HIGHER DIMENSIONAL OBSTRUCTIONS FOR STAR REDUCTIONS

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ABSTRACT. A $*$ -reduction between two equivalence relations is a Baire measurable reduction which preserves generic notions, i.e., preimages of meager sets are meager. We show that a $*$ -reduction between orbit equivalence relations induces generically an embedding between the associated Becker graphs. We introduce a notion of dimension for Polish G -spaces which is generically preserved under $*$ -reductions. For every natural number n we define a free action of S_∞ whose dimension is n on every invariant Baire measurable non-meager set. We also show that the S_∞ -space which induces the equivalence relation $=^+$ of countable sets of reals is ∞ -dimensional on every invariant Baire measurable non-meager set. We conclude that the orbit equivalence relations associated to all these actions are pairwise incomparable with respect to $*$ -reductions.

1. INTRODUCTION

Many classification problems in mathematics can be formally presented as pairs (X, E) , where X is a Polish space and E is an analytic equivalence relation on X . For example, the problem of classifying graph structures on domain ω up to isomorphism, or the problem of classifying self-adjoint operators on a separable Hilbert space up to unitary equivalence, are both instances of this formal setup. In order to compare the relative complexity of two such problems $(X, E), (Y, F)$, one often wants to know whether there exists a map $f: X \rightarrow Y$ such that:

- (1) f is a **reduction** from E to F , i.e., $xEx' \iff f(x)Ff(x')$, for all $x, x' \in X$;
- (2) f preserves some structural properties of (X, E) and (Y, F) ?

In practice, and since otherwise the question trivializes,¹ the reduction f is always assumed to be Borel, or at least Baire measurable. Besides this minimal definability requirement, one often wants f to be sensitive to various other structural properties of (X, E) and (Y, F) . For example, in the context of the topological Vaught conjecture it is useful to consider *faithful reductions*, i.e., Borel reductions f from E to F

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¹By the axiom of choice, there is an “abstract” reduction from E to F if and only if $|X/E| \leq |Y/F|$.

with the additionally property that the F -saturation of the image of any E -invariant Borel subset of X is a Borel subset of Y [FS89, Hjo00, Gao01, Gao05]. In the context of ergodic theory, when X, Y additionally support probability measures μ, ν , and E, F are orbit equivalence relations of measure preserving actions of countable groups, one often works with *orbit equivalences*. These are Borel reductions from E to F which are bijective and measure preserving, i.e., they pull back ν -null sets to μ -null sets [Gab02, Gab00, KM04]. In this paper we introduce and study **-reductions*. These are Baire measurable reductions which are category preserving.²

Definition 1. Let X, Y be Polish spaces and let E, F be equivalence relations on X and Y respectively. A Baire measurable map $f: X \rightarrow Y$ is **category preserving** if for every meager subset M of Y , $f^{-1}(M)$ is a meager subset of X . A ***-reduction** from E to F is a Baire measurable and category preserving reduction from E to F . We write $E \leq_* F$ when such a *-reduction exists.

As with Borel reductions, the relation $\leq_*$ induces a preordering among the various classification problems. This preordering reflects the relative complexity between two such problems from the point of view of *-reductions. Showing that some classification problem *-reduces to another often just amounts to finding a “canonical way” of coding E into F . However, showing negative results predicates upon developing a basic obstruction theory for *-reductions. When it comes to simple Borel and Baire measurable reductions there are many well known descriptive set theoretic and dynamical obstructions [Hjo00, Gao08, LP, FS89]. Similar obstructions have been developed for orbit equivalences. For example Gaboriau’s theory of cost [Gab02, Gab00] implies that two free groups of different rank can never produce orbit equivalent equivalence relations via free and measure preserving actions on a standard measure space. The main goal of this paper is to develop certain obstructions for *-reductions by advancing further some of the techniques introduced in [LP].

We briefly describe here the main ideas; definitions and details can be found in Section 2. Given a Polish G -space X , let E_X^G be the associated orbit equivalence relation on X . In [LP], the authors introduce a digraph structure $\mathcal{B}(X/G)$ on the quotient X/E_X^G based on Becker’s notion of right-embeddings.³ The *right-Becker graph* $\mathcal{B}(X/G)$ has the elements $[x]$ of X/E_X^G as vertices, and an arrow $[x] \rightarrow [y]$ whenever there is a right-Cauchy sequence (g_n) in G so that $g_n y$ converges to x . The main structural result in [LP] ([LP, Proposition 2.8]) states that a Baire measurable reduction f from E_X^G to E_Y^H induces an injective digraph homomorphism from $\mathcal{B}(X_0/G)$ to $\mathcal{B}(Y/H)$, where X_0 is an invariant generic subset of X . Using this result and the fact that the Becker graph associated to the action of a CLI group

²For more information on category preserving maps between Polish spaces one may consult [MT13, Appendix A].

³In [Bec98], Becker studies left-embeddings. Here and in [LP], with an eye on the applications, it is convenient to develop everything in terms of right-embeddings. Analogous results hold for left-embeddings.

H is trivial,⁴ they introduce—in analogy to Hjorth’s turbulence condition—a new dynamical obstruction for classifying E_X^G by CLI-group actions. Here we show that under $*$ -reductions the main structural result from [LP] can be strengthened to the following.

Theorem 2. *Suppose that G, H are Polish groups, X is a Polish G -space, Y is a Polish H -space, and $f: X \rightarrow Y$ is $*$ -reduction from E_X^G to E_Y^H . Then there is a G -invariant dense G_δ subset X_0 of X and an H -invariant non-meager Baire measurable subset Y_0 of Y so that the induced map $[f]$ restricted on X_0/G is an isomorphism from the digraph $\mathcal{B}(X_0/G)$ to the digraph $\mathcal{B}(Y_0/H)$.*

This strengthening allows us to utilize “higher dimensional” properties of the Becker graphs $\mathcal{B}(X/G)$ and $\mathcal{B}(Y/H)$ as obstructions for $*$ -reducing E_X^G to E_Y^H . We say that the *dimension* of the Becker graph $\mathcal{B}(X/G)$ is at least n , if the combinatorial n -cube embeds in $\mathcal{B}(X/G)$. We say that the *generic dimension* of $\mathcal{B}(X/G)$ is at least n if the combinatorial n -cube embeds in $\mathcal{B}(X_0/G)$, for every invariant comeager subset X_0 of X . Similarly, we say the *locally generic dimension* of $\mathcal{B}(X/G)$ is at least n if the combinatorial n -cube embeds in $\mathcal{B}(X_0/G)$, for every invariant non-meager subset X_0 of X with the Baire property.

FIGURE 1. The combinatorial n -cube.

From this point of view, the anti-classification criterion in [LP] corresponds to the fact that CLI groups induce 0-dimensional orbit equivalence relations and hence they cannot reduce—in a Baire measurable fashion—orbit equivalence relations which are generically at least 1-dimensional. In Section 3 we define for every $n > 0$ a free action of S_∞ on a Polish space of countable structures $\text{Mod}_\omega(\widehat{\mathcal{B}}_n)$ which, as we show in Theorem 9, locally generically $(n - 1)$ -dimensional but not n -dimensional. Theorem 2 then implies that the associated classification problems $(\text{Mod}_\omega(\widehat{\mathcal{B}}_n), \simeq_{\text{iso}})$, $n > 0$, are incomparable under $*$ -reductions. The structures in $\text{Mod}_\omega(\widehat{\mathcal{B}}_n)$ are *labeled* versions of certain families of structures that were introduced and studied in [BKL17] for their interesting behavior with respect to *disjoint n -amalgamation*.

In the process of proving Theorem 9 we develop a general method which can be used for computing the generic dimension of other similar problems; see Remark

⁴A group is CLI if it admits a complete and left-invariant metric.

17. In particular, our method implies that $=^+$ is the orbit equivalence relation of an ∞ -dimensional S_∞ -space and therefore incomparable under $*$ -reductions to $(\text{Mod}_\omega(\widehat{\mathcal{B}}_n), \simeq_{\text{iso}})$ for all $n > 0$.

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2. $*$ -REDUCTIONS AND BECKER GRAPHS

In what follows X and Y will be Polish spaces and E and F will be analytic equivalence relations on X and Y respectively. Often these equivalence relations will be *orbit equivalence relations* associated with a Polish group action. Let G be a Polish group. A **Polish G -space** is a Polish space X together with a continuous action $G \times X \rightarrow X$ on X . The associated **orbit equivalence relation** E_X^G on X is defined by $xE_X^G y \iff [x] = [y]$, where $[x]$ denotes the orbit of $x \in X$ under the action. If A is any subset of X , we will denote by $[A]$ the G -saturation of A : $\{x \mid \exists g \in G \, gx \in A\}$. Note that if A is an analytic subset of X , then $[A]$ is analytic.

There are some immediate noticeable differences between the usual definable reductions and $*$ -reductions. For example, since the preimages of a comeager set under a category preserving map is comeager, an equivalence relation E cannot $*$ -reduce to an equivalence relation F with a comeager F -class, unless E has also a comeager E -class. As a consequence, and in contrast to the Borel reduction hierarchy [FS89], countable graph isomorphism is not $\leq_*$ -universal amongst orbit equivalence relations of S_∞ actions. On the other hand, if E and F are orbit equivalence relations both having a comeager equivalence class, then any Baire measurable reduction from E to F can be modified to a $*$ -reduction. This follows simply from the fact that all orbits of a Polish G -space are Borel. The problem becomes more interesting when both E and F have only meager equivalence classes, i.e., when generic properties do not concentrate on a single equivalence class.

In general $*$ -reductions reveal to be much more sensitive to the dynamical aspects of the classification problems under comparison compared to the usual definable reductions. For example, we have the following proposition. Recall that an equivalence relation E on X is **generically ergodic** if every Baire measurable E -invariant subset of X is either meager or comeager.

Proposition 3. *Let E and F be analytic equivalence relations on Polish spaces X and Y respectively. If $E \leq_* F$ and F is generically ergodic then E is generically ergodic.*

Proof. Let f be the $*$ -reduction from E to F and let A be a Baire measurable E -invariant subset of X . Let B be any G_δ subset of A so that $A \setminus B$ is meager. By restricting B further if necessary we can assume that $f \upharpoonright B$ is continuous. Let C be the F -saturation of the image of B under f . Then C is an invariant analytic subset of Y and hence it is either meager or comeager. It follows that A is meager or comeager as well. $\square$

We restrict now our attention entirely to orbit equivalence relations. Recall that a sequence (g_n) in G is Cauchy with respect to some left invariant metric on G if and only if it is Cauchy with respect to any left invariant metric on G ; see [Bec98]. In this case we say that (g_n) is **left-Cauchy**. Similarly we define when (g_n) is right-Cauchy. Let X be a Polish G -space and let $x, x' \in X$. Following Becker [Bec98], we say that x **left-embeds in** x' if there is a left-Cauchy sequence (g_n) so that $(g_n x)$ converges to x' . Similarly we say that x **right-embeds in** x' if there is a right-Cauchy sequence (g_n) so that $(g_n x')$ converges to x .

We recall now some definitions and a result from [LP]. In view of our applications in Section 3 we are going to develop here everything in terms of right embeddings. All results hold equally for left embeddings. The **right Becker digraph** $\mathcal{B}(X/G)$ associated to Polish G -space is a graph on domain $X/G = \{[x] \mid x \in X\}$, whose arrows are precisely all pairs $([x], [x'])$ such that x right-embeds in x' . The main anti-classification result developed in [LP] was a consequence of the following proposition. Recall that an (E, F) -**homomorphism** is any map $f: X \rightarrow Y$ with

$$xEx' \implies f(x)Ff(x').$$

An (E, F) -homomorphism f induces a map $[f]: X/E \rightarrow Y/F$ between the quotients, sending $[x]$ to $[f(x)]$. This map is injective if and only if f is a reduction.

Proposition 4 ([LP], Proposition 2.8). *Let G, H be Polish groups, let X be a Polish G -space, and let Y be a Polish H -space. If $f: X \rightarrow Y$ is Baire measurable (E_X^G, E_Y^H) -homomorphism, then there is a G_δ subset X_0 of X so that the induced map $[f]: X_0/G \rightarrow Y/H$ is a graph homomorphism from the digraph $\mathcal{B}(X_0/G)$ to the digraph $\mathcal{B}(Y/H)$.*

The above proposition was used in [LP] in order to show that certain equivalence relations do not reduce in a Baire measurable fashion to any orbit equivalence relation of a CLI group action. In particular, if H is CLI then $\mathcal{B}(Y/H)$ contains only loops [LP, Lemma 2.7]. Hence if $\mathcal{B}(X/G)$ contains non-trivial edges in every invariant G_δ subset X_0 of X then, by Proposition 4, every Baire measurable (E_X^G, E_Y^H) -homomorphism would fail to be a reduction.

In the context of $*$ -reductions, we can strengthen the conclusion of Proposition 4 so that $[f] \upharpoonright (X_0/G)$ is an embedding of digraphs rather than an injective digraph homomorphism. This is the essence of Theorem 2. For the proof we will need the following minor strengthening of [LP, Lemma 2.5]⁵.

Lemma 5. *Suppose that G, H are Polish groups, X is a Polish G -space, and Y is a Polish H -space. Let $C \subseteq X$ be a G_δ subset of X such that for any $x \in C$ the set $\{g \in G \mid gx \in C\}$ is comeager in G . Let $f: C \rightarrow Y$ be a Baire measurable homomorphism from the equivalence relation $E_X^G \upharpoonright C$ to E_Y^H . Then there is a dense in C , G_δ subset $\tilde{C}$ of C so that:*

⁵This is essentially [Hjo00, Lemma 3.17] modified as in the beginning of the proof of [Hjo00, Theorem 3.18].

- (a) $f|_{\tilde{C}}$ is continuous;
- (b) for any $x \in \tilde{C}$ the set $\{g \in G \mid gx \in \tilde{C}\}$ is comeager in G ;
- (c) for any $x_0 \in \tilde{C}$ and for any open neighborhood W of the identity in H there exists an open neighborhood U of x_0 and an open neighborhood V of the identity of G such that for any $x \in U \cap \tilde{C}$ and for a comeager set of $g \in V$, we have that $f(gx) \in Wf(x)$ and $gx \in \tilde{C}$.

Proof. Let $V \subseteq_1 G$ stand for “ V is an open neighborhood of identity in G ” and let $\forall^* x \in A$ stand for “for a comeager collection of elements in A .”

The proof is exactly the same as in [LP, Lemma 2.5] so we will omit the details. First one needs to show that for any fixed $W \subseteq_1 H$ we have that

$$\forall x_0 \in C \quad \forall^* g_0 \in G \quad \exists V \subseteq_1 G \quad \forall^* g_1 \in V \quad f(g_1 g_0 x_0) \in Wf(g_0 x_0).$$

Then by an application of Kuratowski-Ulam theorem, and since by assumption we have that $\forall x \in C \quad \forall^* g \in G \quad gx \in C$, we get a dense G_δ subset C_0 of C so that

$$\forall x \in C_0 \quad \exists V \subseteq_1 G \quad \forall^* g \in V \quad f(gx) \in Wf(x).$$

Then we use C_0 exactly as in the proof of [LP, Lemma 2.5] to define C_1 and we finish by setting $\tilde{C} := \{x \in C_1 \mid \forall^* g \in G \quad gx \in C_1\}$. $\square$

We now proceed to the proof of Theorem 2.

Proof of Theorem 2. Let $f: X \rightarrow Y$ be as in the statement of the theorem. As in [LP, Lemma 2.8], if $\tilde{C}$ is the set provided by Lemma 5 with $C = X$, then setting $X'_0 = \{x \in X \mid \forall^* g \in G \quad gx \in \tilde{C}\}$ we get the invariant dense G_δ set appearing in the statement of Proposition 4, i.e. we get that $[f]: X'_0/G \rightarrow Y/H$ is homomorphism from $\mathcal{B}(X'_0/G)$ to $\mathcal{B}(Y/H)$. For the convenience of the reader we include a brief sketch: assume that x, x' are in X'_0 and x right-embeds in x' . Since this is a G -invariant property we can assume that x, x' are in $\tilde{C}$. Let (g_n) be a right-Cauchy sequence with $g_n x' \rightarrow x$. As in [LP, Lemma 2.3] we can use properties (b) and (c) of Lemma 5 to slightly modify (g_n) so that $g_n x$ is in $\tilde{C}$ and so that there is a right-Cauchy sequence (h_n) in H with $f(g_n x) = h_n f(x)$, for all $n > 0$. By property (a) of Lemma 5 we get that $h_n f(x) \rightarrow f(x')$.

Since f is by assumption a reduction we also have that $[f]$ is injective. In what follows, we will intersect X'_0 with another invariant dense G_δ set X''_0 with the property that for every $x, x' \in X''_0$, if $f(x)$ right-embeds in $f(x')$, then x right-embeds in x' . The desired sets will then be: $X_0 = X'_0 \cap X''_0$, and $Y_0 = [f(X_0)]$.

Claim. *There exists $D \subseteq Y$ and $h: D \rightarrow X$ so that:*

- (1) D is a G_δ subset of Y with $f^{-1}(D)$ dense G_δ in X ;
- (2) for every $y \in D$ we have that $\{g \in H \mid gy \in D\}$ is a comeager subset of H ;
- (3) h is a Baire measurable reduction from $E_Y^H \upharpoonright D$ to E_X^G , with $[f] \circ [h] = \text{id}$ and $[h] \circ [f|_{f^{-1}(D)}] = \text{id}$.

Proof of Claim. Since f is Baire measurable, we can find a dense G_δ subset C_0 of X so that $f|_{C_0}$ is continuous on C_0 . Let $A = [f(C_0)] \subseteq Y$ be the H -saturation of the image of C_0 under f . Since $F = \{(y, x) \in A \times C_0 \mid yE_Y^H f(x)\}$ is an analytic set, by the Jankov-von Neumann selection theorem [Kec11, Theorem 18.1] there is Baire measurable⁶ map $h': A \rightarrow C_0$ uniformizing F . Let D_0 be a G_δ subset of A so that $B := A \setminus D_0$ is meager. We claim that the set $D = \{y \in D_0 \mid \forall^* g \in H \, gy \in D_0\}$ and the map $h = h'|_D$ are as required. To see this notice that property (3) follows immediately from the fact that h' uniformizes F , and property (2) follows from [Gao08, Proposition 3.2.5(v)]. For property (1), D is G_δ since it is the Vaught transform of a G_δ set intersected with a G_δ set; see [Gao08, Proposition 3.2.7]. Finally, notice that since A is H -invariant we have that

$$f^{-1}(D) = f^{-1}\left(\left\{y \in (Y \setminus B) \mid \forall^* g \in H \, gy \in (Y \setminus B)\right\}\right),$$

and the set $\{y \in (Y \setminus B) \mid \forall^* g \in H \, gy \in (Y \setminus B)\}$ is comeager in Y by Kuratowski-Ulam. Since f preserves category and it is continuous on the dense G_δ subset C_0 of X , $f^{-1}(D)$ is a dense G_δ subset of X as well. $\square$

By Lemma 5 applied to $h: D \rightarrow X$ we can now find a dense G_δ subset $\tilde{D}$ of D satisfying the conclusions of Lemma 5. By the proof of [LP, Proposition 2.8], as described in the first paragraph of this proof, for every $y, y' \in \tilde{D}$, if y right-embeds in y' then $h(y)$ right-embeds in $h(y')$.

Setting now X_0'' to be the Vaught transform $\{x \in X \mid \forall^* g \in G \, gx \in f^{-1}(\tilde{D})\}$ of $f^{-1}(\tilde{D})$ we get the desired sets: $X_0 = X_0' \cap X_0''$ and $Y_0 = [f(X_0)]$. To see this, let $x, x' \in X_0''$ so that $f(x)$ right-embeds in $f(x')$. By replacing x, x' with a generic translate we can assume without the loss of generality that $x, x' \in f^{-1}(\tilde{D})$. Since $f(x), f(x') \in \tilde{D}$ we get that $h(f(x))$ right-embeds in $h(f(x'))$. Property (3) of the above claim implies that x right-embeds in x' . Finally, since $[f(X_0)] = [f(X_0 \cap \tilde{C})]$ and $f|_{\tilde{C}}$ is continuous, we have that Y_0 is analytic and hence Baire measurable. Moreover, since $f^{-1}(Y_0) = X_0$, f is category preserving, and X_0 is comeager in X , Y_0 cannot be meager in Y . $\square$

3. HIGHER DIMENSIONAL OBSTRUCTIONS

For every $n \geq 0$, the **combinatorial n -cube** is the poset category Δ^{n-1} , whose set of objects is the powerset $\mathcal{P}(\{0, \dots, n-1\})$ of $[n] = \{0, \dots, n-1\}$ and whose arrows are precisely the inclusions $\sigma \subseteq \tau$ between subsets σ, τ of $\{0, \dots, n-1\}$. For $n = 0$, the combinatorial n -cube is the one-object poset $\Delta^{-1} = \mathcal{P}(\emptyset) = \{\emptyset\}$.

Forgetting for a moment the composition law between the arrows we can view Δ^{n-1} as a digraph. Let X be a Polish G -space and let C be a non-empty G -invariant subset of X . The **dimension of C** is the largest natural number $n \geq 0$ so that the combinatorial n -cube embeds in $\mathcal{B}(C/G)$, if such n exists; and it is ∞

⁶ $\sigma(\Sigma_1^1)$ -measurable, in particular.

FIGURE 2. Hasse diagram of the combinatorial 3-cube.

otherwise. Notice that for every C as above the dimension of C is at least 0. We say that X is **generically n -dimensional** if n is the largest element in $\{0, 1, \dots\} \cup \{\infty\}$ so that the dimension of every invariant comeager subset X is at least n . Similarly we say that X is **locally generically n -dimensional** if n is the largest element in $\{0, 1, \dots\} \cup \{\infty\}$ so that the dimension of every invariant non-meager subset of X with the Baire property is at least n . If $\dim(X)$ denotes the dimension of X , $\dim_{\forall}^*(X)$ denotes the generic dimension of X , and $\dim_{\exists}^*(X)$ denotes the locally generic dimension of X , then we always have that $\dim(X) \geq \dim_{\forall}^*(X) \geq \dim_{\exists}^*(X)$. While the inequalities are strict in general, the last two quantities are equal whenever X is generically ergodic.

The obstruction, developed in [LP], for classifying orbit equivalence relations by CLI group actions relied on the fact that the image of a combinatorial 1-cube under an injective digraph homomorphism can never be a self-loop. Working with digraph embeddings—rather than just injective homomorphisms—allows us to utilize combinatorial n -cubes as obstructions for classification under $*$ -reductions, even when $n > 1$.

Theorem 6. *Let X be a Polish G -space and let Y be a Polish H space where G and H are Polish groups. Let also $n \in \{0, 1, 2, \dots\} \cup \{\infty\}$.*

- (1) *If X is generically n -dimensional and the dimension of Y is less than n , then E_X^G does not $*$ -reduce to E_Y^H .*
- (2) *If the dimension of X is less than n and Y is locally generically n -dimensional, then E_X^G does not $*$ -reduce to E_Y^H .*

Proof. The proof is a direct consequence of Theorem 2. □

In the rest of this section we develop examples of Polish G -spaces X which are n -dimensional and locally generically n -dimensional. We then use Theorem 6 to deduce that these spaces are pairwise incomparable under $*$ -reductions.

Let $\mathcal{L}$ be a countable language. We consider the space $\mathcal{X}_{\mathcal{L}}$ of $\mathcal{L}$ -structures with domain ω . For any formula $\varphi(x)$ and any tuple a from ω , define

$$[\varphi(a)] = \{A \in \mathcal{X}_{\mathcal{L}} \mid A \models \varphi(a)\}.$$

Then a subbasis for the topology on $\mathcal{X}_{\mathcal{L}}$ is given by the sets of the form $[R(a)]$ and $[\neg R(a)]$ for every n -ary relation symbol R in $\mathcal{L}$ and every n -tuple a from ω , and $[f(a) = b]$ for every n -ary function symbol f in $\mathcal{L}$, every n -tuple a from ω , and every element $b \in \omega$. That is, $\mathcal{X}_{\mathcal{L}}$ is a Polish space homeomorphic to a product of Cantor spaces $2^{(\omega^n)}$ for each n -ary relation symbol in $\mathcal{L}$ and Baire spaces $\omega^{(\omega^n)}$ for each n -ary function symbol in $\mathcal{L}$.

Consider now the Polish group S_{∞} , of all bijections from ω to ω , endowed with the pointwise convergence topology. There is a natural continuous action of S_{∞} on $\mathcal{X}_{\mathcal{L}}$; namely, if $g \in S_{\infty}$ and $A \in \mathcal{X}_{\mathcal{L}}$, then gA is the unique $B \in \mathcal{X}_{\mathcal{L}}$ so that for every tuple $a = (a_0, \dots, a_{n-1})$ in ω and every quantifier free formula φ we have that

$$B \models \varphi(a_0, \dots, a_{n-1}) \iff A \models \varphi(g^{-1}(a_0), \dots, g^{-1}(a_{n-1}))$$

In other words, $gA = B$ if and only if g is an isomorphism $A \rightarrow B$. The orbit equivalence relation on $\mathcal{X}_{\mathcal{L}}$ induced by the logic action is denoted $\simeq_{\text{iso}}$.

Recall that a sequence (g_n) in any Polish group is left-Cauchy if it is Cauchy with respect to some left-invariant metric, and moreover, that this is equivalent with (g_n) being Cauchy with respect to every left-invariant metric; see [Bec98]. In S_{∞} a compatible left-invariant metric is given by $d_l(g, h) = 1/2^m$, where m is the least natural number with $g(m) \neq h(m)$. Elements γ of the left completion of S_{∞} can be identified with injections $\gamma: \omega \rightarrow \omega$ which are not necessarily surjective; see e.g. [Gao98]. Since for every left-invariant metric d_l , in any Polish group, the metric $d_r(g, h) := d_l(g^{-1}, h^{-1})$ is right-invariant, we have that a sequence (g_n) in S_{∞} is right-Cauchy if and only if (g_n^{-1}) is left-Cauchy. Similarly to the logic action, where $gA = B$ if and only if g is an isomorphism from A to B , the following proposition states that right-embeddings from A to B correspond to model-theoretic embeddings from A to B . For left-embeddings the situation is a bit more complicated; see [Bec98].

Proposition 7. *Let $A, B \in \mathcal{X}_{\mathcal{L}}$ and let (g_n) be a right-Cauchy sequence in S_{∞} . Let also $\gamma: \omega \rightarrow \omega$ be the injective map that is the limit of (g_n^{-1}) . The following are equivalent:*

- (1) $(g_n B)$ converges to A ;
- (2) γ is an embedding from A to B .

Proof. Let φ be a quantifier free formula and let a be a tuple in ω . By definition of the logic action, $\varphi(a)$ holds in $g_n B$ for large n if and only if $\varphi(g_n^{-1}a)$ holds in B for large n . Since (g_n^{-1}) is converging to γ , the later is equivalent to $B \models \varphi(\gamma a)$. The rest follows from the fact that $g_n B$ converges to A if and only if for all a and φ as above and for all large enough n we have that $A \models \varphi(a) \iff g_n B \models \varphi(a)$. $\square$

When $\mathcal{K}$ is a class of $\mathcal{L}$ -structures, we write $\text{Mod}_\omega(\mathcal{K})$ for the subspace of $\mathcal{X}_\mathcal{L}$ consisting of structures in $\mathcal{K}$. Abusing notation, we also define

$$[\varphi(a)] = \{A \in \text{Mod}_\omega(\mathcal{K}) \mid A \models \varphi(a)\}.$$

When $\text{Mod}_\omega(\mathcal{K})$ is an invariant subspace of $\mathcal{X}_\mathcal{L}$, the logic action descends to $\text{Mod}_\omega(\mathcal{K})$. We are particularly interested in the case when $\text{Mod}_\omega(\mathcal{K})$ is a G_δ , and hence Polish, subspace of $\mathcal{X}_\mathcal{L}$. Note that $\text{Mod}_\omega(\mathcal{K})$ is an invariant Borel subspace of $\mathcal{X}_\mathcal{L}$ whenever $\mathcal{K}$ is axiomatizable by a countable $\mathcal{L}_{\omega_1, \omega}$ -theory, and it is often possible to check that $\text{Mod}_\omega(\mathcal{K})$ is G_δ by looking at the form of this axiomatization.

Given a countable language $\mathcal{L}$, we denote by $\widehat{\mathcal{L}}$ the language $\mathcal{L} \cup \{P_i \mid i \in \omega\}$, where the P_i are unary relation symbols which do not appear in $\mathcal{L}$. Given a class $\mathcal{K}$ of $\mathcal{L}$ -structures, we denote by $\widehat{\mathcal{K}}$ the class of $\widehat{\mathcal{L}}$ -structures whose reduct to $\mathcal{L}$ is in $\mathcal{K}$ and such that no two elements satisfy exactly the same set of predicates P_i , i.e. the satisfy the following axiom:

$$(A1) \quad \forall x \forall y \left((x \neq y) \rightarrow \bigvee_{i \in \omega} (P_i(x) \not\leftrightarrow P_i(y)) \right).$$

We call the structures in $\widehat{\mathcal{K}}$ **labeled $\mathcal{K}$ -structures**.

Example 8. If $\mathcal{L}_\emptyset$ is the empty language and $\mathcal{S}$ is the class of all sets then every structure in $\text{Mod}_\omega(\widehat{\mathcal{S}})$ is essentially a sequence of distinct reals (elements of 2^ω). Up to isomorphism, such a structure is essentially a countable set of reals. The orbit equivalence relation $\simeq_{\text{iso}}$ on $\text{Mod}_\omega(\widehat{\mathcal{S}})$ is often denoted $=^+$.

For any class of $\mathcal{L}$ -structures $\mathcal{K}$, $\text{Mod}_\omega(\widehat{\mathcal{K}})$ and $\text{Mod}_\omega(\mathcal{K}) \times \text{Mod}_\omega(\widehat{\mathcal{S}})$ are isomorphic as Polish S_∞ -spaces. Note that $\text{Mod}_\omega(\widehat{\mathcal{S}})$ is an invariant G_δ subspace of $\mathcal{X}_{\widehat{\mathcal{L}}_\emptyset}$. Hence, when $\text{Mod}_\omega(\mathcal{K})$ is an invariant G_δ subspace of $\mathcal{X}_\mathcal{L}$, then $\text{Mod}_\omega(\widehat{\mathcal{K}})$ is also an invariant G_δ subspace of $\mathcal{X}_{\widehat{\mathcal{L}}} = \mathcal{X}_\mathcal{L} \times \mathcal{X}_{\widehat{\mathcal{L}}_\emptyset}$.

Besides the fact that they give rise to natural equivalence relations which generalize $=^+$ it is convenient to consider labeled $\mathcal{K}$ -structures for two more reasons. First, the logic action on $\text{Mod}_\omega(\widehat{\mathcal{K}})$ automatically has meager orbits. This is crucial if one wants to apply Theorem 6 on $\text{Mod}_\omega(\widehat{\mathcal{K}})$ in any meaningful way since the existence of a comeager orbit in any G -space X implies that X is generically 0-dimensional. Second, if A and B are labeled $\mathcal{K}$ -structures, then there is at most one embedding $A \rightarrow B$. As a consequence, every diagram of labeled $\mathcal{K}$ -structures and embeddings is automatically commutative. This labeling trick will allow us to work at the level of the embeddability relation (i.e., the information contained within the Becker digraph) without having to keep track of the composition relation between embeddings. We leave it to future work to develop a functorial version of the results in this paper, perhaps within the framework of Polish groupoids [Lup17].

The main examples of logic actions that we will consider below consist of *labeled BKL_n -structures*. Fix $n \geq 1$ and let $\mathcal{L}_n$ be the language which contains a collection

$\{s_i \mid i \in \omega\}$ of n -ary function symbols, and a collection $\{R_j \mid j \in \omega\}$ of n -ary relation symbols. A **BKL $_n$ -structure** is any $\mathcal{L}_n$ -structure A which satisfies the following list of axioms:

- (B1) The n -ary relations $(R_j)_{j \in \omega}$ partition A^n .
- (B2) For any n -tuple $a = (a_0, \dots, a_{n-1})$ satisfying R_j , we have $s_i(a) = a_0$ for all functions s_i with $i > j$.
- (B3) There is no substructure-independent set of size $(n+1)$: If $|B| = n+1$, then there is some $b \in B$ such that b is in the substructure generated by $B \setminus \{b\}$.
- (B4) A is locally finite: For any finite tuple a in A , the substructure generated by a is finite.

We will denote by $\mathcal{B}_n$ and by $\mathcal{B}_n^{\text{fin}}$ the classes of all BKL $_n$ -structures and all finite BKL $_n$ -structures, respectively. Note that the empty structure is a BKL $_n$ -structure.

The BKL $_n$ structures were introduced by Baldwin, Koerwein, and Laskowski in [BKL17], based on a similar construction by Laskowski and Shelah in [LS93]. In [BKL17], the classes $\mathcal{B}_n$ and $\mathcal{B}_n^{\text{fin}}$ were called $\hat{\mathbf{K}}^{n-1}$ and $\mathbf{K}_0^{n-1}$, respectively.

Our goal for the rest of this section is to prove the following theorem.

Theorem 9. *For every $n \geq 1$ the S_∞ -space $\text{Mod}_\omega(\widehat{\mathcal{B}}_n)$ is $(n-1)$ -dimensional and locally generically $(n-1)$ -dimensional.*

We will conclude with the proof of Theorem 9 at the end of the section after we collect the necessary lemmas. First we record the following corollary which is an immediate consequence of Theorem 6 and Theorem 9.

Corollary 10. *There is no $*$ -reduction from $(\text{Mod}_\omega(\widehat{\mathcal{B}}_m), \simeq_{\text{iso}})$ to $(\text{Mod}_\omega(\widehat{\mathcal{B}}_n), \simeq_{\text{iso}})$, when $m \neq n$.*

Recall from the beginning of this section the poset category $\Delta^{n-1} = (\mathcal{P}(\{0, \dots, n-1\}), \subseteq)$ where $[n] = \{0, 1, \dots, n-1\}$ is the terminal object. Consider also the full subcategory $\partial\Delta^{n-1}$ of Δ^{n-1} whose set of objects is $\Delta^{n-1} \setminus \{[n]\}$. We view each class of structures $\mathcal{K}$ as a category whose arrows are embeddings. An **n -cube in $\mathcal{K}$** is a functor $\mathcal{A}$ from Δ^{n-1} to $\mathcal{K}$, i.e., a pair $\mathcal{A} = ((A_\sigma)_\sigma, (f_\tau^\sigma)_{\sigma \subseteq \tau})$, where $(A_\sigma)_\sigma$ is a collection of structures from $\mathcal{K}$, indexed by elements σ of Δ^{n-1} ; together with embeddings $f_\tau^\sigma: A_\sigma \rightarrow A_\tau$, so that each f_σ^σ is the identity map, and for every $\sigma \subseteq \tau \subseteq \rho$ we have that $f_\rho^\tau \circ f_\tau^\sigma = f_\rho^\sigma$. Similarly, a **partial n -cube in $\mathcal{K}$** is a functor from $\partial\Delta^{n-1}$ to $\mathcal{K}$.

An n -cube in $\mathcal{K}$ is **disjoint** if $f_{\sigma \cup \tau}^\sigma(A_\sigma) \cap f_{\sigma \cup \tau}^\tau(A_\tau) = f_{\sigma \cap \tau}^{\sigma \cap \tau}(A_{\sigma \cap \tau})$ for all σ and τ . Similarly, a partial n -cube is disjoint if the same condition holds whenever $\sigma \cup \tau \subsetneq [n]$. The class $\mathcal{K}$ has **disjoint n -amalgamation** if every disjoint partial n -cube can be extended to a disjoint n -cube. Let $\mathcal{A} = ((A_\sigma), (f_\tau^\sigma))$ and $\mathcal{B} = ((B_\sigma), (g_\tau^\sigma))$ be disjoint n -cubes. A **disjoint embedding $\mathcal{A} \rightarrow \mathcal{B}$** is a family of embeddings $h_\sigma: A_\sigma \rightarrow B_\sigma$ such that for all $\sigma \subseteq \tau$, $h_\tau \circ f_\tau^\sigma = g_\tau^\sigma \circ h_\sigma$, and for all σ and τ ,

$$(h_{\sigma \cup \tau} \circ f_{\sigma \cup \tau}^\sigma)(A_\sigma) \cap g_{\sigma \cup \tau}^\tau(B_\tau) = (h_{\sigma \cap \tau} \circ f_{\sigma \cap \tau}^{\sigma \cap \tau})(A_{\sigma \cap \tau}).$$

An n -cube is **reducible** if there are $\sigma, \tau \in \Delta^{n-1}$ with $\sigma \not\subseteq \tau$ and some embedding $f: A_\sigma \rightarrow A_\tau$ so that $f_\rho^\tau \circ f = f_\rho^\sigma$ and $f \circ f_\sigma^{\rho'} = f_\tau^{\rho'}$, for all $\rho' \subseteq \sigma$ and $\tau \subseteq \rho$. Equivalently, an n -cube is reducible if and only if there are $\sigma, \tau \in \Delta^{n-1}$ with $\sigma \not\subseteq \tau$ such that the image of A_σ in $A_{[n]}$ is contained in the image of A_τ in $A_{[n]}$.

Lemma 11. (1) *The class $\mathcal{B}_n^{\text{fin}}$ has disjoint k -amalgamation for all $1 \leq k \leq n$.*
 (2) *If $\mathcal{A}$ is an n -cube in $\mathcal{B}_n$ such that $A_\emptyset$ is infinite, then $\mathcal{A}$ is reducible.*

Proof. [BKL17, Theorem 3.1.1] essentially proves (1) but we repeat the short proof for completeness. Suppose $((A_\sigma)_\sigma, (f_\tau^\sigma)_{\sigma \subseteq \tau})$ is a disjoint partial k -cube in $\mathcal{B}_n^{\text{fin}}$. Let $A_{[k]}$ be the colimit of this diagram in the category of sets. Explicitly, $A_{[k]}$ is the disjoint union of the A_σ for $\sigma \in \partial\Delta^{k-1}$, with $f_\tau^\sigma(a)$ and $f_\tau^{\tau'}(a)$ identified for all $\sigma \subseteq \tau, \tau'$ and $a \in A_\sigma$. For all $\sigma \in \partial\Delta^{k-1}$, we define $f_{[k]}^\sigma$ to be the canonical inclusion $A_\sigma \rightarrow A_{[k]}$. Then the disjointness conditions are satisfied.

It remains to make $A_{[k]}$ into a structure in $\mathcal{B}_n^{\text{fin}}$ by defining the functions s_i and relations R_j on all n -tuples. So let $a = (a_0, \dots, a_{n-1})$ be an n -tuple in $A_{[k]}$. If a is in the image of any A_σ for $\sigma \in \partial\Delta^{k-1}$, there is a unique way to define the s_i and R_j so that $f_{[k]}^\sigma$ is an embedding. So we may assume that a is not contained in the image of any A_σ . Enumerate $A_{[k]}$ as $c_0, \dots, c_N$, set $R_N(a)$, and define $s_i(a) = c_i$ for all $i \leq N$ and $s_i(a) = a_0$ for all $i > N$. Axioms (B1) and (B2) are satisfied by construction, and (B4) is trivially satisfied, since $A_{[k]}$ is finite. For (B3), suppose for contradiction that $B \subseteq A_{[k]}$ is substructure-independent, with $|B| = n + 1$. Let $B = \{b_0, \dots, b_n\}$, and for all $0 \leq i \leq n$, let $b^i = (b_0, \dots, b_{i-1}, b_{i+1}, \dots, b_n)$. If each b^i is contained in the image of some $A_{[k] \setminus \{j\}}$, then since $k \leq n$, there are $i \neq i'$ and j such that b^i and $b^{i'}$ are both contained in the image of $A_{[k] \setminus \{j\}}$, and hence all of B is contained in the image of $A_{[k] \setminus \{j\}}$. This contradicts the fact that $A_{[k] \setminus \{j\}} \in \mathcal{B}_n^{\text{fin}}$. So there is some i such that b^i is an n -tuple not contained in the image of any A_σ . Then $b_i = c_j$ for some $0 \leq j \leq N$, and $s_j(b^i) = b_i$, so B is not substructure-independent.

For (2), assume towards contradiction that there is an irreducible n -cube $\mathcal{A}$ in $\mathcal{B}_n$ such that each structure A_σ in $\mathcal{A}$ is infinite. Since $\mathcal{A}$ is not reducible, for each $0 \leq i < n$, the image of $A_{\{i\}}$ in $A_{[n]}$ is not contained in the image of $A_{[n] \setminus \{i\}}$ in $A_{[n]}$. Pick x_i in $f_{[n]}^{\{i\}}(A_{\{i\}}) \setminus f_{[n]}^{[n] \setminus \{i\}}(A_{[n] \setminus \{i\}})$ for all i . Now by axiom (B4), $\langle x_0, \dots, x_{n-1} \rangle$ is finite, so we can pick some $x_n \in f_{[n]}^\emptyset(A_\emptyset) \setminus \langle x_0, \dots, x_{n-1} \rangle$.

Then the set $\{x_0, \dots, x_n\}$ is substructure-independent. We have already seen that $x_n \notin \langle x_0, \dots, x_{n-1} \rangle$. And for all $0 \leq i < n$, $\langle x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n \rangle \subseteq f_{[n]}^{[n] \setminus \{i\}}(A_{[n] \setminus \{i\}})$, so $x_i \notin \langle x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n \rangle$. This contradicts (B3). $\square$

The statement of Lemma 11(2) also holds for $\widehat{\mathcal{B}}_n$, by taking reducts to $\mathcal{L}_n$. As a consequence, there are no irreducible n -cubes in $\widehat{\mathcal{B}}_n$ all of whose structures lie in $\text{Mod}_\omega(\widehat{\mathcal{B}}_n)$. Most of our remaining work is to prove that, in contrast, $\text{Mod}_\omega(\widehat{\mathcal{B}}_n)$ has many irreducible k -cubes whenever $k < n$. We begin by observing that $\mathcal{B}_n^{\text{fin}}$ is a Fraïssé class when $n \geq 2$ (this was also used in [BKL17]). Then we will find

an irreducible k cube in $\mathcal{B}_n$ for $1 \leq k < n$ such that every structure in the cube is isomorphic to the Fraïssé limit of $\mathcal{B}_n^{\text{fin}}$. Finally, we will use the existence of this k -cube and a Baire category argument to find k -cubes in any comeager subset of $\text{Mod}_\omega(\widehat{\mathcal{B}_n})$.

Lemma 12. (1) *Let A be a finite BKL_n -structure with domain $\{a_0, \dots, a_k\}$. Then there is a finite conjunction of atomic and negated atomic $\mathcal{L}_n$ -sentences with parameters from A , denoted $\theta_A(a_0, \dots, a_k)$, such that for any BKL_n -structure B , $B \models \theta_A(b_0, \dots, b_k)$ if and only if the map $a_i \mapsto b_i$ is an embedding $A \rightarrow B$.*

(2) *When $n \geq 2$, $\mathcal{B}_n^{\text{fin}}$ is a Fraïssé class.*

Proof. For any n -tuple c from A , let j_c be the unique natural number such that $A \models R_{j_c}(c)$, and let $d_c^i = s_i(c)$ for all $i \in \omega$. Then let $\theta_A(a_0, \dots, a_k)$ be the conjunction of the following formulas:

- $a_i \neq a_j$, for all $1 \leq i < j \leq k$.
- $R_{j_c}(c)$, for each n -tuple c from A .
- $s_i(c) = d_c^i$, for each n -tuple c from A and each $i \leq j_c$.

This finite piece of the diagram of A is sufficient to determine the rest of its diagram. Indeed, by axioms (B1) and (B2), $R_{j_c}(c)$ implies $\neg R_j(c)$ for all $j \neq j_c$ and $s_i(c) = c_0$ for all $i > j_c$. This establishes (1).

For (2), it is clear from the definition that $\mathcal{B}_n^{\text{fin}}$ has the hereditary property, and the amalgamation property is Lemma 11(1) in the case $k = 2$. The amalgamation property implies the joint embedding property, since $\mathcal{B}_n^{\text{fin}}$ includes the empty structure. Finally, the fact that $\mathcal{B}_n^{\text{fin}}$ is countable up to isomorphism follows from part (1), since there are only countably many $\mathcal{L}_n$ formulas. $\square$

We denote by $\mathcal{B}_n^*$ the class of all structures in $\mathcal{B}_n$ which are isomorphic to the Fraïssé limit of $\mathcal{B}_n^{\text{fin}}$. These are exactly the countably infinite BKL_n -structures which satisfy the following **extension axiom** for every pair of finite BKL_n -structures $A \subseteq B$ with domains $\{a_0, \dots, a_k\}$ and $\{a_0, \dots, a_k, a_{k+1}, \dots, a_\ell\}$, respectively:

$$\forall x_0, \dots, x_k (\theta_A(x_0, \dots, x_k) \rightarrow \exists x_{k+1}, \dots, x_\ell \theta_B(x_0, \dots, x_\ell)).$$

Lemma 13. *Suppose $1 \leq k < n$. Let $\mathcal{A} = ((A_\sigma), (f_\tau^\sigma))$ be a disjoint k -cube in $\mathcal{B}_n^{\text{fin}}$. For every $\rho \in \Delta^{k-1}$ and every embedding $h: A_\rho \rightarrow B$ in $\mathcal{B}_n^{\text{fin}}$, there is a disjoint k -cube $\mathcal{B} = ((B_\sigma), (g_\tau^\sigma))$ in $\mathcal{B}_n^{\text{fin}}$ and a disjoint embedding $(h_\sigma): \mathcal{A} \rightarrow \mathcal{B}$ such that $B_\rho = B$ and $h_\rho = h$, and further, for all τ such that $\rho \not\subseteq \tau$, we have $B_\tau = A_\tau$, $h_\tau: A_\tau \rightarrow B_\tau$ is the identity map, and $g_\tau^\sigma = f_\tau^\sigma$ for all $\sigma \subseteq \tau$.*

Proof. We define $\mathcal{B}$ and (h_σ) in stages, ensuring that the parts of $\mathcal{B}$ and (h_σ) that we have defined so far are functorial and satisfy the relevant disjointness conditions. Namely, for each τ , we must check:

- (a) For all $\sigma \subseteq \sigma' \subseteq \tau$, g_τ^τ is the identity, and $g_\tau^{\sigma'} \circ g_{\sigma'}^\sigma = g_\tau^\sigma$.
- (b) For all $\sigma \subseteq \tau$, $h_\tau \circ f_\tau^\sigma = g_\tau^\sigma \circ h_\sigma$.

- (c) For all $\sigma \cup \sigma' = \tau$, $g_\tau^\sigma(B_\sigma) \cap g_\tau^{\sigma'}(B_{\sigma'}) = g_\tau^{\sigma \cap \sigma'}(B_{\sigma \cap \sigma'})$.
 (d) For all $\sigma \cup \sigma' = \tau$, $(h_\tau \circ f_\tau^\sigma)(A_\sigma) \cap g_\tau^{\sigma'}(B_{\sigma'}) = (h_\tau \circ f_\tau^{\sigma \cap \sigma'})(A_{\sigma \cap \sigma'})$.

For τ such that $\rho \not\subseteq \tau$, set $B_\tau = A_\tau$, let $h_\tau: A_\tau \rightarrow B_\tau$ be the identity map, and let $g_\tau^\sigma = f_\tau^\sigma$ for $\sigma \subseteq \tau$, as required in the statement of the lemma. Conditions (a), (b), (c), and (d) are satisfied for all such τ ; (a) by functoriality of $\mathcal{A}$, (b) trivially, and (c) and (d) because $\mathcal{A}$ is disjoint. For τ with $\rho \subseteq \tau$, we proceed by induction on the size $m = |\tau \setminus \rho|$.

For $m = 0$, i.e., for $\tau = \rho$, let $h_\tau: A_\tau \rightarrow B_\tau$ be $h: A_\rho \rightarrow B$. Let g_τ^τ be the identity and $g_\tau^\sigma = h \circ f_\tau^\sigma$ for $\sigma \subsetneq \tau$. Condition (a) is trivially satisfied when $\sigma' = \tau$, and letting $\sigma \subseteq \sigma' \subsetneq \tau$, we have $\rho \not\subseteq \sigma'$, so $g_\tau^{\sigma'} \circ g_\tau^\sigma = h \circ f_\tau^{\sigma'} \circ f_\tau^\sigma = h \circ f_\tau^\sigma = g_\tau^\sigma$. Condition (b) is trivially satisfied when $\sigma = \tau$, so it suffices to consider $\sigma \subsetneq \tau$. Then $\rho \not\subseteq \sigma$, and we have $g_\tau^\sigma \circ h_\sigma = h_\tau \circ f_\tau^\sigma$, since h_σ is the identity map.

Condition (c) is trivial when $\sigma = \tau$ or $\sigma' = \tau$, so we may assume $\rho \not\subseteq \sigma$ and $\rho \not\subseteq \sigma'$. Then

$$g_\tau^\sigma(B_\sigma) \cap g_\tau^{\sigma'}(B_{\sigma'}) = h(f_\tau^\sigma(A_\sigma) \cap f_\tau^{\sigma'}(A_{\sigma'})) = h(f_\tau^{\sigma \cap \sigma'}(A_{\sigma \cap \sigma'})) = g_\tau^{\sigma \cap \sigma'}(B_{\sigma \cap \sigma'}).$$

Similarly, condition (d) is trivial when $\sigma' = \tau$, so we may assume $\rho \not\subseteq \sigma'$. Then

$$(h_\tau \circ f_\tau^\sigma)(A_\sigma) \cap g_\tau^{\sigma'}(B_{\sigma'}) = h(f_\tau^\sigma(A_\sigma) \cap f_\tau^{\sigma'}(A_{\sigma'})) = (h_\tau \circ f_\tau^{\sigma \cap \sigma'})(A_{\sigma \cap \sigma'}).$$

Assume now that we have defined B_τ , g_τ^σ , and h_τ for all τ with $|\tau \setminus \rho| < m$ and all $\sigma \subseteq \tau$, such that conditions (a), (b), (c), and (d) are satisfied. We consider τ with $|\tau \setminus \rho| = m$. Let $\{a_0, \dots, a_{m-1}\}$ be an enumeration of the set $\tau \setminus \rho$.

First we define a disjoint partial $(m+1)$ -cube, i.e. a functor $F: \partial\Delta^m \rightarrow \mathcal{B}_n^{\text{fin}}$. Let

$$F(\sigma) = \begin{cases} B_{\rho \cup \{a_i | i \in \sigma, i \neq m\}} & m \in \sigma \\ A_{\rho \cup \{a_i | i \in \sigma\}} & m \notin \sigma. \end{cases}$$

The image of the relation $\sigma \subseteq \sigma'$ under F is defined as the unique choice amongst $f_{F(\sigma')}^{F(\sigma)}$, $g_{F(\sigma')}^{F(\sigma)}$, or $h_{F(\sigma')} \circ f_{F(\sigma')}^{F(\sigma)}$ which makes sense. Functoriality of F follows from conditions (a) and (b) and functoriality of $\mathcal{A}$, and disjointness of the $(m+1)$ -cube follows from conditions (c) and (d) and disjointness of $\mathcal{A}$.

Since $m+1 \leq k+1 \leq n$, $\mathcal{B}_n^{\text{fin}}$ has disjoint $(m+1)$ -amalgamation by Lemma 11, so we can extend F to a disjoint $(m+1)$ -cube $G: \Delta^m \rightarrow \mathcal{B}_n^{\text{fin}}$. Set $B_\tau = G([m+1])$, and let h_τ be the image of the relation $[m] \subseteq [m+1]$ under G . For any subset σ of τ , we define g_τ^σ to be the appropriate embedding in the image of G , if $\rho \subseteq \sigma$, or $h_\tau \circ f_\tau^\sigma$ if not. Now conditions (a) and (b) follow from functoriality of G and induction, and conditions (c) and (d) follow from disjointness of G and induction. $\square$

Lemma 14. *For all $1 \leq k < n$, there is an irreducible k -cube in $\mathcal{B}_n^*$.*

Proof. We build a sequence of disjoint k -cubes in $\mathcal{B}_n^{\text{fin}}$, $\mathcal{A}^i = ((A_\sigma^i), (f_\tau^{\sigma,i}))$ for $i \in \omega$, together with disjoint embeddings $(h_\sigma^i): \mathcal{A}^i \rightarrow \mathcal{A}^{i+1}$ for all i . Then for each σ , we will let C_σ be the directed colimit of the A_σ^i , and for each $\sigma \subseteq \tau$, let g_τ^σ be the natural map $C_\sigma \rightarrow C_\tau$ induced by the maps $f_\tau^{\sigma,i}$ for all i . This defines a k -cube

$\mathcal{C} = ((C_\sigma), (g_\tau^\sigma))$. The idea is to carry out the Fraïssé construction simultaneously for each σ , so that each C_σ is in $\mathcal{B}_n^*$.

We begin with the empty k -cube $\mathcal{A}^0$, where A_σ^0 is the empty structure for all σ . Suppose we have defined $\mathcal{A}^i$, and we are given some ρ , some substructure $B \subseteq A_\rho^i$, and some embedding $e: B \rightarrow B'$ in $\mathcal{B}_n^{\text{fin}}$. By disjoint 2-amalgamation, there is a structure $A_\rho^{i+1} \in \mathcal{B}_n^{\text{fin}}$ and embeddings $h_\rho^i: A_\rho^i \rightarrow A_\rho^{i+1}$ and $e': B \rightarrow A_\rho^{i+1}$ such that $h_\rho^i \upharpoonright B = e' \circ e$ and $h_\rho^i(A_\rho) \cap e'(B) = h(B)$. By Lemma 13, we can extend A_ρ^{i+1} to a disjoint k -cube $\mathcal{A}^{i+1}$ and extend h_ρ^i to a disjoint embedding $(h_\sigma^i): \mathcal{A}^i \rightarrow \mathcal{A}^{i+1}$. Taking care in the course of the construction to handle every embedding from a substructure of some A_ρ^i in this way, we may ensure that each C_σ is in $\mathcal{B}_n^*$.

It remains to show that the k -cube $\mathcal{C}$ is irreducible. Suppose $\sigma \not\subseteq \tau$. We will show that $g_{[k]}^\sigma(C_\sigma) \not\subseteq g_{[k]}^\tau(C_\tau)$. Pick some i such that the extension $h_\sigma^i: A_\sigma^i \rightarrow A_\sigma^{i+1}$ is determined by some proper extension of a substructure of A_σ^i . Since A_σ^{i+1} is defined by disjoint 2-amalgamation, there is some $x_{i+1} \in A_\sigma^{i+1} \setminus h_\sigma^i(A_\sigma^i)$, and further, by Lemma 13, we know that $h_\tau^i: A_\tau^i \rightarrow A_\tau^{i+1}$ is the identity map, since $\sigma \not\subseteq \tau$.

Let $y_{i+1} = f_{[k]}^{i+1, \sigma}(x) \in A_{[k]}^{i+1}$. Then $y_{i+1} \notin f_{[k]}^{i+1, \tau}(A_\tau^{i+1})$. Indeed, if it were, then there would be some $z \in A_\tau^i$ such that $f_{[k]}^{i+1, \tau}(h_\tau^i(z)) = y_{i+1}$, since h_τ^i is the identity map. But then since (h_σ) is a disjoint embedding, x_{i+1} and z would both be in the image of some element in $A_{\sigma \cap \tau}^i$, and in particular x_{i+1} would be in the image of some element of A_σ^i , contradicting the choice of x_{i+1} .

Now for $j > i + 1$, define $x_{j+1} = h_\sigma^j(x_j) \in A_\sigma^{j+1}$ and $y_{j+1} = h_{[k]}^j(y_j) \in A_{[k]}^{j+1}$ by induction. Let x and y be the common images of the x_j and y_j in C_σ and $C_{[k]}$. Then $y = g_{[k]}^\sigma(x)$. We will show by induction on $j \geq i + 1$ that $y_j \notin f_{[k]}^{j, \tau}(A_\tau^j)$, from which it follows that $y \in g_{[k]}^\sigma(C_\sigma) \setminus g_{[k]}^\tau(C_\tau)$.

We have already established the base case, when $j = i + 1$. For the inductive step, if $y_{j+1} = h_{[k]}^j(y_j) \in f_{[k]}^{j+1, \tau}(A_\tau^{j+1})$, then since (h_σ) is a disjoint embedding, then $y_j \in f_{[k]}^{j, \tau}(A_\tau^j)$, contradicting the inductive hypothesis. $\square$

It will be convenient to encode an entire k -cube in a single structure. Fix $1 \leq k < n$, and consider the language

$$\mathcal{L}_n^k = \mathcal{L}_n \cup \{D_\sigma \mid \sigma \in \Delta^{k-1}\},$$

where the D_σ are unary relation symbols. Let $\mathcal{D}_n^k$ be the class of all $\mathcal{L}_n^k$ -structures M such that:

- (1) $D_{[k]}^M = M$.
- (2) Each D_σ^M is an $\mathcal{L}_n$ -substructure of $M \upharpoonright \mathcal{L}_n$.
- (3) $D_\sigma^M \subseteq D_\tau^M$ when $\sigma \subseteq \tau$.
- (4) $(D_\sigma^M)_\sigma$ together with the inclusion maps is an irreducible k -cube in $\mathcal{B}_n^*$.

Lemma 15. (1) $\text{Mod}_\omega(\mathcal{B}_n)$ is an invariant G_δ subspace of $\mathcal{X}_{\mathcal{L}_n}$.

- (2) A basis for the topology on $\text{Mod}_\omega(\mathcal{B}_n)$ is given by the collection of sets $[\theta_A(a_0, \dots, a_k)]$, where A is a finite BKL_n -structure with domain $\{a_0, \dots, a_k\} \subseteq \omega$.
- (3) $\text{Mod}_\omega(\mathcal{B}_n^*)$ is an invariant dense G_δ subspace of $\text{Mod}_\omega(\mathcal{B}_n)$.
- (4) $\text{Mod}_\omega(\widehat{\mathcal{B}_n^*})$ is an invariant dense G_δ subspace of $\text{Mod}_\omega(\widehat{\mathcal{B}_n})$.
- (5) For $1 \leq k < n$, $\text{Mod}_\omega(\mathcal{D}_n^k)$ is an invariant nonempty G_δ subspace of $\mathcal{X}_{\mathcal{L}_n^k}$.
- (6) For $1 \leq k < n$, $\text{Mod}_\omega(\widehat{\mathcal{D}_n^k})$ is an invariant nonempty G_δ subspace of $\mathcal{X}_{\widehat{\mathcal{L}_n^k}}$.

Proof. For (1), we give an axiomatization of $\mathcal{B}_n$ in $\mathcal{L}_{\omega_1, \omega}$:

- $\forall x_0, \dots, x_{n-1} \bigvee_{j \geq 0} R_j(x_0, \dots, x_{n-1})$;
- $\forall x_0, \dots, x_{n-1} \bigwedge_{0 \leq i < j} \neg(R_i(x_0, \dots, x_{n-1}) \wedge R_j(x_0, \dots, x_{n-1}))$;
- $\forall x_0, \dots, x_{n-1} \bigwedge_{0 \leq i < j} (R_i(x_0, \dots, x_{n-1}) \rightarrow s_j(x_0, \dots, x_{n-1}) = x_0)$;
- $\forall x_0, \dots, x_n \bigvee_{0 \leq k \leq n} \bigvee_{t(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n)} t(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n) = x_k$,
where t ranges over all terms in the variables $x_0, \dots, x_n$, with x_k omitted.
- $\forall x_0, \dots, x_k (\bigwedge_{1 \leq i < j \leq k} x_i \neq x_j \rightarrow \bigvee_{A=\{a_0, \dots, a_\ell\}} \exists x_{k+1}, \dots, x_\ell \theta_A(x_0, \dots, x_\ell))$,
where A ranges over all isomorphism types of structures in $\mathcal{B}_n^{\text{fin}}$ with a fixed enumeration $a_0, \dots, a_\ell$, where $k \leq \ell$. Note that this is only a countable disjunction, since each such A corresponds to the finitary $\mathcal{L}_n$ -formula θ_A .

From the form of the axiomatization it follows that $\text{Mod}_\omega(\mathcal{B}_n)$ is G_δ in $\mathcal{X}_{\mathcal{L}_n}$.

For (2), let U be a nonempty open subset of $\text{Mod}_\omega(\mathcal{B}_n)$. We may assume that $U = [\varphi(a)]$, where $\varphi(x)$ is a finite conjunction of atomic and negated atomic $\mathcal{L}_n$ -formulas. Let $A \in [\varphi(a)]$. By axiom (B4), A is locally finite. Let B be the substructure of A generated by the tuple a , and fix an enumeration of B by a tuple b . Then $M \in [\theta_B(b)] \subseteq [\varphi(a)]$.

For (3), the fact that $\text{Mod}_\omega(\mathcal{B}_n^*)$ is an invariant G_δ subspace of $\text{Mod}_\omega(\mathcal{B}_n)$ is clear from the axiomatization by extension axioms. Density follows from (2), since every finite BKL_n -structure with domain a subset of ω is a substructure of a structure in $\text{Mod}_\omega(\mathcal{B}_n^*)$. Then (4) follows immediately.

For (5), note that conditions (1)-(3) in the definition of $\mathcal{D}_n^k$ are closed, the condition that the k -cube is irreducible is open, and the condition that each D_σ^M is in $\mathcal{B}_n^*$ is G_δ , by relativizing the extension axioms for $\mathcal{B}_n^*$ to D_σ . To see that it is nonempty, let $((A_\sigma), (f_\tau^\sigma))$ be the irreducible k -cube in $\mathcal{B}_n^*$ constructed in Lemma 14. We may assume that $M = A_{[k]}$ has domain ω and define the relations D_σ^M as $f_{[k]}^\sigma(A_\sigma)$. Then (6) follows immediately. $\square$

If $M \in \text{Mod}_\omega(\widehat{\mathcal{D}_n^k})$, then for all $\sigma \in \Delta^{k-1}$, we view D_σ^M as a $\widehat{\mathcal{L}_n}$ -substructure of $M \upharpoonright \widehat{\mathcal{L}_n}$. Notice that the domain of D_σ^M is always an infinite subset of ω , since $M \upharpoonright \widehat{\mathcal{L}_n}$ is in $\text{Mod}_\omega(\widehat{\mathcal{B}_{n+1}^*})$. Let $e_\sigma^M: \omega \rightarrow \omega$ enumerate D_σ^M in increasing order. Then there is a unique structure $N \in \text{Mod}_\omega(\widehat{\mathcal{B}_n^*})$ such that $e_\sigma^M: N \rightarrow M$ defines an $\widehat{\mathcal{L}_n}$ -isomorphism between N and D_σ^M . The assignment $M \mapsto N$ defines a map:

$$\phi_\sigma: \text{Mod}_\omega(\widehat{\mathcal{D}_n^k}) \rightarrow \text{Mod}_\omega(\widehat{\mathcal{B}_n^*}).$$

Lemma 16. *For all $1 \leq k < n$, and for every $\sigma \in \Delta^{k-1}$, the map $\phi_\sigma: \text{Mod}_\omega(\widehat{\mathcal{D}}_n^k) \rightarrow \text{Mod}_\omega(\widehat{\mathcal{B}}_n^*)$ is continuous and open.*

Proof. For continuity, let U be an open set in $\text{Mod}_\omega(\widehat{\mathcal{B}}_n^*)$. We may assume U has the form $[\psi(a)]$, where ψ is a finite conjunction of atomic and negated atomic $\widehat{\mathcal{L}}_n$ -formulas and $a = (a_1, \dots, a_n)$ is a tuple from ω with $a_1 < \dots < a_n$. Let $M \in \phi_\sigma^{-1}(U)$. Then $M \models \psi(e_\sigma^M(a))$. Let χ be the conjunction of all formulas of the form $D_\sigma(b)$ and $\neg D_\sigma(b)$ which are satisfied in M , for $b \leq e_\sigma^M(a_n)$, and let $V = [\psi(e_\sigma^M(a)) \wedge \chi]$. Then V is an open set in $\text{Mod}_\omega(\widehat{\mathcal{D}}_n^k)$ and $M \in V \subseteq \phi_\sigma^{-1}(U)$. Indeed, if $M' \in V$, then $e_\sigma^{M'}$ agrees with e_σ^M on the initial segment of ω up to a_n , so $M' \models \psi(e_\sigma^{M'}(a))$, and $\phi_\sigma(M') \in U$.

For openness, let U be an open set in $\text{Mod}_\omega(\widehat{\mathcal{D}}_n^k)$. We may assume U has the form $[\psi(a)]$, where ψ is a finite conjunction of atomic and negated atomic $\widehat{\mathcal{L}}_n^k$ -formulas and $a = (a_1, \dots, a_n)$ is a tuple from ω with $a_1 < \dots < a_n$. Note also that ψ mentions only finitely many of the labeling predicates, $P_{i_1}, \dots, P_{i_\ell}$. Let $N \in \phi_\sigma(U)$, and let $M \in U$ with $\phi_\sigma(M) = N$. Let $b \in \omega$ be the largest element such that $e_\sigma(b)$ is in the tuple a in M , and let $B = \langle 0, \dots, b \rangle$ be the substructure of N generated by the initial segment of ω up to b . Enumerate B as $\{b_1, \dots, b_m\}$.

Recall that $\theta_B(b_1, \dots, b_m)$ determines the $\mathcal{L}_n$ -isomorphism type of B . Let χ be the conjunction of all formulas of the form $P_{i_j}(b_{j'})$ and $\neg P_{i_j}(b_{j'})$ which are satisfied in N , for $1 \leq j \leq \ell$ and $1 \leq j' \leq m$. And let $V = [\theta_B(b_1, \dots, b_m) \wedge \chi]$. Then V is an open set in $\text{Mod}_\omega(\widehat{\mathcal{B}}_n^*)$ and $N \in V \subseteq \phi_\sigma(U)$.

To see this, given $N' \in V$, we will find $M' \in U$ such that $\phi_\sigma(M') = N'$. We obtain M' by modifying our original structure M in stages.

- (1) Since $N' \models \theta_B(b_1, \dots, b_m)$, $M \models \theta_B(e_\sigma^M(b_1), \dots, e_\sigma^M(b_m))$. So e_σ^M restricts to an $\mathcal{L}_n$ -embedding $B \rightarrow D_\sigma^M$. Since $D_\sigma^M \upharpoonright \mathcal{L}_n$ and $N' \upharpoonright \mathcal{L}_n$ are both isomorphic to the Fraïssé limit of $\mathcal{B}_n^{\text{fin}}$, this embedding extends to an $\mathcal{L}_n$ -isomorphism $f: N' \rightarrow D_\sigma^M$ which agrees with e_σ^M on B . We will use this f to re-enumerate M so that ϕ_σ produces a structure which agrees with the target structure N' in the reduct to the language $\mathcal{L}_n$.
- (2) We view the composition $e_\sigma^M \circ f^{-1}: D_\sigma^M \rightarrow D_\sigma^M$ as a permutation of $D_\sigma^M \subseteq \omega$. Extending this permutation by the identity outside D_σ^M defines a permutation $g \in S_\infty$. Let $M_0 = g(M)$ (by the logic action of S_∞ on $\text{Mod}_\omega(\widehat{\mathcal{D}}_n^k)$). Note that g is the identity on the tuple a . Indeed, if $a_i \notin D_\sigma^M$, then $g(a_i) = a_i$ by definition. And if $a_i \in D_\sigma^M$, then there is some b' such that $e_\sigma^M(b') = a_i$. Since $a_i < a_n$, $b' < b$, so $b' \in B$. But since f agrees with e_σ^M on B , $g(a_i) = e_\sigma^M(f^{-1}(a_i)) = a_i$. It follows that $M_0 \in [\psi(a)] = U$. Further, since g fixes D_σ^M set-wise, we have $D_\sigma^{M_0} = D_\sigma^M$ as subsets of ω , and $e_\sigma^{M_0} = e_\sigma^M$. But $g^{-1} \circ e_\sigma^{M_0} = f$ is an $\mathcal{L}_n$ -isomorphism $N' \rightarrow D_\sigma^M$, so $e_\sigma^{M_0}$ is an $\mathcal{L}_n$ -isomorphism $N' \rightarrow D_\sigma^{M_0}$. It follows that $\phi_\sigma(M_0) \upharpoonright \mathcal{L}_n = N' \upharpoonright \mathcal{L}_n$.

- (3) It remains to improve this to equality of $\widehat{\mathcal{L}}_n$ -structures by relabeling M_0 by the predicates P_i . Let $M_1 = M_0 \upharpoonright \mathcal{L}_n^k$, dropping all the labels. We now define the $\widehat{\mathcal{L}}_n$ -structure M' expanding M_0 . Define the labels on the elements of $D_\sigma^{M_0}$ so that $e_\sigma^{M_0}$ is an $\widehat{\mathcal{L}}_n$ -isomorphism. Note that no two elements get exactly the same labels, since this is true in N' . Now assign labels arbitrarily to the remaining elements of M_1 , only making sure that no two elements get exactly the same labels and that for each element a_i of the tuple a , the new label of a_i agrees with its old label on the predicates $P_{i_1}, \dots, P_{i_\ell}$. In fact, this is already the case when a_i is in $D_\sigma^{M_0}$, since $N' \text{ in } V$ and the information about the predicates $P_{i_1}, \dots, P_{i_\ell}$ is included in χ . We have now arranged that $\phi_\sigma(M') = N'$.
- (4) It remains to check that $M' \in U = [\psi(a)]$. This follows from the fact that $M_0 \in [\psi(a)]$, and M' agrees with M_0 on all atomic formulas in $\mathcal{L}_n^k$, as well as the values of all the predicates mentioned in ψ on the tuple a . $\square$

We now have all the ingredients necessary to prove Theorem 9.

Proof of Theorem 9. First we observe that having a k -cube in $\widehat{\mathcal{B}}_n$ with structures lying in some invariant subset C of $\text{Mod}_\omega(\widehat{\mathcal{B}}_n)$ is the same as having an embedding of the combinatorial k -cube in the Becker graphs $\mathcal{B}(C/S_\infty)$. This follows from Proposition 7 and the fact that we are working with labeled structures; see the discussion after the definition of labeled $\mathcal{K}$ -structures. By Lemma 11(2) it follows that the dimension of $\text{Mod}_\omega(\widehat{\mathcal{B}}_n)$ is at most $(n-1)$.

When $n = 1$, it is immediate that $\text{Mod}_\omega(\widehat{\mathcal{B}}_n)$ is locally generically $(n-1)$ dimensional (this just means that every non-meager invariant set with the Baire property is nonempty). So in the remainder of the proof, we assume $n > 1$.

First, we show that $\text{Mod}_\omega(\widehat{\mathcal{B}}_n)$ is generically $(n-1)$ -dimensional. Let $C \subseteq \text{Mod}_\omega(\widehat{\mathcal{B}}_n)$ be a comeager set. Since $\text{Mod}_\omega(\widehat{\mathcal{B}}_n^*)$ is dense G_δ in $\text{Mod}_\omega(\widehat{\mathcal{B}}_n)$, the restriction $C^* = C \cap \text{Mod}_\omega(\widehat{\mathcal{B}}_n^*)$ is comeager in $\text{Mod}_\omega(\widehat{\mathcal{B}}_n)$. For each $\sigma \in \Delta^{k-1}$, let $C_\sigma^* = \phi_\sigma^{-1}(C^*) \subseteq \text{Mod}_\omega(\widehat{\mathcal{D}}_n^k)$. Continuous open maps are category preserving, so each C_σ^* is comeager in $\text{Mod}_\omega(\widehat{\mathcal{D}}_n^k)$. Since this space is nonempty and Polish, $\bigcap_{\sigma \in \Delta^{k-1}} C_\sigma^*$ is nonempty. Let M be a structure in the intersection. We define a k -cube $\mathcal{A}$ by setting $A_\sigma = \phi_\sigma(M)$. In particular, $A_{[k]} = M \upharpoonright \widehat{\mathcal{L}}_n$. To define the connecting maps, let $f_{[k]}^\sigma = e_\sigma^M : A_\sigma \rightarrow A_{[k]}$. Now for any $\sigma \subseteq \tau \subseteq [k]$, the map $f_{[k]}^\sigma$ factors through the map $f_{[k]}^\tau$ by a unique map f_σ^τ . By construction, $\mathcal{A}$ is irreducible and $A_\sigma \in C^* \subseteq C$.

Finally, we claim that $\text{Mod}_\omega(\widehat{\mathcal{B}}_n)$ is generically ergodic when $n > 1$, so that the fact that it is generically $(n-1)$ -dimensional immediately implies that it is locally generically $(n-1)$ -dimensional. By [Gao08, Proposition 6.1.9], it suffices to show that there is a structure in $\text{Mod}_\omega(\widehat{\mathcal{B}}_n)$ with a dense orbit.

Consider the $\widehat{\mathcal{L}}_n$ -theory which axiomatizes labeled BKL_n -structures (axioms (A1), (B1), (B2), (B3), and (B4)) and contains the additional axiom:

$$\bigwedge_{A=\{a_0, \dots, a_k\}} \bigwedge_{(T_0, F_0), \dots, (T_k, F_k)} \exists x_0, \dots, x_k \left(\theta_A(x_0, \dots, x_k) \wedge \bigwedge_{i \leq k} \left(\bigwedge_{j \in T_i} P_j(x_i) \wedge \bigwedge_{j \in F_i} \neg P_j(x_i) \right) \right),$$

where $A = \{a_0, \dots, a_k\}$ varies over all isomorphism types of structures in $\mathcal{B}_n^{\text{fin}}$, and for each i , (T_i, F_i) is a pair of disjoint finite subsets of ω . It is easy to see that there are structures in $\text{Mod}_\omega(\widehat{\mathcal{B}}_n)$ which satisfy this theory and that every such structure has a dense orbit under the S_∞ action. $\square$

Remark 17. Keeping track of the properties of $\mathcal{B}_n$ used in the proof of Theorem 9, we see that for any class of structures $\mathcal{K}$ satisfying the following hypotheses, the S_∞ -space $\text{Mod}_\omega(\widehat{\mathcal{K}})$ is locally generically $(\geq (n-1))$ -dimensional:

- (1) $\mathcal{K}$ is closed under substructure.
- (2) $\text{Mod}_\omega(\mathcal{K})$ is a G_δ -subspace of $\mathcal{X}_{\mathcal{L}}$.
- (3) $\mathcal{K}^{\text{fin}}$ is a Fraïssé class, whose Fraïssé limit is in $\mathcal{K}$.
- (4) $\mathcal{K}^{\text{fin}}$ has disjoint k -amalgamation for all $1 \leq k \leq n$.
- (5) $\mathcal{K}^{\text{fin}}$ is *separable*: this is the condition in Lemma 12(1).

It follows, for example, that if $\mathcal{K}$ is the class $\mathcal{S}$ of all sets or $\mathcal{K}$ is the class of all graphs, then $\text{Mod}_\omega(\widehat{\mathcal{K}})$ is locally generically ∞ -dimensional. In both cases Theorem 6 implies that $\simeq_{\text{iso}}$ on $\text{Mod}_\omega(\widehat{\mathcal{K}})$ is incomparable to $\simeq_{\text{iso}}$ on $\text{Mod}_\omega(\widehat{\mathcal{B}}_n)$, for all $n > 0$. In particular, $=^+$ is incomparable to $(\text{Mod}_\omega(\widehat{\mathcal{B}}_n), \simeq_{\text{iso}})$, for all $n > 0$; see Example 8.

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