Research Notes

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Dependence of Shock-Tube Boundary Layers on Shock Strength

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Using familiar considerations of vorticity transport, the essential behavior of shock-tube boundary layers is explicitly exhibited; the nearly constant functions that cannot be calculated analytically are evaluated numerically and compared with Mirels' correlations.

Internal flows generated by moving shock waves in shock tubes are notoriously sensitive to boundary-layer effects; nonuniformities in the flow behind the shock,[1] shock-wave attenuation,[2] and reduction of testing time[3] are caused by boundary-layer displacement effects. The boundary layer behind a shock wave (or sufficiently far behind an expansion wave) moving at constant speed is steady in a wave-fixed coordinate system except near the contact surface (Fig. 1), but is fundamentally different from the familiar flat-plate Blasius boundary layer, because the velocity at the wall u_w is nonzero; this difference is not trivial because the boundary-layer equations are nonlinear. The behavior of laminar shock-tube boundary layers has previously been studied using an analytical approximation based on the weak-shock limit,[4] and using numerical solutions of the full boundary-layer equations.[5] In Ref. 4, Rott and Hartunian make use of the fact[6] that for weak shock waves fluid velocities are small, so the boundary-layer equation is linear and the flow is identical to that generated by an impulsively accelerated flat plate (Rayleigh problem). They show that the velocity profiles for finite-strength shocks can be collapsed onto the weak-shock profile by normalizing the velocity with the decrement $u_w - u_e$, and by scaling the distance normal to the wall with a "thickness"

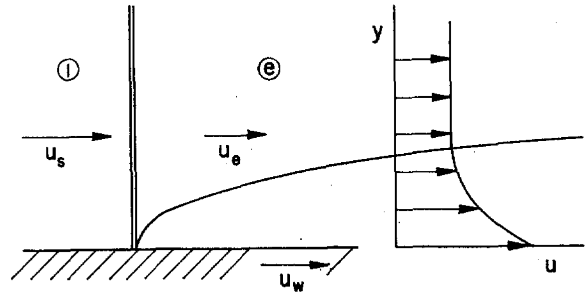

FIG. 1. Shock-tube boundary layer in shock-fixed coordinates.

whose dependence on u_w and u_e (i.e., shock strength) is determined by an integral condition on the profile. In Ref. 5 Mirels summarizes the results of his extensive numerical calculations with empirical correlation formulas for various boundary-layer parameters (thickness, wall shear, etc.) which are accurate for all shock strengths and for a variety of real-gas effects. In this note it is shown that, using familiar considerations of vorticity transport to define the boundary-layer coordinate, the velocity profiles can be reduced to a nearly universal profile, and the essential dependence of boundary-layer parameters on shock strength can be written down explicitly. The slowly varying functions that cannot be calculated analytically are evaluated numerically and compared with Mirels' power-law correlations.

In the wave-fixed coordinate system (Fig. 1) the wall velocity u_w is equal to the velocity of the wave relative to the upstream flow u_s, and the flow velocity outside the boundary layer u_e is given by, e.g., the Rankine–Hugoniot conditions. For a shock wave $u_w/u_e > 1$, for an expansion wave $0 < u_w/u_e < 1$, and for the ordinary Blasius boundary layer $u_w = 0$. The order of magnitude of the boundary-layer thickness δ can be deduced in the usual way by comparing the rate of diffusion of vorticity in the direction normal to the wall with the rate of convection of vorticity in the streamwise direction. Consider, for the moment, the vorticity transport equation for incompressible flow,

$$u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y} = \nu \frac{\partial^2 \zeta}{\partial y^2}, \qquad (1)$$

where $\zeta = -\partial u/\partial y$. It can be seen that the diffusive flux $= O(\nu_e/\delta^2)$, while transport by convection $= O(u^*/x)$ where u^* is some characteristic streamwise velocity. Here, there are *two* such velocities, u_e and u_w, and a reasonable choice for the effective convection velocity in the layer would seem to be the algebraic mean, $u^* = (u_w + u_e)/2$. Thus, balancing convective and diffusive transport,

$$\delta = \left(\frac{2\nu_e x}{u_w + u_e}\right)^{1/2}. \qquad (2)$$

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Equation (2) shows how the thickness of the vorticity boundary layer depends on u_w over the whole range from $u_w = 0$ to $u_w \gg u_e$, and suggests the proper scaling for the normal coordinate y.

Define a boundary-layer variable

$$\eta = \left(\frac{u_w + u_e}{2\nu_e x}\right)^{1/2} \int_0^y \frac{\rho}{\rho_e}\, dy, \tag{3}$$

and a normalized stream function f such that

$$f'(\eta) = \frac{u - u_e}{u_w - u_e}. \tag{4}$$

Then, if $\rho\mu = $ const, the x-momentum equation and boundary conditions become

$$f''' + [\epsilon\eta + (1 - 2\epsilon)f]f'' = 0,$$

$$f'(0) = 1, \qquad f'(\infty) = 0, \qquad f(0) = 0, \tag{5}$$

where $\epsilon = u_e/(u_w + u_e)$. Three limiting cases are of interest:

(1) $\epsilon = 1$ ($u_w = 0$; flat-plate boundary layer). In this case Eq. (5) reduces to the Blasius equation (and boundary conditions) for the dependent variable $g = \eta - f$.

(2) $\epsilon = 1/2$ ($u_w = u_e$; weak shock waves). Here, Eq. (5) reduces to the heat equation and $f' = \text{erfc}\,(\eta/2)$; as is well known,[6] the flow generated by a weak shock wave is identical to that due to the impulsive acceleration of an infinite flat plate (Raleigh problem).

(3) $\epsilon \rightarrow 0$ ($u_w \gg u_e$; *very* strong shock waves). In this case Eq. (5) is again the Blasius equation, but the boundary conditions at the wall and at infinity are different than for the usual flat-plate boundary layer; due to the nonlinearity of the differential equation this difference is nontrivial and the solution is not the same as in Case 1, above.

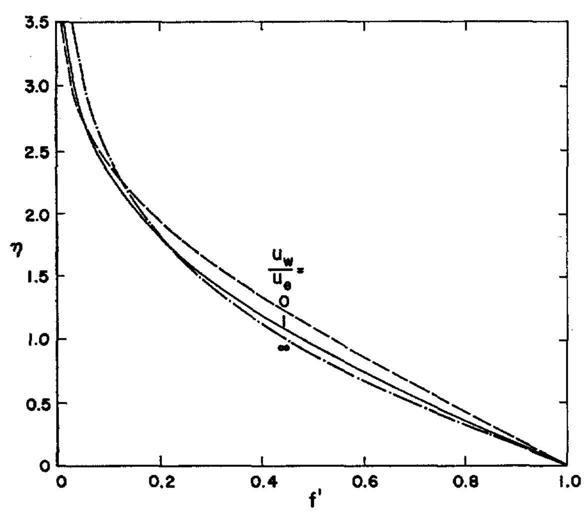

FIG. 2. Normalized velocity profiles for three limiting cases.

The profiles $f'(\eta)$ for these three limiting cases are plotted in Fig. 2. It can be seen that with the definitions Eqs. (3) and (4), there is very little dependence of $f'(\eta)$ on u_w/u_e and that the profiles are very nearly universal.

In considering the energy equation, we assume, for present purposes, a perfect gas and Pr = 1. Then, for the shock tube, where the wall enthalpy equals the upstream enthalpy, $h_w = h_1$, the energy integral is simply

$$h + \frac{u^2}{2} = \text{const}$$

or

$$\frac{h}{h_e} - 1 = \left(\frac{h_w}{h_e} - 1\right)\frac{u^2 - u_e^2}{u_w^2 \quad u_e^2}. \tag{6}$$

Using Eqs. (3), (4), and (6), expressions can be written for the displacement thickness δ^*, wall shear τ_w, and heat transfer q_w in terms of $f(\eta)$, where

$$\delta^* = \int_0^\infty \left(1 - \frac{\rho u}{\rho_e u_e}\right) dy,$$

$$\tau_w = \mu_w\left(\frac{\partial u}{\partial y}\right)_w, \qquad q_w = -k_w\left(\frac{\partial T}{\partial y}\right)_w.$$

Thus,

$$\delta^* = -\delta(U - 1)f(\infty)\left\{1 + \frac{1 - H}{U - 1} \cdot\left[2 - (1 + U)\left(1 + U + \frac{f''(0)}{f(\infty)}\right)\right]\right\}, \tag{7}$$

$$\tau_w = \frac{1}{2}\rho_e u_e^2 \frac{\delta}{x}(U^2 - 1)f''(0), \tag{8}$$

$$q_w = -\tau_w u_w, \tag{9}$$

where, for brevity, $U = u_w/u_e$ and $H = h_w/h_e$.

The quantities $f''(0)$ and $f(\infty)$ have been evaluated for various u_w/u_e by solving Eq. (5) using a numerical shooting method[7] similar to that developed by Smith and Clutter[8]; the results are plotted in Figs. 3 and 4. It can be seen that Eqs. (7)–(9) explicitly exhibit most of the dependence of the boundary-layer parameters on u_w/u_e (i.e., shock strength), and that quite accurate values can be calculated from (7)–(9) assuming simply that $f''(0)$ and $f(\infty)$ are constant. However, very precise values for boundary-layer parameters can be obtained by using Eqs. (7)–(9) together with Figs. 3 and 4. The empirical formulas developed by Mirels[5] have been compared with the present results; his original and improved power-law formulas for what are effectively $f(\infty)$ and $f''(0)$ are also plotted in Figs. 3 and 4.[9]

It has been shown that for shock-tube boundary layers the normal coordinate should be scaled with

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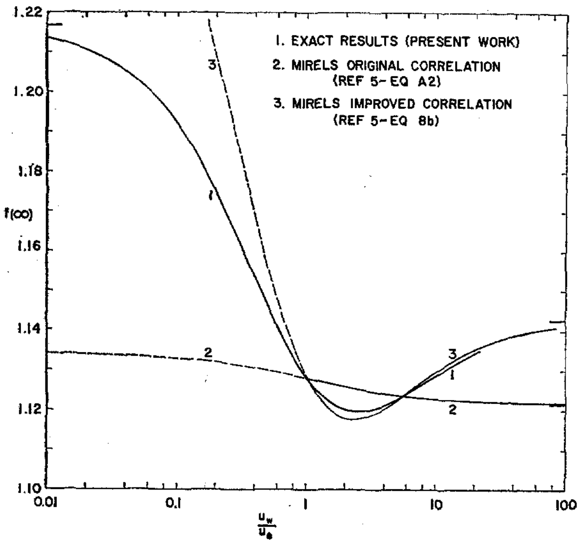

FIG. 3. $f(\infty)$ for shock-tube boundary layers.

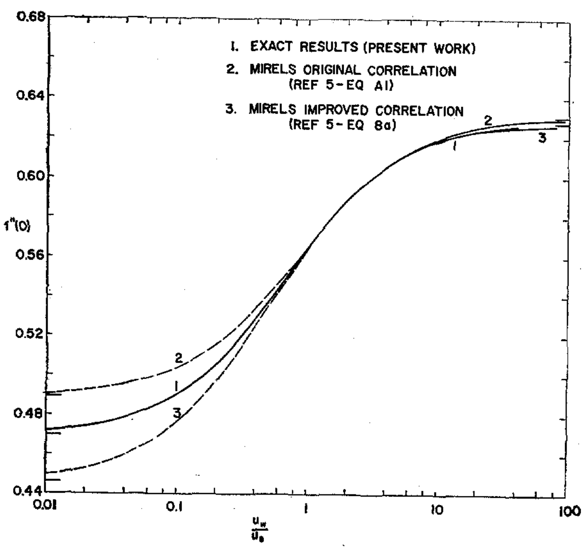

FIG. 4. $f''(0)$ for shock-tube boundary layers.

the *sum* of the wall and free-stream velocities. If this is done, the dependence of boundary-layer parameters on shock strength can be written down explicitly, except for a small contribution that must be evaluated numerically. The results of the present work can easily be extended to fluids in which $\rho\mu \neq$ const and Pr $\neq 1$ (cf. Ref. 5). More interesting is the extension to flows with pressure gradients (due to, e.g., deceleration of the flow behind the shock wave); these effects are presently being studied and will be reported in a subsequent communication.

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[9] These formulas were developed for $1 < u_w/u_e < \infty$, and are not expected to be valid for $0 < u_w/u_e < 1$.

Critical β from Stellarator and Scyllac Expansions

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Toroidal equilibria with helical windings have been studied using two scalings, low β and finite β. A transfer of stability (critical β) found by one expansion is automatically invisible using the other. New stable regions are found with each expansion.

A toroidal configuration with circular axis and a simple periodic helical winding of order l is described by the plasma radius r_0, the coil or conducting wall radius r_1, the helical period $L = 2\pi/k$, the toroidal curvature $\kappa = 1/R$, and the ratio of helical to toroidal field strength at the plasma boundary ϵ. The equilibrium is described by the six dimensionless parameters, $(r_1/r_0)^2 = \rho$, kr_0, κr_0, ϵ, l, and β. For a flat pressure profile ($p =$ const in the plasma, $p = 0$ outside), β is unambiguous. For a parabolic profile, β can be taken as a multiplicative factor.

As an example, the proposed Scyllac design[1] fixes $\kappa r_0 = 1/250$ and $r_1/r_0 \sim 4$ or 5; kr_0, ϵ, and l are flexible; in a straight θ-pinch, $\beta \sim 0.25$ to 1.

The parameters l and k describe a simple, symmetric first-order helical winding, $\cos l(\theta - kz)$. With a conducting wall, these parameters describe the first-order shape of the conductor. More complex windings and shapes can be added at second order, but we restrict the discussion to a single winding, l, at first order.

In a previous paper,[2] it was discovered that equilibria can be found for a large range of parameters at finite β. We now discuss stability (at low and high β) with respect to a certain class of global, approximately $m = 1$, disturbances. Experimental evidence points to these as the only observable magnetohydrodynamic modes.[3] Whether this empirical situation will be confirmed in new experimental parameter ranges is not known and cannot easily be determined theoretically.

There are several differences between the stellarator[4] and scyllac[2] expansions. Here we are primarily concerned with the choice $\beta \sim \epsilon^2$ in the former

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