High-dimensional near-critical percolation and the torus plateau

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Abstract

We consider percolation on $\mathbb{Z}^d$ and on the d-dimensional discrete torus, in dimensions $d \geq 11$ for the nearest-neighbour model and in dimensions $d > 6$ for spread-out models. For $\mathbb{Z}^d$, we employ a wide range of techniques and previous results to prove that there exist positive constants c and C such that the slightly subcritical two-point function and one-arm probabilities satisfy

$$\mathbb{P}_{p_c-\varepsilon}(0 \leftrightarrow x) \leq \frac{C}{\|x\|^{d-2}} e^{-c\varepsilon^{1/2}\|x\|} \quad \text{and} \quad \frac{c}{r^2} e^{-C\varepsilon^{1/2} r} \leq \mathbb{P}_{p_c-\varepsilon}\Big(0 \leftrightarrow \partial[-r,r]^d\Big) \leq \frac{C}{r^2} e^{-c\varepsilon^{1/2} r}.$$

Using this, we prove that throughout the critical window the torus two-point function has a "plateau," meaning that it decays for small x as $\|x\|^{-(d-2)}$ but for large x is essentially constant and of order $V^{-2/3}$ where V is the volume of the torus. The plateau for the two-point function leads immediately to a proof of the torus triangle condition, which is known to have many implications for the critical behaviour on the torus, and also leads to a proof that the critical values on the torus and on $\mathbb{Z}^d$ are separated by a multiple of $V^{-1/3}$. The torus triangle condition and the size of the separation of critical points have been proved previously, but our proofs are different and are direct consequences of the bound on the $\mathbb{Z}^d$ two-point function. In particular, we use results derived from the lace expansion on $\mathbb{Z}^d$, but in contrast to previous work on high-dimensional torus percolation we do not need or use a separate torus lace expansion.

Keywords: percolation; lace expansion; two-point function; one-arm exponent; triangle condition; torus plateau.

MSC2010 Classifications: 05C80, 60K35, 82B27, 82B43.

Contents

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1 Introduction and results

1.1 Introduction

Percolation on $\mathbb{Z}^d$ has been intensively studied by mathematicians and physicists since the 1950s as a fundamental model of a phase transition. Of particular interest is the universal critical behaviour in the vicinity of the critical value p_c. From a mathematical perspective, the critical behaviour has been established for certain 2-dimensional models using the breakthroughs enabled by conformal invariance and the Schramm–Loewner evolution [45,46], and for a wide class of models above the upper critical dimension $d = 6$ using the lace expansion [21,25]. The critical behaviour in intermediate dimensions $d = 3, 4, 5, 6$ remains a major challenge for probability theory, which at present appears not to have adequate tools even to approach the problem. Considerable progress has been also made in the understanding of the finite-size scaling associated with critical percolation on a high-dimensional discrete torus. In this paper, we consider percolation in dimensions $d > 6$, both on $\mathbb{Z}^d$ and on the torus.

The role of $d = 6$ as the upper critical dimension for percolation was first pointed out by Toulouse [48]. The meaning of "upper critical dimension" is that the critical exponents for percolation on $\mathbb{Z}^d$ in dimensions $d > 6$ are predicted to be the same as for percolation on a tree (known as mean-field theory), whereas for $d < 6$ they are not. Critical exponents for $d = 6$ are predicted to have logarithmic corrections to mean-field behaviour [15]. Various one-sided mean-field bounds for critical exponents, such as $\gamma \geq 1$, $\beta \leq 1$, and $\delta \geq 2$, have been proven to hold in all dimensions [2, 3]. In addition, results implying that mean-field behaviour cannot apply in dimensions $d < 6$ have been obtained in [12, 47] (see also [8]). In an important paper in 1984, Aizenman and Newman [2] identified a condition predicted to be valid for $d > 6$—the *triangle condition*—as a sufficient condition for $\gamma = 1$, which is mean-field behaviour for the expected cluster size (also called the susceptibility). Then Barsky and Aizenman [3] proved that the triangle condition also implies that $\beta = 1$ (exponent for the percolation probability) and $\delta = 2$ (exponent for the magnetisation). See also the recent paper [33] for alternative proofs of these results.

In 1990, Hara and Slade [21] derived their lace expansion for bond percolation and used it to verify the triangle condition for the nearest-neighbour model in sufficiently large dimensions ($d \geq 19$ is large enough [22]). Later, Fitzner and van der Hofstad [16] extended this to all $d \geq 11$. An extension to $d > 6$ has not yet been possible, and seems to be impossible without the introduction of some significant new idea, due to the fact that convergence of the lace expansion is proved using a small parameter which is closely related to the triangle diagram and which is not believed to be small in dimensions close to but above 6. On the other hand, since the critical exponents are predicted to be *universal*, meaning that they take the same values for any symmetric short-range model in a given dimension d, it is natural to introduce models with a parameter that *can* be taken to be small in any fixed dimension $d > 6$. This was accomplished in [21], where the triangle condition was proved for a wide variety of sufficiently spread-out models in any dimension $d > 6$. A basic example of a spread-out model is a finite-range model of bond percolation on $\mathbb{Z}^d$ with long bonds, not just nearest-neighbour bonds, and the reciprocal of the degree provides a small parameter for convergence of the lace expansion in any dimension $d > 6$. Related results for long-range models have also been established in [13, 26].

Over the last thirty years, a large literature on high-dimensional percolation has emerged, using the convergence of the lace expansion as a starting point, typically both for sufficiently spread-out models in dimensions $d > 6$ and for the nearest-neighbour model in large enough dimensions. Reviews can be found in [25, 43]. In particular, Hara proved the square root decay of the mass (inverse correlation length) [18]; Hara, van der Hofstad and Slade [20] and Hara [19] proved that the critical two-point function has the Gaussian decay $|x|^{-(d-2)}$; Kozma and Nachmias [36] proved the mean-field behaviour r^{-2} for the one-arm exponent; and Chatterjee and Hanson [11] identified the decay of the critical two-point function in a half-space. We use these results to prove our main results for high-dimensional percolation on $\mathbb{Z}^d$. Following the methodoloy of [31], we also use the OSSS theory of decision trees [41], whose application to statistical mechanical models was pioneered by Duminil-Copin, Raoufi, and Tassion [14], to obtain a new differential inequality which facilitates the transfer of certain estimates at the critical point to estimates at nearby subcritical points.

Our results for $\mathbb{Z}^d$ consist of an upper bound of the form $|x|^{-(d-2)} \exp[-c|p-p_c|^{1/2}|x|]$ for the slightly subcritical two-point function, and upper and lower bounds of the form $r^{-2} \exp[-c|p-p_c|^{1/2} r]$ for the (extrinsic) one-arm probability. We stress for the avoidance of doubt that the inclusion of these sharp exponential factors for $p < p_c$, with the square root in the exponent, requires substantial new ideas and is not a minor extension of the previous results.

In a separate line of research initiated by Borgs, Chayes, van der Hofstad, Slade, and Spencer in [5, 6], the critical behaviour of percolation on a discrete d-dimensional torus has been studied in depth, both for the nearest-neighbour model with d sufficiently large and for sufficiently spread-out models in dimensions $d > 6$. There is a triangle condition for the torus (and indeed for general high-dimensional transient graphs) which implies that percolation on the torus behaves in many respects like the Erdős–Rényi random graph. In particular, the notion of a critical point which is valid for $\mathbb{Z}^d$ is replaced by the notion of a critical scaling window of p values. These ideas are developed in [5, 23–25, 28], and are based on the verification of the triangle condition in high-dimensions via a separate lace expansion on the torus as opposed to on $\mathbb{Z}^d$ [6].

Our first result for the torus is a proof that the torus two-point function has a "plateau." The plateau refers to the fact that the torus two-point function within and slightly below the critical window decays for small x like its $\mathbb{Z}^d$ counterpart before levelling off at a constant value for large x. Related plateaux have been proven to exist for simple random walk (the lattice Green function) for $d > 2$ [44, 49, 50], for weakly self-avoiding walk for $d > 4$ [44], and partially for the Ising model for $d > 4$ [42]. As we show, the plateau for the torus percolation two-point function is highly effective for the analysis of torus percolation (a similar situation applies for weakly self-avoiding walk on a torus for $d > 4$ [40]). In particular, it directly gives a proof of the torus triangle condition, a proof that throughout the critical window the torus susceptibility is of the order of the cube root of the torus volume, and a proof that the $\mathbb{Z}^d$ critical value lies in the critical window for the torus. The triangle condition was proved previously via a separate lace expansion on the torus [6] which we do not need, the behaviour of the torus susceptibility was obtained previously in [5], while the verification that the $\mathbb{Z}^d$ critical value lies in the window was the main topic of [23, 24]. Our work establishes these results directly by applying results on $\mathbb{Z}^d$ rather than via a separate torus lace expansion.

1.2 The models

Let $\mathbb{G} = (\mathbb{V}, \mathbb{B})$ be a finite or infinite graph with vertex set $\mathbb{V}$ and edge (bond) set $\mathbb{B}$. Given $p \in [0, 1]$, we consider independent and identically distributed Bernoulli random variables associated to each bond $b \in \mathbb{B}$, taking the value "open" with probability p and the value "closed" with probability $1 - p$. We denote the probability of an event E by $\mathbb{P}_p(E)$ and the expectation of a random variable X by $\mathbb{E}_p X$.

We consider four different choices of $\mathbb{G}$:

(i) Nearest-neighbour model on $\mathbb{Z}^d$: $\mathbb{V} = \mathbb{Z}^d$ and $\mathbb{B}$ consists of all pairs $\{x, y\}$ with $\|x - y\|_1 = 1$. We assume that $d \geq 11$.

(ii) Spread-out model on $\mathbb{Z}^d$: $\mathbb{V} = \mathbb{Z}^d$ and $\mathbb{B}$ consists of all pairs $\{x, y\}$ with $\|x - y\|_1 \leq L$, for some (large) fixed $L > 1$. We assume that $d > 6$ and L is sufficiently large depending on d.

(iii) Nearest-neighbour model on the torus $\mathbb{T}_r^d$: $\mathbb{V} = (\mathbb{Z}/r\mathbb{Z})^d$ for (large) period $r > 2$ and $\mathbb{B}$ consists of all pairs $\|x - y\|_1 = 1$ with addition mod r. We assume that $d \geq 11$. We write $V = r^d$ for the *volume* of the torus and are interested in the limit $r \to \infty$.

(iv) Spread-out model on the torus $\mathbb{T}_r^d$: $\mathbb{V} = (\mathbb{Z}/r\mathbb{Z})^d$ for (large) period $r > 2L$ with (large) fixed $L > 1$ and $\mathbb{B}$ consists of all pairs $\|x - y\|_1 \leq L$ with addition mod r. We assume that $d > 6$ and L is sufficiently large depending on d.

Notation: We use c, C for positive constants that can vary from line to line. We write $f \sim g$ to mean $\lim f/g = 1$, $f \preceq g$ to mean $f \leq Cg$, $f \succeq g$ to mean $g \preceq f$, and $f \asymp g$ to mean that $f \preceq g \preceq f$, where we require that all constants depend only on the dimension d and the spread-out parameter L. Constants depending on additional parameters will be denoted by subscripts, so that e.g. "$f(n, \lambda) \succeq_\lambda g(n, \lambda)$ for every $n \geq 1$ and $\lambda > 0$" means that for each $\lambda > 0$ there exists a positive constant c_λ such that $f(n,\lambda) \geq c_\lambda g(n,\lambda)$ for every $n \geq 1$. For $a, b \in \mathbb{R}$, we write $a \vee b = \max\{a, b\}$. We write $\Lambda_r^d = [-r, r]^d \cap \mathbb{Z}^d$ for the box of side length $2r+1$ in $\mathbb{Z}^d$, omitting the d when it is unambiguous to do so. The *boundary* $\partial\Lambda_r^d$ of Λ_r^d consists of the points $x \in \mathbb{Z}^d$ with $\|x\|_\infty = r$. To avoid dividing by zero, we use the *Japanese bracket* notation $\langle x \rangle := \|x\|_\infty \vee 1$ for $x \in \mathbb{Z}^d$. Our notational convention is that objects on the torus have a label $\mathbb{T}$, so the two-point function on the torus is written as $\tau_p^{\mathbb{T}}(x)$. Generally, objects without the torus label are for $\mathbb{Z}^d$.

For $\mathbb{Z}^d$, the restrictions on the dimension d and the range L described in (i) and (ii) above are so that previous lace expansion results can be applied. We apply these $\mathbb{Z}^d$ results to the torus under the same restriction. We apply existing results obtained via the lace expansion for $\mathbb{Z}^d$, and do not need to revisit or further develop the expansion itself (nor do we use a separate torus expansion as in [6]). More precisely, our results hold for any $d > 6$ and $L \geq 1$ such that the *two-point function* $\tau_p(x) := \mathbb{P}_p(0 \leftrightarrow x)$ satisfies

$$\tau_{p_c}(x) \asymp \langle x \rangle^{-d+2}. \tag{T}$$

Here and throughout the paper we write p_c for the critical value for percolation on $\mathbb{Z}^d$. The estimate (T) was proven to hold in settings (i) and (ii) above by Hara, van der Hofstad, and Slade [19, 20] and Fitzner and van der Hofstad [16]. Our results also rely crucially on those of Kozma and Nachmias [36] and Chatterjee and Hanson [11], who worked under the same assumptions. We will refer to (i) and (ii) collectively as high-dimensional percolation on $\mathbb{Z}^d$, and to (iii) and (iv) as high-dimensional percolation on the torus.

1.3 Results for $\mathbb{Z}^d$

The two-point function. Our first result, and main tool for all our further results, concerns the transition from exponential to power-law decay for the two-point function.

Theorem 1.1. *Let $d > 6$ and suppose that* (T) *holds on $\mathbb{Z}^d$. There exist positive constants c and C such that*

$$\tau_p(x) \leq \frac{C}{\langle x \rangle^{d-2}} \exp\left[-c(p_c - p)^{1/2}\langle x \rangle\right]. \tag{1.1}$$

for every $p \in (0, p_c]$ and $x \in \mathbb{Z}^d$.

Theorem 1.1 is a partial substantiation, via a one-sided bound, of the generally unproven guiding principle in the scaling theory for critical phenomena in statistical mechanical models on $\mathbb{Z}^d$ that two-point functions near a critical point generically have decay of the form

$$\tau_p(x) \approx \frac{1}{\langle x \rangle^{d-2+\eta}} g(|x|/\xi(p)) \tag{1.2}$$

in some reasonable meaning for "$\approx$", when $\langle x \rangle$ is of roughly the same order as the correlation length $\xi(p)$ and p is close to its critical value p_c. The universal critical exponent η depends on dimension, the correlation length $\xi(p) \approx (1 - p/p_c)^{-\nu}$ diverges as $p \to p_c$ with a dimension-dependent universal critical exponent ν, and g is a function with rapid decay. In high dimensions, $\eta = 0$ and $\nu = \frac{1}{2}$. The role of (1.2) in the derivation of scaling relations between critical exponents, such as Fisher's relation $\gamma = (2 - \eta)\nu$, can be found in [17, Section 9.2].

Let us now summarise how these results compare to previous results. For high-dimensional percolation on $\mathbb{Z}^d$, (T) is known in the more precise asymptotic form

$$\tau_{p_c}(x) \sim A_\tau \frac{1}{\langle x \rangle^{d-2}} \qquad (x \to \infty) \tag{1.3}$$

for some positive constant A_τ, with an explicit error estimate [19, 20]. For all dimensions $d \geq 2$ and for $p < p_c$ there is exponential decay, in the sense that the *mass* (or *inverse correlation length*)

$$m(p) = -\lim_{n\to\infty} \frac{1}{n} \log \tau_p(ne_1) = -\sup_{n\geq 1} \frac{1}{n} \log \tau_p(ne_1), \tag{1.4}$$

is strictly positive for $p < p_c$ [17]. In fact, more is known in general, and it is shown in [17, Proposition 6.47)] that there is a constant c such that

$$cp^d \frac{1}{\|x\|_1^{4d(d-1)}} e^{-m(p)\|x\|_1} \leq \tau_p(x) \leq e^{-m(p)\|x\|_\infty} \tag{1.5}$$

for all $d \geq 2$, $p \in [0, 1]$, $x \in \mathbb{Z}^d$. Hara [18] proved that in high-dimensional percolation on $\mathbb{Z}^d$ the mass satisfies the asymptotic formula

$$m(p) \sim A_m(p_c - p)^{1/2} \qquad (p \to p_c^-) \tag{1.6}$$

for some positive constant A_m. With (1.5), this immediately implies that there is a positive constant c such that

$$\tau_p(x) \leq \exp\left[-c(p_c - p)^{1/2}\|x\|_\infty\right] \tag{1.7}$$

for every $p < p_c$ and $x \in \mathbb{Z}^d$. Theorem 1.1 improves this bound by a polynomial term which is believed to be sharp. No such estimate on the slightly subcritical two-point function had previously been proven for high-dimensional percolation on $\mathbb{Z}^d$.

For weakly self-avoiding walk in dimension $d > 4$, a result analogous to Theorem 1.1 was proved only recently in [44]; that proof does not extend to percolation and our methods are different and do not extend to weakly self-avoiding walk. On the basis of the asymptotic behaviour for the lattice Green function presented in [39], we believe that the precise asymptotic behaviour of the subcritical two-point function for fixed $p < p_c$ and for $d > 6$ takes the form

$$\tau_p(x) \sim A_{p,\hat{x}}\, m(p)^{(d-3)/2} \frac{1}{|x|_p^{(d-1)/2}} e^{-m(p)|x|_p} \qquad (|x|_p \to \infty), \tag{1.8}$$

with $|\cdot|_p$ a p-dependent norm on $\mathbb{R}^d$ (*not* the ℓ_p norm) which interpolates monotonically between the ℓ_1 and ℓ_2 norms as p increases over the interval $(0, p_c)$, and with an amplitude $A_{p,\hat{x}}$ that approaches a nonzero constant (independent of the direction $\hat{x}$) as $p \uparrow p_c$. The *Ornstein–Zernike decay* $\tau_p(n, 0, \ldots, 0) \sim C_p n^{-(d-1)/2} e^{-m(p)n}$ was proved by Campanino, Chayes and Chayes [10] for $p < p_c$ in dimensions $d \geq 2$, but without control of the p-dependence of the constant C_p as $p \to p_c$. From this, by taking n large we see that (1.1) can only hold if the constant c is such that $c(p_c - p)^{1/2}$ is strictly smaller than $m(p)$. The polynomial factors in (1.8) can be rearranged as $(m(p)|x|_p)^{(d-3)/2}|x|_p^{-(d-2)}$, so when $|x|_p$ is comparable to the correlation length $m(p)^{-1}$ the asymptotic estimate (1.8) becomes consistent with (1.1).

The one-arm probability. Our second main result for percolation on $\mathbb{Z}^d$ concerns the probability that the cluster of the origin has a large radius in slightly subcritical percolation. Recall that $\Lambda_r = \Lambda_r^d = [-r,r]^d \cap \mathbb{Z}^d$ is the box of side length $2r+1$ and $\partial\Lambda_r$ is its boundary.

Theorem 1.2. *Let $d > 6$ and suppose that* (T) *holds on $\mathbb{Z}^d$. There exist positive constants c and C such that*

$$\frac{c}{r^2}\exp\left(-C(p_c-p)^{1/2}r\right) \leq \mathbb{P}_p\big(0 \leftrightarrow \partial\Lambda_r\big) \leq \frac{C}{r^2}\exp\left(-c(p_c-p)^{1/2}r\right) \tag{1.9}$$

for every $p_c/2 \leq p \leq p_c$ and $r \geq 1$.

The $p = p_c$ case of this theorem was proven by Kozma and Nachmias [36] and is used crucially in our proof. By (1.7) and a union bound,

$$\mathbb{P}_p\big(0 \leftrightarrow \partial\Lambda_r\big) \leq 2d(2r+1)^{d-1}\exp\left(-c(p_c-p)^{1/2}r\right), \tag{1.10}$$

for every $p < p_c$ and $r \geq 2$; the content of the upper bound of Theorem 1.2 is to identify the correct polynomial prefactor. Similar theorems have been established for the distribution of the *volume* and *intrinsic radius* of slightly subcritical percolation clusters in [32, Section 4]; these proofs apply to arbitrary transitive graphs satisfying the triangle condition and are much easier to prove than Theorem 1.2. See also [32, Section 5] for an overview what is expected to hold for *slightly supercritical* percolation in high dimensions.

1.4 Results for the torus

Percolation on the high-dimensional torus has received much attention in recent years [5,6,23–25,28], with considerable related work on hypercube percolation including [7,25,27,29]. This work has concentrated on the torus susceptibility and on questions with a flavour like those in the literature on the Erdős–Rényi random graph such as the cluster size distribution. Our principal focus here is on the torus two-point function $\tau_p^{\mathbb{T}}(x) := \mathbb{P}_p^{\mathbb{T}}(0 \leftrightarrow x)$ (for $x \in \mathbb{T}_r^d$) and its "plateau."

The *critical value* $p_{\mathbb{T}}$ for the torus is not uniquely defined. In [5, (1.7)], it is defined to be the unique solution, for fixed $\lambda > 0$, of the equation

$$\chi_{\mathbb{T}}(p_{\mathbb{T}}) = \lambda V^{1/3}, \tag{1.11}$$

where $\chi_{\mathbb{T}}(p) := \sum_{x\in\mathbb{T}} \tau_p^{\mathbb{T}}(x)$ is the torus susceptibility. Of course, $p_{\mathbb{T}}$ depends on λ, but only slightly and, as we discuss below, $p_{\mathbb{T}}$ is an effective critical point no matter which λ is chosen. The *torus triangle diagram* is

$$\mathsf{T}_p^{\mathbb{T}}(x) = \sum_{u,v\in\mathbb{T}_r^d} \tau_p^{\mathbb{T}}(u)\tau_p^{\mathbb{T}}(v-u)\tau_p^{\mathbb{T}}(x-v), \tag{1.12}$$

and the *torus triangle condition* is the statement that $\mathsf{T}_{p_{\mathbb{T}}}^{\mathbb{T}}(x)$ (which attains its maximum value when $x = 0$ by [2, Lemma 3.3]) is bounded by a constant independent of r and x. Extensive consequences of the torus triangle condition are derived in [5, 23–25]. These consequences usually require the stronger assumption that

$$\mathsf{T}_{p_{\mathbb{T}}}^{\mathbb{T}}(x) \leq \mathbb{1}(x=0) + a_0 \tag{1.13}$$

for $x \in \mathbb{T}_r^d$ and some small context-dependent constant a_0. We refer to this condition as the *a_0-strong torus triangle condition.* to work under the triangle condition with further work. The (strong) torus triangle condition is proved in [6] using a finite-graph version of the lace expansion under the assumption that either d is very large or $d > 6$ and L is large; we do not need or use the finite-graph lace expansion in this paper and give an alternate derivation of the torus triangle condition using results obtained from the lace expansion on $\mathbb{Z}^d$ together with our plateau estimates for the torus two-point function.

Despite the substantial progress on high-dimensional torus percolation, a detailed analysis of the behaviour of the torus two-point function $\tau_p^{\mathbb{T}}(x)$ within and below the critical window of width $V^{-1/3}$ about $p_{\mathbb{T}}$ has been missing until now. A sizeable physics literature for related models such as the Ising model (in dimensions $d > 4$) predicts the existence of a "plateau" for the torus two-point function, namely that within the critical window $\tau_p^{\mathbb{T}}(x)$ decays like the $\mathbb{Z}^d$ two-point function $\tau_{p_c}(x)$ for a certain volume-dependent range of x values, but beyond this range $\tau_p^{\mathbb{T}}(x)$ levels off at an approximately constant value which exceeds $\tau_{p_c}(x)$. For the Ising model this is discussed, e.g., in [42, 49, 50] and references therein. Different behaviour is expected for free boundary conditions, and has recently been proved for the Ising model in [9]. The plateau has recently been proven to exist for the simple random walk two-point function (lattice Green function) on the torus in all dimensions $d > 2$ [44, 49] and for weakly self-avoiding walk on the torus in dimensions $d > 4$ [44]. The plateau is applied in an essential way to analyse the weakly self-avoiding walk on a torus in dimensions $d > 4$ in [39]. The differences between free, bulk and periodic boundary conditions for percolation have been emphasised in [1], where the focus is on the maximal cluster size rather than the two-point function plateau.

Before stating our results on the torus two-point function, let us first recall some relevant background on the susceptibility. Let $\chi(p) := \sum_{x\in\mathbb{Z}^d} \tau_p(x)$ be the $\mathbb{Z}^d$ susceptibility, which is known [2,16,21] to satisfy the mean-field asymptotics

$$\chi(p) \asymp \frac{1}{1 - p/p_c} \qquad (p \to p_c^-) \tag{1.14}$$

for high-dimensional percolation on $\mathbb{Z}^d$.

By using the estimate (1.1) on the $\mathbb{Z}^d$ two-point function, we prove the following theorem. For notational convenience, we sometimes evaluate a $\mathbb{Z}^d$ two-point function at a point $x \in \mathbb{T}_r^d$, with the understanding that in this case we regard x as a point in $[-r/2, r/2)^d \cap \mathbb{Z}^d$. This occurs in the statement of Theorem 1.3.

Theorem 1.3 (The two-point function plateau)**.** *Let $d > 6$ and suppose that* (T) *holds on $\mathbb{Z}^d$.*

- **Below the scaling window:** *There exist positive constants c_1 and C_1 depending only on d and L such that*

$$\tau_p^{\mathbb{T}}(x) \le \tau_p(x) + C_1 \frac{\chi(p)}{V} \exp\left[-c_1 m(p) r\right] \tag{1.15}$$

  for every $r > 2$, every $x \in \mathbb{T}_r^d$, and every $p \in [0, p_c)$. Moreover, there exist positive constants A_1, A_2, c_2, and M such that

$$\tau_p^{\mathbb{T}}(x) \ge \tau_p(x) + c_2 \frac{\chi(p)}{V} \tag{1.16}$$

  for every $r > 2$, every $x \in \mathbb{T}_r^d$ with $\|x\|_\infty > M$, and every $p \in [p_c - A_1 V^{-2/d}, p_c - A_2 V^{-1/3}]$.

- **Inside the scaling window:** *For each $0 < \delta \le 1$ and $A > 0$, there exist positive constants r_0 and C_3 depending only on d, L, δ, and A such that*

$$\tau_p^{\mathbb{T}}(x) \le (1 + \delta)\tau_{p_c}(x) + \frac{C_3}{V^{2/3}} \tag{1.17}$$

for every $p \in [0, p_c + AV^{-1/3}]$, $r > r_0$, and $x \in \mathbb{T}_r^d$. In fact, the upper bound (1.17) holds for $p \le p_c$ with $\delta = 0$. In addition, there exists a positive constant M depending only on d and L, and a positive c_3 depending on d, L, and A such that

$$\tau_p^{\mathbb{T}}(x) \ge (1-\delta)\tau_{p_c}(x) + \frac{c_3}{V^{2/3}} \tag{1.18}$$

for every $r > r_0$, every $x \in \mathbb{T}_r^d$ with $\|x\|_\infty > M$, and every $p \in [p_c - AV^{-1/3}, p_c + AV^{-1/3}]$.

With a choice $A \ge A_2$, the above theorem gives upper and lower bounds on the two-point function throughout the range $p_c - A_1 V^{-2/d} \le p \le p_c + AV^{-1/3}$. Below the window the bounds involve $\chi(p)/V$, which is infinite at p_c, whereas within the window this constant term is replaced by $V^{-2/3}$.

The upper bound of (1.15) is essentially an immediate consequence of (1.1). The lower bound also uses (1.1), but requires a more involved argument inspired by the method used for weakly self-avoiding walk in [44]. For the estimates inside the scaling window, we also use the critical one-arm result of Kozma and Nachmias [36].

We emphasise that the susceptibility $\chi(p)$ appearing in this theorem is the susceptibility for $\mathbb{Z}^d$, not for the torus. The upper bound (1.17) at $p = p_c$ and with $\delta = 0$ is proved in [28, Theorem 1.7]; we complement the upper bound with a lower bound of the same order, and extend these bounds through the entire scaling window. A more compact though less precise version of (1.17)–(1.18) states that

$$\tau_p^{\mathbb{T}}(x) \asymp_A \frac{1}{\langle x\rangle^{d-2}} + \frac{1}{V^{2/3}} \tag{1.19}$$

for every $r \ge 2$, $x \in \mathbb{T}_r^d$, and $p_c - AV^{-1/3} \le p \le p_c + AV^{-1/3}$. Consequently, for $\langle x\rangle^{d-2} < V^{2/3}$ we have $\tau_p^{\mathbb{T}}(x) \asymp \langle x\rangle^{-(d-2)}$, whereas for $\langle x\rangle^{d-2} > V^{2/3}$ we have $\tau_p^{\mathbb{T}}(x) \asymp V^{-2/3}$. This is the plateau: the torus two-point function levels off at an approximately constant value once x is large enough.

There is in fact a hierarchy of plateaux extending (1.19). Consider $p = p_c - V^{-a}$ with $a \in (\frac{2}{d}, \frac{1}{3}]$. By (1.14) $\chi(p) \asymp V^a$, and by (1.6) $m(p)r \asymp V^{\frac{2}{d}-a} \to 0$ as $r \to \infty$. For such p, the plateau effect occurs as soon as $\langle x\rangle^{d-2} \succeq V^{1-a}$. When $a = \frac{2}{d}$ the constant terms in (1.15)–(1.16) are of order $r^{-(d-2)}$, which is the smallest order that $\tau_p(x)$ can achieve for $x \in \mathbb{T}_r^d$. If $a < \frac{2}{d}$ then $m(p)r \to \infty$ and the plateau effect is absent.

The following corollary of Theorem 1.3 shows that $\chi^{\mathbb{T}}(p) \asymp V^{1/3}$ for p in the scaling window. It shows that the correct transfer of the bounds on the two-point in Theorem 1.3 from below the window into the window is achieved by replacing the $\mathbb{Z}^d$ susceptibility by the torus susceptibility. The corollary reproduces a result of [5, 6] via quite different methods, and without any torus lace expansion.

Corollary 1.4. *For high-dimensional percolation on the torus $\mathbb{T}_r^d$, and for any $A > 0$ and any $p \in [p_c - AV^{-1/3}, p_c + AV^{-1/3}]$,*

$$\chi^{\mathbb{T}}(p) \asymp_A V^{1/3}. \tag{1.20}$$

Proof. This follows immediately by summation of (1.19) over $x \in \mathbb{T}_r^d$. Indeed summation of $\tau_{p_c}(x) \asymp \langle x\rangle^{-(d-2)}$ over the torus yields $r^2 = V^{2/d}$, which is smaller when $d > 6$ than the sum of the constant term which is $V \cdot V^{-2/3} = V^{1/3}$. □

The cube-root divergence in the volume given by Corollary 1.4 for periodic boundary conditions should be contrasted with the situation for free boundary conditions. For free boundary conditions, it is a corollary of the bounds on the finite-volume two-point function in [11, Theorem 1.2] that in our setting of high-dimensional percolation the susceptibility at $p = p_c$ diverges as r^2 rather than $r^{d/3}$. This is one setting where the controversy in the physics literature detailed, e.g., in [50], is rigorously resolved.

The torus triangle condition, stated in the next theorem, also follows relatively easily from Theorem 1.3, again using only the $\mathbb{Z}^d$ lace expansion and without the torus lace expansion from [6]. The results of [5, 23, 24] ultimately all rely on the establishment of the finite-graph triangle condition introduced in [5]. The triangle condition for a high-dimensional torus is proved in [6] by a lace expansion performed on finite graphs. We prove the triangle condition directly and simply from Theorem 1.3.

Theorem 1.5. *Let $d > 6$ and suppose that* (T) *holds on $\mathbb{Z}^d$.*

- ***Torus triangle condition:*** *There exist constants constants C_1, C_2, and λ such that if we define $p_{\mathbb{T}}$ via $\chi^{\mathbb{T}}(p_{\mathbb{T}}) = \lambda V^{1/3}$ then $|p_{\mathbb{T}} - p_c| \leq C_1 V^{-1/3}$ and $\mathsf{T}^{\mathbb{T}}_{p_{\mathbb{T}}}(x) \leq \mathbb{1}(x = 0) + C_2$ for every $r > 2$ and $x \in \mathbb{T}^d_r$.*
- *a_0**-strong torus triangle condition:** Fix any $a_0 > 0$. Then there exist constants d_0 and L_0 depending on a_0 such that if either $d \geq d_0$ or $d > 6$ and $L \geq L_0$ then there exist constants C_1 and λ such that if we define $p_{\mathbb{T}}$ via $\chi^{\mathbb{T}}(p_{\mathbb{T}}) = \lambda V^{1/3}$ then $|p_{\mathbb{T}} - p_c| \leq C_1 V^{-1/3}$ and $\mathsf{T}^{\mathbb{T}}_{p_{\mathbb{T}}}(x) \leq \mathbb{1}(x = 0) + a_0$ for every $r > 2$ and $x \in \mathbb{T}^d_r$.*

The *critical window* for the torus $\mathbb{T}^d_r$ was defined in [5] (see [5, Theorem 1.3]) to consist of the values of p lying within distance of order $V^{-1/3}$ from $p_{\mathbb{T}}$. It was not proven at that time that the critical value p_c for $\mathbb{Z}^d$ lies in the window, and it was not until several years later that this fact was proved [24]. In Theorem 1.3 we have defined the window as being centred at p_c rather than at $p_{\mathbb{T}}$. The following immediate corollary of Theorem 1.5 shows the equivalence of these two definitions of the window. The analysis of [5,24] relied on performing a separate lace expansion on the torus to establish the torus triangle condition, which our proof bypasses.

Corollary 1.6. *In the setting of Theorem 1.5, the $\mathbb{Z}^d$ critical point $p_c = p_c(\mathbb{Z}^d)$ lies in within a distance of order $V^{-1/3}$ from the torus critical point $p_{\mathbb{T}}$.*

Proof. This is an immediate consequence of Theorem 1.5. □

1.5 About the proof

We now give a brief overview of the structure of the paper and the proofs of our main theorems.

- In Sections 2 and 3 we prove our results concerning near-critical percolation on $\mathbb{Z}^d$, Theorems 1.1 and 1.2. Both theorems rely on the notion of *pioneer edges*, which are those through which the cluster of the origin enters some halfspace for the first time.
  - In Section 2 we formulate an estimate on the expected number of pioneer edges for a hyperplane, Theorem 2.3, which we then show to imply Theorems 1.1 and 1.2 in conjunction with the critical two-point estimate (T) and the critical one-arm results of Kozma and Nachmias [36].
  - In Section 3 we prove our main estimate on pioneer edges, Theorem 2.3. This proof has two components: First, in Section 3.1, we apply diagrammatic methods utilising the halfspace two-point function estimates of Chatterjee and Hanson [11] to prove *at criticality* that the expected number of pioneers for the hyperplane $\{x : x_1 = n\}$ is bounded by a constant independent of

n (Proposition 3.1). Using this together with the aforementioned one-arm estimates of Kozma and Nachmias [36], we deduce a power-law tail bound with exponent 3/2 on the *total* number of pioneer edges at criticality (Lemma 3.10). Second, in Section 3.2, we use the theory of decision trees and the OSSS inequality, which we review in Section 3.2.1, to obtain a differential inequality applying to the distribution of the total number of pioneers (Lemma 3.11). This differential inequality is of the same form as that obtained for the distribution of the radius by Menshikov [38] (see also [14]) and for the distribution of volume by Hutchcroft [31]. Using this differential inequality together with our results on the distribution of the number of pioneers at p_c, we deduce sharp estimates on the distribution of the total number of pioneers at $p_c - \varepsilon$ (Proposition 3.9) and conclude the proof of Theorem 2.3.

- In Section 4 we apply our $\mathbb{Z}^d$ results to prove the case of Theorem 1.3 in which p lies below the scaling window. This eventually leads us to easily derive in Section 4.4 the (two versions of the) torus triangle condition presented in Theorem 1.5. In order to apply the $\mathbb{Z}^d$ result of Theorem 1.1 to the torus we crucially rely on a coupling of percolation on $\mathbb{Z}^d$ and on the torus that was first introduced by Benjamini and Schramm [4] and developed extensively in the works of Heydenreich and van der Hofstad [23, 24]. The massive decay of $\tau_p(x)$ for $p < p_c$ then directly gives the upper-bound (1.15) while the lower bound requires fine control of diagrammatic estimates and thus occupies the bulk of this section. Much of this work follows the same general strategy used to analyse self-avoiding walk on the torus in [40] but differs in the details.
- Finally, in Section 5, we prove the part of Theorem 1.3 in which p lies inside the scaling window. The relevant lower bounds are easy consequences of the 'below the scaling window' estimates since $\tau_p^{\mathbb{T}}(x)$ is monotone in p. The upper bounds are proven first at p_c using the coupling between $\mathbb{Z}^d$ and $\mathbb{T}_r^d$ percolation as well as the extrinsic one-arm result from Kozma and Nachmias [36] using a variation on the methods of van der Hofstad and Sapozhnikov [28]. We then extend the result for $p \in (p_c, p_c + AV^{-1/3}]$ by the combination of an elementary coupling of Bernoulli percolation at different probabilities and of the input of the intrinsic one-arm exponent controlled in Kozma and Nachmias [35].

2 Near-critical percolation: Proof of Theorems 1.1–1.2

In this section, we prove Theorems 1.1–1.2 subject to Theorem 2.3, which concerns the expected number of *pioneer edges*.

2.1 Pioneer edges

For each $n \in \mathbb{Z}$, let S_n be the hyperplane $\{(y_1, \ldots, y_d) \in \mathbb{Z}^d : y_1 = n\}$ and let H_n be the halfspace $H_n = \{(y_1, \ldots, y_d) \in \mathbb{Z}^d : y_1 \geq n\}$. We will often write $H = H_0$ to lighten notation.

Definition 2.1. Given $x \in \mathbb{Z}^d$, we call an edge $\{y, z\} \in \mathbb{B}$ an *x-pioneer* if $x_1 < z_1$, $y_1 < z_1$, $\{y, z\}$ is open, and x is connected to y by an open path contained in the half-space $\{(w_1, \ldots, w_d) : w_1 < z_1\}$. That is, $\{y, z\}$ is an x-pioneer if z lies to the right of x and there exists an open path starting at x whose last edge is $\{y, z\}$ with the path lying strictly to the left of z at every previous time. For each $x \in \mathbb{Z}^d$ and $n \geq 1$ we define $\mathcal{P}_x(n)$ to be the set of x-pioneers $\{y, z\}$ with $y_1 < x_1 + n \leq z_1$ and define $\mathcal{P}_x = \bigcup_{n \geq 1} \mathcal{P}_x(n)$ to be the set of all x-pioneers. See Figure 1.

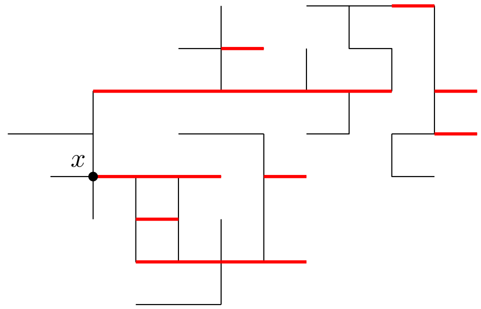

Figure 1: Pioneer edges of a cluster in $\mathbb{Z}^2$. Edges that are pioneers with respect to the distinguished vertex x are thick and red, while other edges are thin and black.

Since we are only interested in finite-range models, we have that $\mathcal{P}_x(n) \cap \mathcal{P}_x(m) = \emptyset$ when $m \geq n + L$ and hence that $\frac{1}{L}\sum_{n\geq 1}|\mathcal{P}_x(n)| \leq |\mathcal{P}_x| \leq \sum_{n\geq 1}|\mathcal{P}_x(n)|$. For each $p \in [0,1]$ and $n \geq 0$ we define $P_p(n) = \mathbb{E}_p|\mathcal{P}_0(n)|$, which may be infinite. We begin by noting that $P_p(n)$ satisfies the following elementary submultiplicativity property.

Lemma 2.2. *$P_p(n+m) \leq P_p(n)\max_{0\leq i\leq L-1} P_p(m-i)$ for every $p \in [0,1]$ and $n \geq 1$ and $m \geq L$.*

It is convenient to simplify the inequality of Lemma 2.2, as follows. Let $e_1 = (1,0,\ldots,0)$ be the basis vector in the positive horizontal direction. First we observe that if A_m denotes the event that each of the m unit-length edges of the horizontal path connecting 0 to me_1 are open then

$$\begin{aligned}\mathbb{E}_p|\mathcal{P}_0(n+m)| &\geq \mathbb{E}_p\Big[|\mathcal{P}_0(n+m)|\mathbb{1}(A_m)\Big] \geq \mathbb{E}_p\Big[|\mathcal{P}_{me_1}(n)|\mathbb{1}(A_m)\Big]\\ &\geq \mathbb{E}_p|\mathcal{P}_{me_1}(n)|p^m = \mathbb{E}_p|\mathcal{P}_0(n)|p^m,\end{aligned} \tag{2.1}$$

where we used the Harris–FKG inequality for the second inequality. A complementary bound can be obtained by a similar argument, with the result that

$$p^m P_p(n) \leq P_p(n+m) \leq p^{-m}P_p(n) \tag{2.2}$$

for every $0 \leq p \leq 1$ and $n,m \geq 1$. Therefore, by Lemma 2.2, we obtain the simplified submuliplicative inequality

$$P_p(n+m) \leq p^{-L+1}P_p(n)P_p(m) \tag{2.3}$$

for every $0 \leq p \leq 1$ and $n,m \geq 1$.

Proof of Lemma 2.2. Suppose that $\{y,z\} \in \mathcal{P}_0(n+m)$ and that $y_1 < z_1$, so that there exists a simple open path γ starting at 0 that has $\{y,z\}$ as its last edge and lies strictly to the left of z at every previous step. Letting $\{a,b\}$ be the first edge crossed by γ as it enters H_n for the first time (where $a_1 < b_1$), we see that the portion of γ up to and including the edge $\{a,b\}$ and the portion of γ after this edge are disjoint witnesses for the events that $\{a,b\} \in \mathcal{P}_0(n)$ and that $\{y,z\} \in \mathcal{P}_b(n+m-b_1)$. It follows by a union bound and the BK inequality that

$$P_p(n+m) = \sum_{y_1<z_1} \mathbb{P}_p\left(\{y,z\} \in \mathcal{P}_0(n+m)\right) \tag{2.4}$$

$$\leq \sum_{a_1<b_1} \mathbb{P}_p\left(\{a,b\} \in \mathcal{P}_0(n)\right) \sum_{y_1<z_1} \mathbb{P}_p\left(\{y,z\} \in \mathcal{P}_0(n+m-b_1)\right). \tag{2.5}$$

Now, if $a_1 < b_1$ and $\{a,b\} \in \mathcal{P}_0(n)$ then we must have that $n \leq b_1 \leq n+L-1$ and the claim follows by translation-invariance. $\square$

Theorems 1.1 and 1.2 will both be deduced from the following theorem.

Theorem 2.3. *Let $d > 6$ and suppose that* (T) *holds. There exist positive constants c and C such that*

$$\exp\left[-C(p_c-p)^{1/2}n\right] \preceq P_p(n) \preceq \exp\left[-c(p_c-p)^{1/2}n\right] \tag{2.6}$$

for every $n \geq 1$ and $p_c/2 \leq p \leq p_c$.

The lower bound of Theorem 2.3 is an easy consequence of (2.3), as follows. First, by Fekete's lemma, $-\lim_{n\to\infty} \frac{1}{n} \log P_p(n)$ is well-defined as an element of $[-\infty,\infty]$ and satisfies

$$-\lim_{n\to\infty} \frac{1}{n} \log P_p(n) = \sup_{n\geq 1} -\frac{1}{n} \log\left[p^{-L+1}P_p(n)\right] \tag{2.7}$$

for every $0 \leq p \leq 1$. Also, $\mathcal{P}_0(n)$ must be nonempty on the event that 0 is connected to ne_1 and hence by Markov's inequality $\tau_{p_c}(ne_1) \leq P_p(n)$ for every $n \geq 1$. Using this together with (1.6) one may verify that the exponential decay rate of $P_p(n)$ is equal to the mass $m(p)$ whenever $p < p_c$, and hence that

$$p^{L-1}\exp\left[-m(p)n\right] \leq P_p(n) \leq \exp\left[-m(p)n + o(n)\right] \tag{2.8}$$

as $n \to \infty$ for each fixed $p < p_c$, where the subexponential correction in the upper bound may depend on the value of $p < p_c$. (Indeed, explicit upper bounds of this form can be deduced from (1.7) by direct summation.) This is of course consistent with Theorem 2.3 since $m(p) \asymp |p-p_c|^{1/2}$ as $p \uparrow p_c$ in the high-dimensional setting [19]. Theorem 2.3 eliminates the sub-exponential term from this upper bound for high-dimensional models at the cost of replacing $m(p)$ with $cm(p)$ for some positive constant c. This will be used to obtain the sharp control of the subexponential terms in Theorems 1.1 and 1.2.

2.2 Proof of Theorems 1.1–1.2

2.2.1 Proof of Theorem 1.1 and the upper bound of Theorem 1.2

We now show how Theorem 2.3 easily implies Theorem 1.1 and the upper bound of Theorem 1.2.

Proof of Theorem 1.1 given Theorem 2.3. We may assume without loss of generality that the point $x \in \mathbb{Z}^d$ satisfies $x_1 = \langle x \rangle \geq 4L \geq 1$. Let $n = \lfloor x_1/2 \rfloor$. Suppose that the origin is connected to x by some simple open path γ, and let $\{a,b\}$ be the edge that is crossed by γ as it enters the halfspace H_n for the first time, with $a_1 < b_1$. Then the portion of γ up to and including the edge $\{a,b\}$ and the portion of γ after this edge are disjoint witnesses for the events $\{a,b\} \in \mathcal{P}_0(n)$ and $\{b \leftrightarrow x\}$. Thus, we have by a union bound and the BK inequality that

$$\mathbb{P}_p(0 \leftrightarrow x) \leq \sum_{a_1<b_1} \mathbb{P}_p\big(\{a,b\} \in \mathcal{P}_0(n)\big)\mathbb{P}_p(b \leftrightarrow x) \preceq \sum_{a_1<b_1} \mathbb{P}_p\big(\{a,b\} \in \mathcal{P}_0(n)\big) \cdot \langle x-b \rangle^{-d+2}, \tag{2.9}$$

for every $0 \leq p \leq p_c$, where we used (T) in the second inequality. Now, if $\{a,b\} \in \mathcal{P}_0(n)$ then we must have that $n \leq b_1 \leq n+L-1$ and hence that $\langle x-b \rangle \geq x_1 - b_1 \geq x_1/4$, so that there exists a positive constant c such that

$$\mathbb{P}_p(0 \leftrightarrow x) \preceq \langle x \rangle^{-d+2} \sum_{a_1 < b_1} \mathbb{P}_p\big(\{a,b\} \in \mathcal{P}_0(n)\big) = \langle x \rangle^{-d+2} P_p(n) \preceq \langle x \rangle^{-d+2} \exp\left[-c(p_c - p)^{1/2} \langle x \rangle\right] \quad (2.10)$$

by Theorem 2.3. This completes the proof of Theorem 1.1. □

Proof of upper bound of Theorem 1.2 given Theorem 2.3. Recall that H_n denotes the halfspace $\{y \in \mathbb{Z}^d : y_1 \geq n\}$. It suffices by symmetry to prove that there exists a positive constant c such that

$$\mathbb{P}_p(0 \leftrightarrow H_{2n}) \preceq \frac{1}{n^2} \exp\left[-c(p_c - p)^{1/2} n\right] \quad (2.11)$$

for every $n \geq 0$ and $p_c/2 \leq p \leq p_c$. Let $n \geq 2L \geq 1$, and suppose that the origin is connected to the halfspace H_{2n} by some simple open path γ. Letting $\{a, b\}$ with $a_1 < b_1$ be the edge that is crossed by γ as it enters the halfspace H_n for the first time, we observe that the portion of γ up to and including the edge $\{a, b\}$ and the portion of γ after this edge are disjoint witnesses for the events $\{a, b\} \in \mathcal{P}_0(n)$ and $\{b \leftrightarrow H_{2n}\}$. Thus, we have by a union bound and the BK inequality that

$$\mathbb{P}_p(0 \leftrightarrow H_{2n}) \leq \sum_{a_1 < b_1} \mathbb{P}_p\big(\{a,b\} \in \mathcal{P}_0(n)\big) \mathbb{P}_p(b \leftrightarrow H_{2n}). \quad (2.12)$$

Since $b_1 \leq n + L - 1 \leq 3n/2$, we deduce by the main result of [36] (i.e., the $p = p_c$ case of Theorem 1.2) that

$$\mathbb{P}_p(0 \leftrightarrow H_{2n}) \preceq n^{-2} \sum_{a_1 < b_1} \mathbb{P}_p\big(\{a,b\} \in \mathcal{P}_0(n)\big) = n^{-2} P_p(n) \preceq n^{-2} \exp\left[-c(p_c - p)^{1/2} n\right] \quad (2.13)$$

as claimed, where we used Theorem 2.3 in the final inequality. □

2.2.2 Proof of lower bound of Theorem 1.2

In this section we apply Theorem 2.3 to prove the lower bound of Theorem 1.2. We give the proof for the nearest-neighbour model, the general finite-range proof being similar but requiring more involved notation.

We begin with some definitions. Recall that S_r denotes the hyperplane $\{x \in \mathbb{Z}^d : x_1 = r\}$ for each $r \in \mathbb{Z}$. For each $-\infty \leq n \leq m \leq \infty$, let $S_{n,m}$ denote the slab $S_{n,m} := \bigcup_{i=n}^m S_i$. For each $r \geq 0$, let X_r be the number of points in the hyperplane S_r that are connected to the origin by an open path lying within the halfspace $S_{-\infty,r}$ and let $Y_r \leq X_r$ be the number of points in the hyperplane $S_r = \{x \in \mathbb{Z}^d : x_1 = r\}$ that are connected to the origin by an open path lying within the slab $S_{-r,r}$. Since we are working with nearest-neighbour models, every edge in $\mathcal{P}_0(r+1)$ must be of the form $\{(r,x),(r+1,x)\}$ for some $x \in \mathbb{Z}^{d-1}$, and the edge $\{(r,x),(r+1,x)\}$ belongs to $\mathcal{P}_0(r+1)$ if and only if it is open and (r,x) is connected to 0 inside the halfspace lying to the left of (r,x). From this it follows that

$$\mathbb{E}_p X_r = \frac{1}{p} \mathbb{E}_p |\mathcal{P}_0(r+1)| \quad (2.14)$$

for every $r \geq 0$ and $0 < p \leq 1$.

Proof of lower bound of Theorem 1.2. Let $r \geq 1$. The lower bound we wish to prove asserts that

$$\mathbb{P}_p\big(0 \leftrightarrow \partial\Lambda_r\big) \geq \frac{c}{r^2} \exp\left(-C(p_c-p)^{1/2}r\right). \tag{2.15}$$

Since $\{Y_r > 0\} \subset \{0 \leftrightarrow \partial\Lambda_r\}$, it suffices to prove that the above lower bound holds with instead $\mathbb{P}_p(Y_r > 0)$ on the left-hand side. We will prove this via the Cauchy–Schwarz inequality

$$\mathbb{P}_p(Y_r > 0) \geq \frac{(\mathbb{E}_p Y_r)^2}{\mathbb{E}_p\left[Y_r^2\right]} \tag{2.16}$$

together with suitable estimates on the first and second moments of Y_r.

It follows from (2.14) and Theorem 2.3 that there exist positive constants c and C such that

$$\exp\left[-C(p_c-p)^{1/2}r\right] \preceq \mathbb{E}_p X_r \preceq \exp\left[-c(p_c-p)^{1/2}r\right] \tag{2.17}$$

for every $p_c/2 \leq p \leq p_c$ and $r \geq 0$. We write $\{x \xleftrightarrow{A} y\}$ to mean that x and y are connected by an open path using only vertices of A. Observe that for each $r \geq 1$ and $x \in S_r$ we have the inclusion of sets

$$\{0 \xleftrightarrow{S_{-\infty,r}} x\} \setminus \{0 \xleftrightarrow{S_{-r,r}} x\} \subseteq \bigcup_{y \in S_{-r}} \{0 \xleftrightarrow{S_{-r,r}} y\} \circ \{y \xleftrightarrow{S_{-\infty,r}} x\}. \tag{2.18}$$

Indeed, if the event on the left-hand side of this inclusion holds, γ is an open path connecting 0 and x in $S_{-\infty,r}$, and y is the first point of S_{-r} visited by γ then the portions of γ before and after visiting y are disjoint witnesses for the events $\{0 \xleftrightarrow{S_{-r,r}} y\}$ and $\{y \xleftrightarrow{S_{-\infty,r}} x\}$ as claimed. It follows by a union bound, the BK inequality, and translation and reflection symmetry that

$$\mathbb{E}_p X_r \leq \mathbb{E}_p Y_r + \mathbb{E}_p Y_r \cdot \mathbb{E}_p X_{2r} \tag{2.19}$$

for every $0 \leq p \leq 1$ and $r \geq 0$. Applying the estimate (2.17) it follows that $\mathbb{E}_p Y_r \asymp \mathbb{E}_p X_r$ for every $0 \leq p \leq p_c$ and $r \geq 0$ and hence that

$$\exp\left[-C(p_c-p)^{1/2}r\right] \preceq \mathbb{E}_p Y_r \preceq \exp\left[-c(p_c-p)^{1/2}r\right] \tag{2.20}$$

for every $p_c/2 \leq p \leq p_c$ and $r \geq 0$.

We turn now to the second moment of the random variable Y_r. Suppose that x and y are two points in S_r both of which are connected to the origin in $S_{-r,r}$. There must exist a point $z \in S_{-r,r}$ such that the events $\{0 \xleftrightarrow{S_{-r,r}} z\}$, $\{z \xleftrightarrow{S_{-r,r}} x\}$, and $\{z \xleftrightarrow{S_{-r,r}} y\}$ all occur disjointly. It follows by a union bound and the BK inequality that

$$\mathbb{E}_p\left[Y_r^2\right] \leq \sum_{k=-r}^{r} \sum_{z \in S_k} \mathbb{P}_p(0 \xleftrightarrow{S_{-r,r}} z) \sum_{x,y \in S_r} \mathbb{P}_p(z \xleftrightarrow{S_{-r,r}} x)\mathbb{P}_p(z \xleftrightarrow{S_{-r,r}} y) \tag{2.21}$$

and hence by (2.17) that

$$\mathbb{E}_p\left[Y_r^2\right] \preceq \sum_{k=-r}^{r}\sum_{z\in S_k}\mathbb{P}_p(0 \xleftrightarrow{S_{-r,r}} z)\exp\left[-2c(p_c-p)^{1/2}(r-k)\right] \tag{2.22}$$

for every $p_c/2 \le p \le p_c$. Our next goal is to bound the resulting sum over z for each $-r \le k \le r$. Suppose that $z \in S_k$ for some $-r \le k \le r$ and suppose that the origin is connected to z by a simple open path in $S_{-r,r}$. By considering the right-most point that this path visits, we see that there must exist $0 \le a \le r$ and $w \in S_a$ such that the events $\{0 \xleftrightarrow{S_{-r,a}} w\}$ and $\{w \xleftrightarrow{S_{-r,a}} z\}$ occur disjointly. Thus, applying a union bound and the BK inequality again as above, we obtain that

$$\begin{aligned}\sum_{z\in S_k}\mathbb{P}_p(0 \xleftrightarrow{S_{-r,r}} z) &\le \sum_{a=k\vee 0}^{r}\sum_{w\in S_a}\sum_{z\in S_k}\mathbb{P}_p(0 \xleftrightarrow{S_{-r,a}} w)\mathbb{P}_p(w \xleftrightarrow{S_{-r,a}} z)\\ &\le \sum_{a=k\vee 0}^{r}\sum_{w\in S_a}\sum_{z\in S_k}\mathbb{P}_p(0 \xleftrightarrow{S_{-\infty,a}} w)\mathbb{P}_p(w \xleftrightarrow{S_{-\infty,a}} z)\end{aligned} \tag{2.23}$$

and a further application of (2.17) gives that

$$\begin{aligned}\sum_{z\in S_k}\mathbb{P}_p(0 \xleftrightarrow{S_{-r,r}} z) &\preceq \sum_{a=k\vee 0}^{r}\exp\left[-c(p_c-p)^{1/2}a-c(p_c-p)^{1/2}(a-k)\right]\\ &\preceq r\exp\left[-c(p_c-p)^{1/2}|k|\right],\end{aligned} \tag{2.24}$$

for every $p_c/2 \le p \le p_c$, $r \ge 1$, and $-r \le k \le r$. Putting these estimates together we obtain that

$$\begin{aligned}\mathbb{E}_p\left[Y_r^2\right] &\preceq r\sum_{k=-r}^{r}\exp\left[-2c(p_c-p)^{1/2}(r-k)-c(p_c-p)^{1/2}|k|\right]\\ &\preceq r^2\exp\left[-c(p_c-p)^{1/2}r\right]\end{aligned} \tag{2.25}$$

for every $p_c/2 \le p \le p_c$ and $r \ge 1$. Putting this together with the lower bound of (2.20), we obtain

$$\mathbb{P}_p(Y_r>0) \ge \frac{(\mathbb{E}_pY_r)^2}{\mathbb{E}_p\left[Y_r^2\right]} \succeq \frac{1}{r^2}\exp\left[-(2C-c)(p_c-p)^{1/2}r\right] \tag{2.26}$$

for every $p_c/2 \le p \le p_c$ and $r \ge 1$. This completes the proof. □

3 Expected number of pioneers: Proof of Theorem 2.3

In this section we complete the proof of Theorems 1.1–1.2 by proving Theorem 2.3.

3.1 The expected number of critical pioneers

3.1.1 The critical case of Theorem 2.3

In this section we prove the $p = p_c$ case of Theorem 2.3.

Proposition 3.1. *Let $d > 6$ and suppose that* (T) *holds. There exist positive constants c and C such that $c \leq P_{p_c}(n) \leq C$ for every $n \geq 1$.*

Note that the lower bound $P_{p_c}(n) \geq p_c^{L-1}$ holds in every dimension by taking $p \uparrow p_c$ in the estimate (2.8) above; the main content of the proposition is that a matching upper bound holds in the high-dimensional case.

To ease notation, we will prove the upper bound of Proposition 3.1 only for nearest-neighbour percolation. The general proof for finite-range models is very similar but substantially more involved as one must introduce various additional summations to most calculations. This assumption will be in force for the remainder of Section 3.1. We write $P(n) = P_{p_c}(n)$ and $\mathbb{P} = \mathbb{P}_{p_c}$ to lighten notation. Recall that H denotes the half-space $\{(n, x) : n \geq 0, x \in \mathbb{Z}^{d-1}\}$. be of the form $\{(n-1, x), (n, x)\}$ for some $x \in \mathbb{Z}^{d-1}$, and that the edge $\{(n-1, x), (n, x)\}$ belongs to $\mathcal{P}_0(n)$ if and only if it is open and $(n-1, x)$ is connected to 0 inside the halfspace lying to the left of $(n-1, x)$. We again write $\{x \xleftrightarrow{A} y\}$ to mean that x and y are connected by an open path using only vertices of A. By (2.14),

$$P(n) = \mathbb{E}|\mathcal{P}_0(n)| = \sum_{x \in \mathbb{Z}^{d-1}} p_c \cdot \mathbb{P}\Big((0,0) \xleftrightarrow{S_{-\infty,n-1}} (n-1, x)\Big). \tag{3.1}$$

By translation and reflection symmetry, this gives (with term-by-term equality)

$$P(n) = \sum_{x \in \mathbb{Z}^{d-1}} p_c \cdot \mathbb{P}\Big((0,x) \xleftrightarrow{H} (n-1, 0)\Big). \tag{3.2}$$

A further translation by $(0, -x)$, followed by replacement of $-x$ by x, gives

$$P(n) = \sum_{x \in \mathbb{Z}^{d-1}} p_c \cdot \mathbb{P}\Big((0,0) \xleftrightarrow{H} (n-1, x)\Big). \tag{3.3}$$

This equality makes it convenient for us to consider for each $n \geq 0$ the quantity

$$\bar{P}(n) := \frac{1}{p_c} P(n+1) = \sum_{x \in \mathbb{Z}^{d-1}} \mathbb{P}\Big((0,x) \xleftrightarrow{H} (n, 0)\Big) = \sum_{x \in \mathbb{Z}^{d-1}} \mathbb{P}\Big((0,0) \xleftrightarrow{H} (n, x)\Big) \tag{3.4}$$

instead of $P(n)$ itself. A very similar proof to that of Lemma 2.2 yields that $\bar{P}$ is submultiplicative in the sense that $\bar{P}(n+m) \leq \bar{P}(n)\bar{P}(m)$ for each $n, m \geq 0$.

In order to upper bound $P(n)$, we will prove a complementary supermultiplicative-type estimate on $\bar{P}(n)$ via diagrammatic methods. For each $n \geq 1$, define

$$\bar{P}^*(n) = \max_{0 \leq k \leq n} \bar{P}(n). \tag{3.5}$$

We deduce Proposition 3.1 from the following two estimates.

Proposition 3.2. *Let $d > 6$ and suppose that* (T) *holds. There exist positive constants $c > 0$ and $\ell \in \mathbb{N}$ such that*

$$\bar{P}^*(2n + \ell) \geq c \bar{P}^*(n)^2 \tag{3.6}$$

for every $n \geq 0$.

Lemma 3.3. *Let $d > 6$ and suppose that* (T) *holds. Then $\bar{P}^*(n) \preceq \log(n+2)$ for every $n \geq 0$.*

We now show how Proposition 3.1 follows from Proposition 3.2 and Lemma 3.3. In brief, Proposition 3.2 implies that if $\bar{P}^*$ is unbounded then it must grow exponentially rapidly. This contradicts Lemma 3.3, so $\bar{P}^*$ must be bounded, as desired.

Proof of Proposition 3.1 given Proposition 3.2 and Lemma 3.3. Let $c > 0$ and $\ell \in \mathbb{N}$ be the constants from Proposition 3.2. If there exists $n \geq \ell$ such that $\bar{P}^*(n) \geq 2/c$ then we have by induction that

$$\bar{P}^*(3^k n) \geq \bar{P}^*(2 \cdot 3^{k-1} n + \ell) \geq \frac{1}{c} 2^{2^k} \tag{3.7}$$

for every $k \geq 0$. This contradicts Lemma 3.3, and so we must in fact have that $\bar{P}^*(n) < 2/c$ for every $n \geq \ell$ and hence for every $n \geq 0$ as claimed. □

We now prove Lemma 3.3, which was used above in the proof of Proposition 3.1 and which will also be used in the proof of Proposition 3.2. The proof is based on the upper bounds

$$\mathbb{P}(x \xleftrightarrow{H} y) \preceq \langle x - y \rangle^{-d+1} \qquad \text{for every } x \in \mathbb{Z}^d \text{ with } x_1 = 0 \text{ and every } y \in H, \tag{3.8}$$

$$\mathbb{P}(0 \xleftrightarrow{H} x) \preceq \langle x - y \rangle^{-d} \qquad \text{for every } x, y \in \mathbb{Z}^d \text{ with } x_1 = y_1 = 0. \tag{3.9}$$

of Chatterjee and Hanson [11, Theorems 7.2 and 1.1(b)], as well as their lower bound [11, Theorem 1.1(b)]

$$\mathbb{P}(x \xleftrightarrow{H} y) \succeq \langle x - y \rangle^{-d+1} \text{ for every } x \in \mathbb{Z}^d \text{ with } x_1 = 0 \text{ and every } y \in H \text{ with } \langle y - x \rangle \leq 2y_1. \tag{3.10}$$

The above bounds are valid for $d > 6$ assuming that (T) holds.

Remark 3.4. For $d > 2$, and given $x, y \in H$ with $y = (y_1, \ldots, y_d)$, we set $\bar{y} = (-y_1, y_2, \ldots, y_d)$. By the method of images (see, e.g., [37, Proposition 8.1.1]), the half-space lattice Green function is given by $G_H(x,y) = G(x,y) - G(x,\bar{y})$ where the unrestricted lattice Green function $G(x,y)$ is asymptotic to a multiple of $|x-y|^{2-d}$. It is natural to assume that the critical two-point function has the same behaviour, which suggests an extended version

$$\mathbb{P}(x \xleftrightarrow{H} y) \preceq \frac{(x_1+1)(y_1+1)}{\langle x - y \rangle^d} \qquad \text{for every } x, y \in H \tag{3.11}$$

of the Chatterjee–Hanson bounds which we believe to be sharp when $x_1 \vee y_1 \leq K\langle x - y \rangle$ for some fixed $K > 0$. If this bound were proven, it would be possible to deduce Proposition 3.1 directly by summation. We do not pursue this further here.

Proof of Lemma 3.3. It suffices to prove that $\bar{P}(n) \leq \log(n+2)$ for each $n \geq 1$. Let $R = (n+1)^d$. By (3.4) and (3.9),

$$\bar{P}(n) = \sum_{x \in \mathbb{Z}^{d-1}} \mathbb{P}\Big((0,x) \xleftrightarrow{H} (n,0)\Big) \preceq \sum_{x \in \Lambda_n^{d-1}} (n+1)^{-d+1} + \sum_{x \in \Lambda_R^{d-1} \setminus \Lambda_n^{d-1}} \langle x \rangle^{-d+1}$$
$$+ \sum_{x \in \mathbb{Z}^{d-1} \setminus \Lambda_R^{d-1}} \mathbb{P}\Big((n,0) \xleftrightarrow{H} (0,x)\Big). \tag{3.12}$$

To control the final term, we use the Harris-FKG inequality, (3.9) and (3.10) to obtain that

$$\sum_{x\in\mathbb{Z}^{d-1}\setminus\Lambda_R^{d-1}} \mathbb{P}\Big((0,x) \xleftrightarrow{H} (n,0)\Big) \leq \sum_{x\in\mathbb{Z}^{d-1}\setminus\Lambda_R^{d-1}} \mathbb{P}\Big((0,0) \xleftrightarrow{H} (n,0)\Big)^{-1} \cdot \mathbb{P}\Big((0,0) \xleftrightarrow{H} (0,x)\Big)$$
$$\preceq \sum_{x\in\mathbb{Z}^{d-1}\setminus\Lambda_R^{d-1}} (n+1)^{d-1}\langle x\rangle^{-d}. \tag{3.13}$$

Putting these bounds together and using that $|\{x \in \mathbb{Z}^{d-1} : \langle x\rangle = r\}| = O((r+1)^{d-2})$ for every $r \geq 0$, we deduce that

$$\bar{P}(n) = \sum_{x\in\mathbb{Z}^{d-1}} \mathbb{P}\Big((0,x) \xleftrightarrow{H} (n,0)\Big) \preceq 1 + \sum_{r=n}^{R} r^{-1} + \sum_{r=R}^{\infty}(n+1)^{d-1}r^{-2}$$
$$\preceq 1 + \log\frac{R+1}{n+1} + \frac{(n+1)^{d-1}}{R} \preceq \log(n+2), \tag{3.14}$$

and the proof is complete. □

3.1.2 Proof of Proposition 3.2

In this section, we prove Proposition 3.2. As a first step, we make the following definition.

Definition 3.5. Let $e_1 = (1, 0, \ldots, 0)$ be the unit vector in the horizontal direction. Recall that for each $k \in \mathbb{Z}$, S_k denotes the hyperplane $S_k = \{(k, x) : x \in \mathbb{Z}^{d-1}\} = \{x \in \mathbb{Z}^d : x_1 = k\}$ and H_k denotes the halfspace $H_k = \bigcup_{i\geq k} S_i$. Given $0 \leq k < n$, $x \in S_k$ and $y \in S_n$, we say that x is a *good pivotal vertex* for the event $\{0 \xleftrightarrow{H_0} y\}$ if the following hold:

1. The edge $\{x, x + e_1\}$ is open.
2. 0 is connected to x in H_0 off of the edge $\{x, x + e_1\}$.
3. $x + e_1$ is connected to y in H_{k+1}.
4. 0 is not connected to y in H_0 off of the edge $\{x, x + e_1\}$.

We claim that if 0 is connected to y in H_0 then for each $0 \leq k < n$ there is at most one good pivotal vertex $x \in S_k$ for the event $\{0 \xleftrightarrow{H_0} y\}$. Indeed, if x is a good pivotal vertex then any open path from 0 to y in H_0 must pass through the edge $\{x, x + e_1\}$. If $x, z \in S_k$ were distinct good pivotal vertices then there would exist simple open paths γ_1 and γ_2 connecting 0 to y in H_0 such that γ_1 visits S_k for the last time at x and γ_2 visits S_k for the last time at z. The concatenation of the portion of γ_1 up until its visit to z with the portion of γ_2 after it visits z would therefore be an open simple path connecting 0 and y in H_0 that avoids x, contradicting the assumption that x is a good pivotal vertex.

The fact that there is at most one good pivotal vertex implies by (3.3) that

$$\bar{P}(n) = \sum_{y\in S_n} \mathbb{P}(0 \xleftrightarrow{H_0} y) \geq \sum_{x\in S_k}\sum_{y\in S_n} \mathbb{P}(0 \xleftrightarrow{H_0} y,\, x \text{ a good pivotal vertex for this event})$$
$$= \frac{p_c}{1-p_c}\sum_{x\in S_k}\sum_{y\in S_n} \mathbb{P}(0 \xleftrightarrow{H_0} x,\; x \overset{H_0}{\not\leftrightarrow} x+e_1,\; \text{and } x+e_1 \xleftrightarrow{H_{k+1}} y) \tag{3.15}$$

for every $0 \le k < n$. By symmetry, we have equivalently that

$$\bar{P}(n+k) \ge \frac{p_c}{1-p_c} \sum_{y \in S_n} \sum_{x \in S_{-k}} \mathbb{P}(x \xleftrightarrow{H_{-k}} 0,\, 0 \overset{H_{-k}}{\not\longleftrightarrow} e_1, \text{ and } e_1 \xleftrightarrow{H_1} y) \tag{3.16}$$

for every $n, k \ge 0$.

To make use of this inequality, we will first prove the following lemma. Like many results in high-dimensional percolation, its proof relies on a bound on the open *triangle diagram*

$$\mathsf{T}_p(x) = \sum_{y,z \in \mathbb{Z}^d} \tau_p(y)\tau_p(z-y)\tau_p(x-z) \tag{3.17}$$

at the critical value $p = p_c$. The triangle diagram was introduced by Aizenman and Newman in 1984 [2] and the finiteness of $\mathsf{T}_{p_c}(x)$ was proved in [21] for sufficiently large d for the nearest-neighbour model and for $d > 6$ for sufficiently spread-out models, and extended in [16] to the nearest-neighbour model in dimensions $d \ge 11$. Although historically the proof of (T) relied on this finiteness of the triangle diagram, a posteriori (T) yields (for $d > 6$)

$$\mathsf{T}_{p_c}(x) \preceq \sum_{y,z \in \mathbb{Z}^d} \langle y \rangle^{2-d} \langle z-y \rangle^{2-d} \langle x-z \rangle^{2-d} \preceq \langle x \rangle^{6-d} \tag{3.18}$$

via the elementary convolution estimate [20, Proposition 1.7].

Indeed, [20, Proposition 1.7] states more generally that for each $a, b > 0$ with $a + b < d$ there exists a constant $C = C(d, a, b)$ such that

$$\sum_{y \in \mathbb{Z}^d} \langle y \rangle^{a-d} \langle x-y \rangle^{b-d} \le C \langle x \rangle^{a+b-d} \tag{3.19}$$

for every $x \in \mathbb{Z}^d$, and it follows by applying this estimate twice that for each $a, b, c > 0$ with $a+b+c < d$ there exists a constant $C = C(d, a, b, c)$ such that

$$\sum_{y,z \in \mathbb{Z}^d} \langle y \rangle^{a-d} \langle z-y \rangle^{b-d} \langle x-z \rangle^{c-d} \le C \langle x \rangle^{a+b+c-d} \tag{3.20}$$

for every $x \in \mathbb{Z}^d$. The following proof will in fact apply (3.19) with $a, b, c = 2 + \varepsilon$ rather than the usual triangle estimate (3.18).

Lemma 3.6. *Let $d > 6$ and suppose that* (T) *holds. There exists a positive constant ℓ such that*

$$\sum_{x \in S_{-n}} \sum_{y \in S_{n+\ell}} \mathbb{P}(x \xleftrightarrow{H_{-n}} 0,\, 0 \overset{H_{-n}}{\not\longleftrightarrow} \ell e_1,\, \ell e_1 \xleftrightarrow{H_\ell} y) \ge \frac{1}{2} \bar{P}(n)^2 \tag{3.21}$$

for every $n \ge 0$ such that $\bar{P}(n) = \bar{P}^*(n)$.

Proof. Fix $n \ge 0$. We follow a variation on the strategy of [35, Lemma 3.2], illustrated in Figure 2. Let $K_{0,n}$ denote the cluster of 0 in H_{-n} and let $\mathscr{C}_0$ be the set of finite connected subsets of $\mathbb{Z}^d$ containing 0.

By conditioning on $K_{0,n}$, we see that

$$\mathbb{P}(x \xleftrightarrow{H_{-n}} 0,\, 0 \overset{H_{-n}}{\not\longleftrightarrow} \ell e_1,\, \ell e_1 \xleftrightarrow{H_\ell} y) = \sum_{\substack{A \in \mathscr{C}_0 \\ A \ni x}} \mathbb{P}(K_{0,n} = A, \ell e_1 \xleftrightarrow{H_\ell} y)\mathbb{1}(\ell e_1 \notin A) \tag{3.22}$$

for each $n, \ell \geq 1$, $x \in S_{-n}$, and $y \in S_{n+\ell}$. Note moreover that if $A \in \mathscr{C}_0$ is such that $y \notin A$ then

$$\mathbb{P}(K_{0,n} = A, \ell e_1 \xleftrightarrow{H_\ell} y)\mathbb{1}(\ell e_1 \notin A) = \mathbb{P}(K_{0,n} = A, \ell e_1 \xleftrightarrow{H_\ell} y \text{ off } A) \tag{3.23}$$

where we write "$\ell e_1 \xleftrightarrow{H_\ell} y$ off A" to mean that there is an open path from ℓe_1 to y in H_ℓ that does not visit any vertex of A, including at its endpoints. Since the events $\{K_{0,n} = A\}$ and $\{\ell e_1 \xleftrightarrow{H_\ell} y \text{ off } A\}$ depend on disjoint sets of edges (namely, those edges with at least one endpoint in A and those edges with neither endpoint in A), these two events are independent and we deduce that

$$\mathbb{P}(K_{0,n} = A, \ell e_1 \xleftrightarrow{H_\ell} y)\mathbb{1}(\ell e_1 \notin A) = \mathbb{P}(K_{0,n} = A)\mathbb{P}(\ell e_1 \xleftrightarrow{H_\ell} y \text{ off } A). \tag{3.24}$$

Next, we observe that

$$\mathbb{P}(\ell e_1 \xleftrightarrow{H_\ell} y \text{ off } A) = \mathbb{P}(\ell e_1 \xleftrightarrow{H_\ell} y) - \mathbb{P}(\ell e_1 \xleftrightarrow{H_\ell} y \text{ only via } A), \tag{3.25}$$

where we write "$\ell e_1 \xleftrightarrow{H_\ell} y$ only via A" to mean that there is an open path from ℓe_1 to y in H_ℓ but every such path must visit a vertex of A. (This holds in particular if ℓe_1 is connected to y in H_ℓ and belongs to the set A.) It follows that

$$\begin{aligned}\mathbb{P}(x \xleftrightarrow{H_{-n}} 0,\, 0 \overset{H_{-n}}{\not\longleftrightarrow} \ell e_1,\, \ell e_1 \xleftrightarrow{H_\ell} y) = &\sum_{\substack{A \in \mathscr{C}_0 \\ A \ni x}} \mathbb{P}(K_{0,n} = A)\mathbb{P}(\ell e_1 \xleftrightarrow{H_\ell} y) \\ &- \sum_{\substack{A \in \mathscr{C}_0 \\ A \ni x}} \mathbb{P}(K_{0,n} = A)\mathbb{P}(\ell e_1 \xleftrightarrow{H_\ell} y \text{ only via } A)\end{aligned} \tag{3.26}$$

and hence that

$$\begin{aligned}\mathbb{P}(x \xleftrightarrow{H_{-n}} 0,\, 0 \overset{H_{-n}}{\not\longleftrightarrow} \ell e_1, \ell e_1 \xleftrightarrow{H_\ell} y) = &\,\mathbb{P}(x \xleftrightarrow{H_{-n}} 0)\mathbb{P}(\ell e_1 \xleftrightarrow{H_\ell} y) \\ &- \sum_{\substack{A \in \mathscr{C}_0 \\ A \ni x}} \mathbb{P}(K_{0,n} = A)\mathbb{P}(\ell e_1 \xleftrightarrow{H_\ell} y \text{ only via } A)\end{aligned} \tag{3.27}$$

for every $\ell \geq 1$, $x \in S_{-n}$, and $y \in S_{n+\ell}$.

Our goal is to prove that the sum over $x \in S_{-n}$ and $y \in S_{n+\ell}$ of the left-hand side of (3.27) is bounded below by $\frac{1}{2}\bar{P}^*(n)^2$, assuming that $\bar{P}(n) = \bar{P}^*(n)$. For the first term on the right-hand side, it follows from (3.4) that

$$\sum_{x \in S_{-n}} \sum_{y \in S_{n+\ell}} \mathbb{P}(x \xleftrightarrow{H_{-n}} 0)\mathbb{P}(\ell e_1 \xleftrightarrow{H_\ell} y) = \bar{P}(n)^2. \tag{3.28}$$

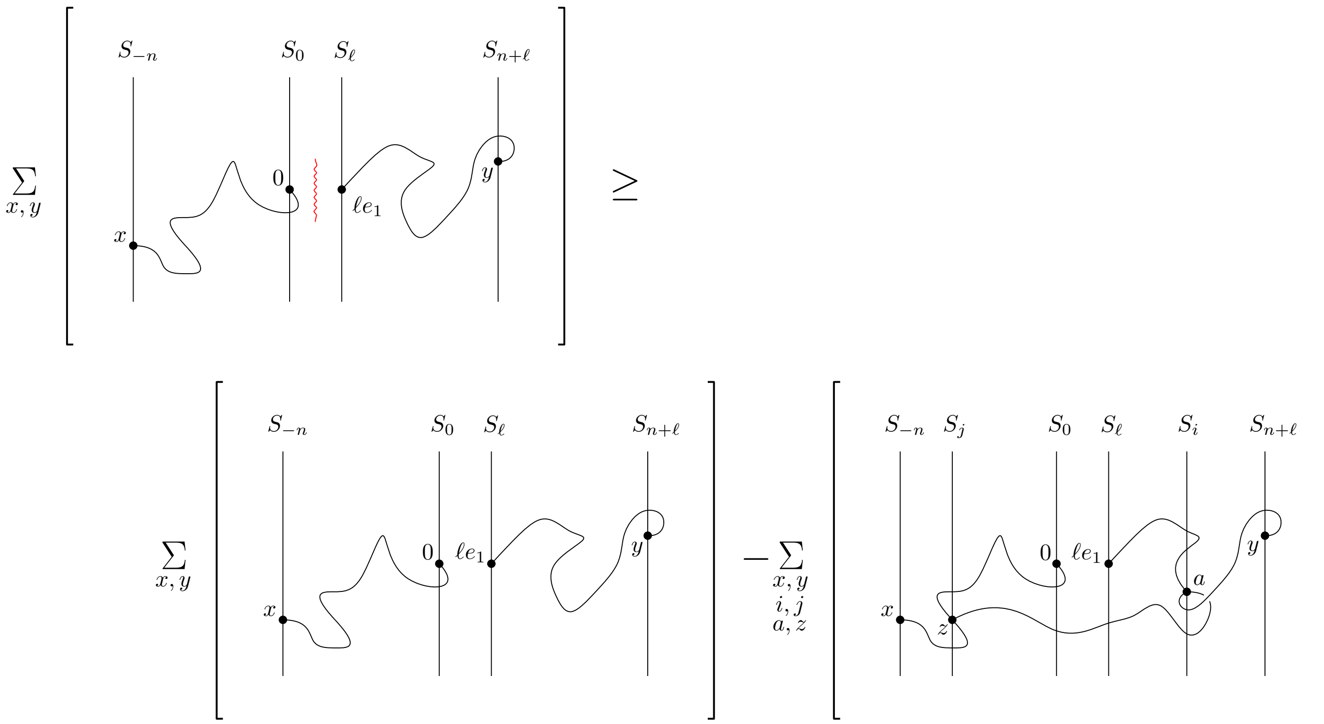

Figure 2: Schematic illustration of the diagrammatic estimate used to prove Lemma 3.6. The squiggly red line indicates that 0 and ℓe_1 are not connected by an open path in the half-space H_{-n}. To prove the lemma, it suffices to prove that the second diagrammatic sum on the right hand side is much smaller than the first when the separation parameter ℓ is large.

It therefore suffices to prove that we can choose ℓ large in order to obtain

$$\sum_{x\in S_{-n}}\sum_{y\in S_{n+\ell}}\sum_{\substack{A\in\mathscr{C}_0\\ A\ni x}}\mathbb{P}(K_{0,n}=A)\mathbb{P}(\ell e_1 \xleftrightarrow{H_\ell} y \text{ only via } A)\leq \frac{1}{2}\bar{P}(n)^2 \tag{3.29}$$

for every $n\geq 0$ such that $\bar{P}(n)=\bar{P}^*(n)$. The remainder of the proof is devoted to establishing (3.29).

As a first step, we observe by the BK inequality that

$$\mathbb{P}(\ell e_1 \xleftrightarrow{H_\ell} y \text{ only via } A)\leq \sum_{a\in A}\mathbb{P}\left(\{\ell e_1 \xleftrightarrow{H_\ell} a\}\circ\{a \xleftrightarrow{H_\ell} y\}\right)\leq \sum_{a\in A}\mathbb{P}\left(\ell e_1 \xleftrightarrow{H_\ell} a\right)\mathbb{P}\left(a \xleftrightarrow{H_\ell} y\right) \tag{3.30}$$

for every $\ell\geq 1$ and $y\in S_{n+\ell}$. Indeed, if the event on the left-hand side occurs then there must exist a simple open path connecting ℓe_1 to y in H_ℓ that passes through A at some point a, and the portions of this path before and after visiting a are disjoint witnesses for the events $\{\ell e_1 \xleftrightarrow{H_\ell} a\}$ and $\{a \xleftrightarrow{H_\ell} y\}$. It follows that

$$\begin{aligned}\sum_{\substack{A\in\mathscr{C}_0\\ A\ni x}}\mathbb{P}(K_{0,n}=A)\mathbb{P}(\ell e_1 \xleftrightarrow{H_\ell} y \text{ only via } A) &\leq \sum_{\substack{A\in\mathscr{C}_0\\ A\ni x}}\mathbb{P}(K_{0,n}=A)\sum_{a\in A}\mathbb{P}\left(\ell e_1 \xleftrightarrow{H_\ell} a\right)\mathbb{P}\left(a \xleftrightarrow{H_\ell} y\right)\\ &= \sum_{a\in H_\ell}\mathbb{P}\left(0 \xleftrightarrow{H_{-n}} x,\, 0 \xleftrightarrow{H_{-n}} a\right)\mathbb{P}\left(\ell e_1 \xleftrightarrow{H_\ell} a\right)\mathbb{P}\left(a \xleftrightarrow{H_\ell} y\right)\end{aligned} \tag{3.31}$$

for each $\ell \geq 1$, $x \in S_{-n}$, and $y \in S_{n+\ell}$. Now, if 0 is connected to both x and a in H_{-n} there must exist $z \in H_{-n}$ such that the events $\{0 \xleftrightarrow{H_{-n}} z\}$, $\{z \xleftrightarrow{H_{-n}} x\}$, and $\{z \xleftrightarrow{H_{-n}} a\}$ all occur disjointly, so it follows by the BK inequality that

$$\mathbb{P}\left(0 \xleftrightarrow{H_{-n}} x,\, 0 \xleftrightarrow{H_{-n}} a\right) \leq \sum_{z \in H_{-n}} \mathbb{P}\left(0 \xleftrightarrow{H_{-n}} z\right) \mathbb{P}\left(z \xleftrightarrow{H_{-n}} x\right) \mathbb{P}\left(z \xleftrightarrow{H_{-n}} a\right). \tag{3.32}$$

We insert (3.32) into (3.31) and insert the result into (3.29). The sums over x and y can then be performed explicitly, since these variables each appear in just one factor. For the sum over x, we use the fact that for $z \in H_j$ with $j \geq -n$ we have

$$\sum_{x \in S_{-n}} \mathbb{P}\left(z \xleftrightarrow{H_{-n}} x\right) \leq \bar{P}(n+j). \tag{3.33}$$

For the sum over y, we use

$$\sum_{y \in S_r} \mathbb{P}(0 \xleftrightarrow{H_{-m}} y) \leq \sum_{y \in S_r} \sum_{k=0}^{m} \sum_{w \in S_{-k}} \mathbb{P}(\{0 \xleftrightarrow{H_{-k}} w\} \circ \{w \xleftrightarrow{H_{-k}} y\}) \leq \sum_{k=0}^{m} \bar{P}(k)\bar{P}(r+k) \tag{3.34}$$

for every $r, m \geq 0$, which follows by decomposing a simple open path from 0 to y according to its left-most point and using the BK inequality. The result is

$$\begin{aligned}
&\sum_{x \in S_{-n}} \sum_{y \in S_{n+\ell}} \sum_{\substack{A \in \mathscr{C}_0 \\ A \ni x}} \mathbb{P}(K_{0,n} = A)\mathbb{P}(\ell e_1 \xleftrightarrow{H_\ell} y \text{ only via } A) \\
&\quad \leq \sum_{i=\ell}^{\infty} \sum_{a \in S_i} \sum_{j=-n}^{\infty} \sum_{z \in S_j} \mathbb{P}\left(0 \leftrightarrow z\right) \bar{P}(n+j) \mathbb{P}\left(z \leftrightarrow a\right) \mathbb{P}\left(\ell e_1 \xleftrightarrow{H_\ell} a\right) \sum_{k=0}^{i} \bar{P}(k)\bar{P}(n-i+k) \\
&\quad \leq \bar{P}^*(n)^2 \sum_{i=\ell}^{\infty} \sum_{a \in S_i} \sum_{j=-n}^{\infty} \sum_{z \in S_j} \mathbb{P}\left(0 \leftrightarrow z\right) \mathbb{P}\left(z \leftrightarrow a\right) \mathbb{P}\left(\ell e_1 \xleftrightarrow{H_\ell} a\right) (i+1)\bar{P}(j \vee 0)\bar{P}^*(i),
\end{aligned} \tag{3.35}$$

where in the last step we used $\bar{P}(k) \leq \bar{P}^*(i)$ and $\bar{P}(n-i+k) \leq \bar{P}^*(n)$, as well as the submultiplicative property of $\bar{P}$ to see that $\bar{P}(n+j) \leq \bar{P}^*(n)\bar{P}(j \vee 0)$.

To estimate the right-hand side of (3.35), we use Lemma 3.3 to bound $\bar{P}(j \vee 0)$ and $\bar{P}^*(i)$, and (T) to bound $\mathbb{P}\left(0 \leftrightarrow z\right)$ and $\mathbb{P}\left(z \leftrightarrow a\right)$. Also, we use the half-space estimate (3.8) to see that

$$\mathbb{P}\left(\ell e_1 \xleftrightarrow{H_\ell} a\right) (i+1)\bar{P}^*(i) \preceq \langle \ell e_1 - a \rangle^{-d+1} (i+1) \log(i \vee 2). \tag{3.36}$$

Here $i \geq \ell$, so $i \leq \ell + \langle \ell e_1 - a \rangle$ and therefore

$$\begin{aligned}
\mathbb{P}\left(\ell e_1 \xleftrightarrow{H_\ell} a\right) (i+1)\bar{P}^*(i) &\preceq \langle \ell e_1 - a \rangle^{-d+2} \log(\langle \ell e_1 - a \rangle \vee 2) + \ell \log(\ell \vee 2) \langle \ell e_1 - a \rangle^{-d+1} \\
&\preceq \langle \ell e_1 - a \rangle^{-d+2+1/4} + \ell^{5/4} \langle \ell e_1 - a \rangle^{-d+1}.
\end{aligned} \tag{3.37}$$

Thus, with the left-hand side of our goal (3.29) temporarily written as $T_{n,\ell}$, using the crude bound

$\bar{P}(j \vee 0) \preceq \log(j \vee 2) \preceq \langle z \rangle^{1/4}$ yields that

$$T_{n,\ell} \preceq \bar{P}^*(n)^2 \sum_{i=\ell}^{\infty} \sum_{a \in S_i} \sum_{j=-n}^{\infty} \sum_{z \in S_j} \langle z \rangle^{-d+2+1/4} \langle z-a \rangle^{-d+2} \langle \ell e_1 - a \rangle^{-d+2+1/4} \\ + \ell^{5/4} \bar{P}^*(n)^2 \sum_{i=\ell}^{\infty} \sum_{a \in S_i} \sum_{j=-n}^{\infty} \sum_{z \in S_j} \langle z \rangle^{-d+2+1/4} \langle z-a \rangle^{-d+2} \langle \ell e_1 - a \rangle^{-d+1} \quad (3.38)$$

from which for $d \geq 7$ (3.20) yields

$$T_{n,\ell} \preceq \bar{P}^*(n)^2 \left(\langle \ell e_1 \rangle^{-d+6+1/2} + \ell^{5/4} \langle \ell e_1 \rangle^{-d+5+1/4} \right) \preceq \ell^{-d+6+1/2} P^*(n)^2 \leq \ell^{-1/2} P^*(n)^2. \quad (3.39)$$

Since this bound holds uniformly over $n \geq 1$ and $\ell \geq 1$, and since the prefactor $\ell^{-1/2}$ tends to zero as $\ell \to \infty$, we deduce that there exists a constant ℓ such that

$$T_{n,\ell} \leq \frac{1}{2} \bar{P}^*(n)^2. \quad (3.40)$$

This proves (3.29) and therefore completes the proof. □

Finally, we deduce Proposition 3.2 from (3.16) and Lemma 3.6. In preparation for this, inspired by [30, Section 4] we define three events and prove a lemma relating them, as follows. Fix any $n \geq 0$, $x \in S_{-n}$ and $y \in S_{n+\ell}$, where ℓ is fixed as in Lemma 3.6. We define the event

$$\mathscr{A}(x,y) = \{x \xleftrightarrow{H_{-n}} 0,\, 0 \overset{H_{-n}}{\nleftrightarrow} \ell e_1,\, \ell e_1 \xleftrightarrow{H_\ell} y\}. \quad (3.41)$$

Then (3.21) can be rewritten more compactly as

$$\sum_{x \in S_{-n}} \sum_{y \in S_{n+\ell}} \mathbb{P}(\mathscr{A}(x,y)) \geq \frac{1}{2} \bar{P}(n)^2 \quad (3.42)$$

for every $n \geq 0$ such that $\bar{P}(n) = \bar{P}^*(n)$.

Let η be the left-directed horizontal geodesic connecting ℓe_1 to 0, and for each $1 \leq i \leq \ell$ let η_i be the ith edge crossed by η. Given a Bernoulli bond percolation configuration ω on $\mathbb{Z}^d$, let ω^i be the configuration obtained from ω by setting

$$\omega^i(e) = \begin{cases} 1 & e \in \{\eta_j : 1 \leq j \leq i\} \\ \omega(e) & e \notin \{\eta_j : 1 \leq j \leq i\}. \end{cases} \quad (3.43)$$

In particular, $\omega^0 = \omega$. For each $1 \leq i \leq \ell$, let $\mathscr{B}_i(x,y)$ be the event that that 0 and ℓe_1 are connected in H_{-n} in ω^i but not in ω^{i-1}, 0 is connected to x in H_{-n} in ω^{i-1}, and ℓe_1 is connected to y in H_ℓ in ω^{i-1}. Finally, for each $1 \leq i \leq \ell$ let

$$\mathscr{C}_i(x,y) = \left\{ x \xleftrightarrow{H_{-n}} (\ell - i)e_1,\, (\ell-i)e_1 \overset{H_{-n}}{\nleftrightarrow} (\ell+1-i)e_1,\, (\ell+1-i)e_1 \xleftrightarrow{H_{\ell+1-i}} y \right\}. \quad (3.44)$$

The events $\mathscr{B}_i(x,y)$ and $\mathscr{C}_i(x,y)$ are depicted in Figure 3.

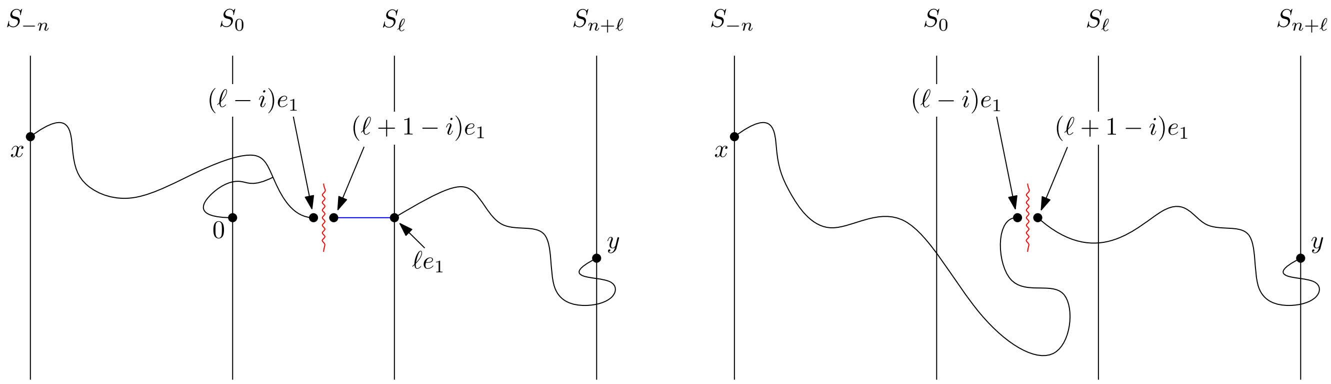

Figure 3: Schematic illustrations of the events $\mathscr{B}_i(x,y)$ (left) and $\mathscr{C}_i(x,y)$ (right). The blue edges represent those that are forced to be open in ω^{i-1}. The squiggly red line indicates that $(\ell-i)e_1$ and $(\ell+1-i)e_1$ lie in distinct clusters in the half-space H_{-n}.

Lemma 3.7. *With the above setup, and with $p = p_c$,*

$$\mathbb{P}\left(\mathscr{A}(x,y)\right) \leq \sum_{i=1}^{\ell} p_c^{-i+1}\mathbb{P}\big(\mathscr{C}_i(x,y)\big). \tag{3.45}$$

Proof. Given a configuration ω, let i be minimal such that 0 and ℓe_1 are connected in ω^i. When the event $\mathscr{A}(x,y)$ holds, i cannot be zero, and hence must be between 1 and ℓ. Since the clusters of 0 and ℓe_1 are both larger in ω^{i-1} than they are in ω, we must have that 0 is connected to x in H_{-n} in ω^{i-1}, and ℓe_1 is connected to y in H_ℓ in ω^{i-1}, which means that $\mathscr{B}_i(x,y)$ holds. It follows that

$$\mathscr{A}(x,y) \subseteq \bigcup_{i=1}^{\ell} \mathscr{B}_i(x,y). \tag{3.46}$$

Since we also have the inclusion of events $\mathscr{C}_i(x,y) \supseteq \mathscr{B}_i(x,y) \cap \{\omega(\eta_j) = 1 \text{ for every } 1 \leq j \leq i-1\}$, and since the two events on the right of this inclusion are independent, we have that

$$\mathbb{P}\big(\mathscr{C}_i(x,y)\big) \geq p_c^{i-1}\mathbb{P}\big(\mathscr{B}_i(x,y)\big). \tag{3.47}$$

With (3.46), this completes the proof. □

Proof of Proposition 3.2. It suffices to prove that there exist positive constants $c > 0$ and $\ell \in \mathbb{N}$ such that $\bar{P}^*(2n+3\ell) \geq c\bar{P}^*(n)^2$ for every $n \geq 0$. Let ℓ be as in Lemma 3.6, and suppose that $n \geq 0$ has $\bar{P}(n) = \bar{P}^*(n)$. Constants in this proof are permitted to depend on ℓ. In view of (3.42), the desired inequality will follow, for such n, if we show that

$$\sum_{x \in S_{-n}} \sum_{y \in S_{n+\ell}} \mathbb{P}\left(\mathscr{A}(x,y)\right) \preceq (\ell+1)\bar{P}^*(2n+\ell-1). \tag{3.48}$$

However this is in fact sufficient for general $n \geq 0$, since we may take $0 \leq n' \leq n$ such that $\bar{P}(n') = \bar{P}^*(n)$ to then deduce that

$$\bar{P}^*(2n+3\ell) \geq \bar{P}^*(2n'+3\ell) \succeq \bar{P}(n')^2 = \bar{P}^*(n)^2 \tag{3.49}$$

for every $n \geq 0$ as claimed.

It remains to prove (3.48). By Lemma 3.7,

$$\mathbb{P}\left(\mathscr{A}(x,y)\right) \preceq \sum_{i=1}^{\ell} \mathbb{P}\big(\mathscr{C}_i(x,y)\big). \tag{3.50}$$

By translation invariance applied to the event $\mathscr{C}_i(x,y)$, this gives

$$\sum_{x\in S_{-n}} \sum_{y\in S_{n+\ell}} \mathbb{P}\left(\mathscr{A}(x,y)\right) \preceq \sum_{i=1}^{\ell} \sum_{x\in S_{-n-\ell+i}} \sum_{y\in S_{n+i-1}} \mathbb{P}\left(x \xleftrightarrow{H_{-n-\ell+i}} 0,\, 0 \overset{H_{-n-\ell+i}}{\not\longleftrightarrow} e_1,\, e_1 \xleftrightarrow{H_1} y\right). \tag{3.51}$$

We have by (3.16) that the right-hand side of (3.51) is bounded above by

$$\sum_{i=1}^{\ell} \frac{1-p_c}{p_c} \bar{P}(2n+\ell-1) \preceq (\ell+1)\bar{P}^*(2n+\ell-1). \tag{3.52}$$

This proves (3.48) and therefore completes the proof. □

3.2 Proof of Theorem 2.3

3.2.1 Randomised algorithms and the OSSS inequality

Our deduction of Theorem 2.3 from Proposition 3.1 relies crucially on the OSSS inequality of O'Donnell, Saks, Schramm, and Servedio [41], which we now briefly review. This inequality has recently been recognised as a powerful and flexible tool in the study of critical and near-critical percolation models following the breakthrough work of Duminil-Copin, Raoufi, and Tassion [14]. We build in particular on the techniques developed to apply this inequality to prove inequalities between critical exponents in [31].

Let $\mathbb{N} = \{1, 2, \ldots\}$, and let E be a countable set. Informally, a *decision tree* is a deterministic procedure for querying the values of $\omega \in \{0,1\}^E$ that starts by querying the value of some fixed element of E and chooses which element of E to query at each subsequent step as a function of the values it has already observed. Formally, a *decision tree* is a function $T : \{0,1\}^E \to E^{\mathbb{N}}$ from subsets of E to infinite E-valued sequences $T = (T_1, T_2, \ldots)$ such that $T_1(\omega) = e_1$ for some $e_1 \in E$ not depending on ω, and such that for each $n \geq 2$ there exists a function $S_n : (E \times \{0,1\})^{n-1} \to E$ such that

$$T_n(\omega) = S_n\left[(T_i, \omega(T_i))_{i=1}^{n-1}\right], \tag{3.53}$$

where we think of $T_n(\omega)$ as the element of E that is queried at time n when given ω as an input to the procedure.

Let μ be a probability measure on $\{0,1\}^E$ and let ω be a random variable with law μ. For each decision tree T and $n \geq 1$ we define $\mathcal{F}_n(T)$ to be the σ-algebra generated by the random variables $\{T_i(\omega) : 1 \leq i \leq n\}$ and define $\mathcal{F}(T) = \bigcup_{n\geq 1} \mathcal{F}_n(T)$. We say that T *computes* a measurable function $f : \{0,1\}^E \to \mathbb{R}$ if $f(\omega)$ is measurable with respect to the μ-completion of the σ-algebra $\mathcal{F}(T)$. This is equivalent by Lévy's 0-1 law to the statement that

$$\mu\left[f(\omega) \mid \mathcal{F}_n(T)\right] \xrightarrow[n\to\infty]{} f(\omega) \qquad \mu\text{-a.s.} \tag{3.54}$$

To allow for exploration algorithms that are naturally described as parallel rather than serial algorithms, it is convenient to introduce the slightly more general notion of *decision forests*. A *decision forest* is defined to be a collection of decision trees $F = \{T^i : i \in I\}$ indexed by a countable set I. Given a decision forest $F = \{T^i : i \in I\}$ and a probability measure μ on $\{0,1\}^E$ we let $\mathcal{F}(F)$ be the smallest σ-algebra containing all of the σ-algebras $\mathcal{F}(T^i)$ and say that a measurable function $f : \{0,1\}^E \to \mathbb{R}$ is *computed by* F if it is measurable with respect to the μ-completion of the σ-algebra $\mathcal{F}(F)$.

Let E be a countable set, let μ be a probability measure on E, and let $F = \{T^i : i \in I\}$ be a decision forest on E. For each $e \in E$, we define the *revealment probability*

$$\delta_e(F,\mu) = \mu\big(\text{there exists } i \in I \text{ and } n \geq 1 \text{ such that } T^i_n(\omega) = e\big), \tag{3.55}$$

so that $\delta_e(F,\mu)$ is the probability that the status of e is ever queried when implementing the decision forest F on a sample from the measure μ. Finally, we define for each probability measure μ on $\{0,1\}^E$ and each pair of measurable functions $f, g : \{0,1\}^E \to \mathbb{R}$ the quantity

$$\mathrm{CoVr}_\mu[f,g] = (\mu \otimes \mu)\left[|f(\omega_1) - g(\omega_2)|\right] - \mu\left[|f(\omega_1) - g(\omega_1)|\right] \tag{3.56}$$

where ω_1, ω_2 are drawn independently from the measure μ. Thus, if f and g are $\{0,1\}$-valued then

$$\mathrm{CoVr}_\mu[f,g] = 2\,\mathrm{Cov}_\mu[f,g] = 2\mu\big(f(\omega) = g(\omega) = 1\big) - 2\mu\big(f(\omega) = 1\big)\mu\big(g(\omega) = 1\big). \tag{3.57}$$

We are now ready to state the version of the OSSS inequality that we will use, which is a special case of [31, Corollary 2.4].

Theorem 3.8 (OSSS for decision forests)**.** *Let E be a finite or countably infinite set and let μ be a product measure on $\{0,1\}^E$. Then for every pair of measurable, μ-integrable functions $f, g : \{0,1\}^E \to \mathbb{R}$ and every decision forest F computing g we have that*

$$\sum_{e \in E} \delta_e(F,\mu)\,\mathrm{Cov}_\mu\left[f, \omega(e)\right] \geq \frac{1}{2}\left|\mathrm{CoVr}_\mu\left[f,g\right]\right|. \tag{3.58}$$

See [14] for an extension of the OSSS inequality to *monotonic measures* such as the law of the Fortuin-Kastelyn random-cluster model.

3.2.2 Differential inequalities for Dini derivatives

In order to discuss how the OSSS inequality leads to differential inequalities in the infinite-volume setting (without any need for finite-volume approximation and limit), it is convenient to introduce the notion of *Dini derivatives*; see, e.g., [34] for further background. The *lower-right Dini derivative* of a function $f : [a,b] \to \mathbb{R}$ is defined to be

$$\left(\frac{d}{dx}\right)_+ f(x) = \liminf_{\varepsilon \downarrow 0} \frac{f(x+\varepsilon) - f(x)}{\varepsilon} \tag{3.59}$$

for each $x \in [a,b)$. In our setting, it is a classical and elementary fact [17, Theorem 2.34] that if A is an event depending on at most finitely many edges then $\mathbb{P}_p(A)$ is a polynomial in p with derivative

$$\frac{d}{dp}\mathbb{P}_p(A) = \frac{1}{p(1-p)} \sum_{e \in E} \operatorname{Cov}\left[\omega(e), \mathbb{1}(A)\right]. \tag{3.60}$$

If A is an *increasing* event depending possibly on infinitely many edges, we still have the lower-right Dini derivative bound

$$\left(\frac{d}{dp}\right)_+ \mathbb{P}_p(A) \geq \frac{1}{p(1-p)} \sum_{e \in E} \operatorname{Cov}\left[\omega(e), \mathbb{1}(A)\right]. \tag{3.61}$$

A detailed proof of this is given in [31, Proposition 2.1]. Thus, the OSSS inequality allows us to prove lower bounds on derivatives of increasing events by exhibiting decision forests that compute these events and have small maximum revealment.

Lower bounds on the lower-right Dini derivatives of monotone functions can often be used in much the same way as bounds on the classical derivative of a differentiable function. For example, the usual logarithmic derivative formula

$$\left(\frac{d}{dx}\right)_+ \log f(x) = \frac{1}{f(x)} \left(\frac{d}{dx}\right)_+ f(x) \tag{3.62}$$

remains valid. Also, if $f : [a,b] \to \mathbb{R}$ is increasing then

$$f(b) - f(a) \geq \int_a^b \left(\frac{d}{dx}\right)_+ f(x) dx. \tag{3.63}$$

Since every measurable function has measurable Dini derivatives [34, Theorem 3.6.5], the above integral is well-defined.

3.2.3 Slightly subcritical pioneers: Proof of Theorem 2.3

Our goal in this section is to study the distribution of the total number of 0-pioneers $|\mathcal{P}_0|$ in critical and slightly subcritical percolation. The main result is the following proposition, which strengthens Theorem 2.3.

Proposition 3.9. *Let $d > 6$ and suppose that* (T) *holds. There exists a positive constant c such that*

$$\mathbb{E}_{p_c-\varepsilon}|\mathcal{P}_0| \preceq \varepsilon^{-1/2} \qquad \textit{and} \qquad \mathbb{P}_{p_c-\varepsilon}\big(|\mathcal{P}_0| \geq k\big) \preceq k^{-2/3} \exp\left[-c\varepsilon^{3/2} k\right] \tag{3.64}$$

for every $0 \leq \varepsilon \leq p_c$ and $k \geq 1$.

The exponential tail bound on $|\mathcal{P}_0|$ is not needed for the proofs of the main theorems but is included since it may be of independent interest. Before proving this proposition, we show how it implies Theorem 2.3.

Proof of Theorem 2.3 given Proposition 3.9. Recall that $P_p(n) = \mathbb{E}_p|\mathcal{P}_0(n)|$. We have already observed that the lower bound of Theorem 2.3 holds, and we have already proved the desired upper bound when

$p = p_c$ in Proposition 3.1. It therefore suffices to prove that there exists a positive constant c such that

$$P_p(n) \preceq \exp\left[-c(p_c - p)^{1/2} n\right] \tag{3.65}$$

for every $n \geq 1$ and $p_c/2 \leq p < p_c$. Fix p in this interval. As discussed below Definition 2.1, $\mathcal{P}_0(n) \cap \mathcal{P}_0(m) = \emptyset$ when $|n - m| \geq L$ and hence by Proposition 3.9

$$\frac{1}{N}\sum_{n=1}^{N} P_p(n) \leq \frac{L}{N}\mathbb{E}_p|\mathcal{P}_0| \preceq \frac{1}{(p_c - p)^{1/2} N} \tag{3.66}$$

for every $N \geq 1$. It follows that there exists a constant C such that there exists n_p with $1 \leq n_p \leq C(p_c - p)^{-1/2}$ such that $P_p(n_p) \leq p^{L-1}/2$. It follows inductively by the submultiplicativity estimate (2.3) that $P_p(kn_p) \leq p^{L-1}2^{-k}$ for every $k \geq 1$. Since we also have that $P_p(n) \leq P_{p_c}(n) \preceq 1$, it follows by another application of (2.3) that $P_p(kn_p + r) \preceq 2^{-k}$ for every $k \geq 1$ and $0 \leq r < n_p$. If now we write arbitrary n as $n = \lfloor \frac{n}{n_p} \rfloor n_p + r$, then we see that the above gives $P_p(n) \preceq 2^{-\lfloor n/n_p \rfloor}$ and the desired exponential estimate follows from the upper bound $n_p \preceq (p_c - p)^{-1/2}$. □

To begin the proof of Proposition 3.9 we first note that Proposition 3.1, together with the result of Kozma and Nachmias [36] that Theorem 1.2 holds for $p = p_c$, yield the following important corollary describing the distribution of the total number of pioneers *at criticality*.

Lemma 3.10. *Let $d > 6$ and suppose that* (T) *holds. Then $\mathbb{P}_{p_c}(|\mathcal{P}_x| \geq k) \preceq k^{-2/3}$ for every $x \in \mathbb{Z}^d$ and $k \geq 0$.*

Proof. It suffices to consider the case $x = 0$. Let $n, k \geq 0$, where n is a parameter we will optimise over shortly. By Markov's inequality,

$$\mathbb{P}_{p_c}(|\mathcal{P}_0| \geq k) \leq \mathbb{P}_{p_c}(|\mathcal{P}_0| \geq k \text{ and } 0 \text{ is not connected to } H_n) + \mathbb{P}_{p_c}(0 \text{ is connected to } H_n) \tag{3.67}$$

$$\leq \frac{1}{k}\mathbb{E}_{p_c}\left[\sum_{i=0}^{n} |\mathcal{P}_0(i)|\right] + \mathbb{P}_{p_c}(0 \text{ is connected to } H_n) \preceq \frac{n}{k} + \frac{1}{n^2}, \tag{3.68}$$

for every $k, n \geq 1$, where the first bound follows from Proposition 3.1 and the second follows from the aforementioned result of Kozma and Nachmias. The claim follows by taking $n = \lceil k^{1/3} \rceil$. □

We now apply the OSSS inequality to deduce Proposition 3.9 from Lemma 3.10. We follow closely the proof of [31, Theorem 1.1]. For each $p \in [0, 1]$ and $h \geq 0$, we write $\mathbb{P}_{p,h}$ for the law of the pair $(\omega, \mathcal{G})$ where ω is distributed as Bernoulli-p bond percolation and $\mathcal{G}$ is a *ghost field* independent of ω, that is, a random subset of the edge set $\mathbb{B}$ in which each edge is included independently at random with inclusion probability $1 - e^{-h}$. We call an edge *green* if it belongs to $\mathcal{G}$. (While it is more standard to consider ghost fields to be random sets of *vertices*, it is more convenient for our purposes to take them to be random sets of edges.) As a first and key step, we use the OSSS inequality to prove a differential inequality.

Lemma 3.11. *The differential inequality*

$$\left(\frac{d}{dp}\right)_{+} \log \mathbb{P}_p(|\mathcal{P}_0| \geq k) \geq \frac{1}{2p(1-p)}\left[\frac{k(1 - e^{-1})}{\sum_{i=0}^{k} \mathbb{P}_p(|\mathcal{P}_0| \geq i)} - 1\right] \tag{3.69}$$

holds for every $k \geq 1$ and $0 < p < 1$.

Proof. By (3.61) and (3.62),

$$\left(\frac{d}{dp}\right)_+ \log \mathbb{P}_p(|\mathcal{P}_0| \geq k) \geq \frac{1}{\mathbb{P}_p(|\mathcal{P}_0| \geq k)} \frac{1}{p(1-p)} \sum_{e \in \mathbb{B}} \mathrm{Cov}\left[\omega(e), \mathbb{1}(|\mathcal{P}_0| \geq k)\right] \tag{3.70}$$

where $\mathbb{B}$ is the set of edges. It therefore suffices to prove that

$$\sum_{e \in \mathbb{B}} \mathrm{Cov}\left[\omega(e), \mathbb{1}(|\mathcal{P}_0| \geq k)\right] \geq \frac{1}{2}\left[\frac{k(1-e^{-1})}{\sum_{i=0}^{k} \mathbb{P}_p(|\mathcal{P}_0| \geq i)} - 1\right] \mathbb{P}_p(|\mathcal{P}_0| \geq k). \tag{3.71}$$

For this we will use Theorem 3.8.

For the setup for Theorem 3.8, we let $(\omega, \mathcal{G})$ have law $\mathbb{P}_{p,h}$, where we think of $\mathbb{P}_{p,h}$ as a product measure on $\{0,1\}^{\mathbb{B} \times \{\text{perc,ghost}\}}$. Consider the two Boolean functions

$$f(\omega, \mathcal{G}) = f(\omega) = \mathbb{1}(|\mathcal{P}_0| \geq k) \qquad \text{and} \qquad g(\omega, \mathcal{G}) = \mathbb{1}(\mathcal{P}_0 \cap \mathcal{G} \neq \emptyset). \tag{3.72}$$

We say that an edge is *horizontal* if its endpoints have distinct horizontal coordinates. We can determine the value of g by first revealing the value of the ghost field at each horizontal edge and then exploring the cluster of each green horizontal edge in the halfspace lying strictly to the left of its rightmost endpoint. This exploration process can be encoded as a decision forest $F = \{T^e : e \in \mathbb{B}\}$ in which the decision tree T^e first queries the status of the ghost field at the edge e, halting if it discovers that $\mathcal{G}(e) = 0$. If the decision tree discovers that $\mathcal{G}(e) = 1$, it next checks whether e is open in ω, halting if it is closed and otherwise exploring the cluster of the leftmost endpoint of e in the halfspace lying strictly to the left of the rightmost endpoint of e. See the proof of [31, Proposition 3.1] to see how such a decision forest may be defined formally. This decision forest clearly computes g.

Its revealments satisfy

$$\delta_{e,\text{perc}}(F, \mathbb{P}_{p,h}) \leq \mathbb{P}_{p,h}(e \in \mathcal{G} \text{ or at least one of the endpoints of } e \text{ has a pioneer in } \mathcal{G}) \tag{3.73}$$

$$\delta_{e,\text{ghost}}(F, \mathbb{P}_{p,h}) = 1 \tag{3.74}$$

for each $e \in \mathbb{B}$. We can bound the revealment probabilities of edges by the union bound

$$\begin{aligned}\delta_{e,\text{perc}}(F, \mathbb{P}_{p,h}) &\leq \mathbb{P}_{p,h}(e \in \mathcal{G}) + 2\mathbb{P}_{p,h}(0 \text{ has a pioneer in } \mathcal{G}) \\ &= 1 - e^{-h} + 2\mathbb{E}_{p,h}\left[1 - e^{-h|\mathcal{P}_0|}\right]. \end{aligned} \tag{3.75}$$

It therefore follows from the OSSS inequality Theorem 3.8 that

$$\begin{aligned}\mathrm{Cov}\left[f,g\right] = \frac{1}{2}|\,\mathrm{CoVr}[f,g]| &\leq \sum_{e \in \mathbb{B}} \delta_{e,\text{perc}}(F)\, \mathrm{Cov}\left[f, \omega(e)\right] + \sum_{x \in \mathbb{Z}^d} \delta_{e,\text{ghost}}(F)\, \mathrm{Cov}\left[f, \mathcal{G}(x)\right] \\ &= \sum_{e \in \mathbb{B}} \delta_{e,\text{perc}}(F)\, \mathrm{Cov}\left[f, \omega(e)\right] \\ &\leq \left(1 - e^{-h} + 2\mathbb{E}_{p,h}\left[1 - e^{-h|\mathcal{P}_0|}\right]\right) \sum_{e \in E} \mathrm{Cov}\left[f, \omega(e)\right], \end{aligned} \tag{3.76}$$

where we used that $f(\omega, \mathcal{G}) = f(\omega)$ is independent of the ghost field $\mathcal{G}$ in the equality on the second line.

On the other hand, we can also compute that

$$\begin{aligned}\operatorname{Cov}[f,g] &= \mathbb{P}_{p,h}\left(|\mathcal{P}_0| \geq k,\ |\mathcal{P}_0 \cap \mathcal{G}| \geq 1\right) - \mathbb{P}_p\left(|\mathcal{P}_0| \geq k\right)\mathbb{P}_{p,h}\left(|\mathcal{P}_0 \cap \mathcal{G}| \geq 1\right) \\ &= \mathbb{E}_p\left[\left(1-e^{-h|\mathcal{P}_0|}\right)\mathbb{1}(|\mathcal{P}_0| \geq k)\right] - \mathbb{E}_p\left[1-e^{-h|\mathcal{P}_0|}\right]\mathbb{P}_p\left(|\mathcal{P}_0| \geq k\right) \\ &\geq (1-e^{-hk})\mathbb{P}_p\left(|\mathcal{P}_0| \geq k\right) - \mathbb{E}_p\left[1-e^{-h|\mathcal{P}_0|}\right]\mathbb{P}_p\left(|\mathcal{P}_0| \geq k\right), \end{aligned} \tag{3.77}$$

so that

$$\sum_{e\in\mathbb{B}} \operatorname{Cov}\left[f,\omega(e)\right] \geq \frac{(1-e^{-hk}) - \mathbb{E}_p\left[1-e^{-h|\mathcal{P}_0|}\right]}{1-e^{-h}+2\mathbb{E}_p\left[1-e^{-h|\mathcal{P}_0|}\right]}\mathbb{P}_p\left(|\mathcal{P}_0| \geq k\right) \tag{3.78}$$

for every $k \geq 1$, $0 \leq p \leq 1$ and $h \geq 0$. The claimed inequality (3.71) follows by taking $h = 1/k$ and using the elementary fact that

$$1-e^{-1/k} + \mathbb{E}_p\left[1-e^{-|\mathcal{P}_0|/k}\right] \leq \frac{1}{k} + \frac{1}{k}\mathbb{E}_p\left[\min\{k,|\mathcal{P}_0|\}\right] = \frac{1}{k}\sum_{i=0}^{k}\mathbb{P}_p(|\mathcal{P}_0| \geq i). \tag{3.79}$$

This completes the proof. □

Proof of Proposition 3.9. We now analyse the differential inequality (3.69) to prove the desired slightly subcritical bounds. We begin with the proof of the inequality $\mathbb{E}_p|\mathcal{P}_0| \preceq (p_c-p)^{-1/2}$. Since $\mathbb{P}_p(|\mathcal{P}_0| \geq k)$ is an increasing function of p, we have by Lemma 3.11 and Lemma 3.10 that there exist positive constants c_1 and C_1 such that

$$\left(\frac{d}{dp}\right)_+ \log \mathbb{P}_p(|\mathcal{P}_0| \geq k) \geq \frac{1}{2p(1-p)}\left[\frac{k(1-e^{-1})}{C_1\sum_{i=1}^{k}(i+1)^{-2/3}} - 1\right] \geq \frac{1}{2p(1-p)}\left[c_1 k^{2/3} - 1\right] \tag{3.80}$$

for every $0 < p \leq p_c$ and $k \geq 1$. Integration of this inequality over the interval $[p,p_c]$, together with (3.63), shows that there exist positive constants c_2 and C_2 such that

$$\begin{aligned}\mathbb{P}_p(|\mathcal{P}_0| \geq k) &\leq \mathbb{P}_{p_c}(|\mathcal{P}_0| \geq k)\exp\left(-\int_p^{p_c}\frac{1}{2q(1-q)}\left[c_1 k^{2/3}-1\right]dq\right) \\ &\leq C_2 k^{-2/3}\exp\left(-c_2(p_c-p)k^{2/3}\right) \end{aligned} \tag{3.81}$$

for every $p_c/2 \leq p \leq p_c$ and $k \geq 1$. It follows by calculus that there exists a positive constant C_3 such that

$$\mathbb{E}_p|\mathcal{P}_0| \leq C_2\sum_{k=1}^{\infty} k^{-2/3}e^{-c_2(p_c-p)k^{2/3}} \leq C_3(p_c-p)^{-1/2} \tag{3.82}$$

for every $p_c/2 \leq p \leq p_c$ as claimed.

Finally, we prove that $\mathbb{P}_p(|\mathcal{P}_0| \geq k) \preceq k^{-2/3}\exp[-c(p_c-p)^{3/2}k]$. The differential inequality (3.69) implies the simplified inequality

$$\left(\frac{d}{dp}\right)_+ \log \mathbb{P}_p(|\mathcal{P}_0| \geq k) \geq \frac{1}{2p(1-p)}\left[\frac{k(1-e^{-1})}{1+\mathbb{E}_p|\mathcal{P}_0|} - 1\right]. \tag{3.83}$$

We again integrate the above inequality, and conclude that there exist positive constants c_3, c_4 and C_3

such that if $0 < \varepsilon \le p_c/4$ then

$$\mathbb{P}_{p_c-2\varepsilon}(|\mathcal{P}_0| \ge k) \le \mathbb{P}_{p_c-\varepsilon}(|\mathcal{P}_0| \ge k) \exp\left(-\int_{p_c-2\varepsilon}^{p_c-\varepsilon} \frac{1}{2q(1-q)}\left[c_3\varepsilon^{1/2}k - 1\right] dq\right)$$
$$\le C_3 k^{-2/3} \exp\left(-c_4\varepsilon^{3/2}k\right) \tag{3.84}$$

for every $k \ge 1$. This completes the proof. □

4 Plateau below the window: Proof of Theorem 1.3

In this section, we apply our bound on the slightly subcritical $\mathbb{Z}^d$ two-point function from Theorem 1.1 to prove the plateau estimates for the torus two-point function below the scaling window in Theorem 1.3. As a corollary, we also prove the torus triangle condition, Theorem 1.5.

4.1 Preliminaries

We start by recording some preliminary estimates that we will use repeatedly in the rest of the section. For each $x \in \mathbb{Z}^d$ and $0 \le p \le 1$, the $\mathbb{Z}^d$ *open bubble* and *open triangle diagrams* $\mathsf{B}_p(x)$ and $\mathsf{T}_p(x)$ are defined by

$$\mathsf{B}_p(x) = \sum_{u\in\mathbb{Z}^d} \tau_p(u)\tau_p(x-u) = (\tau_p * \tau_p)(x), \tag{4.1}$$

$$\mathsf{T}_p(x) = \sum_{u,v\in\mathbb{Z}^d} \tau_p(u)\tau_p(v-u)\tau_p(x-v) = (\tau_p * \tau_p * \tau_p)(x). \tag{4.2}$$

Upper bounds on these two quantities are given in the next lemma. The notation p_c always refers to the critical value for $\mathbb{Z}^d$.

Lemma 4.1. *Let $d > 6$ and suppose that* (T) *holds on $\mathbb{Z}^d$. There exist positive constants C_1, C_2 such that for $x \in \mathbb{Z}^d$ and $p \le p_c$,*

$$\mathsf{B}_p(x) \le \frac{C_1}{\langle x\rangle^{d-4}} e^{-c_1 m(p)\|x\|_\infty}, \tag{4.3}$$

$$\mathsf{T}_p(x) \le \frac{C_2}{\langle x\rangle^{d-6}} e^{-c_1 m(p)\|x\|_\infty}. \tag{4.4}$$

Proof. We insert the bound of Theorem 1.1 into the convolutions defining the bubble and triangle diagrams. By the triangle inequality, the exponential factors are bounded above by an overall factor $e^{-c_1 m(p)|x|}$. For the powers, let $f(x) = \langle x\rangle^{-(d-2)}$. Since $d > 6$ we have by (3.19)–(3.20) that $(f*f)(x) \preceq \langle x\rangle^{-(d-4)}$ and $(f*f*f)(x) \preceq \langle x\rangle^{-(d-6)}$. Together, this gives the desired result. □

Observe that if $x \in \mathbb{T}_r^d$ is regarded as a point in $[-\frac{r}{2}, \frac{r}{2})^d \cap \mathbb{Z}^d$ then $\langle x + ru\rangle \asymp r\langle u\rangle$ uniformly in nonzero $u \in \mathbb{Z}^d$ since

$$\|x + ru\|_\infty \ge \|ru\|_\infty - \frac{r}{2} \ge \|ru\|_\infty - \frac{1}{2}\|ru\|_\infty = \frac{1}{2}\|ru\|_\infty \tag{4.5}$$

and

$$\|x+ru\|_\infty \le \frac{r}{2} + \|ru\|_\infty \le \frac{1}{2}\|ru\|_\infty + \|ru\|_\infty = \frac{3}{2}\|ru\|_\infty. \tag{4.6}$$

The following elementary lemma will be useful with $\nu = cm(p)$.

Lemma 4.2. *Let $r \ge 2$, $a > 0$ and $\nu > 0$. Then*

$$\sum_{u\in\mathbb{Z}^d : u\neq 0} \frac{1}{\|x+ru\|_\infty^{d-a}} e^{-\nu\|x+ru\|_\infty} \preceq_a \frac{1}{\nu^a r^d} e^{-\frac{1}{4}\nu r} \tag{4.7}$$

for every $x \in \mathbb{T}_r^d \equiv [-\frac{r}{2}, \frac{r}{2})^d \cap \mathbb{Z}^d$.

Proof. Let $a > 0$. It follows from (4.5) that for any nonzero $u \in \mathbb{Z}^d$ and $r \ge 2$, $\langle x + ru\rangle \ge \frac{1}{2}\|ru\|_\infty$ and thus

$$\begin{aligned}
\sum_{u\neq 0} \frac{1}{\|x+ru\|_\infty^{d-a}} e^{-\nu\langle x+ru\rangle} &\le \sum_{u\neq 0} \frac{1}{(\frac{1}{2}\|ru\|_\infty)^{d-a}} e^{-\frac{1}{2}\nu\|ru\|_\infty} \\
&\le 2^{a-d} e^{-\frac{1}{4}\nu r} \sum_{N=1}^{\infty} \sum_{u:\|u\|_\infty = N} \frac{1}{\|ru\|_\infty^{d-a}} e^{-\frac{1}{4}\nu\|ru\|_\infty} \\
&\preceq r^{a-d} e^{-\frac{1}{4}\nu r} \sum_{N=1}^{\infty} N^{d-1-d+a} e^{-\frac{1}{4}\nu r N}.
\end{aligned} \tag{4.8}$$

We bound the sum on the right-hand side by an integral to obtain an upper bound which is a constant multiple of

$$r^{a-d} e^{-\frac{1}{4}\nu r} \int_1^\infty u^{a-1} e^{-\frac{1}{4}\nu r u} \mathrm{d}u = \frac{1}{\nu^a r^d} e^{-\frac{1}{4}\nu r} \int_{\nu r}^\infty t^{a-1} e^{-t/4} \mathrm{d}t. \tag{4.9}$$

The integral is uniformly bounded since $a > 0$. This concludes the proof. ☐

Remark 4.3. Bounds expressed in terms of the mass $m(p)$, such as the one in Lemma (4.2) with $\nu = cm(p)$, can also be expressed in terms of the susceptibility $\chi(p)$ since

$$\frac{1}{m(p)^2} \preceq \chi(p). \tag{4.10}$$

To prove (4.10), we first fix any $p_1 \in (0, p_c)$. For $p \le p_1$, since m is decreasing and since $1 = \chi(0) \le \chi(p)$, we have $m(p)^{-2} \le m(p_1)^{-2} \le m(p_1)^{-2}\chi(p)$ and the desired upper bound follows for $p \in (0, p_1]$. We can choose p_1 close enough to p_c that $m(p)^{-2}$ and $\chi(p)$ are comparable for $p \in (p_1, p_c)$, since both are asymptotic to $(1-p/p_c)^{-1}$. In particular for p_1 close enough to p_c there exists C such that $m(p)^{-2} \le C\chi(p)$ for those $p \in [p_1, p_c)$.

The following three estimates will be useful.

Lemma 4.4. *Let $d > 6$ and suppose that* (T) *holds on $\mathbb{Z}^d$. For $x \in \mathbb{T}_r^d$ and $0 \le p < p_c$,*

$$\sum_{u\in\mathbb{Z}^d} \tau_p(x+ru) \le \tau_p(x) + C\frac{\chi(p)}{V} e^{-\frac{c}{4}m(p)r}, \tag{4.11}$$

$$\sum_{u\in\mathbb{Z}^d} \mathsf{B}_p(x+ru) \le \mathsf{B}_p(x) + C\frac{\chi(p)^2}{V} e^{-\frac{c}{4}m(p)r}, \tag{4.12}$$

$$\sum_{u\in\mathbb{Z}^d} \mathsf{T}_p(x+ru) \le \mathsf{T}_p(x) + C\frac{\chi(p)^3}{V} e^{-\frac{c}{4}m(p)r}. \tag{4.13}$$

Proof. For the first inequality, we separate the $w = 0$ term from the sum and apply Theorem 1.1, as well as Lemma 4.2 with $a = 2$, to obtain that there exist positive constants c, C_1, and C_2 such that

$$\sum_{u\in\mathbb{Z}^d} \tau_p(x+ru) \le \tau_p(x) + \sum_{u\neq 0} \frac{C_1}{\langle x+ru\rangle^{d-2}} e^{-cm(p)\|x+ru\|_\infty} \tag{4.14}$$

$$\le \tau_p(x) + C_2\frac{\chi(p)}{V} e^{-\frac{c}{4}m(p)r}. \tag{4.15}$$

For the bubble and triangle diagrams, in place of Theorem 1.1 we instead use the bounds of Lemma 4.1, which modify the power $d-2$ in the above inequality to $d-4$ for the bubble and $d-6$ for the triangle. We then apply Lemma 4.2 and Remark 4.3 with $a = 4$ and with $a = 6$ to complete the proof. $\square$

4.2 Upper bound on the torus two-point function

Proof of (1.15). The proof is as in [44]. For each $0 \le p < p_c$ and $x \in \mathbb{T}_r^d \equiv [-\frac{r}{2}, \frac{r}{2})^d \cap \mathbb{Z}^d$ we define

$$\psi_{r,p}(x) = \sum_{u\in\mathbb{Z}^d : u\neq 0} \tau_p(x+ru), \tag{4.16}$$

which is finite only when $p < p_c$. It follows by a simple coupling argument originating in the work of Benjamini and Schramm [4] and further developed in [23, Proposition 2.1] that

$$\tau_p^{\mathbb{T}}(x) \le \tau_p(x) + \psi_{r,p}(x) \tag{4.17}$$

for every $x \in \mathbb{T}_r^d$ and $0 \le p \le 1$. By Lemma 4.4,

$$\psi_{r,p}(x) \le C\frac{\chi(p)}{V} e^{-\frac{c}{4}m(p)r}, \tag{4.18}$$

and with (4.17) this immediately yields the upper bound (1.15). $\square$

4.3 Lower bound on the torus two-point function below the window

We now turn to the proof of the lower bound (1.16) on the torus two-point function for p below the scaling window. This proof is model dependent and although it follows the general strategy used for weakly self-avoiding walk in [44], it differs in details.

We seek a lower bound of the form $r^{-d}\chi$ for the difference

$$\psi^{\mathbb{T}}_{r,p}(x) = \tau^{\mathbb{T}}_p(x) - \tau_p(x) \qquad (x \in \mathbb{T}^d). \tag{4.19}$$

To this end we first make the decomposition

$$\psi^{\mathbb{T}}_{r,p}(x) = \psi_{r,p}(x) - (\psi_{r,p}(x) - \psi^{\mathbb{T}}_{r,p}(x)). \tag{4.20}$$

We can then deduce the lower bound (1.16) as an immediate consequence of the following two lemmas.

Lemma 4.5. *Let $d > 6$ and suppose that* (T) *holds on $\mathbb{Z}^d$. There exist positive constants A_2 and c_ψ such that if $r \geq 2$ and $p_c - A_2 r^{-2} \leq p < p_c$ then*

$$\psi_{r,p}(x) \geq c_\psi \frac{\chi(p)}{V} \tag{4.21}$$

for every $x \in \mathbb{T}^d_r$.

Lemma 4.6. *Let $d > 6$ and suppose that* (T) *holds on $\mathbb{Z}^d$. Let c_ψ be as in Lemma 4.5. There exist positive constants A_1 and M such that if $r \geq 2$ and $0 \leq p \leq p_c - A_1 V^{-1/3}$ then*

$$\psi_{r,p}(x) - \psi^{\mathbb{T}}_{r,p}(x) \leq \frac{1}{2} c_\psi \frac{\chi(p)}{V} \tag{4.22}$$

for every $x \in \mathbb{T}^d_r$ with $\|x\|_\infty \geq M$.

Proof of (1.16) *subject to Lemmas 4.5 and 4.6.* By definition,

$$\tau^{\mathbb{T}}_p(x) = \tau_p(x) + \psi_{r,p}(x) - [\psi_{r,p}(x) - \psi^{\mathbb{T}}_{r,p}(x)]. \tag{4.23}$$

By Lemmas 4.5 and 4.6, if $\|x\|_\infty \geq M$ and $p_c - A_2 r^{-2} \leq p \leq p_c - A_1 V^{-1/3}$, then we have the lower bound

$$\tau^{\mathbb{T}}_p(x) \geq \tau_p(x) + c_\psi \frac{\chi(p)}{V} - \frac{1}{2} c_\psi \frac{\chi(p)}{V} = \tau_p(x) + \frac{1}{2} c_\psi \frac{\chi(p)}{V}, \tag{4.24}$$

which is the desired estimate. □

4.3.1 Proof of Lemma 4.5

In this section we prove the lower bound on $\psi_{r,p}(x)$ stated in Lemma 4.5. We begin with the following simple observation.

Lemma 4.7. *Let $d > 6$ and suppose that* (T) *holds on $\mathbb{Z}^d$. The inequality*

$$\tau_{p_c}(x) - \tau_p(x) \preceq \frac{p_c - p}{\langle x \rangle^{d-4}}. \tag{4.25}$$

holds for every $p_c/2 \leq p \leq p_c$ and $x \in \mathbb{Z}^d$.

Proof. The proof of (4.25) is a consequence of the following standard differential inequality (cf. [2]). Let $\tau^n_p(x) = \mathbb{P}(0 \leftrightarrow x \text{ inside } [-n,n]^d)$. It is easy to see that for every $n > 0$, $\tau^n_p(x)$ is differentiable in p and that $\tau_p^n(x) \to \tau_p(x)$ as $n \to \infty$. Let $p \in [\frac{1}{2}p_c, p_c]$. By Russo's Formula and the BK inequality (with the sum over the undirected bonds in $[-n,n]^d$) we have

$$\begin{aligned}\frac{d}{dp}\tau_p^n(x) &= \frac{1}{p}\sum_{\{u,v\}} \mathbb{P}_p(\{u,v\} \text{ is pivotal for } 0 \leftrightarrow x \text{ inside } [-n,n]^d,\ \{u,v\} \text{ is open}) \\ &\leq \frac{1}{p}\sum_{\{u,v\}} \mathbb{P}_p(\{0 \leftrightarrow u \text{ inside } [-n,n]^d\} \circ \{u \leftrightarrow x \text{ inside } [-n,n]^d\}) \\ &\preceq (\tau_p * \tau_p)(x). \end{aligned} \tag{4.26}$$

It follows by monotonicity in p and the bound on the bubble from Lemma 4.1 that

$$\frac{d}{dp}\tau_p^n(x) \preceq (\tau_{p_c} * \tau_{p_c})(x) \preceq \frac{1}{\langle x\rangle^{d-4}}. \tag{4.27}$$

Integration of (4.27) over $[p, p_c]$, followed by the limit as $n \to \infty$, gives (4.25). □

We now apply Lemma 4.7 to complete the proof of Lemma 4.5.

Proof of Lemma 4.5. Let $x \in \mathbb{T}_r^d$. To obtain a lower bound on $\psi_{r,p}$, we may sum in (4.16) over only those $u \in \mathbb{Z}^d$ with $\|u\|_\infty \leq R$ with $R \geq 1$ a large number depending on r and $p_c - p$ to be chosen shortly. By (T) and (4.25), there exist positive constants c_1 and C_1 such that, for every $y \in \mathbb{Z}^d$,

$$\tau_p(y) = \tau_{p_c}(y) - \big(\tau_{p_c}(y) - \tau_p(y)\big) \geq \frac{c_1}{\langle y\rangle^{d-2}} - \frac{C_1(p_c-p)}{\langle y\rangle^{d-4}}. \tag{4.28}$$

With this, together with (4.5)–(4.6) we see that there exist positive constants c_2, C_2, C_3 such that

$$\begin{aligned}\sum_{u\in\mathbb{Z}^d:u\neq 0} \tau_p(x+ru) &\geq \sum_{1\leq\|u\|_\infty\leq R} \tau_p(x+ru) \\ &\geq c_1 \sum_{1\leq\|u\|_\infty\leq R} \frac{1}{\|ru\|_\infty^{d-2}} - C_1(p_c-p)\sum_{1\leq\|u\|_\infty\leq R}\frac{1}{\|ru\|_\infty^{d-4}} \\ &\geq \frac{c_2}{r^{d-2}}R^2 - \frac{C_2(p_c-p)}{r^{d-4}}R^4 = \frac{c_2}{r^{d-2}}R^2\left(1 - C_3(p_c-p)r^2R^2\right) \end{aligned} \tag{4.29}$$

for every $r, R \geq 1$ and $0 \leq p < p_c$. Now we choose $R^2 = (2C_3(p_c-p)r^2)^{-1}$, and require $p_c - p \leq A_2 r^{-2}$ with A_2 chosen small enough for R to be indeed greater than 1. This gives

$$\sum_{u\in\mathbb{Z}^d:u\neq 0} \tau_p(x+ru) \geq \frac{c_2R^2}{2r^{d-2}} = \frac{c_3}{(p-p_c)r^d} \succeq \frac{\chi(p)}{V} \tag{4.30}$$

for every $r \geq 2$ and $p \in [p_c - A_2r^{-2}, p_c)$, and completes the proof. □

4.3.2 Proof of Lemma 4.6

We now prove Lemma 4.6, which states that there exist constants M and A_1 such that if $p \leq p_c - A_1V^{-1/3}$ then

$$\psi_{r,p}(x) - \psi_{r,p}^{\mathbb{T}}(x) \leq \frac{1}{2}c_\psi r^{-d}\chi(p) \tag{4.31}$$

for every $x \in \mathbb{T}_r^d$ with $\|x\|_\infty \geq M$, where c_ψ is the constant from Lemma 4.5. In order to prove this, we will prove the following more general inequality.

Lemma 4.8. *Let $d > 6$ and suppose that* (T) *holds on $\mathbb{Z}^d$. There exists a constant C such that the inequality*

$$\psi_{r,p}(x) - \psi_{r,p}^{\mathbb{T}}(x) \leq C\frac{\chi(p)}{V}\left(\mathsf{T}_p(x) + \frac{\chi(p)^3}{V}\right), \tag{4.32}$$

holds for every $p < p_c$ and $x \in \mathbb{T}_r^d$.

Proof of Lemma 4.6 given Lemma 4.8. By taking $p \leq p_c - A_1 V^{-1/3}$, we see from the bound on the susceptibility in (1.14) that $\chi(p) \preceq A_1^{-1} V^{1/3}$, so the term $V^{-1}\chi(p)^3$ can be made as small as desired by taking A_1 sufficiently large. By (4.4), the triangle term $\mathsf{T}_p(x)$ can be made as small as desired by taking $\|x\|_\infty \geq M$ with M sufficiently large. Thus we can choose the constants A_1, M in such a way that the right-hand side of (4.32) is at most $\frac{1}{2}c_\psi \chi(p)/V$. This gives the desired inequality (4.31). □

We turn now to the proof of Lemma 4.8. We build upon the coupling of percolation on $\mathbb{Z}^d$ and $\mathbb{T}_r^d$ developed by Heydenreich and van der Hofstad [23, Proposition 2.1]. With this coupling, they proved that

$$\psi_{r,p}(x) - \psi_{r,p}^{\mathbb{T}}(x) \leq \frac{1}{2}\sum_{u\in\mathbb{Z}^d}\sum_{v\neq u}\mathbb{P}(0 \leftrightarrow x+ru,\, x+rv) + \sum_{u\in\mathbb{Z}^d}\mathbb{P}(\{0 \leftrightarrow x+ru\} \cap \{0 \underset{\mathbb{T}}{\leftrightarrow} x\}^c) \tag{4.33}$$

for every $x \in \mathbb{T}_r^d$, where $\{x \underset{\mathbb{T}}{\leftrightarrow} y\}$ denotes the event that x is connected to y by an open path in $\mathbb{T}_r^d$ in the coupling (see [23, (5.4)]).

The proof of Lemma 4.8 is immediate using the following two lemmas to bound the two terms in (4.33). In the first lemma, there is room to spare by a factor χ in the last term; this is consistent with [23]. Also we see the bubble rather than the triangle, which again has room to spare.

Lemma 4.9. *Let $d > 6$ and suppose that* (T) *holds on $\mathbb{Z}^d$. The inequality*

$$\sum_{u\in\mathbb{Z}^d}\sum_{v\neq u}\mathbb{P}(0 \leftrightarrow x+ru,\, x+rv) \preceq \frac{\chi}{V}\left(\mathsf{B}_p(x) + \frac{\chi^2}{V}\right) \tag{4.34}$$

holds for every $0 \leq p < p_c$, $r > 2$, and $x \in \mathbb{T}_r^d$.

Proof. We use x, y for torus points and u, v, w for translating points in $\mathbb{Z}^d$, and for clarity write the two-point function as $\tau(u,v)$ in place of the usual $\tau_p(v-u)$. By the BK inequality,

$$\begin{aligned}\sum_{u\in\mathbb{Z}^d}\sum_{v\neq u}\mathbb{P}(0 \leftrightarrow x+ru,\, x+rv) &\leq \sum_{z,u\in\mathbb{Z}^d}\sum_{v\neq u}\tau(0,z)\tau(z,x+ru)\tau(z,x+rv)\\ &= \sum_{y\in\mathbb{T}_r^d}\sum_{w\in\mathbb{Z}^d}\tau(0,x+y+rw)\sum_{u\in\mathbb{Z}^d}\tau(y,r(u-w))\sum_{v\neq u}\tau(y,r(v-w))\\ &= \sum_{y\in\mathbb{T}_r^d}\sum_{w\in\mathbb{Z}^d}\tau(0,x+y+rw)\sum_{u\in\mathbb{Z}^d}\tau(y,ru)\sum_{v\neq u}\tau(y,rv), \end{aligned} \tag{4.35}$$

where in the second line we replaced z by $x+y+rw$, and in the third we replaced u by $u+w$ and v by $v+w$. For the sum over v, it follows from Lemma 4.4 that

$$\begin{aligned}\sum_{v\neq u}\tau(y,rv) &= \sum_{v\neq u}\tau(y,rv)(\mathbb{1}_{u=0}+\mathbb{1}_{u\neq 0})\\ &= \mathbb{1}_{u=0}\sum_{v\neq 0}\tau(y,rv)+\mathbb{1}_{u\neq 0}\sum_{v\neq u}\tau(y,rv)\\ &\preceq \mathbb{1}_{u=0}\frac{\chi}{V}+\mathbb{1}_{u\neq 0}\left(\tau(0,y)+\frac{\chi}{V}\right)\preceq \mathbb{1}_{u\neq 0}\tau(0,y)+\frac{\chi}{V}. \end{aligned}\tag{4.36}$$

This leads, using Lemma 4.4 again, to

$$\sum_{u\in\mathbb{Z}^d}\tau(y,ru)\sum_{v\neq u}\tau(y,rv)\preceq\tau(0,y)\frac{\chi}{V}+\frac{\chi}{V}\left(\tau(0,y)+\frac{\chi}{V}\right)\preceq\frac{\chi}{V}\left(\tau(0,y)+\frac{\chi}{V}\right).\tag{4.37}$$

Thus we have an upper bound on (4.35) given by

$$\frac{\chi}{V}\sum_{y\in\mathbb{T}_r^d}\sum_{w\in\mathbb{Z}^d}\tau(0,x+y+rw)\left(\tau(0,y)+\frac{\chi}{V}\right)=\frac{\chi^3}{V^2}+\frac{\chi}{V}\sum_{y\in\mathbb{T}_r^d}\sum_{w\in\mathbb{Z}^d}\tau(0,x+y+rw)\tau(0,y).\tag{4.38}$$

We extend this last sum over y to all of $\mathbb{Z}^d$ and use the inequality for the bubble from Lemma 4.4 to finally get that

$$\sum_{u\in\mathbb{Z}^d}\sum_{v\neq u}\mathbb{P}(0\leftrightarrow x+ru,\, x+rv)\preceq\frac{\chi}{V}\left(\mathsf{B}_p(x)+\frac{\chi^2}{V}\right)\tag{4.39}$$

as claimed. □

Lemma 4.10. *Let $d>6$ and suppose that* (T) *holds on $\mathbb{Z}^d$. The estimate*

$$\sum_{u\in\mathbb{Z}^d}\mathbb{P}(\{0\leftrightarrow x+ru\}\cap\{0\underset{\mathbb{T}}{\leftrightarrow} x\}^c)\preceq\frac{\chi(p)}{V}\left(\mathsf{T}_p(x)+\frac{\chi(p)^3}{V}\right).\tag{4.40}$$

holds for every $0\le p<p_c$, $r>2$, and $x\in\mathbb{T}_r^d$.

Proof. Our starting point is the set inclusion

$$\begin{aligned}\{0\leftrightarrow x+ru\}\cap\{0\underset{\mathbb{T}}{\leftrightarrow} x\}^c\subseteq\bigcup_{z\in\mathbb{Z}^d}\bigcup_{a_1\in\mathbb{T}_r^d}\bigcup_{v_1,v_2\in\mathbb{Z}^d:v_1\neq v_2}&\{0\leftrightarrow z\}\circ\{z\leftrightarrow a+rv_1\}\\ &\circ\{z\leftrightarrow a+rv_2\}\circ\{a+rv_2\leftrightarrow x+ru\}\end{aligned}\tag{4.41}$$

for every $x\in\mathbb{T}_r^d$ and $u\in\mathbb{Z}^d$, which arises from the coupling of torus and $\mathbb{Z}^d$ percolation in [23, Proposition 2.1]. We use a,b,x,y,z for torus points and use u,v,w for translating vectors. It follows from the set inclusion (4.41) together with a union bound and the BK inequality that

$$\begin{aligned}&\mathbb{P}(\{0\leftrightarrow x+ru\}\cap\{0\underset{\mathbb{T}}{\leftrightarrow} x\}^c)\\ &\qquad\le\sum_{a\in\mathbb{T}_r^d}\sum_{z,v_1\in\mathbb{Z}^d}\sum_{v_2\neq v_1}\tau_p(z)\tau_p(a+rv_2-z)\tau_p(a+rv_1-z)\tau_p(x+ru-a-rv_2).\end{aligned}\tag{4.42}$$

We translate to more convenient vertices, as follows. First, we write the $\mathbb{Z}^d$ point $a + rv_2 - x$ uniquely as a torus point y plus rv with $v \in \mathbb{Z}^d$, and similarly for the others, to obtain

$$\begin{aligned} a + rv_2 - x &= y + rv, & y &\in \mathbb{T}_r^d, \quad v \in \mathbb{Z}^d, \\ a + rv_1 - x &= y + rv + rv', & v' &= v_2 - v_1 \neq 0, \\ z - x &= y + z' + ru', & z' &\in \mathbb{T}_r^d, \quad u' \in \mathbb{Z}^d. \end{aligned} \tag{4.43}$$

This gives

$$\begin{aligned} \sum_{u \in \mathbb{Z}^d} \mathbb{P}(\{0 \leftrightarrow x + ru\} \cap \{0 \underset{\mathbb{T}}{\leftrightarrow} x\}^c) \leq \sum_{y, z' \in \mathbb{T}_r^d} \sum_{v, u' \in \mathbb{Z}^d} \tau_p(x + y + z' + ru') \tau_p(-z' + r(v - u')) \\ \times \sum_{v' \neq 0} \tau_p(-z' + r(v' + v - u')) \sum_{u \in \mathbb{Z}^d} \tau_p(-y + r(u - v)). \end{aligned} \tag{4.44}$$

We bound the sums over u and v' with Lemma 4.4 and obtain

$$\sum_{u \in \mathbb{Z}^d} \tau_p(-y + r(u - v)) = \sum_{u \in \mathbb{Z}^d} \tau_p(-y + ru) \leq \tau_p(y) + C \frac{\chi(p)}{V}, \tag{4.45}$$

$$\sum_{v' \neq 0} \tau_p(-z' + r(v' + v - u')) = \sum_{v' \neq v - u'} \tau_p(-z' + rv') \leq \tau_p(z') \mathbb{1}_{v \neq u'} + C \frac{\chi(p)}{V}. \tag{4.46}$$

Then we perform the sum over v, which after translating v by u' is bounded similarly using

$$\begin{aligned} \sum_{v \in \mathbb{Z}^d} \tau_p(-z' + r(v - u')) \Big(\tau_p(z') \mathbb{1}_{v \neq u'} + C \frac{\chi(p)}{V} \Big) &= \sum_{v \in \mathbb{Z}^d} \tau_p(-z' + rv) \Big(\tau_p(z') \mathbb{1}_{v \neq 0} + C \frac{\chi(p)}{V} \Big) \\ &\leq C \frac{\chi(p)}{V} \tau_p(z') + C \frac{\chi(p)}{V} \sum_{v \in \mathbb{Z}^d} \tau_p(-z' + rv) \\ &\preceq \frac{\chi(p)}{V} \tau_p(z') + \frac{\chi(p)^2}{V^2}. \end{aligned} \tag{4.47}$$

This leads to

$$\begin{aligned} &\sum_{u \in \mathbb{Z}^d} \mathbb{P}(\{0 \leftrightarrow x + ru\} \cap \{0 \underset{\mathbb{T}}{\leftrightarrow} x\}^c) \\ &\qquad \preceq \frac{\chi(p)}{V} \sum_{y, z' \in \mathbb{T}_r^d} \sum_{u' \in \mathbb{Z}^d} \tau_p(x + y + z' + ru') \left(\tau_p(y) + \frac{\chi(p)}{V} \right) \left(\tau_p(z') + \frac{\chi(p)}{V} \right). \end{aligned} \tag{4.48}$$

We expand out the brackets and recognise that the term containing the product $\tau_p(w)\tau_p(z')$ obeys

$$\begin{aligned} \sum_{u' \in \mathbb{Z}^d} \sum_{y, z' \in \mathbb{T}_r^d} \tau_p(x + y + z' + ru') \tau_p(z') \tau_p(y) &= \sum_{u' \in \mathbb{Z}^d} \sum_{y, z' \in \mathbb{T}_r^d} \tau_p(z') \tau_p(y + z' - z') \tau_p(x + y + z' + ru') \\ &\leq \sum_{u' \in \mathbb{Z}^d} \mathsf{T}_p(x + ru') \preceq \mathsf{T}_p(x) + \frac{\chi(p)^3}{V}. \end{aligned} \tag{4.49}$$

Meanwhile, the two terms containing exactly one of $\tau_p(w)$ or $\tau_p(z')$ are equal and can be expressed as

$$\frac{\chi(p)}{V}\sum_{u'\in\mathbb{Z}^d}\sum_{y,z'\in\mathbb{T}_r^d}\tau_p(x+y+z'+ru')\tau_p(z') = \frac{\chi(p)}{V}\sum_{w\in\mathbb{Z}^d}\sum_{z'\in\mathbb{T}_r^d}\tau_p(x+z'+w)\tau_p(z') \le \frac{\chi(p)^3}{V}, \tag{4.50}$$

where we extended the sum over z' to all of $\mathbb{Z}^d$ in the last inequality. Finally, the term not containing either $\tau_p(w)$ or $\tau_p(z')$ can be expressed as

$$\frac{\chi(p)^2}{V^2}\sum_{y,z'\in\mathbb{T}_r^d}\sum_{u'\in\mathbb{Z}^d}\tau_p(x+y+z'+ru') = \frac{\chi(p)^2}{V^2}\sum_{z'\in\mathbb{T}_r^d}\sum_{w\in\mathbb{Z}^d}\tau_p(x+z'+w) = \frac{\chi(p)^3}{V}. \tag{4.51}$$

Summation of these contributions gives

$$\sum_{u\in\mathbb{Z}^d}\mathbb{P}(\{0\leftrightarrow x+ru\}\cap\{0\underset{\mathbb{T}}{\leftrightarrow} x\}^c) \preceq \frac{\chi(p)}{V}\Big(\mathsf{T}_p(x)+\frac{\chi(p)^3}{V}\Big), \tag{4.52}$$

and the proof is complete. □

Proof of Lemma 4.8. Lemmas 4.9 and 4.10 give bounds on the two terms on the right-hand side of (4.33), namely

$$\psi_{r,p}(x)-\psi_{r,p}^{\mathbb{T}}(x) \preceq \frac{\chi}{V}\left(\mathsf{B}_p(x)+\frac{\chi^2}{V}\right)+\frac{\chi}{V}\left(\mathsf{T}_p(x)+\frac{\chi^3}{V}\right). \tag{4.53}$$

Since the bubble is bounded above by the triangle and the susceptibility is at least 1, this gives the desired estimate

$$\psi_{r,p}(x)-\psi_{r,p}^{\mathbb{T}}(x) \preceq \frac{\chi}{V}\left(\mathsf{T}_p(x)+\frac{\chi^3}{V}\right) \tag{4.54}$$

and the proof is complete. □

4.4 The torus triangle condition: Proof of Theorem 1.5

To conclude this section, we show how the torus plateau leads to an easy proof of the torus triangle condition.

Proof of Theorem 1.5. Let $p<p_c$ and $x\in\mathbb{T}_r^d$. The open torus triangle diagram is defined by

$$\mathsf{T}_p^{\mathbb{T}}(x) = \sum_{y,z\in\mathbb{T}_r^d}\tau_p^{\mathbb{T}}(y)\tau_p^{\mathbb{T}}(z-y)\tau_p^{\mathbb{T}}(x-z). \tag{4.55}$$

As a first step, we apply (4.17) to obtain

$$\mathsf{T}_p^{\mathbb{T}}(x) \le \sum_{y,z\in\mathbb{T}_r^d}\sum_{u,v,w\in\mathbb{Z}^d}\tau_p(y+ru)\tau_p(z-y+rv)\tau_p(x-z+rw). \tag{4.56}$$

We replace the index v by $v'-u$ and then replace w by $w'-v'$. The above right-hand side becomes (after setting $y' = y + ru$ and $z' = z + rv'$)

$$\sum_{y',z',w' \in \mathbb{Z}^d} \tau_p(y')\tau_p(z'-y')\tau_p(x-z'+rw') = \sum_{w' \in \mathbb{Z}^d} \mathsf{T}_p(x+rw'), \tag{4.57}$$

with $\mathsf{T}_p(x+rw')$ the open $\mathbb{Z}^d$ triangle diagram. By Lemma 4.4, this gives that

$$\mathsf{T}_p^{\mathbb{T}}(x) \le \mathsf{T}_p(x) + C_1\frac{\chi(p)^3}{V} \le \mathsf{T}_{p_c}(x) + C_1\frac{\chi(p)^3}{V}. \tag{4.58}$$

Fix $\varepsilon > 0$ sufficiently small that $\varepsilon^{-1} \ge A_2$, where A_2 is as in Theorem 1.3. Recalling from (1.14) that $\chi \asymp (p_c - p)^{-1}$ and setting $p_0 = p_c - \varepsilon^{-1}V^{-1/3}$, we have that $\chi(p_0) \asymp \varepsilon V^{1/3}$ and hence that

$$\mathsf{T}_p^{\mathbb{T}}(x) \le \mathsf{T}_{p_0}^{\mathbb{T}}(x) \le \mathsf{T}_{p_c}(x) + C_2\varepsilon^3 \tag{4.59}$$

for every $p \le p_0$. On the other hand, the lower bound of (1.16) applies to give

$$\chi^{\mathbb{T}}(p_0) \succeq \sum_{x \in \mathbb{T}_r^d:\ \|x\|_\infty > M} V^{-1}\chi(p_0) \succeq (V - (2M+1)^d)V^{-1}\chi(p) \succeq \varepsilon V^{1/3}. \tag{4.60}$$

Since we also have by the coupling that $\chi^{\mathbb{T}} \le \chi$ it follows that there exist positive constants c_1 and C_3 such that

$$c_1\varepsilon V^{1/3} \le \chi^{\mathbb{T}}(p_0) \le \chi(p_0) \le C_3\varepsilon V^{1/3}. \tag{4.61}$$

A second application of (1.14) yields that there exists a constant $C_4 \ge 1$ such that if we define $p_1 = p_c - C_4\varepsilon^{-1}V^{-1/3}$ then $\chi^{\mathbb{T}}(p_1) \le \chi(p_1) \le c_1\varepsilon V^{1/3}$. It follows by the intermediate value theorem that if we define $\lambda = \lambda(\varepsilon) = c_1\varepsilon$ then the $p_{\mathbb{T}}$ defined by $\chi(p_{\mathbb{T}}) = \lambda V^{1/3}$ satisfies $p_1 \le p_{\mathbb{T}} \le p_0$ and hence that $0 \le p_c - p_{\mathbb{T}} \preceq \varepsilon^{-1}V^{-1/3}$ and $\mathsf{T}_{p_{\mathbb{T}}}^{\mathbb{T}} \le \mathsf{T}_{p_c}(x) + C_2\varepsilon^3$. Taking $\varepsilon = A_2^{-1}$ concludes the proof of the torus triangle condition.

It remains to prove that the stronger version of the triangle condition holds when either $d \gg 6$ or $d > 6$ and $L \gg 1$. It is proved in [21] that for each $a_0 > 0$ there exist d_0 and L_0 such that $\mathsf{T}_{p_c}(x) \le \mathbb{1}(x=0) + a_0/2$ whenever $d \ge d_0$ or $d > 6$ and $L \ge L_0$, and it follows from (4.59) that

$$\mathsf{T}_p^{\mathbb{T}}(x) \le \mathbb{1}(x=0) + \frac{1}{2}a_0 + C_2\varepsilon^3 \tag{4.62}$$

for every $p \le p_0$ under the same conditions. Thus, the claim follows by arguing as in the previous paragraph but now taking ε sufficiently small that $C_2\varepsilon^3 \le a_0/2$. □

5 Plateau within the scaling window: Proof of Theorem 1.3

We now turn to the part of Theorem 1.3 concerning the case that p lies within the scaling window of the torus. The window consists of p values with $|p - p_c| \le AV^{-1/3}$ with A arbitrary but fixed.

5.1 Lower bound in the window

We begin by proving the lower bound (1.18), which follows simply from the monotonicity of $\tau_p^{\mathbb{T}}$ in p, the lower bound below the window, and the comparison of τ_p with τ_{p_c} provided by Lemma 4.7.

Proof of (1.18). Let A_1 and A_2 be the constants from the "below the scaling window" part of Theorem 1.3. Fix $A > 0$. It suffices by monotonicity of the torus two-point function to prove the claimed estimate in the case that $A \geq A_1$ and $p = p_c - AV^{-1/3}$. We denote this value of p by p'. Thus, there exists a constant r_0 depending on A such that $AV^{-1/3} \leq A_2 r^{-2} = A_2 V^{-2/d}$ for every $r \geq r_0$.

By (1.14), $\chi(p') \geq c_\chi A^{-1} V^{1/3}$ for some constant c_χ depending only on d and L. It then follows from (1.16) that there exists a constant M (depending on d and L) such that

$$\tau_{p'}^{\mathbb{T}}(x) \geq \tau_{p'}(x) + \frac{C_2 c_\chi}{AV^{2/3}} \tag{5.1}$$

for every $x \in \mathbb{T}_r^d$ with $\|x\|_\infty \geq M$. By Lemma 4.7, we have moreover that

$$\tau_{p_c}(x) - \tau_{p'}(x) \preceq \frac{A}{V^{1/3}\langle x\rangle^{d-4}} \preceq \frac{Ar^2}{V^{1/3}} \tau_{p_c}(x). \tag{5.2}$$

Since the prefactor $Ar^2V^{-1/3}$ tends to zero as $r \to \infty$, it follows from (5.1)–(5.2) that for each $\delta > 0$ there exists an $r_1 \geq r_0$ depending on A such that

$$\tau_{p'}^{\mathbb{T}}(x) \geq (1-\delta)\tau_{p_c}(x) + \frac{C_2 c_\chi}{AV^{2/3}} \tag{5.3}$$

for every $x \in \mathbb{T}_r^d$ with $\|x\|_\infty \geq M$ and every $r \geq r_1$. This completes the proof. □

5.2 Upper bound in the window

It remains to prove the upper bound. To do so we first consider the case $p = p_c$. We then extend the upper bound to the window $(p_c, p_c + AV^{-1/3}]$ by proving that, in this window, the two-point function changes only up to a multiplicative factor (that can be chosen to be arbitrarily close to one) plus an additive constant term of order $V^{-2/3}$.

At p_c, the upper bound is not new and was proven previously in [28, Theorem 1.7]. However, our proof, which is based on the extrinsic (Euclidean) distance, seems more direct than that of [28] where the intrinsic distance was used. Our proof relies on the extrinsic one-arm exponent estimate

$$\mathbb{P}_{p_c}(0 \leftrightarrow \partial\Lambda_\ell) \asymp \frac{1}{\ell^2} \tag{5.4}$$

of Kozma and Nachmias [36] (i.e. the $p = p_c$ case of Theorem 1.2).

Proposition 5.1. *Let $d > 6$ and suppose that* (T) *holds on $\mathbb{Z}^d$. For $x \in \mathbb{T}_r^d$,*

$$\tau_{p_c}^{\mathbb{T}}(x) \leq \tau_{p_c}(x) + CV^{-2/3}. \tag{5.5}$$

Proof. Fix some large and positive integer M. Since $\mathbb{Z}^d$ covers the torus $\mathbb{T}_r^d$, every path in $\mathbb{T}_r^d$ can be lifted to a path in $\mathbb{Z}^d$ that is unique up to the choice of starting point. For $x, y \in \mathbb{T}_r^d$, we define $E_{\geq\ell}(x,y)$ to be the event that x and y are connected by a simple $\mathbb{T}_r^d$-path that lifts to a $\mathbb{Z}^d$-path of diameter greater than or equal to ℓ, and we define $E_{\leq\ell}(x,y)$ similarly. In addition, we define $A_{\geq\ell}(x)$ to be the event that there exists some simple $\mathbb{T}_r^d$-path starting from x that lifts to a $\mathbb{Z}^d$-path of diameter at least ℓ. We have trivially that $\{0 \underset{\mathbb{T}}{\leftrightarrow} x\} = E_{\leq 3Mr}(0,x) \cup E_{\geq 3Mr}(0,x)$, so that

$$\mathbb{P}^{\mathbb{T}}_{p_c}(0 \underset{\mathbb{T}}{\leftrightarrow} x) \leq \mathbb{P}^{\mathbb{T}}_{p_c}(E_{\leq 3Mr}(0,x)) + \mathbb{P}^{\mathbb{T}}_{p_c}(E_{\geq 3Mr}(0,x)). \tag{5.6}$$

Note that on $E_{\geq 3Mr}(0,x)$, the events $A_{\geq Mr}(0)$ and $A_{\geq Mr}(x)$ must occur disjointly. Also, in the coupling between torus and $\mathbb{Z}^d$ percolation, we have

$$A_{\geq \ell}(0) \subset \{0 \underset{\mathbb{Z}}{\leftrightarrow} \partial\Lambda_\ell\} \qquad \text{and} \qquad A_{\geq \ell}(x) \subset \{x \underset{\mathbb{Z}}{\leftrightarrow} x + \partial\Lambda_\ell\}, \tag{5.7}$$

where we recall that $\Lambda_\ell = [-\ell,\ell]^d \cap \mathbb{Z}^d$. Thus, by the BK inequality on the torus,

$$\begin{aligned}\mathbb{P}^{\mathbb{T}}_{p_c}(E_{\geq 3Mr}(0,x)) &\leq \mathbb{P}^{\mathbb{T}}_{p_c}(A_{\geq Mr}(0) \circ A_{\geq Mr}(x)) \\ &\leq \mathbb{P}^{\mathbb{T}}_{p_c}(A_{Mr}(0))\mathbb{P}^{\mathbb{T}}_{p}(A_{Mr}(x)) \leq \mathbb{P}_{p_c}(0 \leftrightarrow \partial\Lambda_{Mr})^2.\end{aligned} \tag{5.8}$$

Using the one-arm upper bound of (5.4), this gives

$$\mathbb{P}^{\mathbb{T}}_{p_c}(E_{\geq 3Mr}(0,x)) \preceq \frac{1}{M^4 r^4}. \tag{5.9}$$

For the first term of (5.6) we simply use the coupling and a union bound to see that

$$\mathbb{P}^{\mathbb{T}}_{p_c}(E_{\leq 3Mr}(0,x)) \leq \mathbb{P}_{p_c}(\cup_{\|u\|_\infty \leq 3M}\{0 \leftrightarrow x + ru\}) \leq \tau_{p_c}(x) + \sum_{1 \leq \|u\|_\infty \leq 3M} \tau_{p_c}(x+ru). \tag{5.10}$$

The latter sum can be bounded above by an integral over the d-dimensional box of radius $3M$, namely

$$\sum_{1 \leq \|u\|_\infty \leq 3M} \tau_{p_c}(x+ru) \preceq \int_{\|u\|_\infty \leq 3M} \frac{1}{\langle ru \rangle^{d-2}} \mathrm{d}u \preceq M^2 r^{-(d-2)}. \tag{5.11}$$

Together, these bounds imply that there exists a constant C such that

$$\tau^{\mathbb{T}}_{p_c}(x) \leq \tau_{p_c}(x) + CM^2 r^{-(d-2)} + CM^{-4}r^{-4}. \tag{5.12}$$

The choice $M = r^{(d-6)/6}$ gives the desired upper bound $\tau_{p_c}(x) + CV^{-2/3}$ at $p = p_c$. □

Next, we prove an upper bound at the top of the scaling window. For $p \in (p_c, p_c + AV^{-1/3}]$ we use the *intrinsic distance* d_{int}, which is the graph distance on the percolation configuration. If x, y are not connected in the configuration, then $d_{\mathrm{int}}(x,y) = \infty$. Given a percolation configuration, we define the (random) *intrinsic ball* centred at x and of radius ℓ by

$$B_{\mathrm{int}}(x,\ell) = \{y \in \mathbb{T}^d_r : d_{\mathrm{int}}(x,y) \leq \ell\}. \tag{5.13}$$

Thus

$$\{x \leftrightarrow y \text{ by a path of length} \leq \ell\} = \{y \in B_{\mathrm{int}}(x,\ell)\}. \tag{5.14}$$

We write the boundary of the intrinsic ball as

$$\partial B_{\mathrm{int}}(x,\ell) = B_{\mathrm{int}}(x,\ell) \setminus B_{\mathrm{int}}(x,\ell-1) = \{y \in \mathbb{T}^d_r : d_{\mathrm{int}}(x,y) = \ell\}. \tag{5.15}$$

Given a subset g of edges of the edge set $\mathbb{B}$ of $\mathbb{T}_r^d$ or $\mathbb{Z}^d$, we define $B_{\rm int}^g(x,\ell)$ similarly as $B_{\rm int}(x,\ell)$ except that the intrinsic distance from x to y is determined only using paths consisting of edges of g.

Kozma and Nachmias [35] computed the asymptotic behaviour of the critical instrinsic one-arm probability in high dimensions to be

$$\mathbb{P}_{p_c}(\partial B_{\rm int}(0,\ell) \neq \varnothing) \asymp \frac{1}{\ell} \tag{5.16}$$

for every $\ell \geq 1$. In fact, they also proved an extension of the upper bound of (5.16) involving the intrinsic ball restricted to a subgraph g; their proof also extends immediately to the torus as explained in [28, Theorem 2.1(i)] and implies in particular that

$$\max_{g \subset \mathbb{B}(\mathbb{T}_r^d)} \mathbb{P}_{p_c}^{\mathbb{T}}(\partial B_{\rm int}^g(0,\ell) \neq \varnothing) \preceq \frac{1}{\ell}. \tag{5.17}$$

for every $r, \ell \geq 1$. (An important remark is that the proof of (5.17) does not require the lace expansion on $\mathbb{T}_r^d$, but instead uses results for $\mathbb{Z}^d$ along with the coupling of torus and $\mathbb{Z}^d$ percolation—this is discussed in greater detail in the verification of [24, Theorem 4.1(b)]. As such, there is no circular reasoning here, nor an appeal to any result obtained via the torus lace expansion.)

We first isolate two estimates in the following lemma, whose proof uses a standard coupling of percolation at different values of p.

Lemma 5.2. *For $0 \leq p < q \leq 1$, $\ell \geq 1$, and $g \subset \mathbb{B}(\mathbb{T}_r^d)$,*

$$\mathbb{P}_q^{\mathbb{T}}(\partial B_{\rm int}^g(0,\ell) \neq \varnothing) \leq \left(\frac{q}{p}\right)^{\ell} \mathbb{P}_p^{\mathbb{T}}(\partial B_{\rm int}^g(0,\ell) \neq \varnothing), \tag{5.18}$$

$$\mathbb{P}_q^{\mathbb{T}}(x \in B_{\rm int}(0,\ell)) \leq \left(\frac{q}{p}\right)^{\ell} \mathbb{P}_p^{\mathbb{T}}(x \in B_{\rm int}(0,\ell)). \tag{5.19}$$

Proof. Let $0 \leq p < q \leq 1$. We begin with (5.18). Given a subset g of the edge set of $\mathbb{T}_r^d$, we write $R_g(\ell) = \{\partial B_{\rm int}^g(0,\ell) \neq \varnothing\}$. We use the standard coupling of percolation configurations via uniform random variables assigned to each edge of the torus (see [17, p. 11]); these uniform random variables are defined on some probability space $(\Omega, \mathcal{A}, \mathbb{Q})$. We write $\eta_p^{\mathbb{T}}$ for the induced percolation configuration. Since $R_g(\ell)$ depends on the edges inside $\mathbb{T}_r^d$ only, hence on finitely many edges, we can write

$$\begin{aligned} \mathbb{Q}(\eta_p^{\mathbb{T}} \in R_g(\ell), \eta_q^{\mathbb{T}} \in R_g(\ell)) &= \sum_{\omega \in \{0,1\}^{\mathbb{T}_r^d}} \mathbb{Q}(\eta_p^{\mathbb{T}} \in R_g(\ell),\, \eta_q^{\mathbb{T}} = \omega,\, \omega \in R_g(r)) \\ &= \sum_{\omega \in R_g(\ell)} \mathbb{Q}(\eta_p^{\mathbb{T}} \in R_g(\ell) \mid \eta_q^{\mathbb{T}} = \omega) \mathbb{Q}(\eta_q^{\mathbb{T}} = \omega). \end{aligned} \tag{5.20}$$

Since the above left-hand side is at most $\mathbb{Q}(\eta_p^{\mathbb{T}} \in R_g(\ell))$, (5.18) follows once we prove that

$$\mathbb{Q}(\eta_p^{\mathbb{T}} \in R_g(\ell) \mid \eta_q^{\mathbb{T}} = \omega) \geq \left(\frac{p}{q}\right)^{\ell}. \tag{5.21}$$

To prove (5.21), we first observe that on a specific configuration $\omega \in R_g(\ell)$, there exists a deterministic path of open edges of length ℓ inside ω. A fortiori, on the event $\{\eta_q^{\mathbb{T}} = \omega\}$ there exist ℓ independent uniform random variables $U_1, \cdots, U_\ell$ attached to these open edges such that $U_i \leq q$ for all $1 \leq i \leq \ell$. For

$\{\eta_p^{\mathbb{T}} \in R_g(\ell)\}$ to occur, it is enough that $U_i \le p$ for all $1 \le i \le \ell$ which gives

$$\mathbb{Q}(\eta_p^{\mathbb{T}} \in R_g(\ell) \mid \eta_q^{\mathbb{T}} = \omega) \ge \mathbb{Q}\Big(\bigcap_{i=1}^{\ell}\{U_i \le p\} \mid \eta_q^{\mathbb{T}} = \omega\Big). \tag{5.22}$$

Since $U_1, \cdots, U_\ell$ are independent of the other uniform random variables, the above right-hand side is equal to

$$\mathbb{Q}\Big(\bigcap_{i=1}^{\ell}\{U_i \le p\} \mid \eta_q^{\mathbb{T}} = \omega\Big) = \mathbb{Q}\Big(\bigcap_{i=1}^{\ell}\{U_i \le p\} \mid \bigcap_{i=1}^{\ell}\{U_i \le q\}\Big) = \left(\frac{p}{q}\right)^{\ell}, \tag{5.23}$$

which proves (5.21) and hence completes the proof of (5.18).

The proof of (5.19) is almost identical, using the fact that on $\{x \in B_{\rm int}(0,\ell)\}$ there must exist a path of length less than or equal to ℓ connecting 0 to x. Then keeping the uniform random variables along this path open upon reducing q to p gives the result. □

We now prove (1.17) in the following proposition.

Proposition 5.3. *Let $d > 6$ and suppose that* (T) *holds on $\mathbb{Z}^d$. For $x \in \mathbb{T}_r^d$, $A > 0$, $p \in (p_c, p_c + AV^{-1/3}]$, and for any fixed $\delta \in (0,1]$, there exists a constant C_A such that*

$$\tau_p^{\mathbb{T}}(x) \le e^{\delta}\tau_{p_c}(x) + C_A\delta^{-2}V^{-2/3}. \tag{5.24}$$

Proof. Let $x \in \mathbb{T}_r^d$, $A > 0$ and $\delta > 0$. By the monotonicity of $\tau_p^{\mathbb{T}}(x)$ in p and the independence of the upper bound on p, it suffices to prove (5.24) for $p = p_c + \varepsilon$ with $\varepsilon = AV^{-1/3}$. We set $\gamma = \lceil \delta/\varepsilon \rceil$ and begin with the decomposition

$$\begin{aligned}\mathbb{P}_{p_c+\varepsilon}^{\mathbb{T}}(0 \leftrightarrow x) &= \mathbb{P}_{p_c+\varepsilon}^{\mathbb{T}}(x \in \partial B_{\rm int}(0,\ell) \text{ for some } \ell) \\ &\le \mathbb{P}_{p_c+\varepsilon}^{\mathbb{T}}(x \in B_{\rm int}(0,3\gamma)) + \mathbb{P}_{p_c+\varepsilon}^{\mathbb{T}}(x \in \partial B_{\rm int}(0,\ell) \text{ for some } \ell \ge 3\gamma).\end{aligned} \tag{5.25}$$

For the first term in (5.25), we use (5.19) and Proposition 5.1 to see that

$$\begin{aligned}\mathbb{P}_{p_c+\varepsilon}^{\mathbb{T}}(x \in B_{\rm int}(0,3\gamma)) &\le \left(\frac{p_c+\varepsilon}{p_c}\right)^{3\gamma}\mathbb{P}_{p_c}^{\mathbb{T}}(x \in B_{\rm int}(0,3\gamma)) \\ &\le e^{3\gamma\varepsilon/p_c}\mathbb{P}_{p_c}^{\mathbb{T}}(0 \leftrightarrow x) \\ &\le e^{3\delta/p_c}(\tau_{p_c}(x) + CV^{-2/3}).\end{aligned} \tag{5.26}$$

We consider now the second term in (5.25). For this we argue as in the proof of [28, (1.11)] (see also [35, Lemma 2.6]). We first observe that on the event $\{x \in \partial B_{\rm int}(0,\ell) \text{ for some } \ell \ge 3\gamma\}$, the two events $\{\partial B_{\rm int}(0,\gamma) \ne \varnothing\}$ and $\{\partial B_{\rm int}(x,\gamma) \ne \varnothing\}$ must both occur. However, these two events do not necessarily occur disjointly. Indeed, in order for $d_{\rm int}(0,y)$ to be large there must not only be a long open path from 0 to y but also *no shorter path*; the sets of closed edges guaranteeing that no such short path exists may be shared in the common realisation of the two events. On the other hand, the set of vertices in $B_{\rm int}(0,\gamma)$ and in $B_{\rm int}(x,\gamma)$ are disjoint. To deal with this situation, we define G to be the random graph whose edge set consists of all edges which touch $B_{\rm int}(x,\gamma-1)$ (the vertex set of G consists of the vertices incident to at least one edge in this set); these are exactly the edges needed to determine the random set $B_{\rm int}(x,\gamma)$. From now on we identify subgraphs of the torus with subsets of $\mathbb{B}(\mathbb{T}_r^d)$ and write g^c for the complement of a subgraph $g \subseteq \mathbb{B}(\mathbb{T}_r^d)$. Since $B_{\rm int}(0,\gamma)$ and $B_{\rm int}(x,\gamma)$ are disjoint, we see that

$$\begin{aligned}\mathbb{P}^{\mathbb{T}}_{p_c+\varepsilon}(x \in \partial B_{\rm int}(0,\ell) \text{ for some } \ell \geq 3\gamma) &\leq \mathbb{P}^{\mathbb{T}}_{p_c+\varepsilon}(\partial B^{G^c}_{\rm int}(0,\gamma) \neq \varnothing,\ \partial B_{\rm int}(x,\gamma) \neq \varnothing) \\ &= \sum_{g \subset \mathbb{B}(\mathbb{T}_r^d)} \mathbb{P}^{\mathbb{T}}_{p_c+\varepsilon}(\partial B^{g}_{\rm int}(0,\gamma) \neq \varnothing,\ \partial B_{\rm int}(x,\gamma) \neq \varnothing,\ G^c = g).\end{aligned} \tag{5.27}$$

By definition of G, the events $\{\partial B^g_{\rm int}(0,\gamma) \neq \varnothing\}$ and $\{G^c = g,\ \partial B_{\rm int}(x,\gamma) \neq \varnothing\}$ depend on different edges (namely those of g and g^c respectively), and hence

$$\begin{aligned}\mathbb{P}^{\mathbb{T}}_{p_c+\varepsilon}(x \in \partial B_{\rm int}(0,\ell) \text{ for some } \ell \geq 3\gamma) &\leq \sum_{g \subset \mathbb{B}(\mathbb{T}_r^d)} \mathbb{P}^{\mathbb{T}}_{p_c+\varepsilon}(\partial B^{g}_{\rm int}(0,\gamma) \neq \varnothing)\mathbb{P}^{\mathbb{T}}_{p_c+\varepsilon}(\partial B_{\rm int}(x,\gamma) \neq \varnothing,\ G^c = g) \\ &\leq \max_{g \subset \mathbb{B}(\mathbb{T}_r^d)} \mathbb{P}^{\mathbb{T}}_{p_c+\varepsilon}(\partial B^{g}_{\rm int}(0,\gamma) \neq \varnothing)\mathbb{P}^{\mathbb{T}}_{p_c+\varepsilon}(\partial B_{\rm int}(x,\gamma) \neq \varnothing) \\ &\leq \max_{g \subset \mathbb{B}(\mathbb{T}_r^d)} \mathbb{P}^{\mathbb{T}}_{p_c+\varepsilon}(\partial B^{g}_{\rm int}(0,\gamma) \neq \varnothing)^2.\end{aligned} \tag{5.28}$$

Now, we have by (5.18) with $p = p_c$ and $q = p_c + \varepsilon$ and by (5.17) that

$$\mathbb{P}^{\mathbb{T}}_{p_c+\varepsilon}(\partial B^{g}_{\rm int}(0,\gamma) \neq \varnothing) \leq \left(\frac{p_c+\varepsilon}{p_c}\right)^{\gamma} \mathbb{P}^{\mathbb{T}}_{p_c}(\partial B^{g}_{\rm int}(0,\gamma) \neq \varnothing) \preceq \left(\frac{p_c+\varepsilon}{p_c}\right)^{\gamma} \frac{1}{\gamma} \tag{5.29}$$

and hence that

$$\mathbb{P}^{\mathbb{T}}_{p_c+\varepsilon}(x \in \partial B_{\rm int}(0,\ell) \text{ for some } \ell \geq 3\gamma) \preceq \frac{1}{\gamma^2}\left(\frac{p_c+\varepsilon}{p_c}\right)^{2\gamma} \preceq \frac{\varepsilon^2}{\delta^2} e^{2\delta/p_c}. \tag{5.30}$$

Altogether, by (5.26) and (5.30) we therefore have

$$\tau^{\mathbb{T}}_p(x) \leq e^{3\delta/p_c}\tau_{p_c}(x) + C\delta^{-2}e^{3\delta/p_c}V^{-2/3}. \tag{5.31}$$

Finally, we replace δ by $\delta' = 3\delta/p_c$, and we may obtain in this way any $\delta' \in (0,1]$. for $\delta' \in [0,1]$, we obtain This gives the desired result and the proof is complete. □

Acknowledgements

This work was carried out primarily while TH was a Senior Research Associate at the University of Cambridge, during which time he was supported by ERC starting grant 804166 (SPRS). The work of EM and GS was supported in part by NSERC of Canada.

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