exist positive constants a and A depending only on d, L, α, c , and C such that*

$$aL^{-(d-\alpha)n} \leq \frac{1}{|\Lambda_n|} \sum_{x \in \Lambda_n \setminus \Lambda_{n-1}} \mathbb{P}_{\beta_c}(0 \overset{\Lambda_n}{\longleftrightarrow} x) \leq \frac{1}{|\Lambda_n|} \sum_{x \in \Lambda_n} \mathbb{P}_{\beta_c}(0 \leftrightarrow x) \leq AL^{-(d-\alpha)n}$$

for every $n \geq 1$.

Note that this theorem implies in particular that $\beta_c < \infty$ and that there are no infinite clusters at β_c , recovering the results of [13, 32]. For radially symmetric J , the averaged two-point function estimate of Theorem 1.1 can immediately be upgraded to a pointwise estimate via symmetry, which we now state. Given two points $x, y \in \mathbb{H}_L^d$ we write

$$\Lambda(x, y) = \left\{ z \in \mathbb{H}_L^d : \|z - x\| \leq \|y - x\| \right\} = \left\{ z \in \mathbb{H}_L^d : \|z - y\| \leq \|y - x\| \right\}$$

for the smallest ultrametric ball containing both x and y .

Corollary 1.2. *Let $J : \mathbb{H}_L^d \rightarrow [0, \infty)$ be a radially symmetric, integrable function, let $0 < \alpha < d$, and suppose that there exist positive constants c and C such that $c\|x\|^{-d-\alpha} \leq J(x) \leq C\|x\|^{-d-\alpha}$ for every $x \in \mathbb{H}_L^d \setminus \{0\}$. Then there exist positive constants a and A depending only on d, L, α, c , and C such that*

$$a\langle x - y \rangle^{-d+\alpha} \leq \mathbb{P}_{\beta_c}(x \overset{\Lambda(x,y)}{\longleftrightarrow} y) \leq \mathbb{P}_{\beta_c}(x \leftrightarrow y) \leq A\langle x - y \rangle^{-d+\alpha}$$

for every $x, y \in \mathbb{H}_L^d$.

Proof of Corollary 1.2 given Theorem 1.1. Since \mathbb{H}_L^d is distance-transitive, $\mathbb{P}_{\beta_c}(0 \overset{\Lambda_n}{\longleftrightarrow} x)$ depends only on n and $\|x\|$, so that if $\|x\| = L^n$ then we have by Theorem 1.1 that

$$\begin{aligned} \mathbb{P}_{\beta_c}(0 \overset{\Lambda_n}{\longleftrightarrow} x) &= \frac{1}{|\Lambda_n \setminus \Lambda_{n-1}|} \sum_{y \in \Lambda_n \setminus \Lambda_{n-1}} \mathbb{P}_{\beta_c}(0 \overset{\Lambda_n}{\longleftrightarrow} y) \geq aL^{-(d-\alpha)n} \quad \text{and} \\ \mathbb{P}_{\beta_c}(0 \leftrightarrow x) &= \frac{1}{|\Lambda_n \setminus \Lambda_{n-1}|} \sum_{y \in \Lambda_n \setminus \Lambda_{n-1}} \mathbb{P}_{\beta_c}(0 \leftrightarrow y) \leq \frac{L^d}{L^d - 1} AL^{-(d-\alpha)n} \end{aligned}$$

as claimed. □

Remark 1.3. For long-range percolation on \mathbb{Z}^d with $d \geq 2$, it is predicted that the critical two-point function satisfies the same asymptotics given by Corollary 1.2 for α strictly smaller than the crossover value $\alpha_c = \alpha_c(d)$, while for $\alpha > \alpha_c$ the critical two-point function should obey the same asymptotic decay as for nearest neighbour percolation on \mathbb{Z}^d . See [30, Section 1.3] for details. Here we see that the behaviour on the hierarchical lattice is rather simpler, and indeed is more

closely analogous to long-range percolation on the *one-dimensional* lattice \mathbb{Z} . This is related to the fact that \mathbb{H}_L^d and $\mathbb{H}_{L^d}^1$ are related by a bijection that transforms distances by a d th root, so that long-range percolation on \mathbb{H}_L^d with $J(x) = \langle x \rangle^{-d-\alpha}$ is equivalent to long-range percolation on $\mathbb{H}_{L^d}^1$ with $J(x) = \langle x \rangle^{-1-\alpha/d}$. As such, the dimension d does not really make any difference to the model besides a change of parameterization.

Further critical exponents. It would be very interesting to compute further critical exponents of the model beyond those describing the two-point function. The next most accessible of these critical exponents is probably the exponent δ , which is conjectured to describe the tail of the volume of a critical cluster via

$$\delta = - \lim_{n \rightarrow \infty} \frac{\log n}{\log \mathbb{P}_{\beta_c}(|K| \geq n)} \quad \text{so that} \quad \mathbb{P}_{\beta_c}(|K| \geq n) = n^{-1/\delta \pm o(1)} \quad \text{as } n \rightarrow \infty,$$

where $|K|$ denotes the cluster of the origin. (It is a part of the conjecture that this limit is well-defined.) Using Theorem 1.1, heuristic hyperscaling arguments (see e.g. [30, Sections 1.3 and 2]) predict that, under the hypotheses of Theorem 1.1,

$$\delta = \begin{cases} 2 & \text{if } 0 < \alpha \leq d/3 \\ \frac{d+\alpha}{d-\alpha} & \text{if } d/3 < \alpha < d. \end{cases} \quad (1.1)$$

In particular, it is expected that the model should have *mean-field critical behaviour* if and only if $\alpha < d/3$, with polylogarithmic corrections to this behaviour when $\alpha = d/3$.

While we have not been able to prove (1.1), Theorem 1.1 does yield some interesting partial progress on the problem. Indeed, when $\alpha < d/3$ we easily verify from Theorem 1.1 that the model satisfies the *triangle condition* of Aizenman and Newman [2], which is known to imply that many critical exponents exist and take their mean-field values [2, 4, 33, 34]. (These proofs are usually written for Euclidean lattices but apply equally well in the hierarchical case.) Since the triangle condition and its consequences are well-known, we do not go into them in detail here but refer the reader to e.g. [22, Chapter 10.3] and [26] for background.

Corollary 1.4. *Let $J : \mathbb{H}_L^d \rightarrow [0, \infty)$ be a radially symmetric, integrable function, let $0 < \alpha < d$, and suppose that there exist constants c and C such that $c\|x\|^{-d-\alpha} \leq J(x) \leq C\|x\|^{-d-\alpha}$ for every $x \in \mathbb{H}_L^d \setminus \{0\}$. If $\alpha < d/3$ then the triangle condition*

$$\nabla_{\beta_c} := \sum_{x, y \in \mathbb{H}_L^d} \mathbb{P}_{\beta_c}(0 \leftrightarrow x) \mathbb{P}_{\beta_c}(x \leftrightarrow y) \mathbb{P}_{\beta_c}(y \leftrightarrow 0) < \infty$$

holds and the model exhibits mean-field critical behaviour. In particular, we have that

$$\begin{aligned} \mathbb{P}_{\beta_c}(|K| \geq n) &\asymp n^{-1/2} && \text{for every } n \geq 1, \\ \mathbb{E}_{\beta}|K| &\asymp (\beta_c - \beta)^{-1} && \text{for every } 0 < \beta < \beta_c, \text{ and} \\ \mathbb{P}_{\beta}(|K| = \infty) &\asymp \max\{\beta - \beta_c, 1\} && \text{for every } \beta > \beta_c, \end{aligned}$$

where \asymp denotes an equality holding to within multiplication by positive constants.

We remark that, following the breakthrough work of Hara and Slade [24], the triangle condition

has been verified for a number of high-dimensional Euclidean percolation models using a technique known as the *lace expansion* [9, 12, 18, 24, 25, 27], which is surveyed in [26] and [37]. We expect that it should also be possible to prove a version of Corollary 1.4 using the lace expansion, but that this would be much more involved than the proof we have given and would also need slightly stronger hypotheses, for example that the constant L is large.

In the case $\alpha > d/3$, we are able to deduce a sharp *lower bound* on the exponent δ from Theorem 1.1 together with the rigorous hyperscaling inequality of [30, Theorem 2.1].

Corollary 1.5. *Let $J : \mathbb{H}_L^d \rightarrow [0, \infty)$ be a symmetric, integrable function, let $0 < \alpha < d$, and suppose that there exist constants c and C such that $c\|x\|^{-d-\alpha} \leq J(x) \leq C\|x\|^{-d-\alpha}$ for every $x \in \mathbb{H}_L^d \setminus \{0\}$. If $\alpha > d/3$ and the exponent δ is well-defined then*

$$\delta \geq \frac{d + \alpha}{d - \alpha} > 2.$$

Thus, the model does not have mean-field critical behaviour when $\alpha > d/3$.

Remark 1.6. We are not aware of any conjectured values of the critical exponents β , γ , and Δ describing *near critical* percolation on the hierarchical lattice when $\alpha > d/3$. (See e.g. [22, Chapter 9] for definitions of these exponents.) It would be very interesting even to have even heuristic calculations of these exponents in this regime.

2 Proof

2.1 The maximum cluster size

Fix $d \geq 1$, $L \geq 2$ and $0 < \alpha < d$. For each $n \geq 1$ and $x \in \mathbb{H}_L^d$ let $K_n(x)$ be the cluster of x in the ultrametric ball $\Lambda_n(x)$ (i.e., the set of $y \in \Lambda_n(x)$ connected to x by an open path contained in $\Lambda_n(x)$) and write $K_n = K_n(0)$ to lighten notation. We will also write $K(x)$ for the cluster of x in \mathbb{H}_L^d and write $K = K(0)$. In this section, we study the distribution of the size of the *largest* cluster in an ultrametric ball

$$|K_n^{\max}| := \max\{|K_n(x)| : x \in \Lambda_n\}.$$

(Note that this is a slight abuse of notation since the largest cluster might not be unique, in which case K_n^{\max} is not well-defined as a set.) Following [30, Section 2], we define the **typical value** of $|K_n^{\max}|$ to be

$$M_n = M_n(\beta) := \min\left\{m \geq 1 : \mathbb{P}_\beta(|K_n^{\max}| \geq m) \leq \frac{1}{e}\right\}$$

for each $\beta \geq 0$. Note that $M_n(\beta)$ is an increasing, continuous function of β for each $n \geq 0$. Note also that $M_n \geq 2$, so that

$$\mathbb{P}_\beta\left(|K_n^{\max}| \geq \frac{1}{2}M_n\right) \geq \mathbb{P}_\beta(|K_n^{\max}| \geq M_n - 1) \geq \frac{1}{e} \tag{2.1}$$

and hence by symmetry that

$$\mathbb{E}_\beta |K_n| = \frac{1}{|\Lambda_n|} \sum_{x \in \Lambda_n} \mathbb{E}_\beta |K_n(x)| \geq \frac{M_n^2}{4|\Lambda_n|} \mathbb{P}_\beta \left(|K_n^{\max}| \geq \frac{1}{2} M_n \right) \geq \frac{M_n^2}{4eLdn} \quad (2.2)$$

for every $\beta \geq 0$ and $n \geq 0$.

The following theorem is a special case of [30, Theorem 2.2] and shows in particular that the maximum cluster size $|K_n^{\max}|$ is of the same order as its typical value M_n with high probability.

Theorem 2.1. *Let $J : \mathbb{H}_L^d \rightarrow [0, \infty)$ be a symmetric, integrable function, let $\beta \geq 0$, and let $M_n = M_n(\beta)$ for each $n \geq 0$. The inequalities*

$$\mathbb{P}_\beta \left(|K_n^{\max}| \geq \lambda M_n \right) \leq \exp \left(-\frac{1}{9} \lambda \right) \quad \text{and} \quad \mathbb{P}_\beta \left(|K_n^{\max}| < \varepsilon M_n \right) \leq 27\varepsilon \quad (2.3)$$

hold for every $n \geq 0$, $\lambda \geq 1$ and $0 < \varepsilon \leq 1$. Moreover, the inequality

$$\mathbb{P}_\beta \left(|K_n| \geq \lambda M_n \right) \leq \mathbb{P}_\beta \left(|K_n| \geq M_n \right) \exp \left(1 - \frac{1}{9} \lambda \right) \quad (2.4)$$

holds for every $n \geq 0$ and $\lambda \geq 1$.

The main goal of this section is to prove an upper bound on the growth of $M_n(\beta_c)$. Theorem 2.1, and specifically the inequality (2.4), will allow us to make use of these asymptotics to study the growth of $\mathbb{E}_{\beta_c} |K_n|$ in the next subsection.

Proposition 2.2. *Let $J : \mathbb{H}_L^d \rightarrow [0, \infty)$ be a symmetric, integrable function, let $0 < \alpha < d$, and suppose that there exists a positive constant c such that $J(x) \geq c\|x\|^{-d-\alpha}$ for every $x \in \mathbb{H}_L^d \setminus \{0\}$. Then there exists a constant $A = A(d, L, \alpha, c)$ such that*

$$M_n(\beta) \leq \frac{A}{\beta} L^{(d+\alpha)n/2}$$

for every $0 \leq \beta \leq \beta_c$ and $n \geq 0$.

Since $M_n(\beta) \geq 2$ for every $n \geq 0$ and $\beta \geq 0$, Proposition 2.2 has the following immediate corollary. (Indeed, taking $n = 0$ gives that $\beta_c \leq A/2$ where A is the constant from Proposition 2.2.)

Corollary 2.3. *Let $J : \mathbb{H}_L^d \rightarrow [0, \infty)$ be a symmetric, integrable function, let $0 < \alpha < d$, and suppose that there exists a positive constant c such that $J(x) \geq c\|x\|^{-d-\alpha}$ for every $x \in \mathbb{H}_L^d \setminus \{0\}$. Then there exists a constant $\beta_1 = \beta_1(d, L, \alpha, c)$ such that $\beta_c \leq \beta_1$.*

We will deduce Proposition 2.2 from the sharpness of the phase transition together with the following renormalization lemma.

Lemma 2.4 (Renormalization of the maximum cluster size). *Let $J : \mathbb{H}_L^d \rightarrow [0, \infty)$ be a symmetric, integrable function, let $0 < \alpha < d$, and suppose that there exists a positive constant c such that $J(x) \geq c\|x\|^{-d-\alpha}$ for every $x \in \mathbb{H}_L^d \setminus \{0\}$. There exist constants $\ell = \ell(d, L, \alpha, c)$ and*

$A = A(d, L, \alpha, c)$ such that the implication

$$\left(M_n(\beta)^2 \geq \frac{A}{\beta} L^{(d+\alpha)n} \right) \Rightarrow \left(M_{n+\ell}(\beta)^2 \geq \frac{A}{\beta} L^{(d+\alpha)(n+\ell)} \right)$$

holds for every $\beta > 0$ and $n \geq 0$.

Proof of Lemma 2.4. Since $\alpha < d$, there exists ℓ_0 such that $\frac{1}{8}L^{d\ell} \geq L^{(d+\alpha)\ell/2}$ for every $\ell \geq \ell_0$. The set $\Lambda_{n+\ell}$ contains $L^{d\ell}$ disjoint copies of Λ_n , each of which contains a cluster of size at least $M_n - 1 \geq \frac{1}{2}M_n$ independently with probability at least $1/e > 1/4$ by (2.1). Letting $\mathcal{A}_{n,\ell}$ be the event that at least $\frac{1}{4}L^{d\ell}$ of these copies of Λ_n contains a cluster of size at least $\frac{1}{2}M_n$, it follows by standard concentration estimates for binomial random variables (or indeed by the weak law of large numbers) that we may take $\ell \geq \ell_0$ to be a sufficiently large constant that $\mathbb{P}(\mathcal{A}_{n,\ell}) \geq 9/10$ for every $n \geq 0$. We now fix $\ell = \ell(d, L, \alpha)$ to be one such constant.

Fix $n \geq 0$ and let \mathcal{F}_n be the σ -algebra generated by those edges of \mathbb{H}_L^d that have endpoints at distance at most L^n . Condition on \mathcal{F}_n and suppose that $\mathcal{A}_{n,\ell}$ holds, noting that $\mathcal{A}_{n,\ell}$ is measurable with respect to \mathcal{F}_n . Since $\mathcal{A}_{n,\ell}$ holds there exist at least $\frac{1}{4}L^{d\ell}$ copies of Λ_n in $\Lambda_{n+\ell}$ containing a cluster of size at least $\frac{1}{2}M_n$. Pick one such cluster within each of these copies, and call these clusters K^1, \dots, K^m where $\frac{1}{4}L^{d\ell} \leq m \leq L^{d\ell}$. We have by the definitions if $\mathcal{A}_{n,\ell}$ holds then

$$\begin{aligned} \mathbb{P}_\beta(K^i \xleftrightarrow{\Lambda_{n+\ell}} K^j \mid \mathcal{F}_n) &\geq \mathbb{P}_\beta(\exists \text{ an open edge connecting } K^i \text{ to } K^j \mid \mathcal{F}_n) \\ &= 1 - \exp \left[-\beta \sum_{x \in K^i} \sum_{y \in K^j} J(y-x) \right] \geq 1 - \exp \left[-\frac{c\beta}{4} M_n^2 L^{-(d+\alpha)(n+\ell)} \right] \end{aligned}$$

for every $1 \leq i < j \leq m$. It follows by a union bound that

$$\begin{aligned} \mathbb{P}_\beta \left(|K_{n+\ell}^{\max}| \geq \frac{1}{2}M_n \cdot \frac{1}{4}L^{d\ell} \mid \mathcal{F}_n \right) &\geq \mathbb{P}_\beta \left(|K_{n+\ell}^{\max}| \geq \sum_{i=1}^m |K^i| \mid \mathcal{F}_n \right) \mathbb{1}(\mathcal{A}_{n,\ell}) \\ &\geq \left(1 - \sum_{1 \leq i < j \leq m} \mathbb{P}_\beta(K^i \xleftrightarrow{\Lambda_{n+\ell}} K^j \mid \mathcal{F}_n) \right) \mathbb{1}(\mathcal{A}_{n,\ell}) \\ &\geq \left(1 - L^{2d\ell} \exp \left[-\frac{c\beta}{4} M_n^2 L^{-(d+\alpha)(n+\ell)} \right] \right) \mathbb{1}(\mathcal{A}_{n,\ell}) \end{aligned}$$

and hence that

$$\mathbb{P}_\beta \left(|K_{n+\ell}^{\max}| \geq \frac{1}{2}M_n \cdot \frac{1}{4}L^{d\ell} \right) \geq \frac{9}{10} \left(1 - L^{2d\ell} \exp \left[-\frac{c\beta}{4} M_n^2 L^{-(d+\alpha)(n+\ell)} \right] \right). \quad (2.5)$$

It follows that there exists a constant $A = A(d, L, \alpha, c)$ such that if $M_n^2 \geq \frac{A}{\beta} L^{(d+\alpha)n}$ then

$$\mathbb{P}_\beta \left(|K_{n+\ell}^{\max}| \geq \frac{1}{2}M_n \cdot \frac{1}{4}L^{d\ell} \right) \geq \frac{9}{10} \left(1 - L^{2d\ell} \exp \left[-\frac{cA}{4} L^{-(d+\alpha)\ell} \right] \right) > \frac{1}{e}$$

and hence that

$$M_{n+\ell} \geq \frac{1}{8} L^{d\ell} M_n \geq L^{(d+\alpha)\ell/2} M_n \geq \sqrt{\frac{A}{\beta}} L^{(d+\alpha)(n+\ell)} \quad (2.6)$$

as desired. \square

We now deduce Proposition 2.2 from Lemma 2.4. The proof will use the *sharpness of the phase transition*, a fundamental result in percolation theory originally due to Menshikov [35] and Aizenman and Barsky [1] which is known to hold for arbitrary transitive weighted graphs [1, 16, 29]. We state the theorem for the hierarchical lattice only as this is the only case relevant to us.

Theorem 2.5 (Sharpness of the phase transition). *Let $d \geq 1$, $L \geq 2$, and let $J : \mathbb{H}_L^d \rightarrow [0, \infty)$ be a symmetric, integrable function. Then $\mathbb{E}_\beta |K| < \infty$ for every $0 \leq \beta < \beta_c$.*

Proof of Proposition 2.2. Let A be the constant from Lemma 2.4. We have by sharpness of the phase transition that if $\beta < \beta_c$ then $\mathbb{E}_\beta |K| = \sup_n \mathbb{E}_\beta |K_n| < \infty$, and it follows from (2.2) that

$$\limsup_{n \rightarrow \infty} L^{-dn} M_n(\beta)^2 \leq \limsup_{n \rightarrow \infty} 4e \mathbb{E}_\beta |K_n| < \infty \quad \text{for every } 0 \leq \beta < \beta_c. \quad (2.7)$$

On the other hand, if there were to exist $0 \leq \beta < \beta_c$ and $n \geq 0$ such that $M_n(\beta)^2 \geq \frac{A}{\beta} L^{(d+\alpha)n}$ then we would have inductively by Lemma 2.4 that $M_{n+i\ell}(\beta)^2 \geq \frac{A}{\beta} L^{(d+\alpha)(n+i\ell)}$ for every $i \geq 1$, which would contradict (2.7). Thus, we must instead have that

$$M_n(\beta)^2 < \frac{A}{\beta} L^{(d+\alpha)n}$$

for every $n \geq 0$ and $0 \leq \beta < \beta_c$. The claim follows by taking the limit as $\beta \uparrow \beta_c$. \square

Remark 2.6. We conjecture that the inequality of Proposition 2.2 is of the correct order if and only if $\alpha > d/3$. It is certainly not of the correct order when $\alpha < d/3$. Indeed, it is a consequence of the *tree-graph inequality* method of Aizenman and Newman [2] (see in particular [22, Equation 6.99]) that

$$\mathbb{P}_\beta(|K_n| \geq m) \leq \frac{\sqrt{2} \mathbb{E}_\beta |K_n|}{m} \exp \left[-\frac{m}{4(\mathbb{E}_\beta |K_n|)^2} \right] \quad (2.8)$$

for every $0 \leq \beta < \infty$, $n \geq 1$, and $m \geq 1$, and hence by Markov's inequality that

$$\mathbb{P}_\beta(|K_n^{\max}| \geq m) \leq \frac{1}{m} \sum_{x \in \Lambda_n} \mathbb{P}_\beta(|K_n(x)| \geq m) \leq \frac{\sqrt{2} L^{dn} \mathbb{E}_\beta |K_n|}{m^2} \exp \left[-\frac{m}{4(\mathbb{E}_\beta |K_n|)^2} \right] \quad (2.9)$$

for every $0 \leq \beta < \infty$, $n \geq 1$, and $m \geq 1$. Applying Theorem 1.1 we deduce that there exist positive constants A_1 and a such that

$$\mathbb{P}_{\beta_c}(|K_n^{\max}| \geq m) \leq \frac{A_1 L^{(d+\alpha)n}}{m^2} \exp \left[-a L^{-2\alpha n} m \right] \quad (2.10)$$

for every $n \geq 1$ and $m \geq 1$. It follows that there exists a constant A_2 such that

$$M_n(\beta_c) \leq A_2 n L^{2\alpha n} \quad (2.11)$$

for every $n \geq 1$, which is of lower order than the bound of Proposition 2.2 when $\alpha < d/3$.

2.2 Upper bounds on the restricted two-point function

We now apply the results of Section 2.1 to study the expected volume of the cluster of the origin within an ultrametric ball.

Proposition 2.7. *Let $J : \mathbb{H}_L^d \rightarrow [0, \infty)$ be a symmetric, integrable function, let $0 < \alpha < d$, and suppose that there exists a positive constant c such that $J(x) \geq c\|x\|^{-d-\alpha}$ for every $x \in \mathbb{H}_L^d \setminus \{0\}$. Then there exists a positive constant $A = A(d, L, \alpha, c)$ such that*

$$\mathbb{E}_\beta |K_n| = \sum_{x \in \Lambda_n} \mathbb{P}_\beta(0 \overset{\Lambda_n}{\longleftrightarrow} x) \leq \frac{A}{\beta_c} L^{\alpha n}$$

for every $0 < \beta \leq \beta_c$ and $n \geq 0$.

We will deduce Proposition 2.7 from the following renormalization lemma.

Lemma 2.8 (Renormalization of the restricted susceptibility). *Let $J : \mathbb{H}_L^d \rightarrow [0, \infty)$ be a symmetric, integrable function, let $0 < \alpha < d$, and suppose that there exists a positive constant c such that $J(x) \geq c\|x\|^{-d-\alpha}$ for every $x \in \mathbb{H}_L^d \setminus \{0\}$. Then there exists a positive constant $a = a(d, L, \alpha, c)$ such that*

$$\mathbb{E}_\beta |K_{n+1}| \geq \sum_{x \in \Lambda_{n+1} \setminus \Lambda_n} \mathbb{P}_\beta(0 \overset{\Lambda_{n+1}}{\longleftrightarrow} x) \geq a\beta L^{-\alpha n} (\mathbb{E}_\beta |K_n|)^2$$

for every $0 < \beta \leq \beta_c$ and $n \geq 0$.

In order to prove this lemma, we first use Proposition 2.2 and Theorem 2.1 to prove the following supporting technical estimate. We write $a \wedge b = \min\{a, b\}$.

Lemma 2.9 (Truncating at the typical maximum). *Let $J : \mathbb{H}_L^d \rightarrow [0, \infty)$ be a symmetric, integrable function, let $0 < \alpha < d$, and suppose that there exists a positive constant c such that $J(x) \geq c\|x\|^{-d-\alpha}$ for every $x \in \mathbb{H}_L^d \setminus \{0\}$. Then there exists a positive constant $a = a(d, L, \alpha, c)$ such that*

$$\mathbb{E}_\beta \left[|K_n| \wedge (\lambda L^{(d+\alpha)n/2}) \right] \geq a\lambda \mathbb{E}_\beta |K_n|$$

for every $0 < \beta \leq \beta_c$, $0 < \lambda \leq 1$ and $n \geq 0$.

Proof of Lemma 2.9. Let A be the constant from Proposition 2.2. Fix $n \geq 0$ and let $N = \lfloor 99M_n \rfloor$. Then we have that

$$\mathbb{E}_\beta |K_n| = \sum_{k=1}^{\infty} \mathbb{P}_\beta(|K_n| \geq k) \quad \text{and} \quad \mathbb{E}_\beta [|K_n| \wedge N] = \sum_{k=1}^N \mathbb{P}_\beta(|K_n| \geq k),$$

and hence by Theorem 2.1 that

$$\mathbb{E}_\beta |K_n| - \mathbb{E}_\beta [|K_n| \wedge N] = \sum_{k=N+1}^{\infty} \mathbb{P}_\beta(|K_n| \geq k) \leq e\mathbb{P}_\beta(|K_n| \geq M_n) \sum_{k=N+1}^{\infty} e^{-k/9M_n}.$$

Since $N + 1 \geq 99M_n$, we can therefore compute by Markov's inequality that

$$\mathbb{E}_\beta |K_n| - \mathbb{E}_\beta [|K_n| \wedge N] \leq \frac{e^{1-11}}{1 - e^{-1/9M_n}} \mathbb{P}_\beta(|K_n| \geq M_n) \leq \frac{e^{-10}}{(1 - e^{-1/9M_n})M_n} \mathbb{E}_\beta |K_n| \leq \frac{1}{2} \mathbb{E}_\beta |K_n|,$$

where the final inequality follows by calculus since $M_n \geq 2$. (Indeed, the optimal constant here is much smaller than $1/2$.) It follows that

$$\mathbb{E}_\beta [|K_n| \wedge N] \geq \frac{1}{2} \mathbb{E}_\beta |K_n|, \quad (2.12)$$

and the claim follows from Proposition 2.2 together with the trivial inequality $\mathbb{E}_\beta [|K_n| \wedge (\lambda N)] \geq \lambda \mathbb{E}_\beta [|K_n| \wedge N]$, which holds for every $\lambda \in [0, 1]$. \square

We now apply Lemma 2.9 to prove Lemma 2.8. For each $n \geq 0$ we write \mathcal{C}_n for the set of clusters of the percolation configuration inside Λ_n , so that \mathcal{C}_n is a random set of disjoint subsets of Λ_n whose union is the entire ultrametric ball Λ_n . Note that if f is any function from subsets of Λ_n to \mathbb{R} that is isometry-invariant in the sense that $f(\gamma A) = f(A)$ for every $A \subseteq \Lambda_n$ and any isometry of Λ_n then we have by symmetry that

$$\mathbb{E}_\beta [f(K_n)] = \frac{1}{|\Lambda_n|} \sum_{x \in \Lambda_n} \mathbb{E}_\beta [f(K_n(x))] = \frac{1}{|\Lambda_n|} \mathbb{E}_\beta \left[\sum_{C \in \mathcal{C}_n} |C| f(C) \right], \quad (2.13)$$

where the second equality follows by linearity of expectation.

Proof of Lemma 2.9. For each $n \geq 0$ write $h(n) = c\beta L^{-(d+\alpha)n}$, so that if $n \geq 1$ and x, y satisfy $\|x - y\| = L^n$ then there is an edge connecting x and y with probability at least $1 - e^{-h(n)}$. Fix $n \geq 0$. The ultrametric ball Λ_{n+1} can be decomposed into L^d copies of Λ_n . Call these copies $\Lambda_n^1, \dots, \Lambda_n^{L^d}$, with the origin belonging to $\Lambda_n^1 = \Lambda_n$, and for each $1 \leq i \leq L^d$ let \mathcal{C}_n^i be the set of clusters of the restriction of the percolation configuration to Λ_n^i . (That is, $x, y \in \Lambda_n^i$ belong to the same element of \mathcal{C}_n^i if they are connected by a path inside Λ_n^i .) Let \mathcal{F}_n be the σ -algebra generated by all edges whose endpoints have distance at most L^n in \mathbb{H}_L^d and define

$$\tilde{K}_{n+1} = \left\{ x \in \Lambda_{n+1} \setminus \Lambda_n : 0 \overset{\Lambda_{n+1}}{\longleftrightarrow} x \right\}.$$

Conditional on \mathcal{F}_n , each cluster that belongs to \mathcal{C}_n^i for some $2 \leq i \leq L^d$ is connected to K_n by an edge with probability at least $1 - e^{-h(n+1)|K_n||C|}$, so that

$$\mathbb{E}_\beta [|K_{n+1}| \mid \mathcal{F}_n] \geq \sum_{i=2}^{L^d} \sum_{C \in \mathcal{C}_n^i} |C| \left(1 - \exp \left[-h(n+1)|K_n||C| \right] \right).$$

Using the inequality $1 - e^{-t} \geq (1 - e^{-1})(t \wedge 1)$ it follows that

$$\begin{aligned} \mathbb{E}_\beta \left[|\tilde{K}_{n+1}| \mid \mathcal{F}_n \right] &\geq (1 - e^{-1})h(n+1) \sum_{i=2}^{L^d} \sum_{C \in \mathcal{C}_n^i} |C| \left(\frac{1}{h(n+1)} \wedge |K_n| |C| \right) \\ &\geq (1 - e^{-1})h(n+1) \sum_{i=2}^{L^d} \sum_{C \in \mathcal{C}_n^i} |C| \left(\frac{1}{\sqrt{h(n+1)}} \wedge |C| \right) \left(\frac{1}{\sqrt{h(n+1)}} \wedge |K_n| \right). \end{aligned}$$

Taking expectations and using that \mathcal{C}_n^i and K_n are independent for every $2 \leq i \leq L^d$ yields that

$$\begin{aligned} \mathbb{E}_\beta |\tilde{K}_{n+1}| &\geq (1 - e^{-1})h(n+1) \mathbb{E}_\beta \left[\frac{1}{\sqrt{h(n+1)}} \wedge |K_n| \right] \sum_{i=2}^{L^d} \mathbb{E}_\beta \left[\sum_{C \in \mathcal{C}_n^i} |C| \left(\frac{1}{\sqrt{h(n+1)}} \wedge |C| \right) \right] \\ &= (1 - e^{-1})(L^d - 1)h(n+1)L^{dn} \mathbb{E}_\beta \left[\frac{1}{\sqrt{h(n+1)}} \wedge |K_n| \right]^2, \end{aligned}$$

where the equality in the second line follows from (2.13). The claim now follows from Lemma 2.9 since $h(n+1)^{-1/2}$ is of order $L^{(d+\alpha)n/2}$. \square

It remains to deduce Proposition 2.7 from Lemma 2.4.

Proof of Proposition 2.7. Let a be the constant from Lemma 2.8 and let $\beta < \beta_c$. If there were to exist $n \geq 0$ such that

$$\mathbb{E}_\beta |K_n| \geq \frac{L^{\alpha(n+1)}}{a\beta}$$

then we would have by induction that

$$\mathbb{E}_\beta |K_{m+1}| \geq a\beta L^{-\alpha m} \left(\frac{L^{\alpha(m+1)}}{a\beta} \right)^2 = \frac{L^{\alpha(m+2)}}{a\beta}$$

for every $m \geq n$, contradicting the fact that $\mathbb{E}_\beta |K| = \sup_{m \geq 1} \mathbb{E}_\beta |K_m| < \infty$ by sharpness of the phase transition (Theorem 2.5). Thus, we must instead have that

$$\mathbb{E}_\beta |K_n| < \frac{L^{\alpha(n+1)}}{a\beta}$$

for every $0 \leq \beta < \beta_c$ and $n \geq 0$, and the claim follows by continuity of $\mathbb{E}_\beta |K_n|$ as before. \square

2.3 Upper bounds on the unrestricted two-point function

We now deduce upper bounds on the *unrestricted* two-point function $\mathbb{P}_\beta(0 \leftrightarrow x)$ from the corresponding upper bounds on the *restricted* two-point function $\mathbb{P}_\beta(0 \xleftrightarrow{\Lambda_n} x)$ proven in Proposition 2.7.

Proposition 2.10. *Let $J : \mathbb{H}_L^d \rightarrow [0, \infty)$ be a symmetric, integrable function, let $0 < \alpha < d$, and suppose that there exist positive constants c and C such that $c\|x\|^{-d-\alpha} \leq J(x) \leq C\|x\|^{-d-\alpha}$ for*

every $x \in \mathbb{H}_L^d \setminus \{0\}$. Then there exists a positive constant $A = A(d, L, \alpha, c, C)$ such that

$$\frac{1}{|\Lambda_n|} \sum_{x \in \Lambda_n} \mathbb{P}_{\beta_c}(0 \leftrightarrow x) \leq \frac{A}{\beta_c} L^{-(d-\alpha)n}$$

for every $n \geq 0$.

The proof of this proposition will employ the *BK inequality* and notion of the *disjoint occurrence* of events; we refer the reader to e.g. [22, Chapter 2.3] for relevant background.

Proof of Proposition 2.10. We have by Proposition 2.7 that there exists a positive constant $A_1 = A_1(d, L, \alpha, c)$ such that

$$\sum_{x \in \Lambda_n} \mathbb{P}_{\beta_c}(0 \xleftrightarrow{\Lambda_n} x) \leq \frac{A_1}{\beta_c} L^{\alpha n}. \quad (2.14)$$

Fix $n \geq 0$ and $x \in \mathbb{H}_L^d$ with $\langle x \rangle = L^n$. We have trivially that

$$\mathbb{P}_{\beta_c}(0 \leftrightarrow x) \leq \mathbb{P}_{\beta_c}(0 \xleftrightarrow{\Lambda_n} x) + \sum_{k=0}^{\infty} \mathbb{P}_{\beta_c}(\{0 \xleftrightarrow{\Lambda_{n+k+1}} x\} \setminus \{0 \xleftrightarrow{\Lambda_{n+k}} x\}). \quad (2.15)$$

Let $k \geq 0$ and suppose that the event $\{0 \xleftrightarrow{\Lambda_{n+k+1}} x\} \setminus \{0 \xleftrightarrow{\Lambda_{n+k}} x\}$ holds, so that there exists a simple path γ connecting 0 to x that visits Λ_{n+k+1} but not Λ_{n+k+2} . Let z be the first point of Λ_{n+k+1} visited by γ and let y be the point of Λ_{n+k} visited immediately before γ visits z . Then the portions of γ up to first visiting y , the edge $\{y, z\}$, and the portion of γ after first visiting z are disjoint witnesses for the events $\{0 \xleftrightarrow{\Lambda_{n+k}} y\}$, $\{\{y, z\} \text{ open}\}$, and $\{z \xleftrightarrow{\Lambda_{n+k+1}} x\}$. Thus, we have by a union bound and the BK inequality that

$$\begin{aligned} & \mathbb{P}_{\beta_c}(\{0 \xleftrightarrow{\Lambda_{n+k+1}} x\} \setminus \{0 \xleftrightarrow{\Lambda_{n+k}} x\}) \\ & \leq \sum_{y \in \Lambda_{n+k}} \sum_{z \in \Lambda_{n+k+1} \setminus \Lambda_{n+k}} \mathbb{P}_{\beta_c} \left(\{0 \xleftrightarrow{\Lambda_{n+k}} y\} \circ \{\{y, z\} \text{ open}\} \circ \{z \xleftrightarrow{\Lambda_{n+k+1}} x\} \right) \\ & \leq \sum_{y \in \Lambda_{n+k}} \sum_{z \in \Lambda_{n+k+1} \setminus \Lambda_{n+k}} \mathbb{P}_{\beta_c}(0 \xleftrightarrow{\Lambda_{n+k}} y) \mathbb{P}_{\beta_c}(\{y, z\} \text{ open}) \mathbb{P}_{\beta_c}(z \xleftrightarrow{\Lambda_{n+k+1}} x) \end{aligned}$$

for every $k \geq 0$. Using that $\{y, z\}$ is open with probability $1 - e^{-\beta_c J(y-z)} \leq C\beta_c L^{-(d+\alpha)(n+k+1)}$, we deduce from (2.14) that

$$\begin{aligned} & \mathbb{P}_{\beta_c}(\{0 \xleftrightarrow{\Lambda_{n+k+1}} x\} \setminus \{0 \xleftrightarrow{\Lambda_{n+k}} x\}) \\ & \leq C\beta_c L^{-(d+\alpha)(n+k+1)} \sum_{y \in \Lambda_{n+k}} \mathbb{P}_{\beta_c}(0 \xleftrightarrow{\Lambda_{n+k}} y) \sum_{z \in \Lambda_{n+k+1}} \mathbb{P}_{\beta_c}(z \xleftrightarrow{\Lambda_{n+k+1}} x) \\ & \leq \frac{CA_1^2}{\beta_c} L^{-(d+\alpha)(n+k+1)} L^{\alpha(n+k)} L^{\alpha(n+k+1)} \leq \frac{CA_1^2}{\beta_c} L^{-(d-\alpha)(n+k+1)} \end{aligned}$$

for every $k \geq 0$. Substituting this inequality into (2.15), summing over $k \geq 0$ and using Corol-

lary 2.3 yields that there exists a constant $A_2 = A_2(d, L, \alpha, c, C)$ such that

$$\mathbb{P}_{\beta_c}(0 \leftrightarrow x) \leq \mathbb{P}_{\beta_c}(0 \xleftrightarrow{\Lambda_n} x) + \frac{A_2}{\beta_c} \langle x \rangle^{-d+\alpha} \quad (2.16)$$

for every $n \geq 0$ and every $x \in \mathbb{H}_L^d$ with $\langle x \rangle = L^n$. The claim follows by summing over $x \in \Lambda_n$ and applying (2.14) again. \square

Remark 2.11. All the results of Sections 2.1 and 2.2 apply equally well to long-range percolation on \mathbb{Z}^d as defined in [30]. In order to generalize the upper bound Theorem 1.1 to this setting, it would suffice to compare critical connection probabilities inside a box and inside the full space as we have done here. Unfortunately we are not aware of any good techniques to do this for long-range percolation on \mathbb{Z}^d at present.

2.4 Lower bounds

We now prove the lower bounds of Theorem 1.1. We begin with the following proposition.

Proposition 2.12. *Let $J : \mathbb{H}_L^d \rightarrow [0, \infty)$ be a symmetric, integrable function, let $0 < \alpha < d$, and suppose that there exists a constant C such that $J(x) \leq C\|x\|^{-d-\alpha}$ for every $x \in \mathbb{H}_L^d \setminus \{0\}$. Then*

$$\beta_c \geq \frac{L^\alpha - 1}{C} \quad \text{and} \quad \mathbb{E}_{\beta_c} |K_n| = \sum_{x \in \Lambda_n} \mathbb{P}_{\beta_c}(0 \xleftrightarrow{\Lambda_n} x) \geq L^{\alpha n}$$

for every $n \geq 0$.

Proof of Proposition 2.12. Following Duminil-Copin and Tassion [16], we define for each finite set S and $\beta \geq 0$ the quantity

$$\phi_\beta(S) = \sum_{x \in S} \sum_{y \notin S} \left(1 - e^{-\beta J(y-x)}\right) \mathbb{P}_\beta(0 \xleftrightarrow{S} x).$$

It is proven in [16] that the critical parameter β_c can be characterised alternatively as

$$\beta_c = \inf \left\{ \beta \geq 0 : \phi_\beta(S) \geq 1 \text{ for every finite } S \right\}.$$

Since $\phi_\beta(S)$ is a continuous function of β when S is finite, it follows in particular that $\phi_{\beta_c}(S) \geq 1$ for every finite set S . Using the inequality $1 - e^{-x} \leq x$ we have for each $x \in \Lambda_n$ that

$$\sum_{y \notin \Lambda_n} \left(1 - e^{-\beta_c J(y-x)}\right) \leq \sum_{r=n}^{\infty} \sum_{y \in \Lambda_{r+1} \setminus \Lambda_r} \left(1 - e^{-\beta_c J(y-x)}\right) \leq \sum_{r=n}^{\infty} \sum_{y \in \Lambda_{r+1} \setminus \Lambda_r} \beta_c J(y-x)$$

for every $n \geq 0$ and $x \in \Lambda_n$ and hence that

$$\sum_{y \notin \Lambda_n} \left(1 - e^{-\beta_c J(y-x)}\right) \leq C\beta_c \sum_{r=n}^{\infty} L^{d(r+1)} L^{-(r+1)(d+\alpha)} = \frac{C\beta_c}{1 - L^{-\alpha}} L^{-\alpha(n+1)}$$

for every $n \geq 0$ and $x \in \Lambda_n$ by our assumptions on J . Thus, we deduce that

$$\sum_{x \in \Lambda_n} \mathbb{P}_{\beta_c}(0 \xleftrightarrow{\Lambda_n} x) \geq \frac{1 - L^{-\alpha}}{C\beta_c} L^{\alpha(n+1)} \phi_{\beta_c}(\Lambda_n) \geq \frac{1 - L^{-\alpha}}{C\beta_c} L^{\alpha(n+1)} \quad (2.17)$$

for every $n \geq 0$. Noting that the left hand side is 1 when $n = 0$ it follows in particular that

$$\beta_c \geq \frac{L^\alpha - 1}{C} \quad \text{and hence that} \quad \sum_{x \in \Lambda_n} \mathbb{P}_{\beta_c}(0 \xleftrightarrow{\Lambda_n} x) \geq L^{\alpha n} \quad (2.18)$$

for every $n \geq 0$ as claimed. \square

We now have all the ingredients in place to complete the proof of our main theorem.

Proof of Theorem 1.1. The upper bound follows immediately from Proposition 2.10 together with the lower bound on β_c provided by Proposition 2.12. The lower bound follows from Proposition 2.12 together with Lemma 2.8 which yields that there exist positive constants a_1 and a_2 such that

$$\sum_{x \in \Lambda_{n+1} \setminus \Lambda_n} \mathbb{P}_{\beta_c}(0 \xleftrightarrow{\Lambda_{n+1}} x) \geq a_1 \beta_c L^{-\alpha n} (\mathbb{E}_{\beta_c} |K_n|)^2 \geq a_2 L^{\alpha n}$$

for every $n \geq 0$ as claimed. \square

2.5 Consequences for the tail of the volume

We now apply the results of the previous sections to prove Corollaries 1.4 and 1.5, which establish respectively that the model has mean-field critical behaviour when $\alpha < d/3$ and does *not* have mean-field critical behaviour when $\alpha > d/3$.

Proof of Corollary 1.4. We want to prove that the diagrammatic sum $\nabla_{\beta_c} := \sum_{x, y \in \mathbb{H}_L^d} \mathbb{P}_{\beta_c}(0 \leftrightarrow x) \mathbb{P}_{\beta_c}(x \leftrightarrow y) \mathbb{P}_{\beta_c}(y \leftrightarrow 0)$ is finite. For each $n \geq 0$ define

$$\nabla_n := \sum_{x, y \in \Lambda_n} \mathbb{P}_{\beta_c}(0 \leftrightarrow x) \mathbb{P}_{\beta_c}(x \leftrightarrow y) \mathbb{P}_{\beta_c}(y \leftrightarrow 0).$$

Then we have by symmetry that $\nabla_0 = 1$ and

$$\begin{aligned} \nabla_{n+1} &= \nabla_n + \sum_{x, y \in \Lambda_{n+1} \setminus \Lambda_n} \mathbb{P}_{\beta_c}(0 \leftrightarrow x) \mathbb{P}_{\beta_c}(x \leftrightarrow y) \mathbb{P}_{\beta_c}(y \leftrightarrow 0) \\ &\quad + 2 \sum_{x \in \Lambda_n} \sum_{y \in \Lambda_{n+1} \setminus \Lambda_n} \mathbb{P}_{\beta_c}(0 \leftrightarrow x) \mathbb{P}_{\beta_c}(x \leftrightarrow y) \mathbb{P}_{\beta_c}(y \leftrightarrow 0) \end{aligned}$$

for every $n \geq 0$. We have by Corollary 1.2 that there exists a constant A such that

$$\begin{aligned} \sum_{x, y \in \Lambda_{n+1} \setminus \Lambda_n} \mathbb{P}_{\beta_c}(0 \leftrightarrow x) \mathbb{P}_{\beta_c}(x \leftrightarrow y) \mathbb{P}_{\beta_c}(y \leftrightarrow 0) &\leq A^2 L^{-2(d-\alpha)(n+1)} \sum_{x, y \in \Lambda_{n+1}} \mathbb{P}_{\beta_c}(x \leftrightarrow y) \\ &\leq A^3 L^{-2(d-\alpha)(n+1)} L^{d(n+1)} L^{\alpha(n+1)} = A^3 L^{(3\alpha-d)(n+1)} \end{aligned}$$

and

$$\begin{aligned} \sum_{x \in \Lambda_n} \sum_{y \in \Lambda_{n+1} \setminus \Lambda_n} \mathbb{P}_{\beta_c}(0 \leftrightarrow x) \mathbb{P}_{\beta_c}(x \leftrightarrow y) \mathbb{P}_{\beta_c}(y \leftrightarrow 0) &\leq A^2 L^{-2(d-\alpha)(n+1)} \sum_{x \in \Lambda_n} \mathbb{P}_{\beta_c}(0 \leftrightarrow x) \\ &\leq A^3 L^{(3\alpha-d)(n+1)} \end{aligned}$$

for every $n \geq 0$. It follows by induction that

$$\nabla_{n+1} \leq \nabla_n + 3A^3 L^{(3\alpha-d)(n+1)} \leq 3A^3 \sum_{k=0}^{n+1} L^{(3\alpha-d)(k+1)} \quad (2.19)$$

for every $n \geq 0$, and hence that $\nabla_{\beta_c} = \lim_{n \rightarrow \infty} \nabla_n < \infty$ when $\alpha < d/3$ as claimed. \square

Remark 2.13. In forthcoming work [28] we give a new, more quantitative derivation of mean-field critical behaviour from the triangle condition, which allows us to deduce from (2.19) that mean-field critical behaviour holds up to polylogarithmic factors when $\alpha = d/3$.

We now prove Corollary 1.5. The proof will rely on the following special case of the rigorous hyperscaling inequality of [30, Theorem 2.1], which is a consequence of Theorem 2.1.

Theorem 2.14. *There exists a universal constant C such that the following holds. Let $d \geq 1$, $L \geq 2$, and let $J : \mathbb{H}_L^d \rightarrow [0, \infty)$ be symmetric and integrable. Let $\beta \geq 0$, and suppose that there exist constants $A < \infty$ and $0 \leq \theta \leq 1/2$ such that $\mathbb{P}_\beta(|K| \geq n) \leq An^{-\theta}$ for every $\lambda > 0$. Then*

$$\sum_{x \in \Lambda} \mathbb{P}_\beta(0 \leftrightarrow x) \leq CA^{2/(1+\theta)} |\Lambda|^{(1-\theta)/(1+\theta)}$$

for every every finite set $\Lambda \subseteq \mathbb{H}_L^d$.

Proof of Corollary 1.5. Let $\alpha > d/3$ and suppose for contradiction that the exponent δ is well-defined and satisfies $\delta < (d+\alpha)/(d-\alpha)$. Thus, if we fix θ such that $(d-\alpha)/(d+\alpha) < \theta < (1/\delta) \wedge (1/2)$ then there exists a positive constant A_1 such that

$$\mathbb{P}_{\beta_c}(|K| \geq n) \leq A_1 n^{-\theta}$$

for every $n \geq 1$. It follows Theorem 2.14 that there exists a constant A_2 such that

$$\mathbb{E}_{\beta_c} |K_n| \leq A_2 |\Lambda_n|^{(1-\theta)/(1+\theta)} = A_2 L^{d(1-\theta)/(1+\theta)n} \quad (2.20)$$

for every $n \geq 0$. Since $\theta > (d-\alpha)/(d+\alpha)$ we have that $(1-\theta)/(1+\theta) < \alpha/d$ so that (2.20) contradicts Theorem 1.1 when n is sufficiently large. \square

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