

EXISTENCE OF OPTIMIZERS IN A SOBOLEV INEQUALITY FOR VECTOR FIELDS

RUPERT L. FRANK AND MICHAEL LOSS

ABSTRACT. We consider the minimization problem corresponding to a Sobolev inequality for vector fields and show that minimizing sequences are relatively compact up to the symmetries of the problem. In particular, there is a minimizer. An ingredient in our proof is a version of the Rellich–Kondrachov compactness theorem for sequences satisfying a nonlinear constraint.

1. INTRODUCTION AND MAIN RESULT

Motivated in part by the problem of stability of matter in magnetic fields and the influence of zero modes on this [12, 26], in our previous paper [11] we studied the problem of the minimal $3/2$ -norm of magnetic fields supporting a zero mode. A zero mode is a non-trivial solution to the spinor equation

$$\sigma \cdot (-i\nabla)\psi = \sigma \cdot A\psi \quad \text{in } \mathbb{R}^3,$$

where $\psi : \mathbb{R}^3 \rightarrow \mathbb{C}^2$ is a spinor and $A : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is the vector potential with magnetic field $B = \nabla \wedge A$, the curl of A . Moreover, σ denotes the vector of Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The canonical quantity that determines whether a solution is possible is $\|B\|_{3/2}$ and the aim of the work in [11] is to determine the optimal value for this quantity for a zero mode to exist. One approach explained in [11] is to consider the chain of inequalities

$$C_s \|\psi\|_3 \leq \|\sigma \cdot (-i\nabla)\psi\|_{3/2} = \|\sigma \cdot A\psi\|_{3/2} = \| |A| \psi \|_{3/2} \leq \|A\|_3 \|\psi\|_3,$$

which shows that necessarily $\|A\|_3 \geq C_s$, the best constant in the inequality

$$\|\sigma \cdot (-i\nabla)\psi\|_{3/2} \geq C_s \|\psi\|_3 \tag{1}$$

The validity of this inequality can be shown, for instance, using the Hardy–Littlewood–Sobolev inequality [23, Theorem 4.3], although the value of the constant C_s is not known.

© 2021 by the authors. This paper may be reproduced, in its entirety, for non-commercial purposes.

Partial support through US National Science Foundation grants DMS-1363432 and DMS-1954995 (R.L.F.) and DMS-1856645 (M.L.) is acknowledged.

Loss and Yau [26] found examples of zero modes and it is of interest in this connection that the spinor [26]

$$\psi = \frac{I + i\sigma \cdot x}{(1 + |x|^2)^{3/2}} \eta, \quad (2)$$

where $\eta \in \mathbb{C}^2$ is any constant spinor, is a solution of the Euler-Lagrange equation associated with the inequality (1). The existence of an optimizer for (1) follows using ideas from concentration compactness and, while not completely standard, does not pose real problems. A solution of our problem is furnished by the inequality

$$C_v \|A\|_3 \leq \|\nabla \wedge A\|_{3/2} \quad (3)$$

valid for all A that satisfy the condition $\nabla \cdot (|A|A) = 0$. The reason for this condition on the divergence will be explained below. As a consequence, $C_s C_v \leq \|B\|_{3/2}$ is a necessary condition for the existence of a zero mode.

Of interest is that the field

$$A(x) = \frac{3}{(1 + |x|^2)^2} [(1 - |x|^2)w + 2x \cdot wx + 2w \wedge x] \quad (4)$$

where $w \in \mathbb{R}^3$ is a constant vector, satisfies $\nabla \cdot (|A|A) = 0$ and is a solution of the formal Euler-Lagrange equation associated with the best constant in (3). One should also note that (2) and (4) satisfy the zero mode equation if we require that $w = \langle \eta, \sigma \eta \rangle$ (see [26]). These considerations lend credence to the conjecture that the spinor (2) together with the vector field (4) are optimizers for their respective Sobolev inequalities and, moreover, yield the minimal value for $\|B\|_{3/2}$.

This motivates the study of the optimal constants for the two Sobolev inequalities (1) and (3) in more detail and, as a first step, we shall prove the existence of optimizers. One should emphasize that with no symmetry results available, it is far from clear how to show that (2) and (4) are indeed optimizers of their respective inequalities. We leave this to future investigations.

There is an abundance of results on the existence of minimizers in optimization problems related to Sobolev inequalities and, as mentioned before, the existence of minimizers in our Sobolev inequality (1) for spinor fields can be shown by an adaptation of these existing techniques. Perhaps surprisingly, the case of vector fields is significantly harder and the combination of quasilinearity, nonlocality and vectorvaluedness take it outside of the scope of standard methods. While we carry out our analysis only for inequality (3), we believe that the arguments are more general and can be useful in related problems.

Let us be more specific. We consider

$$\mathcal{Y} := \{A \in L^3(\mathbb{R}^3, \mathbb{R}^3) : \nabla \wedge A \in L^{3/2}(\mathbb{R}^3)\}, \quad (5)$$

endowed with the norm

$$\|A\|_{\mathcal{Y}} := \|A\|_3 + \|\nabla \wedge A\|_{3/2}.$$

Here $\nabla \wedge \cdot$ denotes the curl operator, understood in distributional sense. We will also make use of the seminorm

$$\|A\|_3 := \inf_{\varphi \in \dot{W}^{1,3}(\mathbb{R}^3)} \|A - \nabla \varphi\|_3,$$

where $\dot{W}^{1,3}(\mathbb{R}^3)$ denotes the space of all real functions $\varphi \in L^1_{\text{loc}}(\mathbb{R}^3)$ such that $\nabla \varphi \in L^3(\mathbb{R}^3, \mathbb{R}^3)$. Our interest lies in the minimization problem

$$S := \inf_{0 \neq A \in \mathcal{Y}} \frac{\|\nabla \wedge A\|_{3/2}^{3/2}}{\|A\|_3^{3/2}}.$$

Note that both numerator and denominator in this minimization problem vanishes precisely when A is a gradient field. The validity of the Sobolev inequality, that is, the fact that $S > 0$, follows from the Helmholtz decomposition in $L^{3/2}(\mathbb{R}^3, \mathbb{R}^3)$ and the Sobolev inequality in $\dot{W}^{1,3/2}(\mathbb{R}^3, \mathbb{R}^3)$; see Lemma 4.

Our main result is that the infimum defining S is attained. In fact, we prove the stronger result that all suitably normalized minimizing sequences for S are relatively compact in \mathcal{Y} , up to symmetries. The non-compact symmetries of the minimization problem S are the translation and dilation symmetries, as well as a gauge symmetry, to be discussed momentarily. The problem has also a remarkable conformal symmetry, but this will not play a major role in our arguments. As shown in Lemma 5, the gauge invariance can be broken by choosing for a given $A \in L^3(\mathbb{R}^3, \mathbb{R}^3)$ an $A' \in L^3(\mathbb{R}^3, \mathbb{R}^3)$ with $\nabla \cdot (|A'|A') = 0$, where $\nabla \cdot$ denotes the divergence operator, understood in distributional sense. Then $\|A'\|_3 = \|A\|_3$ and for our minimization problem it is natural to work in this gauge.

The following is our main result.

Theorem 1. *Let $(A_n) \subset \mathcal{Y}$ be a minimizing sequence for S , normalized such that $\nabla \cdot (|A_n|A_n) = 0$ and $\|A_n\|_3 = 1$. Then there are $\lambda_n \in (0, \infty)$ and $a_n \in \mathbb{R}^3$ such that, along a subsequence,*

$$\lambda_n A_n(\lambda_n(x - a_n))$$

converges in \mathcal{Y} to a minimizer for S . In particular, there is a minimizer for S .

Using the tools developed in the proof of Theorem 1 we will also be able to prove existence of an optimizer for the problem that motivated our study, namely that of finding the minimal $L^{3/2}$ -norm of magnetic fields admitting a zero mode. We define

$$\Sigma := \inf \left\{ \|\nabla \wedge A\|_{3/2} : A \in \mathcal{Y} \text{ and } \exists 0 \neq \psi \in L^3(\mathbb{R}^3, \mathbb{C}^2) \text{ with } \sigma \cdot (-i\nabla - A)\psi = 0 \right\}$$

The equation $\sigma \cdot (-i\nabla - A)\psi = 0$ is understood in distributional sense in \mathbb{R}^3 and, as shown in [11], the requirement $\psi \in L^3$ can be replaced by $\psi \in L^p$ for any $3/2 < p < \infty$. We choose the L^3 norm here because it appears naturally in the proof.

Theorem 2. *Let $(A_n) \subset \mathcal{Y}$ be a minimizing sequence for Σ , normalized such that $\nabla \cdot (|A_n|A_n) = 0$. Then there are $\lambda_n \in (0, \infty)$ and $a_n \in \mathbb{R}^3$ such that, along a subsequence,*

$$\lambda_n A_n(\lambda_n(x - a_n))$$

converges in \mathcal{Y} to a minimizer \tilde{A} for Σ . In particular, there is a minimizer for Σ . Moreover, if $(\psi_n) \subset \dot{W}^{1,3/2}(\mathbb{R}^3, \mathbb{C}^2)$ is a corresponding sequence of zero modes, normalized such that $\|\psi_n\|_3 = 1$, then

$$\lambda_n \psi_n(\lambda_n(x - a_n))$$

converges weakly in $\dot{W}^{1,3/2}(\mathbb{R}^3, \mathbb{C}^2)$ to $\tilde{\psi} \neq 0$ and one has $\sigma \cdot (-i\nabla + \tilde{A})\tilde{\psi} = 0$.

We return now to the discussion of Theorem 1. The relative compactness statement in this theorem is the analogue of a theorem of Lions [24] concerning the Sobolev inequality in $\dot{W}^{1,p}(\mathbb{R}^d)$ for $1 < p < d$. In this case, the existence of minimizers and, indeed, the identification of minimizers and the optimal constants are due to Rodemich [27], Aubin [2] and Talenti [30].

As we will argue now, in our setting there are significant differences to the scalar setting of $\dot{W}^{1,p}(\mathbb{R}^d)$ and the difficulties to be overcome are the combination of quasilinearity (coming from the power $3/2$ of the derivative term), nonlocality (coming from the seminorm $\|\cdot\|_3$) and vector-valuedness (of the objects to be optimized over). While existing methods can deal with any one or two of these difficulties, it is not clear to us whether they can deal with all three of them.

Let us be more specific concerning the various standard methods to prove existence of minimizers. In each case we give exemplary references, without being exhaustive. Symmetrization-based methods [29, 21] seem to be unable to deal with the vector-valuedness and, in particular, the curl operator and the divergence constraint. Lieb's method of the missing mass [22, 5, 6, 12] uses typically the Hilbert space nature of the underlying space, or, if not, needs some additional ingredients [22, 10]. In the framework of Lions' concentration compactness principle [24, 25], it is unclear how to deal with the nonlocal and nonlinear dependence of the term $\nabla\varphi$ on A in the definition of the seminorm $\|A\|_3$. Finally, Yamabe's method of subcritical approximations [33, 31, 1] relies on an L^∞ -boundedness result, whose analogue for the curl system is not clear to us.

While not obvious to us, it might very well be possible that one of these methods can be used to prove Theorem 1. We, however, choose a different approach. In a first step we show that any minimizing sequence has a subsequence which, up to these symmetries, has a nontrivial weak limit, and in a second step we show that this subsequence, in fact, converges strongly. The first step relies on an improved Sobolev inequality, where on the right side $\|\nabla \wedge A\|_{3/2}$ is replaced by its geometric mean with a certain Besov-space norm of $\nabla \wedge A$. In the setting of $\dot{H}^s(\mathbb{R}^d)$ such inequalities appear

in [16] and build the basis of profile decompositions [15]. The improved Sobolev inequality implies rather directly the existence of a nontrivial weak limit point up to symmetries.

In a second step we slightly alter the minimizing sequence using Ekeland's variational principle in order to deduce that the minimizing sequence satisfies the Euler–Lagrange equation with a small inhomogeneity. From this equation we can deduce that the curls of the elements of the minimizing sequence converge in L^p_{loc} for any $p < 3/2$. Together with a nonlinear Rellich–Kondrachov theorem, discussed below, this allows us to conclude that the weak limit of the minimizing sequence satisfies the Euler–Lagrange equation and, consequently, that the minimizing sequence converges strongly in \mathcal{Y} . The basic idea behind this second step can be traced back to [13] where the idea is briefly sketched on p. 448 in the case of scalar functions and in an unconstrained variational principle. A related argument appears also in [14]. The vector-valuedness and justifying almost everywhere convergence, however, require several new ingredients compared to the scalar case.

Probably the most significant among these is a nonlinear Rellich–Kondrachov theorem, which says that if $\nabla \wedge A_n \rightharpoonup \nabla \wedge A$ in $L^{3/2}$ and if $\nabla \cdot (|A_n|A_n) = \nabla \cdot (|A|A) = 0$, then $A_n \rightarrow A$ in L^q_{loc} for any $q < 3$. In particular, a subsequence converges almost everywhere. The importance here is that the constraint $\nabla \cdot (|A_n|A_n) = 0$ is nonlinear and, indeed, if it would be replaced by the linear constraint $\nabla \cdot A_n$ the conclusion would follow from the standard form of the Rellich–Kondrachov theorem. Our proof of the nonlinear variant is surprisingly complicated and makes use of deep results by Iwaniec [18] on solutions of quasilinear equations. We emphasize that the almost everywhere convergence of minimizing sequences is an ingredient in essentially every proof of existence of minimizers of Sobolev-type inequalities. Therefore, even if a different proof of existence of a minimizer could be found, our nonlinear Rellich–Kondrachov theorem is likely to play a fundamental role in such a proof as well.

As we mentioned before, we explain our technique in the context of one specific inequality, which we find interesting in view of our work in [11]. The methods, however, are much more general and are applicable in a variety of settings. Some immediate extensions concern the generalization of the vector field inequality to arbitrary dimensions $d \geq 3$,

$$\inf_{\varphi \in \dot{W}^{1,d}} \|A - \nabla \varphi\|_d \lesssim \|\nabla \wedge A\|_{d/2}.$$

Also, the case of general exponents $1 < p < d$ on the right side and $q = dp/(d-p)$ on the left side should be doable, after some changes in the proof of the nonlinear Rellich–Kondrachov theorem, which currently uses the conformal invariance in the $q = d/2$ case. One could consider this problem in a wider context by considering k -forms ω on

\mathbb{R}^d or on any Riemannian manifold. Then the Sobolev-type inequality is of the form

$$\inf \|\omega - d\varphi\|_q \lesssim \|d\omega\|_p$$

where the infimum is taken over $k - 1$ forms φ and $q = dp/(d - p)$. The conformal invariant case corresponds to $p = \frac{d}{k+1}$ and $q = \frac{d}{k}$. We stay with the case $d = 3$ and $k = 1$ since the mathematical issues that concern us are of considerable difficulty and these difficulties would only be obfuscated in treating the general case.

We add that the method works in the case of spinor fields, mentioned at the beginning of this introduction. In this case, however, there is no analogue of the nonlinear divergence constraint and therefore the standard form of the Rellich–Kondrachov theorem suffices. Moreover, in the absence of this constraint Lions’ concentration compactness method is applicable and gives the relative compactness of minimizing sequences in a standard way. If one insists on using the method in the present paper, there is only a minor change in the application of Ekeland’s theorem because the underlying Banach space is complex. We briefly comment on this in Subsection 7.2.

2. SOME PRELIMINARY RESULTS

In this section we collect some simple results that we will use repeatedly in this paper. We begin with the Helmholtz decomposition in L^p . While this is valid for any $1 < p < \infty$, we only state it for $p = 3/2$, the only case that we will be using.

Lemma 3. *Let $A \in \mathcal{Y}$. Then there are $\tilde{A} \in \dot{W}^{1,3/2}(\mathbb{R}^3, \mathbb{R}^3)$ and $\varphi \in \dot{W}^{1,3}(\mathbb{R}^3)$ with*

$$A = \tilde{A} + \nabla\varphi \quad \text{and} \quad \nabla \cdot \tilde{A} = 0.$$

This decomposition is unique, up to adding a constant to φ . Moreover,

$$\|\nabla \otimes \tilde{A}\|_{3/2} \lesssim \|\nabla \wedge A\|_{3/2} \quad \text{and} \quad \|\nabla\varphi\|_3 \lesssim \|A\|_3.$$

We include the proof, since we will need the explicit construction later on.

Proof. Let

$$\varphi(x) := \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{A(y) \cdot (x - y)}{|x - y|^3} dy.$$

It follows from an endpoint case of the Hardy–Littlewood–Sobolev inequality [28] that $\varphi \in BMO(\mathbb{R}^3)$ with $\|\varphi\|_{BMO} \lesssim \|A\|_{L^3}$. Moreover, by Calderón–Zygmund theory we know that φ is weakly differentiable with $\|\nabla\varphi\|_{L^3} \lesssim \|A\|_{L^3}$. Moreover, in the sense of distributions,

$$-\Delta\varphi = -\nabla \cdot A.$$

Let $\tilde{A} := A - \nabla\varphi$ and $B := \nabla \wedge A$. Then, in the sense of distributions,

$$\nabla \wedge \tilde{A} = B \quad \text{and} \quad \nabla \cdot \tilde{A} = 0.$$

We conclude that

$$\tilde{A}(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{B(y) \wedge (x - y)}{|x - y|^3} dy. \quad (6)$$

Indeed, the left side belongs to L^3 by the above discussion and the right side belongs to L^3 by the Hardy–Littlewood–Sobolev inequality and the assumption $B \in L^{3/2}$. Since the curl and the divergence of the right side, in the sense of distributions, coincide with those of the left side, we conclude by a simple Liouville theorem that both sides are equal. Note that this argument also gives the uniqueness of the decomposition.

Using Calderón–Zygmund theory one deduces that \tilde{A} is weakly differentiable with $\|\nabla \otimes \tilde{A}\|_{L^{3/2}} \lesssim \|\nabla \wedge A\|_{L^{3/2}}$, as claimed. \square

As an application of the Helmholtz decomposition in $L^{3/2}$, we obtain the validity of the Sobolev inequality.

Lemma 4. $S > 0$

Proof. Given $A \in \mathcal{Y}$, let \tilde{A} and φ be as in Lemma 3. The usual Sobolev inequality in $\dot{W}^{1,3/2}$ implies

$$\begin{aligned} \|\nabla \wedge A\|_{3/2} &= \|\nabla \wedge \tilde{A}\|_{3/2} \gtrsim \|\nabla \otimes \tilde{A}\|_{3/2} \gtrsim \|\tilde{A}\|_3 \\ &= \|A - \nabla \varphi\|_3 \geq \|A\|_3. \end{aligned}$$

Thus, $S > 0$. \square

The next lemma discusses the natural choice of the gauge in our problem.

Lemma 5. *For any $A \in L^3(\mathbb{R}^3, \mathbb{R}^3)$ there is a $\varphi_0 \in \dot{W}^{1,3}(\mathbb{R}^3)$, unique up to an additive constant, such that $\|A - \nabla \varphi_0\|_3 = \|A\|_3$. Moreover, $\nabla \cdot (|A - \nabla \varphi_0|(A - \nabla \varphi_0)) = 0$. Conversely, if $A' \in L^3(\mathbb{R}^3, \mathbb{R}^3)$ satisfies $\nabla \cdot (|A'|A') = 0$ in \mathbb{R}^3 , then $\|A'\|_3 = \|A'\|_3$.*

Proof. The existence of φ_0 follows easily from the fact that $\|\cdot\|_3$ is convex and that

$$\mathcal{M} := \{A \in L^3(\mathbb{R}^3, \mathbb{R}^3) : \nabla \wedge A = 0\} \quad (7)$$

is closed in $L^3(\mathbb{R}^3, \mathbb{R}^3)$. The latter follows from standard properties of the distributional curl. Note also that $\nabla \varphi \in \mathcal{M}$ for all $\varphi \in \dot{W}^{1,3}(\mathbb{R}^3)$. Uniqueness of $\nabla \varphi_0$ follows from the strict convexity of $\|\cdot\|_3$, and the equation $\nabla \cdot (|A - \nabla \varphi_0|(A - \nabla \varphi_0)) = 0$ arises as the Euler–Lagrange equation of the minimization problem.

Now assume that $A' \in L^3(\mathbb{R}^3, \mathbb{R}^3)$ satisfies $\nabla \cdot (|A'|A') = 0$ in \mathbb{R}^3 and let $\varphi \in \dot{W}^{1,3}(\mathbb{R}^3)$. Applying the inequality $f(1) \geq f(0) + f'(0)$ for any convex function on $[0, 1]$ to $f(t) := \|A' - \nabla \varphi\|_3$, we obtain $\|A' - \nabla \varphi\|_3 \geq \|A'\|_3$. Taking the infimum over φ gives $\|A'\|_3 \geq \|A'\|_3$, and the reverse inequality is trivial. \square

3. A NONLINEAR RELICH–KONDRACHOV LEMMA

In this section we present the technical main result of our paper. To motivate it, we note that if $\tilde{A}_n, \tilde{A} \in \mathcal{Y}$ with $\nabla \wedge \tilde{A}_n \rightharpoonup \nabla \wedge \tilde{A}$ in $L^{3/2}(\mathbb{R}^3, \mathbb{R}^3)$ and $\nabla \cdot \tilde{A}_n = \nabla \cdot \tilde{A} = 0$, then $\tilde{A}_n \rightarrow \tilde{A}$ in $L^p_{\text{loc}}(\mathbb{R}^3, \mathbb{R}^3)$ for any $p < 3$. This is a consequence of the usual Rellich–Kondrachov lemma for scalar functions, applied to each component of \tilde{A} , since by the Helmholtz decomposition in $L^{3/2}$ (Lemma 3) the boundedness of $\nabla \wedge \tilde{A}_n$ in $L^{3/2}$ together with $\nabla \cdot \tilde{A}_n = 0$ implies boundedness of $\nabla \otimes \tilde{A}_n$ in $L^{3/2}$.

The following theorem says that the same conclusion remains true if the linear constraint $\nabla \cdot \tilde{A}_n = 0$ is replaced by a nonlinear constraint $\nabla \cdot (|A_n|A_n) = 0$. Our proof of this result is rather involved and takes up this and the following section.

Theorem 6. *Let $A_n, A \in \mathcal{Y}$ with $\nabla \wedge A_n \rightharpoonup \nabla \wedge A$ in $L^{3/2}(\mathbb{R}^3, \mathbb{R}^3)$ and $\nabla \cdot (|A_n|A_n) = \nabla \cdot (|A|A) = 0$. Then $A_n \rightarrow A$ in $L^q_{\text{loc}}(\mathbb{R}^3, \mathbb{R}^3)$ for any $q < 3$.*

While a direct proof of this result on \mathbb{R}^3 should be possible, we use the conformal invariance of the relevant norms and prove the corresponding result on \mathbb{S}^3 . We denote by $L^p(\Lambda^k \mathbb{S}^3)$ the space of p -integrable k -forms on \mathbb{S}^3 .

Theorem 7. *Let $\alpha_n, \alpha \in L^3(\Lambda^1 \mathbb{S}^3)$ with $d\alpha_n \rightharpoonup d\alpha$ in $L^{3/2}(\Lambda^2 \mathbb{S}^3)$ and $d^*(|\alpha_n|\alpha_n) = d^*(|\alpha|\alpha) = 0$. Then $\alpha_n \rightarrow \alpha$ in $L^q(\Lambda^1 \mathbb{S}^3)$ for any $q < 3$.*

Proof of Theorem 6 given Theorem 7. Let $\mathcal{S} : \mathbb{R}^3 \rightarrow \mathbb{S}^3$ be the (inverse) stereographic projection,

$$\mathcal{S}_j(x) = \frac{2x_j}{1+x^2}, \quad j = 1, 2, 3, \quad \mathcal{S}_4(x) = \frac{1-x^2}{1+x^2}.$$

To a vector field A on \mathbb{R}^3 we associate the vector field α on \mathbb{S}^3 by

$$A(x) = (D\mathcal{S}(x))^T \alpha(\mathcal{S}(x)).$$

Identifying α with a one-form on \mathbb{S}^3 , we see that

$$\inf_{\varphi} \int_{\mathbb{R}^3} |A - \nabla \varphi|^3 dx = \inf_{\Phi} \int_{\mathbb{S}^3} |\alpha - d\Phi|^3 d\omega$$

and

$$\int_{\mathbb{R}^3} |\nabla \wedge A|^{3/2} dx = \int_{\mathbb{S}^3} |d\alpha|^{3/2} d\omega,$$

where ω is the uniform surface measure on the sphere. Similarly, the weak convergence of $\nabla \wedge A_n$ in $L^{3/2}(\mathbb{R}^3, \mathbb{R}^3)$ is equivalent to weak convergence of $d\alpha_n$ in $L^{3/2}(\Lambda^2 \mathbb{S}^3)$ and the condition $\nabla \cdot (|A_n|A_n) = 0$, which arises as the Euler equation of the above minimization problem with respect to φ , is equivalent to $d^*(|\alpha_n|\alpha_n) = 0$. Thus, we are in the situation of Theorem 7 and we conclude that for any $q < 3$,

$$\int_{\mathbb{R}^3} |A_n - A|^q \left(\frac{2}{1+x^2} \right)^{3-q} dx = \int_{\mathbb{S}^3} |\alpha - \alpha_n|^q d\omega \rightarrow 0.$$

Since the weight on the left side is bounded away from zero on every bounded set, this implies the $L^q_{\text{loc}}(\mathbb{R}^3, \mathbb{R}^3)$ convergence of (A_n) . \square

Thus, it remains to prove Theorem 7. As a preparation we recall the Helmholtz decomposition in $L^{3/2}$ on \mathbb{S}^3 , analogous to that in Lemma 3 in \mathbb{R}^3 . Since $d^*\alpha_n$ has integral zero, the solvability of the Poisson problem implies that there is a function u_n on \mathbb{S}^3 such that $d^*du_n = d^*\alpha_n$. Thus,

$$\tilde{\alpha}_n := \alpha_n - du_n$$

satisfies

$$d\tilde{\alpha}_n = d\alpha_n \quad \text{and} \quad d^*\tilde{\alpha}_n = 0. \quad (8)$$

Similarly, we define u and $\tilde{\alpha}$.

We recall the inequality

$$\|\xi\|_{W^{1,3/2}(\Lambda^1\mathbb{S}^3)} \leq C \left(\|d\xi\|_{L^{3/2}(\Lambda^2\mathbb{S}^3)} + \|d^*\xi\|_{L^{3/2}(\mathbb{S}^3)} \right); \quad (9)$$

see, for instance, [20, Thm. 4.11] together with the fact that there are no harmonic one-forms on \mathbb{S}^3 .

Inequality (9), applied to $\tilde{\alpha}_n$, implies that $(\tilde{\alpha}_n)$ is bounded in $W^{1,3/2}(\Lambda^1\mathbb{S}^3)$ and therefore, after passing to a subsequence, we may assume that $(\tilde{\alpha}_n)$ converges weakly in $W^{1,3/2}(\Lambda^1\mathbb{S}^3)$. By passing to the limit in (8) we see that the limit of $\tilde{\alpha}_n$, which we temporarily denote by α' , satisfies.

$$d\alpha' = d\alpha, \quad d^*\alpha' = 0.$$

Thus, $d(\alpha' - \tilde{\alpha}) = 0$ and $d^*(\alpha' - \tilde{\alpha}) = 0$. Since there are no harmonic one-forms, we conclude that $\alpha' = \tilde{\alpha}$. Thus, $\tilde{\alpha}_n \rightarrow \tilde{\alpha}$ in $W^{1,3/2}(\Lambda^1\mathbb{S}^3)$.

A quick aside: Here we extracted a subsequence, whereas we stated Theorem 7 for the full sequence. To deduce the theorem as stated we note that the proof really shows that any subsequence has a further subsequence such that the conclusion holds, and this proves that the conclusion holds, indeed, along the full sequence.

Next, by the usual Rellich–Kondrachov lemma mentioned at the beginning of this section, $\tilde{\alpha}_n \rightarrow \tilde{\alpha}$ in $L^p(\Lambda^1\mathbb{S}^3)$ for any $p < 3$. Thus, to prove the theorem we need to show that $du_n \rightarrow du$ in $L^p(\Lambda^1\mathbb{S}^3)$ for any $p < 3$. To prove this, we recall the equations satisfied by α_n and α , namely,

$$d^*(|du_n + \tilde{\alpha}_n|(du_n + \tilde{\alpha}_n)) = 0 \quad \text{and} \quad d^*(|du + \tilde{\alpha}|(du + \tilde{\alpha})) = 0.$$

We think of this as an equation for du_n for given $\tilde{\alpha}_n$. The key step in the proof is the following inequality, which says that the solution u_n depends, in some sense, continuously on the data $\tilde{\alpha}_n$. This is easy in the topology of L^3 , but rather deep for L^p with $p < 3$.

Lemma 8. *There are absolute constants $C < \infty$ and $\varepsilon_* > 0$ such that, if $0 \leq \varepsilon \leq \varepsilon_*$ and if $\varphi_1, \varphi_2 \in W^{1,3-3\varepsilon}(\mathbb{S}^3)$ and $\xi_1, \xi_2 \in L^{3-3\varepsilon}(\Lambda^1\mathbb{S}^3)$ satisfy*

$$d^*(|d\varphi_1 - \xi_1|(d\varphi_1 - \xi_1)) = 0 \quad \text{and} \quad d^*(|d\varphi_2 - \xi_2|(d\varphi_2 - \xi_2)) = 0,$$

then

$$\|d\varphi_1 - d\varphi_2\|_{L^{3-3\varepsilon}(\Lambda^1\mathbb{S}^3)}^2 \leq C (\|\xi_1 - \xi_2\|_{L^{3-3\varepsilon}(\Lambda^1\mathbb{S}^3)} M + \varepsilon^2 M^2) \quad (10)$$

with

$$M := \|\xi_1\|_{L^{3-3\varepsilon}(\Lambda^1\mathbb{S}^3)} + \|\xi_2\|_{L^{3-3\varepsilon}(\Lambda^1\mathbb{S}^3)}.$$

This lemma is the analogue of a result in [17] for the closely related equation $d^*(|d\varphi|d\varphi) = d\psi$. We defer its proof to the following section.

The ‘problem’ with the bound (10) is the term ε^2 on the right side, which only becomes small when $\varepsilon \rightarrow 0$. However, in our application where $\xi_1 = -\tilde{\alpha}_n$ and $\xi_2 = -\tilde{\alpha}$, we cannot expect convergence of the first term on the right side of (10) at $\varepsilon = 0$.

To go around this impasse, we follow [17] and deduce from (10) a bound in the grand Lebesgue space $L^{\theta,3}(\mathbb{S}^3)$. This space (which depends on a parameter $\theta > 0$, which only plays a minor role in what follows) strictly contains $L^3(\mathbb{S}^3)$. The second ingredient, which is due to [7], is the observation that the Rellich–Kondrachov theorem remains valid in this space. Combining these two ingredients it will be easy to complete the proof of Theorem 7.

We now present the details of this argument. For $\theta > 0$ we denote by $L^{\theta,3}(\mathbb{S}^3)$ the set of (equivalence classes of) measurable functions f on \mathbb{S}^3 for which

$$\|f\|_{L^{\theta,3}(\mathbb{S}^3)} := \sup_{0 < \delta \leq 2} \left(\delta^{\frac{\theta}{3}} |\mathbb{S}^3|^{-\frac{1}{3-\delta}} \|f\|_{L^{3-\delta}(\mathbb{S}^3)} \right).$$

is finite. The factor $|\mathbb{S}^3|^{-\frac{1}{3-\delta}}$ normalizes the measure on \mathbb{S}^3 , but is not really important.

Corollary 9. *There is an absolute constant $C < \infty$ such that if $\varphi_1, \varphi_2 \in W^{1,3}(\mathbb{S}^3)$ and $\xi_1, \xi_2 \in L^3(\Lambda^1\mathbb{S}^3)$ satisfy*

$$d^*(|d\varphi_1 - \xi_1|(d\varphi_1 - \xi_1)) = 0 \quad \text{and} \quad d^*(|d\varphi_2 - \xi_2|(d\varphi_2 - \xi_2)) = 0,$$

then for any $0 < \theta \leq 3$,

$$\|d\varphi_1 - d\varphi_2\|_{L^{\theta,3}(\Lambda^1\mathbb{S}^3)} \leq C \|\xi_1 - \xi_2\|_{L^{\theta,3}(\Lambda^1\mathbb{S}^3)}^{1-\frac{\theta}{3}} (M')^{1+\frac{\theta}{3}} \quad (11)$$

with

$$M' := \|\xi_1\|_{L^{\theta,3}(\Lambda^1\mathbb{S}^3)} + \|\xi_2\|_{L^{\theta,3}(\Lambda^1\mathbb{S}^3)}.$$

Proof. We abbreviate $\|\cdot\|_{\theta,3} := \|\cdot\|_{L^{\theta,3}(\mathbb{S}^3)}$. Since $\|f\|_{L^{3-3\varepsilon}(\mathbb{S}^3)} \leq (3\varepsilon)^{-\frac{\theta}{3}} |\mathbb{S}^3|^{\frac{1}{3-3\varepsilon}} \|f\|_{\theta,3}$, the bound (10) implies

$$\|d\varphi_1 - d\varphi_2\|_{L^{3-3\varepsilon}(\mathbb{S}^3)}^2 \leq C (3\varepsilon)^{-\frac{2\theta}{3}} |\mathbb{S}^3|^{\frac{2}{3-3\varepsilon}} (\|\xi_1 - \xi_2\|_{\theta,3} M' + \varepsilon^2 (M')^2) \quad (12)$$

for all $0 < \varepsilon \leq \varepsilon_*$ and $\theta > 0$. Now given a parameter $0 < \delta \leq \min\{\varepsilon_*, \frac{2}{3}\}$, we set

$$\varepsilon := \delta \frac{\|\xi_1 - \xi_2\|_{\theta,3}^{1/2}}{(M')^{1/2}}.$$

Note that, in view of the explicit expression of M' , we have $\varepsilon \leq \delta$ and therefore,

$$\begin{aligned} |\mathbb{S}^3|^{-\frac{2}{3-3\delta}} \|d\varphi_1 - d\varphi_2\|_{L^{3-3\delta}(\mathbb{S}^3)}^2 &\leq |\mathbb{S}^3|^{-\frac{2}{3-3\varepsilon}} \|d\varphi_1 - d\varphi_2\|_{L^{3-3\varepsilon}(\mathbb{S}^3)}^2 \\ &\leq C(3\varepsilon)^{-\frac{2\theta}{3}} (\|\xi_1 - \xi_2\|_{\theta,3} M' + \varepsilon^2 (M')^2) \\ &= C(3\delta)^{-\frac{2\theta}{3}} \|\xi_1 - \xi_2\|_{\theta,3}^{1-\frac{\theta}{3}} (M')^{1+\frac{\theta}{3}} (1 + \delta^2) \\ &\leq C'(3\delta)^{-\frac{2\theta}{3}} \|\xi_1 - \xi_2\|_{\theta,3}^{1-\frac{\theta}{3}} (M')^{1+\frac{\theta}{3}} \end{aligned}$$

with $C' = C(1 + \min\{\varepsilon_*^2, \frac{4}{9}\})$. Moreover, in case $\varepsilon_* < \frac{2}{3}$, we bound for $\varepsilon_* < \delta \leq \frac{2}{3}$,

$$\begin{aligned} |\mathbb{S}^3|^{-\frac{2}{3-3\delta}} \|d\varphi_1 - d\varphi_2\|_{L^{3-3\delta}(\mathbb{S}^3)}^2 &\leq |\mathbb{S}^3|^{-\frac{2}{3-3\varepsilon_*}} \|d\varphi_1 - d\varphi_2\|_{L^{3-3\varepsilon_*}(\mathbb{S}^3)}^2 \\ &\leq C'(3\varepsilon_*)^{-\frac{2\theta}{3}} \|\xi_1 - \xi_2\|_{\theta,3}^{1-\frac{\theta}{3}} (M')^{1+\frac{\theta}{3}} \\ &\leq C'(3\varepsilon_*/2)^{-\frac{2\theta}{3}} (3\delta)^{-\frac{2\theta}{3}} \|\xi_1 - \xi_2\|_{\theta,3}^{1-\frac{\theta}{3}} (M')^{1+\frac{\theta}{3}}. \end{aligned}$$

To summarize, we have for all $0 < \delta \leq \frac{2}{3}$,

$$|\mathbb{S}^3|^{-\frac{2}{3-3\delta}} \|d\varphi_1 - d\varphi_2\|_{L^{3-3\delta}(\mathbb{S}^3)}^2 \leq C_\theta (3\delta)^{-\frac{2\theta}{3}} \|\xi_1 - \xi_2\|_{\theta,3}^{1-\frac{\theta}{3}} (M')^{1+\frac{\theta}{3}}$$

with $C_\theta := C' \max\{1, (3\varepsilon_*/2)^{-\frac{2\theta}{3}}\}$. This implies

$$\|d\varphi_1 - d\varphi_2\|_{\theta,3}^2 \leq C_\theta \|\xi_1 - \xi_2\|_{\theta,3}^{1-\frac{\theta}{3}} (M')^{1+\frac{\theta}{3}}.$$

Since $\theta \leq 3$, we have $C_\theta \leq C_3$ and we obtain the claimed bound. \square

As we mentioned already, the second ingredient in the proof of Theorem 7 is a version of the Rellich–Kondrachov lemma in grand Lebesgue spaces. This appears as [7], but we give a self-contained and elementary proof.

Lemma 10. *Assume that $v_n \rightarrow 0$ in $W^{1,3/2}(\mathbb{S}^3)$. Then $v_n \rightarrow 0$ in $L^{\theta,3}(\mathbb{S}^3)$ for any $\theta > 0$.*

Proof. For any $\delta_0 > 0$, we bound, using Hölder's inequality,

$$\begin{aligned} \|v_n\|_{L^{\theta,3}(\mathbb{S}^3)} &\leq \sup_{0 < \delta \leq \delta_0} \left(\delta^{\frac{\theta}{3}} |\mathbb{S}^3|^{-\frac{1}{3-\delta}} \|v_n\|_{3-\delta} \right) + \sup_{\delta_0 \leq \delta \leq 2} \left(\delta^{\frac{\theta}{3}} |\mathbb{S}^3|^{-\frac{1}{3-\delta}} \|v_n\|_{3-\delta} \right) \\ &\leq \delta_0^{\frac{\theta}{3}} |\mathbb{S}^3|^{-\frac{1}{3}} \|v_n\|_3 + 2^{\frac{\theta}{3}} |\mathbb{S}^3|^{-\frac{1}{3-\delta_0}} \|v_n\|_{3-\delta_0}. \end{aligned}$$

By the ordinary Rellich–Kondrachov lemma, we have $v_n \rightarrow 0$ in $L^{3-\delta_0}(\mathbb{S}^3)$, so

$$\limsup_{n \rightarrow \infty} \|v_n\|_{L^{\theta,3}(\mathbb{S}^3)} \leq \delta_0^{\frac{\theta}{3}} |\mathbb{S}^3|^{-\frac{1}{3}} \limsup_{n \rightarrow \infty} \|v_n\|_3.$$

Since (v_n) is bounded in $L^3(\mathbb{S}^3)$ by Sobolev and since $\delta_0 > 0$ can be chosen arbitrarily small, we obtain the assertion. \square

We are now in position to complete the proof of Theorem 7. Indeed, Lemma 10 implies that $\tilde{\alpha}_n \rightarrow \tilde{\alpha}$ in $L^{\theta,3}(\mathbb{S}^3)$ for any $\theta > 0$. Thus, by Corollary 9, $du_n \rightarrow du$ in $L^{\theta,3}(\Lambda^1\mathbb{S}^3)$ for any $\theta > 0$. Since $\|f\|_{L^q(\mathbb{S}^3)} \leq C_{q,\theta}\|f\|_{L^{\theta,3}(\mathbb{S}^3)}$ for any $q < 3$ and $\theta > 0$, we conclude that $du_n \rightarrow du$ in $L^q(\Lambda^1\mathbb{S}^3)$ for any $q < 3$. Since $\alpha_n = \tilde{\alpha}_n + du_n$ and $\tilde{\alpha}_n \rightarrow \tilde{\alpha}$ in $L^q(\mathbb{S}^3)$ for any $q < 3$, this proves the assertion. \square

4. NONLINEAR HELMHOLTZ DECOMPOSITION

Our goal in this section is to prove Lemma 8. The key ingredient is a nonlinear version of the Helmholtz decomposition due to Iwaniec [18]. A simplified proof of an improved result appears in [19] in the case of Euclidean space and the result for Riemannian manifolds is in [20, Proof of Thm. 8.8]. We only state the special case of the result that we need.

Theorem 11. *There is an absolute constant $C < \infty$ such that for any $0 \leq \varepsilon < \frac{1}{3}$ and any $\varphi \in W^{1,3-3\varepsilon}(\mathbb{S}^3)$ there are $\psi \in W^{1,\frac{3-3\varepsilon}{1-3\varepsilon}}(\mathbb{S}^3)$ and $\gamma \in L^{\frac{3-3\varepsilon}{1-3\varepsilon}}(\Lambda^1\mathbb{S}^3)$ such that*

$$|d\varphi|^{-3\varepsilon}d\varphi = d\psi + \gamma, \quad d^*\gamma = 0$$

and

$$\|\gamma\|_{L^{\frac{3-3\varepsilon}{1-3\varepsilon}}(\Lambda^1\mathbb{S}^3)} \leq C\varepsilon \|d\varphi\|_{L^{3-3\varepsilon}(\mathbb{S}^3)}^{1-3\varepsilon}.$$

With this theorem at our disposal, we now turn to the proof of Lemma 8. As we already mentioned, our proof is analogous to the proof of a similar result for a related equation in [17].

Proof of Lemma 8. According to Theorem 11, for any $0 \leq \varepsilon < \frac{1}{3}$ we can find $\psi \in W^{1,\frac{3-3\varepsilon}{1-3\varepsilon}}(\mathbb{S}^3)$ and $\gamma \in L^{\frac{3-3\varepsilon}{1-3\varepsilon}}(\Lambda^1\mathbb{S}^3)$ such that

$$|d\varphi_1 - d\varphi_2|^{-3\varepsilon}(d\varphi_1 - d\varphi_2) = d\psi + \gamma, \quad d^*\gamma = 0$$

and, with the obvious abbreviation for the norm,

$$\|\gamma\|_{L^{\frac{3-3\varepsilon}{1-3\varepsilon}}} \leq C\varepsilon \|d\varphi_1 - d\varphi_2\|_{L^{3-3\varepsilon}}^{1-3\varepsilon}.$$

Testing the equations for φ_1 and φ_2 against ψ and subtracting them from each other, we get

$$\begin{aligned} & \int_{\mathbb{S}^3} \langle |d\varphi_1 - \xi_1|(d\varphi_1 - \xi_1) - |d\varphi_2 - \xi_2|(d\varphi_2 - \xi_2), |d\varphi_1 - d\varphi_2|^{-3\varepsilon}(d\varphi_1 - d\varphi_2) \rangle d\omega \\ &= \int_{\mathbb{S}^3} \langle |d\varphi_1 - \xi_1|(d\varphi_1 - \xi_1) - |d\varphi_2 - \xi_2|(d\varphi_2 - \xi_2), \gamma \rangle d\omega. \end{aligned} \quad (13)$$

We will bound the right side from above and the left side from below.

Using

$$||x|x - |y|y| \leq (|x| + |y|)|x - y| \quad \text{for all } x, y \in \mathbb{R}^n, \quad (14)$$

(which can be proved, for instance, by squaring both sides) we get

$$\begin{aligned} & |\langle |d\varphi_1 - \xi_1|(d\varphi_1 - \xi_1) - |d\varphi_2 - \xi_2|(d\varphi_2 - \xi_2), \gamma \rangle| \\ & \leq (|d\varphi_1 - \xi_1| + |d\varphi_2 - \xi_2|) |d\varphi_1 - \xi_1 - d\varphi_2 + \xi_2| |\gamma| \\ & \leq (|d\varphi_1| + |d\varphi_2| + |\xi_1| + |\xi_2|) (|d\varphi_1 - d\varphi_2| + |\xi_1 - \xi_2|) |\gamma|. \end{aligned}$$

By Hölder's inequality,

$$\begin{aligned} & \left| \int_{\mathbb{S}^3} \langle |d\varphi_1 - \xi_1|(d\varphi_1 - \xi_1) - |d\varphi_2 - \xi_2|(d\varphi_2 - \xi_2), \gamma \rangle d\omega \right| \\ & \leq \mu (\|d\varphi_1 - d\varphi_2\|_{3-3\varepsilon} + \|\xi_1 - \xi_2\|_{3-3\varepsilon}) \|\gamma\|_{\frac{3-3\varepsilon}{1-3\varepsilon}} \\ & \leq C\mu\varepsilon (\|d\varphi_1 - d\varphi_2\|_{3-3\varepsilon} + \|\xi_1 - \xi_2\|_{3-3\varepsilon}) \|d\varphi_1 - d\varphi_2\|_{3-3\varepsilon}^{1-3\varepsilon} \end{aligned}$$

with

$$\mu := \|d\varphi_1\|_{3-3\varepsilon} + \|d\varphi_2\|_{3-3\varepsilon} + \|\xi_1\|_{3-3\varepsilon} + \|\xi_2\|_{3-3\varepsilon}.$$

We now turn to the left side in (13). It is elementary to see that

$$\begin{aligned} & \langle |x - a|(x - a) - |y - b|(y - b), (x - y) \rangle \\ & \geq \frac{1}{2}|x - y|^3 - (|x| + |y| + |a| + |b|) |a - b| |x - y| \\ & \quad \text{for all } x, y, a, b \in \mathbb{R}^n. \end{aligned} \quad (15)$$

We provide the details at the end of this proof. It follows from this inequality that

$$\begin{aligned} & \langle |d\varphi_1 - \xi_1|(d\varphi_1 - \xi_1) - |d\varphi_2 - \xi_2|(d\varphi_2 - \xi_2), |d\varphi_1 - d\varphi_2|^{-3\varepsilon}(d\varphi_1 - d\varphi_2) \rangle \\ & \geq \frac{1}{2}|d\varphi_1 - d\varphi_2|^{3-3\varepsilon} - (|d\varphi_1| + |d\varphi_2| + |\xi_1| + |\xi_2|) |\xi_1 - \xi_2| |d\varphi_1 - d\varphi_2|^{1-3\varepsilon}. \end{aligned}$$

Therefore, by Hölder's inequality,

$$\begin{aligned} & \int_{\mathbb{S}^3} \langle |d\varphi_1 - \xi_1|(d\varphi_1 - \xi_1) - |d\varphi_2 - \xi_2|(d\varphi_2 - \xi_2), |d\varphi_1 - d\varphi_2|^{-3\varepsilon}(d\varphi_1 - d\varphi_2) \rangle d\omega \\ & \geq \frac{1}{2}\|d\varphi_1 - d\varphi_2\|_{3-3\varepsilon}^{3-3\varepsilon} - \mu \|\xi_1 - \xi_2\|_{3-3\varepsilon} \|d\varphi_1 - d\varphi_2\|_{3-3\varepsilon}^{1-3\varepsilon}. \end{aligned}$$

Combining the bounds on both sides of (13), we obtain

$$\begin{aligned} \frac{1}{2}\|d\varphi_1 - d\varphi_2\|_{3-3\varepsilon}^{3-3\varepsilon} & \leq C\mu\varepsilon (\|d\varphi_1 - d\varphi_2\|_{3-3\varepsilon} + \|\xi_1 - \xi_2\|_{3-3\varepsilon}) \|d\varphi_1 - d\varphi_2\|_{3-3\varepsilon}^{1-3\varepsilon} \\ & \quad + \mu \|\xi_1 - \xi_2\|_{3-3\varepsilon} \|d\varphi_1 - d\varphi_2\|_{3-3\varepsilon}^{1-3\varepsilon}, \end{aligned}$$

which is the same as

$$\frac{1}{2}\|d\varphi_1 - d\varphi_2\|_{3-3\varepsilon}^2 \leq C\mu\varepsilon (\|d\varphi_1 - d\varphi_2\|_{3-3\varepsilon} + \|\xi_1 - \xi_2\|_{3-3\varepsilon}) + \mu \|\xi_1 - \xi_2\|_{3-3\varepsilon}. \quad (16)$$

Absorbing the term $\|d\varphi_1 - d\varphi_2\|_{3-3\varepsilon}$ on the right side into the left side gives

$$\|d\varphi_1 - d\varphi_2\|_{3-3\varepsilon}^2 \leq C' (\|\xi_1 - \xi_2\|_{3-3\varepsilon}\mu + \varepsilon^2\mu^2) \quad (17)$$

with an absolute constant $C' < \infty$.

This is almost the claimed bound, except that we need to replace μ by M . This is where the restriction on ε comes in. We return to (16) in the special case where $d\varphi_2 = \xi_2 = 0$, that is,

$$\frac{1}{2}\|d\varphi_1\|_{3-3\varepsilon}^2 \leq C\varepsilon (\|d\varphi_1\|_{3-3\varepsilon} + \|\xi_1\|_{3-3\varepsilon})^2 + (\|d\varphi_1\|_{3-3\varepsilon} + \|\xi_1\|_{3-3\varepsilon}) \|\xi_1\|_{3-3\varepsilon}.$$

We restrict ourselves to $\varepsilon \leq 1/(4C) =: \varepsilon_*$. Then the term $\|d\varphi_1\|_{3-3\varepsilon}^2$ on the right side can be absorbed into the left side. Of course, all the factors $\|d\varphi_1\|_{3-3\varepsilon}$ on the right side can be absorbed as well. In this way, we finally arrive at

$$\|d\varphi_1\|_{3-3\varepsilon} \leq C'' \|\xi_1\|_{3-3\varepsilon} \quad \text{for all } 0 \leq \varepsilon \leq \varepsilon_*$$

with an absolute constant $C'' < \infty$. This, together with a similar bound for $d\varphi_2$, gives $\mu \leq (1 + C'')M$, which, when inserted into (17), completes the proof. \square

Proof of (15). Let $c := (a + b)/2$ and write

$$\langle |x - a|(x - a) - |y - b|(y - b), (x - y) \rangle = I_0 + I_1 + I_2$$

with

$$\begin{aligned} I_0 &:= \langle |x - c|(x - c) - |y - c|(y - c), (x - y) \rangle, \\ I_1 &:= \langle |x - a|(x - a) - |x - c|(x - c), (x - y) \rangle, \\ I_2 &:= \langle |y - c|(y - c) - |y - b|(y - b), (x - y) \rangle. \end{aligned}$$

To bound I_0 from below, we note that

$$\langle |X|X - |Y|Y, (X - Y) \rangle = \frac{1}{2} ((|X| + |Y|)|X - Y|^2 + \mathcal{R}) \quad (18)$$

with

$$\mathcal{R} := |X|^3 + |Y|^3 - |X||Y|(|X| + |Y|) \geq 0. \quad (19)$$

This follows from $|X|^2|Y| \leq \frac{2}{3}|X|^3 + \frac{1}{3}|Y|^3$ and $|X||Y|^2 \leq \frac{1}{3}|X|^3 + \frac{2}{3}|Y|^3$. Applying (18), (19) with $X = x - c$ and $Y = y - c$ gives

$$I_0 \geq \frac{1}{2} (|x - c| + |y - c|) |x - y|^2 \geq \frac{1}{2} |x - y|^3.$$

To bound I_1 from above, we bound, using (14),

$$\begin{aligned} |I_1| &\leq ||x - a|(x - a) - |x - c|(x - c)| |x - y| \\ &\leq (|x - a| + |x - c|) |a - c| |x - y| \\ &\leq (2|x| + |a| + |c|) |a - c| |x - y| \\ &\leq (|x| + \frac{3}{4}|a| + \frac{1}{4}|b|) |a - b| |x - y|. \end{aligned}$$

This and the corresponding bound on I_2 give

$$|I_1 + I_2| \leq (|x| + |y| + |a| + |b|) |a - b| |x - y|,$$

which yields the claimed bound. \square

5. ANOTHER RELICH–KONDRACHOV LEMMA

This section is a short digression and its content is not needed for the proof of Theorem 1. We present a different Rellich–Kondrachov lemma for vector fields which might prove useful in other applications.

We need to introduce a gauge-invariant local L^2 (semi)norm. Let $\Omega \subset \mathbb{R}^3$ be an open set and define, for $A \in L^2(\Omega, \mathbb{R}^3)$,

$$\|A\|_{2\Omega} := \inf_{\varphi \in \dot{H}^1(\Omega)} \|A - \nabla \varphi\|_{L^2(\Omega, \mathbb{R}^3)}.$$

The main result of this section is as follows.

Proposition 12. *Suppose that $\nabla \wedge A_n \rightharpoonup \nabla \wedge A$ in $L^{3/2}(\mathbb{R}^3, \mathbb{R}^3)$. Then for any open set $\Omega \subset \mathbb{R}^3$ of finite measure, $\|A_n - A\|_{2\Omega} \rightarrow 0$.*

For the proof of this proposition, it is useful to express $\|A\|_{2\Omega}$ by duality. For $B \in L^2(\Omega, \mathbb{R}^3)$ we say that

$$\nabla \cdot B = 0 \quad \text{in } \Omega \quad \text{and} \quad \nu \cdot B = 0 \quad \text{on } \partial\Omega$$

if $\int_{\Omega} \nabla \varphi \cdot B \, dx = 0$ for any $\varphi \in \dot{H}^1(\Omega)$. Clearly, if $B \in C^1(\Omega) \cap C(\overline{\Omega})$ and $\partial\Omega$ is Lipschitz, this definition coincides with the classical one.

Lemma 13. *For any $A \in L^2(\Omega, \mathbb{R}^3)$,*

$$\|A\|_{2\Omega} = \sup \left\{ \int_{\Omega} A \cdot B \, dx : \|B\|_{L^2(\Omega, \mathbb{R}^3)} \leq 1, \nabla \cdot B = 0 \text{ in } \Omega, \nu \cdot B = 0 \text{ on } \partial\Omega \right\}. \quad (20)$$

Proof. For any B as on the right side of (20) and any $\varphi \in H^1(\Omega)$, we have

$$\int_{\Omega} A \cdot B \, dx = \int_{\Omega} (A - \nabla \varphi) \cdot B \, dx \leq \|A - \nabla \varphi\|_{L^2(\Omega, \mathbb{R}^3)}.$$

Taking the infimum over φ and the supremum over B we obtain \geq in (20).

Conversely, by convexity there is a $\varphi_* \in H^1(\Omega)$ such that

$$\|A - \nabla \varphi_*\|_{L^2(\Omega, \mathbb{R}^3)} = \|A\|_{2\Omega}.$$

The Euler–Lagrange equation corresponding to this minimization problem is

$$\int_{\Omega} \nabla \varphi \cdot (A - \nabla \varphi_*) \, dx = 0 \quad \text{for all } \varphi \in \dot{H}^1(\Omega),$$

that is, $B_* := A - \nabla \varphi_*$ satisfies $\nabla \cdot B_* = 0$ in Ω and $\nu \cdot B_* = 0$ on $\partial\Omega$. If $B_* \equiv 0$ then $\|A\|_{2\Omega} = 0$ and \leq in (20) holds trivially. Otherwise, $B_*/\|B_*\|_{L^2(\Omega)}$ is an admissible candidate for the right side in (20) and we have

$$\int_{\Omega} A \cdot \frac{B_*}{\|B_*\|_{L^2(\Omega, \mathbb{R}^3)}} \, dx = \int_{\Omega} (A - \nabla \varphi_*) \cdot \frac{B_*}{\|B_*\|_{L^2(\Omega, \mathbb{R}^3)}} \, dx = \|A\|_{2\Omega}.$$

This proves \leq in (20). □

Lemma 14. *Let $\eta \in C^1(\mathbb{R}^3)$ be nonnegative function with sufficiently fast decay and $\int_{\mathbb{R}^3} \eta dx = 1$ and set $\eta_\varepsilon(x) := \varepsilon^{-3} \eta(\frac{x}{\varepsilon})$. Then*

$$\|A - \eta_\varepsilon \star A\|_{2\mathbb{R}^3} \leq C_\eta \sqrt{\varepsilon} \|\nabla \wedge A\|_{3/2}.$$

Proof. We use Lemma 13 with $\Omega = \mathbb{R}^3$ and consider $B \in L^2(\mathbb{R}^3, \mathbb{R}^3)$ with $\|B\|_2 \leq 1$ and $\nabla \cdot B = 0$ in \mathbb{R}^3 . By Plancherel (with the normalization of the Fourier transform as, for instance, in [23]), we have

$$\int_{\mathbb{R}^3} (A - \eta_\varepsilon \star A) \cdot B dx = \int_{\mathbb{R}^3} (1 - \hat{\eta}_\varepsilon(k)) \overline{\hat{A}(k)} \cdot \hat{B}(k) dk.$$

Since $k \cdot \hat{B}(k) = 0$ we can write

$$\overline{\hat{A}(k)} \cdot \hat{B}(k) = \frac{(k \wedge \overline{\hat{A}(k)}) \cdot (k \wedge \hat{B}(k))}{|k|^2},$$

and hence

$$\int_{\mathbb{R}^3} (A - \eta_\varepsilon \star A) \cdot B dx = \int_{\mathbb{R}^3} \frac{(1 - \hat{\eta}_\varepsilon(k))}{|k|^{1/2}} \frac{(k \wedge \overline{\hat{A}(k)})}{|k|^{1/2}} \cdot \frac{(k \wedge \hat{B}(k))}{|k|} dk,$$

which is bounded above by

$$C \sqrt{\varepsilon} \left(\int_{\mathbb{R}^3} \left| \frac{(k \wedge \overline{\hat{A}(k)})}{|k|^{1/2}} \right|^2 dk \right)^{1/2}.$$

Here we used the fact that, because of the sufficiently fast decay of η , we have $\sup |\xi|^{-1/2} |1 - \hat{\eta}(\xi)| < \infty$. The square of the last factor is, by Plancherel and Hardy–Littlewood–Sobolev,

$$C \left(\nabla \wedge A, \frac{1}{|x|^2} \star \nabla \wedge A \right) \leq C \|\nabla \wedge A\|_{3/2}^2.$$

This is the claimed inequality. \square

Finally, we are in position to prove the main result of this section.

Proof of Proposition 12. We will be using the following two properties of the seminorm $\|\cdot\|_{2\Omega}$. First, we have

$$\|A_1 + A_2\|_{2\Omega} \leq \|A_1\|_{2\Omega} + \|A_2\|_{2\Omega}.$$

This follows directly from the definition of the seminorm. Second, we have for $A \in L^2(\mathbb{R}^3, \mathbb{R}^3)$,

$$\|A\|_{2\Omega} \leq \|A\|_{2\mathbb{R}^3}.$$

This follows from the duality lemma 13 and the fact that if $B \in L^2(\Omega, \mathbb{R}^3)$ satisfies $\nabla \cdot B = 0$ in Ω and $\nu \cdot B = 0$ on $\partial\Omega$, then its extension \tilde{B} by zero to \mathbb{R}^3 satisfies $\nabla \cdot \tilde{B} = 0$ in \mathbb{R}^3 .

Using these two facts, together with Lemma 14 we can bound

$$\begin{aligned} \|A_n - A\|_{2\Omega} &\leq \|A_n - \eta_\varepsilon \star A_n\|_{2\Omega} + \|\eta_\varepsilon \star A_n - \eta_\varepsilon \star A\|_{2\Omega} + \|\eta_\varepsilon \star A - A\|_{2\Omega} \\ &\leq \|A_n - \eta_\varepsilon \star A_n\|_{2\mathbb{R}^3} + \|\eta_\varepsilon \star A_n - \eta_\varepsilon \star A\|_{2\Omega} + \|\eta_\varepsilon \star A - A\|_{2\mathbb{R}^3} \\ &\leq 2C\sqrt{\varepsilon} + \|\eta_\varepsilon \star A_n - \eta_\varepsilon \star A\|_{2\Omega}. \end{aligned}$$

We need to show that every subsequence has a further subsequence along which, for every fixed $\varepsilon > 0$, $\|\eta_\varepsilon \star A_n - \eta_\varepsilon \star A\|_{2\Omega} \rightarrow 0$. Since $\nabla \wedge A_n \rightharpoonup \nabla \wedge A$ in $L^{3/2}(\mathbb{R}^3, \mathbb{R}^3)$ we have, by the Sobolev inequality (Lemma 4),

$$\int_{\mathbb{R}^3} A_n \cdot B \, dx \rightarrow \int_{\mathbb{R}^3} A \cdot B \, dx \quad \text{for any } B \in L^{3/2}(\mathbb{R}^3, \mathbb{R}^3) \text{ with } \nabla \cdot B = 0. \quad (21)$$

Moreover, since $\|A_n - A\|_3$ is bounded, there is a sequence Φ_n such that $\|A_n - A - \nabla \Phi_n\|_3$ is bounded. Now for the given subsequence, there is a further subsequence along which $A_n - A - \nabla \Phi_n \rightharpoonup F$ in $L^3(\mathbb{R}^3, \mathbb{R}^3)$. It follows from (21) that $\int_{\mathbb{R}^3} F \cdot B \, dx = 0$ for all $B \in L^{3/2}(\mathbb{R}^3, \mathbb{R}^3)$ with $\nabla \cdot B = 0$, that is, $F = \nabla \Phi$. Thus, $A_n - A - \nabla \varphi_n \rightharpoonup 0$ in $L^3(\mathbb{R}^3, \mathbb{R}^3)$ with $\varphi_n := \Phi_n - \Phi$. Thus, $\eta_\varepsilon \star A_n - \eta_\varepsilon \star A - \eta_\varepsilon \star \nabla \varphi_n$ converges pointwise to zero and is bounded uniformly in n . Thus, by dominated convergence, $\eta_\varepsilon \star A_n - \eta_\varepsilon \star A - \eta_\varepsilon \star \nabla \varphi_n \rightarrow 0$ in $L^2(\Omega)$ and so,

$$\|\eta_\varepsilon \star A_n - \eta_\varepsilon \star A\|_{2\Omega} \leq \|\eta_\varepsilon \star A_n - \eta_\varepsilon \star A - \eta_\varepsilon \star \nabla \varphi_n\|_{L^2(\Omega, \mathbb{R}^3)} \rightarrow 0.$$

This proves the proposition. \square

6. NONZERO WEAK LIMIT

Our goal in the section is to show that a minimizing sequence has a nonzero weak limit up to symmetries. In the language of concentration compactness, we exclude ‘vanishing’.

6.1. An improved inequality. Our goal in this section is to prove the following proposition.

Proposition 15. *For any $A \in \mathcal{V}$,*

$$\|A\|_3 \lesssim \|\nabla \wedge A\|_{3/2}^{1/2} \left(\sup_{t>0} t \|e^{t\Delta} \nabla \wedge A\|_{L^\infty(\mathbb{R}^3, \mathbb{R}^3)} \right)^{1/2}.$$

Note that, since $e^{t\Delta}$ is convolution with a function whose L^3 norm is proportional to t^{-1} , one has, by Hölder,

$$\sup_{t>0} t \|e^{t\Delta} \nabla \wedge A\|_{L^\infty(\mathbb{R}^3, \mathbb{R}^3)} \lesssim \|\nabla \wedge A\|_{3/2}.$$

In this sense the inequality ‘improves’ upon the Sobolev inequality in Lemma 4.

Proof. We set $B := \nabla \wedge A$ and define φ as in the proof of Lemma 3. Then $A - \nabla\varphi = \tilde{A}$ is given by (6). In this formula, we write

$$\frac{1}{4\pi} \frac{x-y}{|x-y|^3} = -\frac{1}{4\pi} \nabla_x \frac{1}{|x-y|} = -\int_0^\infty dt (4\pi t)^{-3/2} \nabla_x e^{-(x-y)^2/4t},$$

where we used $(-\Delta)^{-1} = \int_0^\infty dt e^{t\Delta}$. Thus, with a parameter T to be determined,

$$A(x) - \nabla\varphi(x) = I_<(x) + I_>(x),$$

where

$$\begin{aligned} I_<(x) &:= -\int_0^T dt (4\pi t)^{-3/2} \int_{\mathbb{R}^3} dy B(y) \wedge \nabla_x e^{-(x-y)^2/4t}, \\ I_>(x) &:= -\int_T^\infty dt (4\pi t)^{-3/2} \int_{\mathbb{R}^3} dy B(y) \wedge \nabla_x e^{-(x-y)^2/4t}. \end{aligned}$$

Clearly,

$$\begin{aligned} |I_<(x)| &\leq \int_0^T dt (4\pi t)^{-3/2} \int_{\mathbb{R}^3} dy |B(y)| |\nabla_x e^{-(x-y)^2/4t}| \\ &= \int_0^T dt t^{-2} \int_{\mathbb{R}^3} dy |B(y)| k((x-y)/\sqrt{t}) \end{aligned}$$

with

$$k(z) := (4\pi)^{-3/2} (|z|/2) e^{-z^2/4}.$$

Let \tilde{k} be the monotone hull of k , that is,

$$\tilde{k}(z) := \begin{cases} (4\pi)^{-3/2} (1/\sqrt{2}) e^{-1/2} & \text{if } |z| < \sqrt{2}, \\ (4\pi)^{-3/2} (|z|/2) e^{-z^2/4} & \text{if } |z| \geq \sqrt{2}. \end{cases}$$

Then, by the layer cake formula

$$\begin{aligned} \int_{\mathbb{R}^3} t^{-3/2} k((x-y)/\sqrt{t}) |B(y)| dy &\leq \int_{\mathbb{R}^3} t^{-3/2} \tilde{k}((x-y)/\sqrt{t}) |B(y)| dy \\ &= \int_0^\infty d\kappa t^{-3/2} \int_{\mathbb{R}^3} \mathbb{1}_{\{\tilde{k} > \kappa\}}((x-y)/\sqrt{t}) |B(y)| dy \\ &\leq \int_0^\infty d\kappa |\{\tilde{k} > \kappa\}| B^*(x) \\ &= \|\tilde{k}\|_{L^1(\mathbb{R}^3)} B^*(x) \end{aligned}$$

with the maximal function

$$B^*(x) := \sup_{r>0} \frac{3}{4\pi r^3} \int_{\{|x-y|<r\}} |B(y)| dy.$$

Thus,

$$|I_<(x)| \leq \|\tilde{k}\|_1 B^*(x) \int_0^T dt t^{-1/2} = 2\|\tilde{k}\|_{L^1(\mathbb{R}^3)} B^*(x) T^{1/2}.$$

On the other hand, in order to estimate $I_>$ we use the semi-group property $e^{t\Delta} = e^{t\Delta/2}e^{t\Delta/2}$ to write

$$(4\pi t)^{-3/2} \int_{\mathbb{R}^3} dy B(y) \wedge \nabla_x e^{-(x-y)^2/4t} = (2\pi t)^{-3/2} \int_{\mathbb{R}^3} dy (e^{t\Delta/2} B)(y) \wedge \nabla_x e^{-(x-y)^2/2t}.$$

Thus,

$$\begin{aligned} |I_>(x)| &\leq \int_T^\infty dt (2\pi t)^{-3/2} \int_{\mathbb{R}^3} dy |(e^{t\Delta/2} B)(y)| |\nabla_x e^{-(x-y)^2/2t}| \\ &= \int_T^\infty dt \int_{\mathbb{R}^3} dy |(e^{t\Delta/2} B)(y)| (t/2)^{-2} k((x-y)/\sqrt{t/2}). \end{aligned}$$

Now

$$\begin{aligned} \int_{\mathbb{R}^3} |(e^{t\Delta/2} B)(y)| (t/2)^{-2} k((x-y)/\sqrt{t/2}) dy &\leq \|e^{t\Delta/2} B\|_\infty (t/2)^{-1/2} \|k\|_{L^1(\mathbb{R}^3)} \\ &\leq M(t/2)^{-3/2} \|k\|_1 \end{aligned}$$

with $M := \sup_{t>0} t \|e^{t\Delta} B\|_\infty$. Thus,

$$|I_>(x)| \leq \int_T^\infty dt M(t/2)^{-3/2} \|k\|_{L^1(\mathbb{R}^3)} = M \|k\|_{L^1(\mathbb{R}^3)} 4(T/2)^{-1/2}.$$

To summarize, we have shown that

$$|A(x) - \nabla\varphi(x)| \leq 2\|\tilde{k}\|_{L^1(\mathbb{R}^3)} B^*(x) T^{1/2} + M \|k\|_{L^1(\mathbb{R}^3)} 4(T/2)^{-1/2}.$$

Optimizing in T , we get

$$|A(x) - \nabla\varphi(x)| \leq 2^{11/4} \|\tilde{k}\|_{L^1(\mathbb{R}^3)}^{1/2} \|k\|_{L^1(\mathbb{R}^3)}^{1/2} M^{1/2} B^*(x)^{1/2}.$$

We raise this inequality to the third power and use the fact that the maximal function is a bounded operator on $L^{3/2}(\mathbb{R}^3)$. This proves the claimed inequality. \square

Remark 16. The analogue of Proposition 15 for the Sobolev inequality for spinor fields mentioned in the introduction is

$$\|\psi\|_3 \lesssim \|\sigma \cdot (-i\nabla)\psi\|_{3/2}^{1/2} \left(\sup_{t>0} t \|e^{t\Delta} \sigma \cdot (-i\nabla)\psi\|_\infty \right)^{1/2}, \quad \psi \in \dot{W}^{1,3/2}(\mathbb{R}^3, \mathbb{C}^2). \quad (22)$$

This is proved in exactly the same way as Proposition 15.

6.2. Nonzero weak limit up to symmetries. In this subsection we use the improved inequality in Proposition 15 to show that, up to translations and dilations, one can extract from every minimizing sequence for S a subsequence which has a nontrivial weak limit.

Proposition 17. *Let $(A_n) \subset \mathcal{Y}$ be a sequence with*

$$\|A\|_3 = 1 \quad \text{and} \quad \|\nabla \wedge A_n\|_{3/2} \lesssim 1.$$

Then there are $\lambda_n \in (0, \infty)$ and $a_n \in \mathbb{R}^3$ such that a subsequence of

$$\lambda_n^2 (\nabla \wedge A_n)(\lambda_n(x - a_n))$$

converges weakly in $L^{3/2}(\mathbb{R}^3, \mathbb{R}^3)$ to some $\tilde{B} \neq 0$.

Proof. We write $B_n := \nabla \wedge A_n$. Applying the improved inequality from Proposition 15 we infer that

$$\varepsilon := \liminf_{n \rightarrow \infty} \sup_{t > 0} t \|e^{t\Delta} B_n\|_{L^\infty(\mathbb{R}^3, \mathbb{R}^3)} > 0.$$

Thus, for all sufficiently large n there are $t_n > 0$ and $x_n \in \mathbb{R}^3$ such that

$$t_n |(e^{t_n \Delta} B_n)(x_n)| \geq \varepsilon/2,$$

that is,

$$\left| \int_{\mathbb{R}^3} (4\pi)^{-3/2} e^{-x^2/4} \tilde{B}_n(x) dx \right| \geq \varepsilon/2 \quad (23)$$

for

$$\tilde{B}_n(x) := t_n B_n(\sqrt{t_n}x + x_n).$$

Since $\|\tilde{B}_n\|_{3/2} = \|B_n\|_{3/2} = 1$, weak compactness implies that a subsequence of \tilde{B}_n converges weakly in $L^{3/2}(\mathbb{R}^3, \mathbb{R}^3)$ to some \tilde{B} . Since $e^{-x^2/4} \in L^3(\mathbb{R}^3)$, it follows from (23) that $\tilde{B} \neq 0$, as claimed. \square

7. APPLYING EKELAND'S VARIATIONAL PRINCIPLE

In the previous section, in order to get a nonzero weak limit along a subsequence, we only used $\limsup_{n \rightarrow \infty} (\|\nabla \wedge A_n\|_{3/2} / \|A_n\|_3) > 0$. We did *not* use the fact that A_n is minimizing. Our goal in this and the next section is to upgrade the weak convergence to strong convergence, and we do this by using $\lim_{n \rightarrow \infty} (\|\nabla \wedge A_n\|_{3/2} / \|A_n\|_3) = S^{2/3}$.

More specifically, in this section we show that the minimizing sequence can be slightly altered to satisfy a version of the Euler–Lagrange equation with a small inhomogeneity. This will be achieved through Ekeland's variational principle. In the next section, we will study this approximated Euler–Lagrange equation in more detail.

Proposition 18. *Let $(A_n) \subset \mathcal{Y}$ be a minimizing sequence for S with $\|A_n\|_3 = 1$. Then there is a sequence $(A'_n) \subset \mathcal{Y}$ with $\nabla \cdot (|A'_n| A'_n) = 0$ and $\|A'_n\|_3 = 1$ for all n such that*

$$\nabla \wedge A'_n - \nabla \wedge A_n \rightarrow 0 \quad \text{in } L^{3/2}(\mathbb{R}^3, \mathbb{R}^3)$$

and

$$\nabla \wedge (|\nabla \wedge A'_n|^{-1/2} \nabla \wedge A'_n) - S|A'_n| A'_n = \nabla \wedge r_n \quad \text{with } r_n \rightarrow 0 \text{ in } L^3(\mathbb{R}^3, \mathbb{R}^3).$$

Our goal in this section will be to prove this proposition.

7.1. Differentiability of the norm. The following result implies that $A \mapsto \|A\|_3^3$ is Fréchet differentiable and gives a formula for its derivative.

Lemma 19. *Let $A \in L^3(\mathbb{R}^3, \mathbb{R}^3)$ with $\nabla \cdot (|A|A) = 0$. Then for all $F \in L^3(\mathbb{R}^3, \mathbb{R}^3)$,*

$$\left| \|A + F\|_3^3 - \|A\|_3^3 - 3 \int_{\mathbb{R}^3} |A|A \cdot F \, dx \right| \lesssim \|A\|_3^{3/2} \|F\|_3^{3/2} + \|F\|_3^3.$$

Proof. Note that according to Lemma 5, the assumption on A implies $\|A\|_3 = \|A\|_3$. Moreover, since the claimed inequality is invariant under adding a gradient to F , we may also assume that $\|F\|_3 = \|F\|_3$. We have

$$\|A + F\|_3^3 \leq \|A + F\|_3^3 \leq \|A\|_3^3 + 3 \int_{\mathbb{R}^3} |A|A \cdot F \, dx + \text{const} (\|A\|_3 \|F\|_3^2 + \|F\|_3^3).$$

Since the remainder on the right side is bounded by a constant times $\|A\|_3^{3/2} \|F\|_3^{3/2} + \|F\|_3^3$, this proves one of the two claimed inequalities. For the converse inequality, we choose φ such that $\|A + F\|_3 = \|A + F - \nabla\varphi\|_3$ and bound

$$\begin{aligned} \|A + F\|_3^3 &\geq \|A - \nabla\varphi\|_3^3 + 3 \int_{\mathbb{R}^3} |A - \nabla\varphi|(A - \nabla\varphi) \cdot F \, dx \\ &\quad - \text{const} (\|A - \nabla\varphi\|_3 \|F\|_3^2 + \|F\|_3^3). \end{aligned}$$

To bound the right side, we use $\|A - \nabla\varphi\|_3^3 \geq \|A\|_3^3 = \|A\|_3^3$ and

$$\int_{\mathbb{R}^3} |A - \nabla\varphi|(A - \nabla\varphi) \cdot F \, dx \geq \int_{\mathbb{R}^3} |A|A \cdot F \, dx - 2\|\nabla\varphi\|_3 \|A\|_3 \|F\|_3 - \|\nabla\varphi\|_3^2 \|F\|_3.$$

In this way we arrive at

$$\begin{aligned} \|A + F\|_3^3 &\geq \|A\|_3^3 + 3 \int_{\mathbb{R}^3} |A|A \cdot F \, dx \\ &\quad - \text{const} (\|A\|_3 \|F\|_3^2 + \|F\|_3^3 + \|\nabla\varphi\|_3 \|A\|_3 \|F\|_3 + \|\nabla\varphi\|_3^2 \|F\|_3) \end{aligned}$$

and it remains to bound $\|\nabla\varphi\|_3$. We note that

$$\nabla \cdot (|A + F - \nabla\varphi|(A + F - \nabla\varphi)) = 0 \quad \text{and} \quad \nabla \cdot (|A|A) = 0.$$

Using the \mathbb{R}^3 -version of Lemma 8 corresponding to $\varepsilon = 0$ (which can be proved by the same argument and is, in fact, much simpler since for $\varepsilon = 0$ Theorem 11 is not needed) we obtain

$$\|\nabla\varphi\|_3^2 \lesssim \|F\|_3 (\|A + F\|_3 + \|A\|_3).$$

Inserting this into the above bound, we obtain the lower bound

$$\|A + F\|_3^3 \geq \|A\|_3^3 + 3 \int_{\mathbb{R}^3} |A|A \cdot F \, dx - \text{const} (\|A\|_3^{3/2} \|F\|_3^{3/2} + \|F\|_3^3),$$

which concludes the proof. \square

7.2. Applying Ekeland's variational principle. In order to prove Proposition 18 we apply Ekeland's variational principle in the following setting. We recall that \mathcal{Y} and \mathcal{M} were defined in (5) and (7), respectively, and we set

$$X := \mathcal{Y}/\mathcal{M}$$

endowed with the norm $\|\nabla \wedge A\|_{3/2}$. Using standard properties of weak derivatives it is easy to see that this is, indeed, a norm and that X endowed with this norm is complete. We claim that the dual space of X is

$$X^* = \{\nabla \wedge B : B \in L^3(\mathbb{R}^3, \mathbb{R}^3)\} \quad (24)$$

with norm

$$\|\nabla \wedge B\|_{X^*} = \inf_{\varphi \in \dot{W}^{1,3}(\mathbb{R}^3)} \|B - \nabla \varphi\|_3,$$

in the sense that every functional $\Phi \in X^*$ is of the form

$$\Phi([A]) = \int_{\mathbb{R}^3} B \cdot (\nabla \wedge A) dx \quad \text{for } [A] \in X,$$

and conversely, any functional of this form defines an element of X^* . This can again be shown by standard arguments.

From Ekeland's variational principle we will deduce the following lemma, which is the core of the proof of Proposition 18.

Lemma 20. *Let $A \in \mathcal{Y}$ with $\|A\|_3 = 1$. Then for any $\delta > 0$ there are $A' \in \mathcal{Y}$ and $\lambda \in \mathbb{R}$ such that*

$$\begin{aligned} \nabla \cdot (|A'|A') &= 0, & \|A'\|_3 &= \|A'\|_3 = 1, \\ \|\nabla \wedge A'\|_{3/2} &\leq \|\nabla \wedge A\|_{3/2}, & \|\nabla \wedge (A' - A)\|_{3/2} &\leq \delta, \end{aligned}$$

and

$$\|\nabla \wedge (|\nabla \wedge A'|^{-1/2} \nabla \wedge A') - \lambda |A'|A'\|_{X^*} \leq 3 \left(\|\nabla \wedge A\|_{3/2}^{3/2} - S \right) \delta^{-1}.$$

Proof of Proposition 18 given Lemma 20. In the situation of Proposition 18 we apply Lemma 20 to $A = A_n$ with the choice $\delta^2 = \varepsilon_n := \|\nabla \wedge A_n\|_{3/2}^{3/2} - S$. We obtain sequences $(A'_n) \subset \mathcal{Y}$ and $(\lambda_n) \subset \mathbb{R}$ such that

$$\begin{aligned} \nabla \cdot (|A'_n|A'_n) &= 0, & \|A'_n\|_3 &= 1, \\ \|\nabla \wedge (A'_n - A_n)\|_{3/2} &\leq \sqrt{\varepsilon_n} \end{aligned} \quad (25)$$

and

$$\|\nabla \wedge (|\nabla \wedge A'_n|^{-1/2} \nabla \wedge A'_n) - \lambda_n |A'_n|A'_n\|_{X^*} \leq 3\sqrt{\varepsilon_n}. \quad (26)$$

Since (A_n) is minimizing, we have $\varepsilon_n \rightarrow 0$ and therefore, by (25), $\nabla \wedge (A'_n - A_n) \rightarrow 0$ in $L^{3/2}$, as claimed.

Let us show that in the almost-Euler–Lagrange equation (26), we can replace λ_n by S . Indeed, since $([A'_n])$ is bounded in X by (25), when testing (26) against $[A'_n]$ we obtain

$$\int_{\mathbb{R}^3} |\nabla \wedge A'_n|^{3/2} dx - \lambda_n \rightarrow 0. \quad (27)$$

Here we also used $\|A'_n\|_3 = 1$. According to (25) and the minimizing property of A_n we have $\int_{\mathbb{R}^3} |\nabla \wedge A'_n|^{3/2} dx \rightarrow S$. Therefore (27) implies $\lambda_n \rightarrow S$ and

$$\begin{aligned} & \|\nabla \wedge (|\nabla \wedge A'_n|^{-1/2} \nabla \wedge A'_n) - S|A'_n|A'_n\|_{X^*} \\ & \leq \|\nabla \wedge (|\nabla \wedge A'_n|^{-1/2} \nabla \wedge A'_n) - \lambda_n|A'_n|A'_n\|_{X^*} + |\lambda_n - S| \| |A'_n|A'_n \|_{X^*} \rightarrow 0. \end{aligned}$$

Here we used the fact that $|A'_n|A'_n$ is uniformly bounded in X^* . Indeed, the Sobolev inequality in Lemma 4 implies by duality that $(L^3/\mathcal{M})^* \subset X^*$ continuously and so,

$$\| |A'_n|A'_n \|_{X^*} \lesssim \| |A'_n|A'_n \|_{(L^3/\mathcal{M})^*} \leq \| |A'_n|A'_n \|_{3/2} = \|A'_n\|_3^2 = 1.$$

To complete the proof, we recall the characterization of X^* in (24), which implies that there is an $r_n \in L^3(\mathbb{R}^3, \mathbb{R}^3)$ such that

$$\nabla \wedge (|\nabla \wedge A'_n|^{-1/2} \nabla \wedge A'_n) - S|A'_n|A'_n = \nabla \wedge r_n.$$

After subtracting a gradient, we can assume that $\nabla \cdot (r_n/r_n) = 0$ and then

$$\|r_n\|_3 = \|\nabla \wedge (|\nabla \wedge A'_n|^{-1/2} \nabla \wedge A'_n) - S|A'_n|A'_n\|_{X^*} \rightarrow 0.$$

This completes the proof of the proposition. \square

It remains to prove Lemma 20. This is almost an immediate consequence of [9, Theorem 3.1], except that it is not obvious to us that the functional $A \mapsto \|A\|_3$ is *continuously* Fréchet differentiable. Its Fréchet differentiability is a consequence of Lemma 19. While it might be possible to show the continuity of its Fréchet derivative, we think it is easier to redo in our setting the reduction of [9, Theorem 3.1] to [9, Theorem 1.1]. The observation is that because of the homogeneity of $A \mapsto \|A\|_3$, one does not need its continuous Fréchet differentiability. In fact, only its Gateaux differentiability suffices. Our proof also shows that the same method works in the case of *complex* Banach spaces. This substantiates our claim in the introduction that the same method works for the Sobolev inequality for spinor fields.

Proof of Lemma 20. We consider the metric space $Z := \{[A] \in X : \|A\|_3 = 1\}$ with the metric induced by the norm in X . As a consequence of the Sobolev inequality in Lemma 4, this metric space is complete. In Z we consider the functional $F([A]) := \|\nabla \wedge A\|_{3/2}$, which is well-defined and continuous. Now given $A \in \mathcal{Y}$ with $\|A\|_3 = 1$ and $\delta > 0$, we deduce from [9, Theorem 1.1] that there is an $[A'] \in Z$ such that

$$\|\nabla \wedge A'\|_{3/2} \leq \|\nabla \wedge A\|_{3/2}, \quad \|\nabla \wedge (A' - A)\|_{3/2} \leq \delta,$$

and such that for any $[A'] \neq [A''] \in Z$,

$$\|\nabla \wedge A''\|_{3/2}^{3/2} > \|\nabla \wedge A'\|_{3/2}^{3/2} - \frac{\varepsilon}{\delta} \|\nabla \wedge (A'' - A')\|_{3/2} \quad (28)$$

with $\varepsilon := \|\nabla \wedge A\|_{3/2}^{3/2} - S$. According to Lemma 5 we can fix the gauge of A' such that $\nabla \cdot (|A'|A') = 0$ and then $\|A'\|_3 = 1$.

It remains to prove the last inequality in the statement of the lemma. Let $F \in \mathcal{Y}$ with

$$\int_{\mathbb{R}^3} |A|A \cdot F \, dx = 0. \quad (29)$$

We apply (28) with $A'' = (A' + tF)/\|A' + tF\|_3$, where $t \geq 0$ is such that $\|A' + tF\|_3 \neq 0$. We conclude that

$$\frac{\|\nabla \wedge (A' + tF)\|_{3/2}^{3/2}}{\|A' + tF\|_3^{3/2}} \geq \|\nabla \wedge A'\|_{3/2}^{3/2} - \frac{\varepsilon}{\delta} \frac{t}{\|A' + tF\|_3} \left\| \nabla \wedge \left(F - \frac{\|A' + tF\|_3 - 1}{t} A' \right) \right\|_{3/2}.$$

Now we use the Gateau differentiability of $\|\cdot\|_{3/2}$ and $\|\cdot\|_3$. Note that Lemma 19 and (29) imply that

$$\|A' + tF\|_3 = 1 + o(t).$$

Thus, we obtain

$$\frac{3}{2} \int_{\mathbb{R}^3} |\nabla \wedge A'|^{-1/2} (\nabla \wedge A') \cdot (\nabla \wedge F) \, dx \geq -\frac{\varepsilon}{\delta} \|\nabla \wedge F\|_{3/2}. \quad (30)$$

By a simple abstract result (see Lemma 21 below), from the fact that for any $F \in \mathcal{Y}$, (29) implies (30), we deduce the existence of a $\lambda \in \mathbb{R}$ such that the last inequality in the statement of Lemma 20 holds. \square

The following lemma holds for normed spaces over \mathbb{K} , where either $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. In the real case it is a special case of [9, Lemma 3.3]. If one were to apply our techniques to the case of spinor fields, one would need the complex case.

Lemma 21. *Let X be a normed space and let $F, G \in X^*$ and $\rho > 0$ such that for any $x \in X$ with $\langle G, x \rangle = 0$ one has $\operatorname{Re} \langle F, x \rangle \geq -\rho \|x\|$. Then there is a $\lambda \in \mathbb{K}$ such that*

$$\|F - \lambda G\| \leq \rho.$$

Proof. By applying the assumption to x times a constant of absolute value one, we see that $|\langle F, x \rangle| \leq \rho \|x\|$ for all $x \in X$ with $\langle G, x \rangle = 0$. Thus, $\|F|_{\ker G}\| \leq \rho$. By Hahn–Banach, there is an $\tilde{F} \in X^*$ such that $\tilde{F}|_{\ker G} = F|_{\ker G}$ and $\|\tilde{F}\| = \|F|_{\ker G}\|$. In particular, $\|\tilde{F}\| \leq \rho$. By construction, $\ker G \subset \ker(\tilde{F} - F)$. Thus, by a well-known algebraic lemma [4, Lemma 3.2] there is a $\lambda \in \mathbb{K}$ such that $\tilde{F} - F = \lambda G$. Thus, $\|F - \lambda G\| = \|\tilde{F}\| \leq \rho$, as claimed. \square

8. STUDY OF THE APPROXIMATE EULER–LAGRANGE EQUATION

In this section we study solutions A'_n to the equations

$$\nabla \wedge (|\nabla \wedge A'_n|^{-1/2}(\nabla \wedge A'_n)) - S|A'_n|A'_n = \nabla \wedge r_n \quad \text{with } r_n \rightarrow 0 \text{ in } L^3(\mathbb{R}^3, \mathbb{R}^3) \quad (31)$$

satisfying

$$\nabla \wedge A'_n \rightharpoonup B \quad \text{in } L^{3/2}(\mathbb{R}^3, \mathbb{R}^3). \quad (32)$$

The functions A'_n are not necessarily those constructed in Proposition 18, although this is the application that we have in mind. Note that (31) and (32) imply

$$\|A'_n\|_3 \lesssim 1. \quad (33)$$

Indeed, (31) implies $\nabla \cdot (|A'_n|A'_n) = 0$ and therefore, by Lemma 5, $\|A'_n\|_3 = \|A'_n\|_3$. The boundedness of $(\nabla \wedge A'_n)$ in $L^{3/2}$, which follows from (32), and Lemma 4 give (33).

Our goal in the two subsections of this section is to prove two lemmas concerning the derivative and the nonderivative terms, respectively, on the left side of (31).

8.1. The truncation argument. We will prove the following convergence result.

Lemma 22. *In the situation (31), (32), we have $\nabla \wedge A_n \rightarrow B$ in $L^p_{\text{loc}}(\mathbb{R}^3, \mathbb{R}^3)$ for any $p < 3/2$.*

This lemma is in the spirit of convergence theorems for quasilinear equations due to Boccardo and Murat [3] and, in the case of systems, Dal Maso and Murat [8]. While our equation does not satisfy the assumptions in [8], after choosing an appropriate gauge we can follow their argument rather closely.

Proof. We prove that every subsequence has a further subsequence along which we have convergence in $L^p_{\text{loc}}(\mathbb{R}^3, \mathbb{R}^3)$ for any $p < 3/2$. This clearly implies the lemma.

We abbreviate $B_n := \nabla \wedge A'_n$. We apply Lemma 3 to A'_n and obtain \tilde{A}_n such that $\nabla \wedge \tilde{A}_n = B_n$ and $\nabla \cdot \tilde{A}_n = 0$. Moreover, the bound in Lemma 3 together with the $L^{3/2}$ -boundedness of (B_n) resulting from (32) implies that \tilde{A}_n is bounded in $\dot{W}^{1,3/2}(\mathbb{R}^3, \mathbb{R}^3)$. Thus, after passing to a subsequence, we have $\tilde{A}_n \rightharpoonup \tilde{A}$ in $\dot{W}^{1,3/2}(\mathbb{R}^3, \mathbb{R}^3)$. The first consequence of this convergence is that $B_n = \nabla \wedge \tilde{A}_n \rightharpoonup \nabla \wedge \tilde{A}$ in $L^{3/2}(\mathbb{R}^3, \mathbb{R}^3)$ and therefore $\nabla \wedge \tilde{A} = B$. The second consequence is that by the Rellich–Kondrachov lemma, $\tilde{A}_n \rightarrow \tilde{A}$ in $L^q_{\text{loc}}(\mathbb{R}^3, \mathbb{R}^3)$ for any $q < 3$. Moreover, after passing to a further subsequence, $\tilde{A}_n \rightarrow \tilde{A}$ almost everywhere.

Let $\psi \in C^1(\mathbb{R}^3, \mathbb{R}^3)$ with $\psi(y) = y$ for $|y| \leq 1$ and $\psi(y) = 0$ for $|y| \geq 2$. For $\delta > 0$ we set $\psi_\delta(y) := \delta\psi(y/\delta)$. Let $\chi \in C^1_c(\mathbb{R}^3)$ and let $(\delta_n) \subset (0, \infty)$ be a bounded sequence

to be specified later and multiply equation (31) by $\chi\psi_{\delta_n}(\tilde{A}_n - \tilde{A})$ to obtain

$$\begin{aligned} & \int_{\mathbb{R}^3} \chi (|B_n|^{-1/2} B_n - |B|^{-1/2} B) \cdot (\nabla \wedge \psi_{\delta_n}(\tilde{A}_n - \tilde{A})) dx \\ &= - \int_{\mathbb{R}^3} |B_n|^{-1/2} B_n \cdot (\nabla \chi \wedge \psi_{\delta_n}(\tilde{A}_n - \tilde{A})) dx - \int_{\mathbb{R}^3} \chi |B|^{-1/2} B \cdot (\nabla \wedge \psi_{\delta_n}(\tilde{A}_n - \tilde{A})) dx \\ & \quad + S \int_{\mathbb{R}^3} \chi |A'_n| A'_n \cdot \psi_{\delta_n}(\tilde{A}_n - \tilde{A}) dx + \int_{\mathbb{R}^3} r_n \cdot \nabla \wedge (\chi \psi_{\delta_n}(\tilde{A}_n - \tilde{A})) dx. \end{aligned}$$

It is not difficult to see that, independently of the choice of (δ_n) , $\nabla \wedge \psi_{\delta_n}(\tilde{A}_n - \tilde{A}) \rightarrow 0$ in $L^{3/2}(\mathbb{R}^3, \mathbb{R}^3)$. This implies that the second term on the right side tends to zero as $n \rightarrow \infty$. Moreover, since $\nabla \wedge \psi_{\delta_n}(\tilde{A}_n - \tilde{A})$ is bounded in $L^{3/2}(\mathbb{R}^3)$ and r_n tends to zero in $L^3(\mathbb{R}^3, \mathbb{R}^3)$, the fourth term on the right side tends to zero as $n \rightarrow \infty$. Thus,

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} \chi (|B_n|^{-1/2} B_n - |B|^{-1/2} B) \cdot (\nabla \wedge \psi_{\delta_n}(\tilde{A}_n - \tilde{A})) dx \\ & \leq \limsup_{n \rightarrow \infty} \left(\|B_n\|_{3/2}^{1/2} \|\nabla \chi\|_{3/2} \|\psi_{\delta_n}(\tilde{A}_n - \tilde{A})\|_{\infty} + S \|A'_n\|_3^2 \|\chi\|_3 \|\psi_{\delta_n}(\tilde{A}_n - \tilde{A})\|_{\infty} \right) \\ & \leq (\Gamma \|\nabla \chi\|_{3/2} + S \Gamma' \|\chi\|_3) M \limsup_{n \rightarrow \infty} \delta_n \end{aligned} \quad (34)$$

with $M := \sup |\psi|$, $\Gamma := \limsup_{n \rightarrow \infty} \|B_n\|_{3/2}^{1/2}$ and $\Gamma' := \limsup_{n \rightarrow \infty} \|A'_n\|_3^2$. Note that Γ and Γ' are finite by (32) and (33). We bound

$$\begin{aligned} & \int_{\mathbb{R}^3} \chi (|B_n|^{-1/2} B_n - |B|^{-1/2} B) \cdot (\nabla \wedge \psi_{\delta_n}(\tilde{A}_n - \tilde{A})) dx \\ & \geq \int_{\{|\tilde{A}_n - \tilde{A}| \leq \delta_n\}} \chi e_n dx - \|\chi\|_{\infty} M' \int_{\{\delta_n < |\tilde{A}_n - \tilde{A}| \leq 2\delta_n\}} h_n dx \end{aligned} \quad (35)$$

with $M' := \sup |\nabla \otimes \psi|$ and

$$\begin{aligned} e_n &:= (|B_n|^{-1/2} B_n - |B|^{-1/2} B) \cdot (B_n - B), \\ h_n &:= (|B_n|^{1/2} + |B|^{1/2}) |\nabla \otimes (\tilde{A}_n - \tilde{A})|. \end{aligned}$$

Here we used $|\nabla \wedge F| \leq |\nabla \otimes F|$ and $|\nabla \otimes \psi(G)| \leq M' |\nabla \otimes G|$.

By Lemma 3, with C denoting the implicit constant in the first bound there,

$$\begin{aligned} \int_{\mathbb{R}^3} h_n dx & \leq \| |B_n|^{1/2} + |B|^{1/2} \|_3 \|\nabla \otimes (\tilde{A}_n - \tilde{A})\|_{3/2} \\ & \leq C \left(\|B_n\|_{3/2}^{1/2} + \|B\|_{3/2}^{1/2} \right) (\|B_n\|_{3/2} + \|B\|_{3/2}), \end{aligned}$$

so $\limsup_{n \rightarrow \infty} \|h_n\|_1 \leq 4C\Gamma^3$ and, in particular, h_n is bounded in L^1 .

We fix two parameters $0 < \varepsilon < \varepsilon'$ and choose the sequence (δ_n) depending on those parameters as follows. We have

$$\int_{\varepsilon}^{\varepsilon'} \int_{\{\delta < |\tilde{A}_n - \tilde{A}| \leq 2\delta\}} h_n dx \frac{d\delta}{\delta} \leq \int_{\mathbb{R}^3} h_n \int_{|\tilde{A}_n(x) - \tilde{A}(x)|/2}^{|\tilde{A}_n(x) - \tilde{A}(x)|} \frac{d\delta}{\delta} dx = (\ln 2) \|h_n\|_1$$

and

$$\int_{\varepsilon}^{\varepsilon'} \frac{d\delta}{\delta} = \ln \frac{\varepsilon'}{\varepsilon}.$$

Thus, for each n there is a $\delta_n \in [\varepsilon, \varepsilon']$ such that

$$\int_{\{\delta_n < |\tilde{A}_n - \tilde{A}| \leq 2\delta_n\}} h_n dx \leq \frac{(\ln 2) \|h_n\|_1}{\ln(\varepsilon'/\varepsilon)}.$$

From now on, we work with this choice of δ_n .

It is elementary to see that

$$(|v|^{-1/2}v - |w|^{-1/2}w) \cdot (v - w) \geq (|v|^2 + |w|^2)^{-1/4} |v - w|^2 \quad \text{for all } v, w \in \mathbb{R}^3, \quad (36)$$

and therefore

$$e_n \geq (|B_n|^2 + |B|^2)^{-1/4} |B_n - B|^2 \geq 0. \quad (37)$$

Assuming, in addition, that $\chi \geq 0$, we can bound, using the choice of δ_n and (35),

$$\begin{aligned} \int_{\{|\tilde{A}_n - \tilde{A}| \leq \varepsilon\}} \chi e_n dx &\leq \int_{\{|\tilde{A}_n - \tilde{A}| \leq \delta_n\}} \chi e_n dx \\ &\leq \int_{\mathbb{R}^3} \chi (|B_n|^{-1/2} B_n - |B|^{-1/2} B) \cdot (\nabla \wedge \psi_{\delta_n}(\tilde{A}_n - \tilde{A})) dx \\ &\quad + \frac{(\ln 2) \|\chi\|_{\infty} M' \|h_n\|_1}{\ln(\varepsilon'/\varepsilon)}. \end{aligned}$$

Thus, in view of (34),

$$\begin{aligned} \limsup_{n \rightarrow \infty} \int_{\{|\tilde{A}_n - \tilde{A}| \leq \varepsilon\}} \chi e_n dx &\leq (\Gamma \|\nabla \chi\|_{3/2} + S\Gamma' \|\chi\|_3) M \limsup_{n \rightarrow \infty} \delta_n \\ &\quad + \frac{(\ln 2) \|\chi\|_{\infty} M' \limsup_{n \rightarrow \infty} \|h_n\|_1}{\ln(\varepsilon'/\varepsilon)} \\ &\leq (\Gamma \|\nabla \chi\|_{3/2} + S\Gamma' \|\chi\|_3) M\varepsilon' + \frac{4C\Gamma^3(\ln 2) \|\chi\|_{\infty} M'}{\ln(\varepsilon'/\varepsilon)}. \end{aligned}$$

Now let $0 < \theta < 1$ and bound

$$\int_{\mathbb{R}^3} \chi e_n^{\theta} dx \leq \|\chi\|_1^{1-\theta} \left(\int_{\{|\tilde{A}_n - \tilde{A}| \leq \varepsilon\}} \chi e_n dx \right)^{\theta} + \|\chi\|_{\infty}^{\theta} \left(\int_{\mathbb{R}^3} e_n dx \right)^{\theta} \left(\int_{\{|\tilde{A}_n - \tilde{A}| > \varepsilon\}} \chi dx \right)^{1-\theta}.$$

Since e_n is bounded in $L^1(\mathbb{R}^3)$ and since $\tilde{A}_n \rightarrow \tilde{A}$ almost everywhere, dominated convergence implies that

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} \chi e_n^{\theta} dx \leq \|\chi\|_1^{1-\theta} \limsup_{n \rightarrow \infty} \left(\int_{\{|\tilde{A}_n - \tilde{A}| \leq \varepsilon\}} \chi e_n dx \right)^{\theta}.$$

Inserting the bound on the limsup on the right side, we obtain

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} \chi e_n^{\theta} dx \leq \|\chi\|_1^{1-\theta} \left((\Gamma \|\nabla \chi\|_{3/2} + S\Gamma' \|\chi\|_3) M\varepsilon' + \frac{4C\Gamma^3(\ln 2) \|\chi\|_{\infty} M'}{\ln(\varepsilon'/\varepsilon)} \right)^{\theta}.$$

Letting first $\varepsilon \rightarrow 0$ and then $\varepsilon' \rightarrow 0$, we find

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} \chi e_n^\theta dx = 0,$$

and therefore $e_n^\theta \rightarrow 0$ in $L_{\text{loc}}^1(\mathbb{R}^3)$. According to the following lemma, this implies $B_n \rightarrow B$ in $L_{\text{loc}}^p(\mathbb{R}^3, \mathbb{R}^3)$ for any $p < 3/2$. This completes the proof. \square

Lemma 23. *Let E be a set of finite measure and let $(F_n) \subset L^{3/2}(E, \mathbb{R}^3)$ be bounded. Assume that for some $F \in L^{3/2}(E, \mathbb{R}^3)$ and some $\theta > 0$, one has*

$$\left((|F_n|^{-1/2} F_n - |F|^{-1/2} F) \cdot (F_n - F) \right)^\theta \rightarrow 0 \quad \text{in } L^1(E).$$

Then $F_n \rightarrow F$ in $L^p(E)$ for any $1 \leq p < 3/2$.

Proof. We show that any subsequence has a further subsequence along which $F_n \rightarrow F$ in $L^p(E)$ for any $p < 3/2$. This clearly implies the lemma.

The assumption implies that, given a subsequence, there is a further subsequence along which one has $(|F_n|^{-1/2} F_n - |F|^{-1/2} F) \cdot (F_n - F) \rightarrow 0$ almost everywhere in E . By (36), this implies $F_n \rightarrow F$ almost everywhere on E . By Vitali's convergence theorem, this, together with the boundedness of (F_n) in $L^{3/2}$, implies the assertion. \square

Looking back at the proof of Lemma 22, one might wonder why we passed from A'_n to \tilde{A}_n . This was needed at two places. First, in the bound on $\int \chi e_n^\theta dx$ we used the fact that $\tilde{A}_n \rightarrow \tilde{A}$ almost everywhere and second, in the bound on $\nabla \wedge \psi_{\delta_n}(\tilde{A}_n - \tilde{A})$, we needed the full Jacobi matrix $\nabla \otimes (\tilde{A}_n - \tilde{A})$ and not only $\nabla \wedge (\tilde{A}_n - \tilde{A})$. While we could get around the first item by appealing to Theorem 6, we do not know how perform the truncation in the second item in the gauge of Lemma 5.

8.2. Application of the nonlinear Rellich–Kondrachov lemma. Independently of the convergence result in the previous subsection about $\nabla \wedge A'_n$, we will now prove a convergence result for A'_n . This is where the nonlinear Rellich–Kondrachov lemma in Theorem 6 enters.

Lemma 24. *In the situation (31), (32), there is an $A' \in \mathcal{Y}$ with $\nabla \wedge A' = B$ and $\nabla \cdot (|A'|A') = 0$ such that $A'_n \rightharpoonup A'$ in $L^3(\mathbb{R}^3, \mathbb{R}^3)$ and $A'_n \rightarrow A'$ in $L_{\text{loc}}^q(\mathbb{R}^3, \mathbb{R}^3)$ for any $q < 3$.*

Proof. Let $A' \in \mathcal{Y}$ with $\nabla \wedge A' = B$ and $\nabla \cdot (|A'|A') = 0$. Such an A' exists, for define \tilde{A} by (6), which belongs to L^3 by Hardy–Littlewood–Sobolev, and then apply Lemma 5 to pass from \tilde{A} to A' by changing the gauge.

Note that by (31) we have $\nabla \cdot (|A'_n|A'_n) = 0$. Therefore, by Theorem 6, we have that $A'_n \rightarrow A'$ in $L_{\text{loc}}^q(\mathbb{R}^3, \mathbb{R}^3)$ for any $q < 3$.

On the other hand, since (A'_n) is bounded in L^3 by (33), a subsequence converges weakly in L^3 to some A . Because of the L_{loc}^q convergence to A we conclude that

$A = A'$. Applying this argument to a sub-subsequence of an arbitrary subsequence, we obtain the claimed weak convergence in L^3 of the full sequence. \square

9. COMPLETION OF THE PROOF

We are now in position to prove our main result, Theorem 1. Let $(A_n) \subset \mathcal{Y}$ be a minimizing sequence for S with $\|A_n\|_3 = 1$. By Proposition 17, after passing to a subsequence and after a translation and dilation, which we do not reflect in the notation, we have $\nabla \wedge A_n \rightharpoonup B$ in $L^{3/2}$ for some $B \neq 0$. According to Proposition 18, there is a sequence $(A'_n) \subset \mathcal{Y}$ with $\nabla \cdot (|A'_n|A'_n) = 0$ and $\|A'_n\|_3 = 1$ for all n such that

$$\nabla \wedge A'_n - \nabla \wedge A_n \rightarrow 0 \quad \text{in } L^{3/2}(\mathbb{R}^3, \mathbb{R}^3) \quad (38)$$

and

$$\nabla \wedge (|\nabla \wedge A'_n|^{-1/2} \nabla \wedge A'_n) - S|A'_n|A'_n = \nabla \wedge r_n \quad \text{with } r_n \rightarrow 0 \text{ in } L^3(\mathbb{R}^3, \mathbb{R}^3). \quad (39)$$

It follows from Lemmas 22 and 24 that there is an $A' \in \mathcal{Y}$ with $\nabla \wedge A' = B$ and $\nabla \cdot (|A'|A') = 0$ such that $A'_n \rightharpoonup A'$ in $L^3(\mathbb{R}^3, \mathbb{R}^3)$ and such that for all $F \in C_c^1(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} |A'_n|A'_n \cdot F \, dx \rightarrow \int_{\mathbb{R}^3} |A'|A' \cdot F \, dx \quad (40)$$

and

$$\int_{\mathbb{R}^3} |\nabla \wedge A'_n|^{-1/2} (\nabla \wedge A'_n) \cdot (\nabla \wedge F) \, dx \rightarrow \int_{\mathbb{R}^3} |\nabla \wedge A'|^{-1/2} (\nabla \wedge A') \cdot (\nabla \wedge F) \, dx. \quad (41)$$

Because of (40) and (41) we deduce from (39) that

$$\nabla \wedge (|\nabla \wedge A'|^{-1/2} (\nabla \wedge A')) - S|A'|A' = 0. \quad (42)$$

Testing (42) with A' , we obtain

$$\|\nabla \wedge A'\|_{3/2}^{3/2} - S\|A'\|_3^3 = 0$$

and therefore

$$\frac{\|\nabla \wedge A'\|_{3/2}^{3/2}}{\|A'\|_3^{3/2}} = S\|A'\|_3^{3/2} \leq S \quad (43)$$

since, by weak convergence, $\|A'\|_3 \leq \liminf_{n \rightarrow \infty} \|A'_n\|_3 = 1$. Note also that $A' \neq 0$ since $B = \nabla \wedge A' \neq 0$. By definition of S , (43) implies that A' is a minimizer for S and that

$$\|A'\|_3 = 1 \quad \text{and} \quad \|\nabla \wedge A'\|_{3/2} = S^{3/2}.$$

Equality in the lower semicontinuity inequalities

$$\|A'\|_3 \leq \lim_{n \rightarrow \infty} \|A'_n\|_3 = 1 \quad \text{and} \quad \|\nabla \wedge A'\|_{3/2} \leq \lim_{n \rightarrow \infty} \|\nabla \wedge A'_n\|_{3/2} = S^{3/2}$$

implies, by [23, Theorem 2.11], that (A'_n) and $(\nabla \wedge A'_n)$ converge strongly to A' in L^3 and to $\nabla \wedge A'$ in $L^{3/2}$, respectively.

Let us now pass from the sequence (A'_n) to the original sequence (A_n) . Because of (38) we also have that $\nabla \wedge A_n \rightarrow \nabla \wedge A'$ strongly in $L^{3/2}$.

Let us assume that the gauge of the A_n was fixed as in Lemma 5 by requiring $\nabla \cdot (|A_n|A_n) = 0$. Of course, this condition is preserved under the translations and dilations that are performed in the above proof. It follows from Theorem 6 that $A_n \rightarrow A'$ in $L^q_{\text{loc}}(\mathbb{R}^3, \mathbb{R}^3)$ for any $q < 3$. This, together with the boundedness of A_n in L^3 , implies by the same argument as in the proof of Lemma 24 that $A_n \rightarrow A'$ in L^3 . From $\|A_n\|_3 = 1$ and $\|A'\|_3 = 1$ we deduce as before that $A_n \rightarrow A'$ strongly in L^3 . This concludes the proof of Theorem 1.

10. PROOF OF THEOREM 2

Let $(A_n) \subset \mathcal{Y}$ be a minimizing sequence for Σ , normalized such that $\nabla \cdot (|A_n|A_n) = 0$, and let $\psi_n \in L^3(\mathbb{R}^3, \mathbb{C}^2)$ be a corresponding sequence such that $\sigma \cdot (-i\nabla - A_n)\psi_n = 0$. By homogeneity we may assume that $\|\psi_n\|_3 = 1$. Since (A_n) is a minimizing sequence, $\|\nabla \wedge A_n\|_{3/2}$ is bounded and therefore, by the Sobolev inequality (Lemma 4), $\|A_n\|_3 = \|A_n\|_3 \lesssim \|\nabla \wedge A_n\|_{3/2} \lesssim 1$. Moreover, by the zero mode equation, we have

$$\|\sigma \cdot (-i\nabla)\psi_n\|_{3/2} = \|\sigma \cdot A_n\psi_n\|_{3/2} = \| |A_n|\psi_n \|_{3/2} \leq \|A_n\|_3 \|\psi_n\|_3 \lesssim 1.$$

Applying the improved inequality (22) to our sequence, we deduce that

$$\sup_{t>0} t \|e^{t\Delta} \sigma \cdot (-i\nabla)\psi_n\|_\infty \gtrsim 1.$$

Thus, by the same argument as in Proposition 17, after translations and dilations, we can pass to a subsequence that converges weakly in $\dot{W}^{1,3/2}(\mathbb{R}^3, \mathbb{C}^2)$ to a limit $\tilde{\psi} \neq 0$. By the usual Rellich–Kondrachov theorem we infer that it converges strongly in $L^q_{\text{loc}}(\mathbb{R}^3, \mathbb{C}^2)$ for any $q < 3$.

We translate and rescale the A_n accordingly and note that these operations preserve the zero mode equation. By passing to another subsequence, we can ensure that $(\nabla \wedge A_n)$ converges weakly in $L^{3/2}(\mathbb{R}^3, \mathbb{R}^3)$ to some \tilde{B} . By the same argument as in the proof of Lemma 24 we deduce that there is an $\tilde{A} \in \mathcal{Y}$ with $\nabla \wedge \tilde{A} = \tilde{B}$ and $\nabla \cdot (|\tilde{A}|\tilde{A}) = 0$ such that $A_n \rightarrow \tilde{A}$ in $L^3(\mathbb{R}^3, \mathbb{R}^3)$ and $A_n \rightarrow \tilde{A}$ in $L^q_{\text{loc}}(\mathbb{R}^3, \mathbb{R}^3)$ for any $q < 3$. This step uses our nonlinear Rellich–Kondrachov theorem.

Consequently, $\sigma \cdot A_n\psi_n \rightarrow \sigma \cdot \tilde{A}\tilde{\psi}$ in $L^q_{\text{loc}}(\mathbb{R}^3, \mathbb{R}^3)$ for any $q < 3/2$. This allows us to pass to the limit in the distributional formulation of the zero mode equation and to conclude that $\sigma \cdot (-i\nabla - \tilde{A})\tilde{\psi} = 0$.

By weak convergence, one has

$$\|\nabla \wedge \tilde{A}\|_{3/2} \leq \liminf_{n \rightarrow \infty} \|\nabla \wedge A_n\|_{3/2} = \Sigma. \quad (44)$$

By definition of Σ and the fact that \tilde{A} admits a zero mode $\tilde{\psi} \neq 0$, we deduce that equality holds in (44). This means that \tilde{A} is a minimizer and that the convergence of $(\nabla \wedge A_n)$ to $\nabla \wedge \tilde{A}$ is strong in $L^{3/2}(\mathbb{R}^3, \mathbb{R}^3)$. This completes the proof of Theorem 2. \square

REFERENCES

- [1] T. Aubin, *Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire*. J. Math. Pures Appl. (9) **55** (1976), no. 3, 269–296.
- [2] T. Aubin, *Problèmes isopérimétriques et espaces de Sobolev*. J. Differential Geometry **11** (1976), no. 4, 573–598.
- [3] L. Boccardo, F. Murat, *Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations*. Nonlinear Anal. **19** (1992), no. 6, 581–597.
- [4] H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011.
- [5] H. Brezis, E. H. Lieb, *A relation between pointwise convergence of functions and convergence of functionals*. Proc. Amer. Math. Soc. **88** (1983), no. 3, 486–490.
- [6] H. Brezis, E. H. Lieb, *Minimum action solutions of some vector field equations*. Comm. Math. Phys. **96** (1984), no. 1, 97–113.
- [7] F. Cobos, T. Kühn, *Extrapolation results of Lions–Peetre type*. Calc. Var. Partial Differential Equations **49** (2014), no. 1-2, 847–860.
- [8] G. Dal Maso, F. Murat, *Almost everywhere convergence of gradients of solutions to nonlinear elliptic systems*. Nonlinear Anal. **31** (1998), no. 3-4, 405–412.
- [9] I. Ekeland, *On the variational principle*. J. Math. Anal. Appl. **47** (1974), 324–353.
- [10] R. L. Frank, E. H. Lieb, *A compactness lemma and its application to the existence of minimizers for the liquid drop model*. SIAM J. Math. Anal. **47** (2015), no. 6, 4436–4450.
- [11] R. L. Frank, M. Loss, *Which magnetic fields support a zero mode?* Preprint (2020), arXiv:2012.13646.
- [12] J. Fröhlich, E. H. Lieb, M. Loss, *Stability of Coulomb systems with magnetic fields. I. The one-electron atom*. Comm. Math. Phys. **104** (1986), no. 2, 251–270.
- [13] J. P. García Azorero, I. Peral Alonso, *Hardy inequalities and some critical elliptic and parabolic problems*. J. Differential Equations **144** (1998), no. 2, 441–476.
- [14] M. Gazzini, R. Musina, *On a Sobolev-type inequality related to the weighted p -Laplace operator*. J. Math. Anal. Appl. **352** (2009), no. 1, 99–111.
- [15] P. Gérard, *Description du défaut de compacité de l’injection de Sobolev*. ESAIM Control Optim. Calc. Var. **3** (1998), 213–233.
- [16] P. Gérard, Y. Meyer, F. Oru, *Inégalités de Sobolev précisées*. Séminaire sur les Équations aux Dérivées Partielles, 1996–1997, Exp. No. IV, 11 pp., École Polytech., Palaiseau, 1997.
- [17] L. Greco, T. Iwaniec, C. Sbordone, *Inverting the p -harmonic operator*. Manuscripta Math. **92** (1997), no. 2, 249–258.
- [18] T. Iwaniec, *p -harmonic tensors and quasiregular mappings*. Ann. of Math. (2) **136** (1992), no. 3, 589–624.
- [19] T. Iwaniec, C. Sbordone, *Weak minima of variational integrals*. J. Reine Angew. Math. **454** (1994), 143–161.
- [20] T. Iwaniec, C. Scott, B. Stroffolini, *Nonlinear Hodge theory on manifolds with boundary*. Ann. Mat. Pura Appl. (4) **177** (1999), 37–115.
- [21] E. H. Lieb, *Existence and uniqueness of the minimizing solution of Choquard’s nonlinear equation*. Studies in Appl. Math. **57** (1976/77), no. 2, 93–105.
- [22] E. H. Lieb, *Sharp constants in the Hardy–Littlewood–Sobolev and related inequalities*. Ann. of Math. (2) **118** (1983), no. 2, 349–374.

- [23] E. H. Lieb, M. Loss, *Analysis*. Second edition. Graduate Studies in Mathematics, 14. American Mathematical Society, Providence, RI, 2001.
- [24] P.-L. Lions, *The concentration-compactness principle in the calculus of variations. The limit case. I*. Rev. Mat. Iberoamericana 1 (1985), no. 1, 145–201.
- [25] P.-L. Lions, *The concentration-compactness principle in the calculus of variations. The limit case. II*. Rev. Mat. Iberoamericana 1 (1985), no. 2, 45–121.
- [26] M. Loss, H.-Z. Yau, *Stability of Coulomb systems with magnetic fields. III. Zero energy bound states of the Pauli operator*. Comm. Math. Phys. **104** (1986), no. 2, 283–290.
- [27] E. Rodemich, *The Sobolev inequalities with best possible constants*. Analysis Seminar at California Institute of Technology, 25 pp., 1966.
- [28] E. M. Stein, A. Zygmund, *Boundedness of translation invariant operators on Hölder spaces and L^p -spaces*. Ann. of Math. (2) **85** (1967), 337–349.
- [29] W. A. Strauss, *Existence of solitary waves in higher dimensions*. Comm. Math. Phys. **55** (1977), no. 2, 149–162.
- [30] G. Talenti, *Best constant in Sobolev inequality*. Ann. Mat. Pura Appl. (4) **110** (1976), 353–372.
- [31] N. S. Trudinger, *Remarks concerning the conformal deformation of Riemannian structures on compact manifolds*. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3) **22** (1968), 265–274.
- [32] M. Willem, *Minimax theorems*. Progress in Nonlinear Differential Equations and their Applications, 24. Birkhäuser Boston, Inc., Boston, MA, 1996.
- [33] H. Yamabe, *On a deformation of Riemannian structures on compact manifolds*. Osaka Math. J. **12** (1960), 21–37.

(Rupert L. Frank) MATHEMATISCHES INSTITUT, LUDWIG-MAXIMILIANS UNIVERSITÄT MÜNCHEN, THERESIENSTR. 39, 80333 MÜNCHEN, GERMANY, AND MUNICH CENTER FOR QUANTUM SCIENCE AND TECHNOLOGY, SCHELLINGSTR. 4, 80799 MÜNCHEN, GERMANY, AND MATHEMATICS 253-37, CALTECH, PASADENA, CA 91125, USA

Email address: r.frank@lmu.de

(Michael Loss) SCHOOL OF MATHEMATICS, GEORGIA INSTITUTE OF TECHNOLOGY, ATLANTA, GA 30332-0160, USA

Email address: loss@math.gatech.edu