Supplemental Material for “Measurement-induced phase transition in the monitored Sachdev-Ye-Kitaev model”

Shao-Kai Jian, Chunxiao Liu, Xiao Chen, Brian Swingle, and Pengfei Zhang

Condensed Matter Theory Center and Joint Quantum Institute, Department of Physics, University of Maryland, College Park, MD 20742, USA
Department of Physics, University of California Santa Barbara, Santa Barbara, CA 93106, USA
Department of Physics, Boston College, Chestnut Hill, MA 02467, USA
Department of Physics, Brandeis University, Waltham, Massachusetts 02453, USA
Institute for Quantum Information and Matter and Walter Burke Institute for Theoretical Physics, California Institute of Technology, Pasadena, CA 91125, USA

1. DERIVATION OF THE EFFECTIVE ACTION AND THE SADDLE-POINT EQUATION OF THE MONITORED SYSTEM

As stated in the main text, the measurement on the four contours can be cast into (note that the measurement operator is Hermitian)

\[ \sum_{\nu} w_{\nu}(K_{\nu}^{x,i})^{\otimes 2} \otimes (K_{\nu}^{x,i})^{\otimes 2} \]

\[ = (1 - p)I^{\otimes 4} + p(M_{1}^{x,i})^{\otimes 4} + (M_{2}^{x,i})^{\otimes 4} \]

\[ \approx (1 - p) + p \left( 1 - \frac{s^2}{2} \sum_{a=1}^{4} \pi_{x,i}^{+} + \delta t \mu \sum_{a=1}^{4} \pi_{x,i}^{+} \right) \]

\[ \approx \exp \frac{\delta t \mu}{2} \sum_{a} i \pi_{x,i}^{a} \psi_{x,a,i}^{a}, \]

where we have used the relation \( \pi_{x,a,j}^{+} + \pi_{x,a,j}^{-} = 1 \) and also introduced \( \alpha = 1, ..., 4 \) to denote the four copies of the tensor product. To derive the above equation, we assume \( s \ll 1 \) and keep orders up to \( O(s^2) \). In the last line we introduce \( \mu = ps^2/\delta t \) and when the above limit is taken, \( \mu \) is kept fixed. All the constants are neglected because they will not affect the dynamics. We arrive at (6) in the main text.

The derivation of \( G_{x}^{\alpha} \) action was firstly derived in \(^1\), and we outline the major steps in the derivation and refer the details to that paper. Let us look at a single chain first. The action on the four contours reads

\[ -I = \int dt \sum_{\alpha} \left( \sum_{x,i} \psi_{x,i}^{\alpha}(t) \delta(t_{1} - t_{2}) \partial_{x,i} \psi_{x,i}^{\alpha}(t_{2}) \right), \]

where \( \alpha = 1, ..., 4 \) denote the four contours, and \( H[\psi] = H \) given in (1). Performing an average over the Gaussian variable using (2), it becomes

\[ -I = \sum_{\alpha, \beta, x} \int dt_{1} dt_{2} \left( \frac{1}{2} \psi_{x,i}^{\alpha}(t_{1}) \delta \psi_{x,i}^{\beta}(t_{2}) \delta(t_{1} - t_{2}) \partial_{x,i} \psi_{x,i}^{\beta}(t_{2}) \right) \]

\[ + (-1)^{\alpha + \beta + 1} \frac{i^2}{4N} \sum_{i,j} \psi_{x,i}(t_{1}) \psi_{x+1,j}(t_{2}) \psi_{x,i}(t_{2}) \psi_{x+1,j}(t_{2}) \]

\[ + (-1)^{\alpha + \beta + 1} \frac{U\delta(t_{1} - t_{2})}{8qNq^{-1}} \sum_{j_{1}, ..., j_{q}} \psi_{x,j_{1}}(t_{1}) ... \psi_{x,j_{q}}(t_{1}) \psi_{x,j_{1}}(t_{2}) ... \psi_{x,j_{q}}(t_{2}) \right), \]

Now we introduce the bilocal fields \( G_{x}^{\alpha \beta}(t_{1}, t_{2}) \) and \( \Sigma_{x}^{\alpha \beta}(t_{1}, t_{2}) \). The bilocal field \( G_{x}^{\alpha \beta}(t_{1}, t_{2}) \) is the correlation function of Majorana fermions,

\[ G_{x}^{\alpha \beta}(t_{1}, t_{2}) = \frac{1}{N} \sum_{i} \psi_{x,i}^{\alpha}(t_{1}) \psi_{x,i}^{\beta}(t_{2}), \]
and $\Sigma_\alpha^\beta(t_1, t_2)$ is the self-energy of Majorana fermions, and is introduced through the following identity,

$$1 = \int d\Sigma \exp \left[ \int dt_1 dt_2 \left( -\frac{N}{2} \Sigma_\alpha^\beta(t_1, t_2) \left( G_x^\alpha(t_1, t_2) - \frac{1}{N} \sum_i \psi_\alpha^{i+}(t_1) \psi_\alpha^{i-}(t_2) \right) \right) \right].$$  \hspace{1cm} (S11)

By multiplying this identity and using $G_x^\alpha$ to rewrite the coupling, we have

$$-I = \sum_{\alpha, \beta, x} \int dt_1 dt_2 \left( -\frac{1}{2} \psi_\alpha^{i+}(t_1)(-1)^{\alpha+1}\delta^{\alpha\beta}\delta(t_1 - t_2) \partial_{t_2} - \Sigma_\alpha^\beta(t_1, t_2) \psi_\alpha^{i+}(t_2) \right) - \frac{N}{2} \Sigma_\alpha^\beta(t_1, t_2) G_x^\alpha(t_1, t_2) \delta(t_1 - t_2) \right) \right).$$

Now the action is quadratic in the Majorana field, so we can integrate over the Majorana fermions and get the $G-\Sigma$ action,

$$-\frac{I}{N} = \frac{1}{2} \text{Tr} \log \left( (-1)^{\alpha+1} \delta_t - \Sigma_x \right) - \frac{1}{2} \int \Sigma_x^\alpha G_x^\alpha + \int \delta(t - t') \left[ -\frac{(-1)^{\alpha+\beta}}{4} \left( J G_x^\alpha G_x^{a+1} + \frac{U}{2q} (2G_x^\alpha q) \right) \right].$$

A generalization to left and right chains, i.e., $G_x^{\alpha, ab}(t_1, t_2) = \frac{1}{N} \sum_i \psi_\alpha^{a,i}(t_1) \psi_\alpha^{b,i}(t_2)$ is straightforward. Then combining with the measurement part, we arrive at the effective action in (8). The saddle-point equation followed from (8) reads

$$\left[ G_x^{-1} \right]_{\alpha \beta} = (-1)^{\alpha+1} \delta^{\alpha\beta} \delta \partial_{t_2} - \Sigma_{ab, x}^\alpha,$$

$$\Sigma_{ab, x}^\alpha = \delta(t - t') \left[ \frac{(-1)^{\alpha+\beta}}{2} \left( J(G_x^{\alpha, ab}_{x} + G_x^{\alpha, ab}_{x+1}) + U(2G_x^{\alpha, ab}_{x} q) \right) \right] + i\mu \delta^{\alpha L} \delta_{ab R} - \delta^{\alpha R} \delta_{ab L}. $$

We consider the homogeneous solution in real space, i.e., $G_x^{\alpha, ab}(t_1, t_2) = \bar{G}_x^{\alpha, ab}$ and $\Sigma_{ab, x}^{\alpha} = \Sigma_{ab}^{\alpha}$. To get the solution, we focus on two contours, $\alpha, \beta = 1, 2$, because the boundary condition in $\text{Tr}(\rho)^2$ is to connect 1 to 2 and connect 3 to 4 separately. If the evolution is unitary, the correlation between two contours will be $G_{12}^{aa}(t, t) = -\frac{1}{2}$ because the forward and backward evolution cancels. The effect of non-Hermitian couplings is to decrease this correlation, and therefore, we assume the correlation is given by $G_{12}^{aa}(t, t) = -\frac{\zeta}{2}$. (S13) shows that the self-energy is a function of time difference and proportional to Dirac delta function. It is convenient to work in frequency space $\Sigma_{ab}^{\alpha}(t_1, t_2) = \int \frac{d\omega}{2\pi} \Sigma_{\omega}^{\alpha, ab}(\omega)e^{-i\omega(t_1 - t_2)}$. According to (S13), we have $\Sigma(\omega) = -\frac{1}{2}(J\zeta + U\zeta^{-1})i\sigma^y - \frac{\zeta}{2} \tau^y$, where $\sigma$ ($\tau$) acts on the 1, 2 contours (the $L, R$ chains). Using (S12), the Green’s function at the equal time reads

$$G(t, t) = \frac{1}{2} \left( -\frac{J\zeta^2 + U\zeta^{q}}{(J\zeta^2 + U\zeta^{q})^2 + \mu^2 \zeta^2} i\sigma^y + \frac{\mu^2 \zeta}{(J\zeta^2 + U\zeta^{q})^2 + \mu^2 \zeta^2} \tau^y \right).$$

Requiring $G_{12}^{aa}(t, t) = -\frac{\zeta}{2}$, we have

$$(1 - \zeta^2)(J + U\zeta^{-2})^2 = \mu^2,$$

which leads to (15) in the main text. For noninteracting case, $U = 0$, the solution is $\zeta = \sqrt{J^2 - \mu^2}$, which is (9) in the main text.

**II. DERIVATION OF GOLDSTONE MODE EFFECTIVE ACTION**

We consider the fluctuation away from the saddle point solution (9) at $\mu < J$. First notice that $G_x^{\alpha, ab}_{L, R, x}$ is a linear term in the action, so it can be integrated out to enforce $\Sigma_{a, L, R, x} = \frac{i\mu}{\zeta}$. Then we consider the fluctuations $G_x^{\alpha, ab}_{a, x}(t_1, t_2) = \bar{G}_x^{\alpha, ab}_{a, x}(t_1, t_2) + \delta G_x^{\alpha, ab}_{a, x}(t_1, t_2)$ and $\Sigma_{a, x}^{\alpha}(t_1, t_2) = \bar{\Sigma}_{a, x}^{\alpha}(t_1, t_2) + \delta \Sigma_{a, x}^{\alpha}(t_1, t_2)$, $a = L, R$. Expanding the $\text{Tr} \log$ term in (8) leads to the kernel of $\delta \Sigma$, i.e.,

$$\frac{1}{2} \text{Tr} \log \left( (-1)^{\alpha+1} \delta_t - \Sigma_x \right) \approx \frac{1}{4} \int \frac{d\omega d\Omega}{(2\pi)^2} \sum_{\alpha, \beta, \gamma, \delta} \delta \Sigma_{a, x}^{\alpha, \beta}(\omega) G_x^{\gamma, \delta}(\omega) \bar{G}_{a, x}(\omega + \Omega) \delta \Sigma_{a, x}^{\gamma, \delta}(-\Omega),$$

where $\delta \Sigma_{a, x}^{\alpha}(t) = \int \frac{d\omega}{2\pi} \delta \Sigma_{a, x}^{\alpha}(\omega)e^{-i\omega t}$. The kernel can be brought into decoupled sectors by a basis transformation. The calculation is straightforward but tedious, so we only present the main result. The interesting part is the following,

$$\frac{1}{2} \int \Omega_x(\Omega) \left( \left( -i \sqrt{1 - \frac{(\frac{J}{J + \zeta})^2}{J^2 + \mu^2}} \Omega \right) - i \sqrt{1 - \frac{(\frac{J}{J + \zeta})^2}{J^2 + \mu^2}} \Omega \right) \delta \Omega(-\Omega),$$

(S20)
where \( \int_{\Omega} \equiv \int \frac{dt}{2\pi}, \sigma_x \equiv \sum_a \left( \frac{1}{2}(\delta \Sigma^{13}_{1a,ax} + \delta \Sigma^{24}_{2a,ax}, \frac{1}{2}(\delta \Sigma^{14}_{1a,ax} + \delta \Sigma^{23}_{2a,ax}) \right) \) decouples from the other fluctuations. The reason to consider the above component is that \( \sigma_{2,x} \) corresponds to the Goldstone mode and \( \sigma_{1,x} \) cannot be neglected because it couples to the Goldstone mode.

Next we expand the other terms in (8). The second term in (8), i.e.,

\[
- \frac{1}{2} \sum_{a,x} \int dt dt' \Sigma_{a,x}^\alpha(t, t') G_{a,x}^\beta(t, t'),
\]

leads to the coupling between \( \sigma_{x} \) and \( \delta G_{a,x} \) and the term

\[
\sum_{ab,x} \int dt dt' \delta(t - t') \left[ - \frac{1}{4}(\delta \Sigma^{13}_{1a,ax} + \delta \Sigma^{24}_{2a,ax} + \delta \Sigma^{14}_{1a,ax} + \delta \Sigma^{23}_{2a,ax}) \right] \delta_{G_{a,x}}^\alpha(t, t') G_{a,x}^\beta(t, t'),
\]

contributes to the kinetic term of \( \delta G_{a,x} \). Here we are interested in noninteracting case so \( U = 0 \). Integrating over \( \sigma_{x} \), we arrive at

\[
- \frac{I_{\text{eff}}}{N} = \frac{1}{2} \sum_k \int_{\Omega} \phi_k(\Omega) \left( \frac{J}{\Omega - \mu} + J_k \right) \phi_{-k}(-\Omega),
\]

where \( \phi_x \equiv \sum_a \left( \frac{1}{2}(\delta \Sigma^{13}_{1a,ax} + \delta \Sigma^{24}_{2a,ax} + \delta \Sigma^{14}_{1a,ax} + \delta \Sigma^{23}_{2a,ax}) \right) \phi_{x,a} \equiv \sum_a \left( \frac{1}{2}(\delta \Sigma^{13}_{1a,ax} + \delta \Sigma^{24}_{2a,ax} + \delta \Sigma^{14}_{1a,ax} + \delta \Sigma^{23}_{2a,ax}) \right) \phi_{x,a}. \]

It seems the action is intact at \( \mu = J \) contradicting our proposal that transition occurs at \( \mu = J \). However we should note that \( \phi_{x,a} \) is related to the Goldstone mode nontrivially. Namely (13) indicates the \( \phi_{x,a} \) is related to the Goldstone mode \( \theta(t) \) nontrivially by

\[
\phi_{x,a}(t) = \sqrt{1 - \left( \frac{\mu}{J} \right)^2} \theta_x(t).
\]

Using this relation, we finally have the effective theory for the Goldstone mode \( \theta \) in (14),

\[
- \frac{I_{\text{eff}}}{N} = \frac{1}{2} \left( 1 - \left( \frac{\mu}{J} \right)^2 \right) \sum_k \int_{\Omega} \theta_k(\Omega) \left( \frac{J}{\Omega^2} + J_k \right) \theta_{-k}(-\Omega).
\]

It is also interesting to look at \( \phi_{1,x} = \sum_x \frac{1}{2}(\delta \Sigma^{13}_{1a,ax} + \delta \Sigma^{24}_{2a,ax}) \). The correlation function of \( \phi_{1,x} \) becomes algebraic, i.e.,

\[
\langle \phi_{1,\alpha}(\Omega) \phi_{1,\beta}(-\Omega) \rangle \approx \frac{1}{2} \frac{\Omega^2}{\Omega^2 + \mu^2} \left( \frac{J - \mu^2}{J^2 + \mu^2} \right) \delta_{\alpha,-\beta}, \quad \langle \phi_{1,\alpha}(t) \phi_{1,\beta}(t) \rangle \propto \frac{1}{r^2},
\]

where the first equation is obtained from (S23) with an expansion at \( k \approx 0 \). The second equation is the equal time correlation function in the real space. It is obtained from the first equation by the Fourier transform \( \phi_{1,\alpha}(t) \equiv \int \frac{dk d\Omega}{(2\pi)^2} \phi_{1,\alpha}(\Omega) e^{ikr - i\Omega t} \), where a continuum limit of lattice is made. The power-law correlations can be observed in equal-time squared correlation function of fermions.

### III. DERIVATION OF Z4 EFFECTIVE ACTION

We consider the fluctuation away from the symmetric saddle point solution (9) at \( \mu \geq J \). The derivation is similar to the effective Goldstone action shown in the previous section. Again we consider fluctuations, \( G_{x,a}^\alpha(t_1, t_2) = \hat{G}_{x,a}^\alpha(t_1, t_2) + \delta G_{x,a}^\alpha(t_1, t_2) \) and \( \Sigma_{x,a}^\alpha(t_1, t_2) = \hat{\Sigma}_{x,a}^\alpha + \delta \Sigma_{x,a}^\alpha \), \( a = L, R \). Expanding \( \delta \Sigma \) in the trace log term in (8) and the result is (S28). It can again be brought into decoupled sectors. Because they serve as an order parameter we focus on the components \( (\delta \Sigma_{12}^\alpha, \delta \Sigma_{14}^\alpha, \delta \Sigma_{12}^\alpha, \delta \Sigma_{23}^\alpha) \) whose kernel is given by

\[
K = \frac{\mu}{4(\mu^2 + \Omega^2)} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},
\]

(S28)
where the first matrix is in the basis of these four components and the second is in the basis of the $L$ and $R$ chains. It is apparent that there are four zero modes and integrating them out will lead to the following constraints,

$$\delta G_{12}^{2} = \delta G_{11}^{12}, \quad \delta G_{34}^{2} = \delta G_{34}^{34}, \quad \delta G_{44}^{14} = \delta G_{44}^{144}, \quad \delta G_{22}^{3} = \delta G_{22}^{33}. \quad (S29)$$

Thus there are four independent fields left ($\delta G_{12}^{2}, \delta G_{34}^{2}, \delta G_{44}^{14}, \delta G_{22}^{3}$) where we suppress the subscript.

Now it is a straightforward task to integrate out the rest fluctuations with nonzero kernel in (S28). The nontrivial sector that are related to the $C_4$ symmetry is span by $(\phi_1, \phi_2) = (\delta G_{12}^{12} + \delta G_{34}^{34}, \delta G_{44}^{14} + \delta G_{22}^{3})$ which transforms as a vector under the $C_4$ operator. The effective theory reads

$$I_{\text{eff}}^L = \sum_{i=1,2} \sum_k \int \phi_i,k(\Omega) \left( \frac{\Omega^2}{2} k^2 + \frac{\mu^2}{2} \right) \phi_i,-k(-\Omega) + \sum_x \int_t \left( -\frac{z^2}{m^2} (\phi_{1,x}^2 + \phi_{2,x}^2) + V(\phi_{1,x}^2 + \phi_{2,x}^2)^{\nu/2} \right), \quad (S30)$$

where $f_i = \int dt$ and we include the last term which should be obtained by expanding the trace log term to higher orders for stability of the theory. We will focus on $q = 4$ in the following. In this case $V \approx \zeta$ near the transition point inferred from the condition (16) of the parameter $\zeta$ for continuous transitions at $\mu = J$ and the fact that the saddle point solution depends only on $\mu$ for $\mu \geq J$. This leads to (17) in the main text.

We have expanded the action around the symmetric saddle point solution, but we may also wish to explore the ordered phase by expanding around the asymmetric saddle point solution. Similar to we have done to get the noninteracting case (S23), we have the following effective action

$$-\frac{I_{\text{eff}}}{N} = \frac{1}{2} \sum_k \int \phi_k(\Omega) \left( \frac{(y^2J^2 + \mu^2)^{\nu/2}}{2y^2J^2} - J_k \frac{i\sqrt{y^2J^2 + \mu^2} \Omega}{yJ} \right) \phi_{-k}(-\Omega), \quad (S31)$$

where the parameter $y$ is given by

$$y = \frac{y}{\sqrt{y^2 + \mu^2}} + U \left( \frac{y}{\sqrt{y^2 + \mu^2}} \right)^{-1}. \quad (S32)$$

For small interaction strength, $y$ is given by

$$y = \sqrt{1 - \mu^2} + U(1 - \mu^2)^{\nu/2} + O(U^2). \quad (S33)$$

Now we expect the Goldstone mode acquires a finite mass proportional to $U$ due to the lowering of symmetry from $O(2)$ to $C_4$. Indeed by integrating out the $\varphi_1$ we have

$$-\frac{I_{\text{eff}}}{N} = \frac{1}{2} \sum_k \int \varphi_{2,k}(\Omega) \left( -\sqrt{y^2J^2 + \mu^2} + J_k \frac{\Omega^2}{\sqrt{y^2J^2 + \mu^2} - y^2J^2 + \mu^2} \right) \varphi_{2,-k}(-\Omega). \quad (S34)$$

The mass $J - \sqrt{y^2J^2 + \mu^2}$ vanishes when $U = 0$, and we restore (S24). For small interactions, the mass can be simplified as $J - \sqrt{y^2J^2 + \mu^2} \approx U(1 - \mu^2)^{\nu/2}$. It is apparent that the interaction reduces the symmetry and renders the Goldstone mode gaped.

**IV. NUMERICAL CALCULATION OF RÉNYI ENTROPY AND MUTUAL INFORMATION**

In this section, we give more details about how to numerically calculate the Rényi entropy in Fig.3 and 4. The calculations have been discussed in many previous works. We discuss the main changes in this paper, and refer the details of the calculation to these works. The starting point is the following expression of Rényi-2 entropy,

$$\exp(-S_{\lambda}^{(2)}) = \frac{\mathbb{E} \text{Tr}(\rho_A^2)}{\text{Tr}(\rho_A)^2}, \quad (S35)$$

where $\rho$ is unnormalized density matrix ($\rho_A$ is unnormalized reduced density matrix of subsystem $A$), and $\mathbb{E}$ denotes average of Brownian variables and the quantum trajectories. Both the numerator and denominator can be written as a path integral, where the effective action is,

$$-\frac{I[G_x, \Sigma_x; F_x]}{N} = \frac{1}{2} \text{Tr} \log (\delta_{ab} F_x(s,s') - \Sigma_{ab,x}(s,s')) - \frac{1}{2} \int dsds' \Sigma_{ab,x}(s,s') G_{ab,x}(s,s') + \frac{U}{2q} (2G_{ab,x}(s,s'))^q + \frac{i\mu}{2} \delta(s-s') G_{LR}(s,s'), \quad (S36)$$
where $0 < s, s' < 4T$ is a label of four contours: the forward contours are $s \in (0, T) \cup (2T, 3T)$ and the backward contours are $s \in (T, 2T) \cup (3T, 4T)$. The bilocal fields $G$ and $\Sigma$ do not have the contour index $\alpha$ or $\beta$ since $s$ and $s'$ label the contours. The function $f(s)$ characterizes the arrow of times in the unitary evolution,

$$f(s) = \begin{cases} i, & s \in (0, T) \cup (2T, 3T) \\ -i, & s \in (T, 2T) \cup (3T, 4T) \end{cases}$$

(S37)

$g(s, s') = \sum_{\alpha=0}^{1} \delta(|s - s'| - 2\alpha T) + \sum_{\alpha=1}^{3} \delta(s + s' - 2\alpha T)$ is the Brownian correlation on the four contours. $F_x(s, s')$ is a function that will be specified later.

Because of the large-$N$ structure, we can solve the Dyson-Schwinger equation numerically and calculate the onshell action. The Dyson-Schwinger equation followed from the action reads

$$[G_x^{-1}]_{ab}(s, s') = \delta_{ab}F_x(s, s') - \Sigma_{ab,x}(s, s'),$$

(S38)

$$\Sigma_{ab,x}(s, s') = g(s, s')[\frac{f(s)f(s')\delta_{ab}}{2} + \frac{i\mu\delta(s - s')\delta_{aL}\delta_{bR} - \delta_{aR}\delta_{bL}}{2}],$$

(S39)

Then the Rényi entropy is given by

$$\exp(-S^{(2)}_A) = \frac{\text{ETr}(\rho^2)}{\text{ETr}(\rho)} \approx \frac{e^{-I_{\text{onshell}}(A)}}{e^{-I_{\text{onshell}}(\emptyset)}},$$

(S40)

where $I_{\text{onshell}}(A)$ and $I_{\text{onshell}}(\emptyset)$ are defined as

$$I_{\text{onshell}}(A) = I[\bar{G}_x, \bar{\Sigma}_x; F_{A,x}], \quad I_{\text{onshell}}(\emptyset) = I[\bar{G}_x, \bar{\Sigma}_x; F_{\emptyset,x}].$$

(S41)

Here $\bar{G}_x, \bar{\Sigma}_x$ denotes the numerical solutions from iteration, and $F_{A,x}$ and $F_{\emptyset,x}$ are defined in the following to account for the boundary condition.

The difference between the numerator and denominator in (S40) is exactly the boundary condition. To be concrete, we consider infinite temperature thermofield double state\cite{5,9}. Because of the Brownian nature of the model, we consider infinite temperature thermofield double state, the details of which can be found in\cite{9}. As we are interested in Rényi entropy of subsystem $A$, the twist boundary condition is applied in subsystem $A$. To incorporate the different boundary conditions, the $F$ is chosen differently. We define

$$(F^{(0)})^{-1}(s, s') = \frac{1}{2} \text{sgn}(s - s'), \quad s, s' \in (0, 2T) \text{ or } s, s' \in (2T, 4T),$$

(S42)

$$(F^{(1)})^{-1}(s, s') = \frac{1}{2} \text{sgn}(s - s'), \quad s, s' \in (T, 3T) \text{ or } s, s' \in (0, T) \cup (3T, 4T).$$

(S43)

In the denominator, there is no twist boundary condition, while in the numerator, the twist boundary condition is implemented in subsystem $A$. To account for these boundary conditions, the function $F$ is given by

$$F_{\emptyset,x}(s, s') = F^{(0)}(s, s'), \quad \forall x,$$

(S44)

$$F_{A,x}(s, s') = \begin{cases} F^{(0)}(s, s'), & x \notin A \\ F^{(1)}(s, s'), & x \in A \end{cases}.$$  

(S45)

Having specified all the functions in (S38), the Dyson-Schwinger equation can be solved numerically by discretizing $s, s'$, and iterating the equation, the details of which can be found in\cite{1,5,9}. The discretization implemented in our calculation is 400 to 800, which is converged. Other parameters are specified in the caption of each figure. To calculate the mutual information, we use the definition $I_{AB} = S^{(2)}_{AB} - S^{(2)}_A - S^{(2)}_B$, and calculate the Rényi entropy on the right-hand side. The quantity $S^{(2)}_{AB}$ is given as

$$\exp(-S^{(2)}_{AB}) \approx \frac{e^{-I_{\text{onshell}}(A \cup B)}}{e^{-I_{\text{onshell}}(\emptyset)}},$$

(S46)

where $I_{\text{onshell}}(A \cup B) = I[\bar{G}_x, \bar{\Sigma}_x; F_{A \cup B,x}]$,

$$F_{A \cup B,x}(s, s') = \begin{cases} F^{(0)}(s, s'), & x \notin A \cup B \\ F^{(1)}(s, s'), & x \in A \cup B \end{cases}.$$  

(S47)
They contribute equally to this work.

* chenaad@bc.edu
† bswingle@umd.edu
‡ pzhang93@caltech.edu