

BERS SLICES IN FAMILIES OF UNIVALENT MAPS

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ABSTRACT. We construct embeddings of Bers slices of ideal polygon reflection groups into the classical family of univalent functions Σ . This embedding is such that the conformal mating of the reflection group with the anti-holomorphic polynomial $z \mapsto \bar{z}^d$ is the Schwarz reflection map arising from the corresponding map in Σ . We characterize the image of this embedding in Σ as a family of univalent rational maps. Moreover, we show that the limit set of every Kleinian reflection group in the closure of the Bers slice is naturally homeomorphic to the Julia set of an anti-holomorphic polynomial.

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1. INTRODUCTION

In the 1980s, Sullivan proposed a dictionary between Kleinian groups and rational dynamics that was motivated by various common features shared by them [Sul85, SM98]. However, the dictionary is not an automatic procedure to translate results in one setting to those in the other, but rather an inspiration for results and proof techniques. Several efforts to draw more direct connections between Kleinian groups and rational maps have been made in the last few decades (for example, see [BP94, McM95, LM97, Pil03, BL20]). Amongst these, the questions of exploring dynamical relations between limit sets of Kleinian groups and Julia sets of rational maps, and binding together the actions of these two classes of conformal dynamical systems in the same dynamical plane play a central role in the current paper.

The notion of mating has its roots in the work of Bers on simultaneous uniformization of two Riemann surfaces. The simultaneous uniformization theorem allows one to mate two Fuchsian groups to obtain a quasiFuchsian group [Ber60]. In the world of conformal dynamics, Douady and Hubbard introduced the notion of mating two polynomials to produce a rational map [Dou83]. In each of these mating constructions, the key idea is to combine two “similar” conformal dynamical systems

to produce a richer conformal dynamical system in the same class. Examples of “hybrid dynamical systems” that are *conformal matings* of Kleinian reflection groups and anti-holomorphic rational maps (anti-rational for short) were constructed in [LLMM18a, LLMM18b] as *Schwarz reflection maps* associated with *univalent rational maps*. Roughly speaking, this means that the dynamical planes of the Schwarz reflection maps in question can be split into two invariant subsets, on one of which the map behaves like an anti-rational map, and on the other, its grand orbits are equivalent to the grand orbits of a group.

In the current paper, we further explore the aforementioned framework for mating Kleinian reflection groups with anti-rational maps, and show that all Kleinian reflection groups arising from (finite) circle packings satisfying a “necklace” condition can be mated with the anti-polynomial \bar{z}^d . A *necklace Kleinian reflection group* is the group generated by reflections in the circles of a finite circle packing whose contact graph is *2-connected* and *outerplanar*; i.e., the contact graph remains connected if any vertex is deleted, and has a face containing all the vertices on its boundary. The simplest example of a necklace Kleinian reflection group is given by reflections in the sides of a regular ideal $(d+1)$ -gon in the unit disk \mathbb{D} (see Definitions 2.11, 2.15). This group, which we denote by Γ_{d+1} , can be thought of as a base point of the space of necklace groups generated by $(d+1)$ circular reflections. In fact, all necklace groups (generated by $(d+1)$ circular reflections) can be obtained by taking the closure of suitable quasiconformal deformations of Γ_{d+1} in an appropriate topology. This yields the *Bers compactification* $\overline{\beta(\Gamma_{d+1})}$ of the group Γ_{d+1} (see Definitions 2.20, 2.24).

To conformally mate a necklace group Γ in $\overline{\beta(\Gamma_{d+1})}$ with an anti-polynomial, we associate a piecewise Möbius reflection map ρ_Γ to Γ that is *orbit equivalent* to Γ and enjoys Markov properties when restricted to the limit set (see Definition 2.29 and the following discussion). For Γ_{d+1} , the associated map $\rho_{\Gamma_{d+1}}$, restricted to its limit set, is topologically conjugate to the anti-polynomial \bar{z}^d on its Julia set. This yields our fundamental dynamical connection between a Kleinian limit set and a Julia set. Furthermore, the existence of the above topological conjugacy allows one to topologically glue the dynamics of ρ_Γ on its “filled limit set” with the dynamics of \bar{z}^d on its filled Julia set. In the spirit of the classical mating theory, it is then natural to seek a conformal realization of such a *topological mating* (see Subsection 2.3 for the definition of conformal mating). We remark that the aforementioned topological conjugacy is not quasisymmetric since it carries parabolic fixed points to hyperbolic fixed points, and hence, classical conformal welding techniques cannot be applied to construct the desired conformal matings.

The definition of the map ρ_Γ (in particular, the fact that it fixes the boundary of its domain of definition) immediately tells us that a conformal realization of the above topological mating must be an anti-meromorphic map defined on (the closure of) a simply connected domain fixing the boundary of the domain pointwise. A characterization of such maps now implies that such an anti-meromorphic map would be the Schwarz reflection map arising from a univalent rational map [AS76, Lemma 2.3] (see Subsection 2.1 for the precise definitions). This observation leads us to the space Σ_d^* of univalent rational maps. Indeed, the fact that each member f of Σ_d^* has an order d pole at the origin translates to the fact that the associated Schwarz reflection map σ_f has a super-attracting fixed point of local degree d (note that \bar{z}^d also has such a super-attracting fixed point in its filled Julia set). On the other hand, the space Σ_d^* has a lot in common with the groups in the Bers compactification $\overline{\beta(\Gamma_{d+1})}$ too. In fact, for $f \in \Sigma_d^*$, the complement of $f(\mathbb{D}^*)$ resembles

the bounded part of the fundamental domain of a necklace group in $\overline{\beta(\Gamma_{d+1})}$ (compare Figures 3 and 6). Using a variety of conformal and quasiconformal techniques, we prove that this resemblance can be used to construct a homeomorphism between the space Σ_d^* of univalent rational maps and the Bers compactification $\overline{\beta(\Gamma_{d+1})}$, and the Schwarz reflection maps arising from Σ_d^* are precisely the conformal matings of groups in $\overline{\beta(\Gamma_{d+1})}$ with the anti-polynomial \bar{z}^d .

Theorem A. For each $f \in \Sigma_d^*$, there exists a unique $\Gamma_f \in \overline{\beta(\Gamma_{d+1})}$ such that the Schwarz reflection map σ_f is a conformal mating of Γ_f with $z \mapsto \bar{z}^d$. The map

$$\begin{aligned} \Sigma_d^* &\rightarrow \overline{\beta(\Gamma_{d+1})} \\ f &\mapsto \Gamma_f \end{aligned}$$

is a homeomorphism.

We remark that when $d = 2$, both the spaces Σ_d^* and $\overline{\beta(\Gamma_{d+1})}$ are singletons, and the conformal mating statement of Theorem A is given in [LLMM18a, Theorem 1.1].

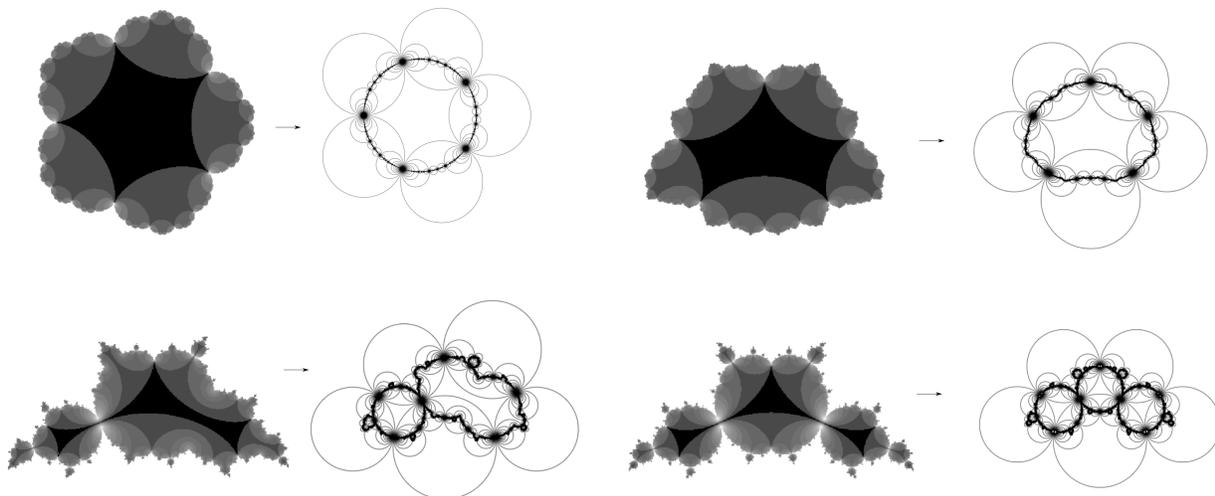


FIGURE 1. Illustrated is the mapping of Theorem A. The dynamical planes of Schwarz reflection maps of elements in Σ_4^* are illustrated next to the limit sets of the corresponding reflection groups in $\beta(\Gamma_5)$. The top-left entry corresponds to the base points $z \mapsto z - 1/(4z^4)$ and Γ_5 in Σ_4^* , $\beta(\Gamma_5)$ respectively. The bottom-left and bottom-right dynamical planes lie on the boundary of the parameter spaces.

It is worth mentioning that the homeomorphism between the parameter spaces appearing in Theorem A has a geometric interpretation. To see this, let us first note that just like the group Γ_{d+1} is a natural base point in its Bers compactification, the map $f_0(z) = z - 1/dz^d$ can be seen as a base point of Σ_d^* . In fact, the complement of $f_0(\mathbb{D}^*)$ is a $(d+1)$ -gon that is conformally isomorphic to the (closure of the) bounded part of the fundamental domain of Γ_{d+1} . The pinching

deformation technique for the family Σ_d^* , as developed in [LMM19], then shows that all other members of Σ_d^* can be obtained from f_0 by quasiconformally deforming $\mathbb{C} \setminus f_0(\mathbb{D}^*)$ and letting various sides of this $(d+1)$ -gon touch. Analogously, all groups in $\overline{\beta(\Gamma_{d+1})}$ can be obtained from Γ_{d+1} by quasiconformally deforming the fundamental domain and letting the boundary circles touch (this also has the interpretation of pinching suitable geodesics on a $(d+1)$ -times punctured sphere). This suggests that one can define analogues of *Fenchel-Nielsen coordinates* on $\text{int } \Sigma_d^*$ and $\beta(\Gamma_{d+1})$ (the latter is just a real slice of the Teichmüller space of $(d+1)$ -times punctured spheres) using extremal lengths of path families connecting various sides of the corresponding $(d+1)$ -gons. The homeomorphism of Theorem A is geometric in the sense that it respects these coordinates on $\text{int } \Sigma_d^*$ and $\beta(\Gamma_{d+1})$ (compare the proof of Theorem 3.6).

We also note that the boundary of the Bers slice $\beta(\Gamma_{d+1})$ is considerably simpler than Bers slices of Fuchsian groups; more precisely, all groups on $\partial\beta(\Gamma_{d+1})$ are geometrically finite (or equivalently, cusps), and are obtained by pinching a special collection of curves on a $(d+1)$ -times punctured sphere (see the last paragraph of Subsection 2.2). It is this feature of the Bers slices of reflection groups that is responsible for continuity of the dynamically defined map from Σ_d^* to $\beta(\Gamma_{d+1})$. This should be contrasted with the usual Fuchsian situation where the natural map from one Bers slice to another typically does not admit a continuous extension to the Bers boundaries (see [KT90]).

We now turn our attention to the other theme of the paper. This is related to the parallel notion of *laminations* that appears in the study of Kleinian groups and polynomial dynamics. The limit set of each group Γ in $\beta(\Gamma_{d+1})$ is topologically modeled as the quotient of the limit set of Γ_{d+1} by a *geodesic lamination* that is invariant under the reflection map $\rho_{\Gamma_{d+1}}$ (see Proposition 2.36 and Remark 4.25). Due to the existence of a topological conjugacy between $\rho_{\Gamma_{d+1}}$ and \bar{z}^d , this geodesic lamination can be “pushed forward” to obtain a \bar{z}^d -invariant equivalence relation on the unit circle (such equivalence relations are known as polynomial laminations in holomorphic dynamics). Using classical results from holomorphic dynamics, we show that this \bar{z}^d -invariant equivalence relation is realized as the lamination of the Julia set of a degree d anti-polynomial. This leads to our second main result.

Theorem B. Let $\Gamma \in \overline{\beta(\Gamma_{d+1})}$. Then there exists a critically fixed anti-polynomial p of degree d such that the dynamical systems

$$\begin{aligned} \rho_\Gamma : \Lambda(\Gamma) &\rightarrow \Lambda(\Gamma), \\ p : \mathcal{J}(p) &\rightarrow \mathcal{J}(p) \end{aligned}$$

are topologically conjugate.

We remark that the proof proceeds by showing that the systems in Theorem B are both topologically conjugate to the Schwarz reflection map of an appropriate element of Σ_d^* acting on its limit set (see Theorem C). This implies, in particular, that all the three fractals; namely, the Julia set of the anti-polynomial p , the limit set of the necklace group Γ , and the limit set of the Schwarz reflection map of an appropriate element of Σ_d^* , are homeomorphic. However, the incompatibility of the structures of cusp points on these fractals imply that they are not quasiconformally equivalent; i.e., there is no global quasiconformal map carrying one fractal to another (compare Figures 3, 6, and 8). Theorem B plays an important role in the recent work [LMMN20], where limit sets of necklace reflection groups are shown to be conformally removable. One of the main steps in the

proof is to show that the topological conjugacy between $p|_{\mathcal{J}(p)}$ and $\rho_\Gamma|_{\Lambda(\Gamma)}$ (provided by Theorem B) can be extended to a David homeomorphism of the sphere.

Let us now briefly outline the organization of the paper. Section 2 collects fundamental facts and known results about the objects studied in the paper. More precisely, in Subsection 2.1, we recall the definitions of the basic dynamical objects associated with the space Σ_d^* of univalent rational maps. Subsection 2.2 introduces the class of reflection groups that will play a key role in the paper. Here we define the Bers slice $\beta(\Gamma_d)$ of the regular ideal polygon reflection group Γ_d (following the classical construction of Bers slices of Fuchsian groups), and describe its compactification $\overline{\beta(\Gamma_d)}$ in a suitable space of discrete, faithful representations. To each reflection group $\Gamma \in \overline{\beta(\Gamma_d)}$, we then associate the reflection map ρ_Γ that is *orbit equivalent* to the group (this mimics a construction of Bowen and Series [BS79]). Using the reflection group ρ_Γ , we formalize the notion of *conformal mating* of a reflection group and an anti-polynomial in Subsection 2.3. Section 3 proves half of Theorem A; here we prove that there is a natural homeomorphism between the spaces Σ_d^* and $\overline{\beta(\Gamma_d)}$. We should mention that the results of Section 3 depend on some facts about the space Σ_d^* (and the associated Schwarz reflection maps) whose proofs are somewhat technical and hence deferred to Section 4. A recurring difficulty in our study is the unavailability of normal family arguments since Schwarz reflection maps are not defined on all of $\widehat{\mathbb{C}}$. After proving some preliminary results about the topology of the limit set of a Schwarz reflection map arising from Σ_d^* in Subsections 4.1 and 4.2, we proceed to the proofs of the statements about Σ_d^* that are used in Section 3 (more precisely, Lemma 4.14, Proposition 4.19, and 4.20). The rest of Section 4 is devoted to the proof of the conformal mating statement of Theorem A. This completes the proof of our first main theorem. Finally, in Section 5, we use the theory of Hubbard trees for anti-holomorphic polynomials to prove Theorem B.

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2. PRELIMINARIES

Notation 2.1. We denote by \mathbb{D}^* the exterior unit disc $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. The Julia set of a holomorphic or anti-holomorphic polynomial $p : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ will be denoted by $\mathcal{J}(p)$, and its filled Julia set by $\mathcal{K}(p)$.

2.1. The Space Σ_d^* and Schwarz Reflection Maps.

Definition 2.2. We will denote by Σ_d^* the following class of rational maps:

$$\Sigma_d^* := \left\{ f(z) = z + \frac{a_1}{z} + \cdots + \frac{a_d}{z^d} : a_d = -\frac{1}{d} \text{ and } f|_{\mathbb{D}^*} \text{ is conformal.} \right\}.$$

Note that for each $d \geq 2$, the space Σ_d^* can be regarded as a slice of the space of schlicht functions:

$$\Sigma := \left\{ f(z) = z + \frac{a_1}{z} + \cdots + \frac{a_d}{z^d} + \cdots : f|_{\mathbb{D}^*} \text{ is conformal} \right\}.$$

We endow Σ_d^* with the topology of coefficient-wise convergence. Clearly, this topology is equivalent to that of uniform convergence on compact subsets of \mathbb{D}^* .

Definition 2.3. Given $f \in \Sigma_d^*$, we define the associated *Schwarz reflection map* $\sigma_f : f(\mathbb{D}^*) \rightarrow \widehat{\mathbb{C}}$ by the following diagram:

$$\begin{array}{ccc}
\mathbb{D}^* & \xrightarrow{z \mapsto 1/\bar{z}} & \mathbb{D} \\
f^{-1} \uparrow & & \downarrow f \\
f(\mathbb{D}^*) & \xrightarrow{\sigma_f} & \widehat{\mathbb{C}}
\end{array}$$

The map $\sigma_f : \sigma_f^{-1}(f(\mathbb{D}^*)) \rightarrow f(\mathbb{D}^*)$ is a proper branched covering map of degree d (branched only at ∞), and $\sigma_f : \sigma_f^{-1}(\text{int } f(\mathbb{D}^*)^c) \rightarrow \text{int } f(\mathbb{D}^*)^c$ is a degree $(d+1)$ covering map.

We also note that ∞ is a super-attracting fixed point of σ_f ; more precisely, ∞ is a fixed critical point of σ_f of multiplicity $(d-1)$.

Definition 2.4. Let $f \in \Sigma_d^*$. We define the *basin of infinity* for σ_f as

$$\mathcal{B}_\infty(\sigma_f) := \{z \in \widehat{\mathbb{C}} : \sigma_f^{on}(z) \xrightarrow{n \rightarrow \infty} \infty\}.$$

Remark 2.5. Let $f \in \Sigma_d^*$. Since σ_f has no critical point other than ∞ in $\mathcal{B}_\infty(\sigma_f)$, the proof of [Mil06, Theorem 9.3] may be adapted to show the existence of a *Böttcher coordinate* for σ_f : a conformal map

$$(1) \quad \phi_{\sigma_f} : \mathbb{D}^* \rightarrow \mathcal{B}_\infty(\sigma_f) \text{ such that } \phi_{\sigma_f}^{-1} \circ \sigma_f \circ \phi_{\sigma_f}(u) = \bar{u}^d, \forall u \in \mathbb{D}^*.$$

Since

$$\sigma_f(z) = -\frac{\bar{z}^d}{d} + O(\bar{z}^{d-1}) \text{ as } z \rightarrow \infty,$$

we may choose ϕ_{σ_f} such that

$$(2) \quad \phi'_{\sigma_f}(\infty) = d^{\frac{1}{d-1}} e^{\frac{i\pi}{d+1}}.$$

As in [Mil06, Theorem 9.3], any Böttcher coordinate for σ_f is unique up to multiplication by a $d+1^{\text{st}}$ root of unity. Thus, (2) determines a *unique* Böttcher coordinate ϕ_{σ_f} which we will henceforth refer to as the Böttcher coordinate for σ_f .

The set $\widehat{\mathbb{C}} \setminus f(\mathbb{D}^*)$ is called the *droplet*, or *fundamental tile*, and is denoted by $T(\sigma_f)$. By [LMM19, Proposition 2.8] and [LM14, Lemma 2.4], the curve $\partial T = f(\mathbb{T})$ has $(d+1)$ distinct cusps and at most $(d-2)$ double points. The *desingularized droplet* $T^o(\sigma_f)$ is defined as

$$T^o(\sigma_f) := T(\sigma_f) \setminus \{\zeta : \zeta \text{ is a cusp or double point of } f(\mathbb{T})\}.$$

Definition 2.6. The *tiling set* $\mathcal{T}_\infty(\sigma_f)$ is defined as:

$$\mathcal{T}_\infty(\sigma_f) := T^o(\sigma_f) \cup \left\{ z \in \widehat{\mathbb{C}} : \sigma_f^{on}(z) \in T^o(\sigma_f) \text{ for some } n \geq 1 \right\}.$$

Lastly, we define the *limit set* of σ_f by $\Lambda(\sigma_f) := \partial \mathcal{T}_\infty(\sigma_f)$.

For more details on the space Σ_d^* and the associated Schwarz reflection maps, we refer the readers to [LMM19].

2.2. Reflection Groups and the Bers Slice.

Notation 2.7. We denote by $\text{Aut}^\pm(\widehat{\mathbb{C}})$ be the group of all Möbius and anti-Möbius automorphisms of $\widehat{\mathbb{C}}$.

Definition 2.8. A discrete subgroup Γ of $\text{Aut}^\pm(\widehat{\mathbb{C}})$ is called a *Kleinian reflection group* if Γ is generated by reflections in finitely many Euclidean circles.

Remark 2.9. For a Euclidean circle C , consider the upper hemisphere $S \subset \mathbb{H}^3 := \{(x, y, t) \in \mathbb{R}^3 : t > 0\}$ such that $\partial S \cap \partial\mathbb{H}^3 = C$. Reflection in the Euclidean circle C extends naturally to reflection in S , and defines an orientation-reversing isometry of \mathbb{H}^3 . Hence, a Kleinian reflection Γ group can be thought of as a 3-dimensional hyperbolic reflection group.

Since a Kleinian reflection group is discrete, by [VS93, Part II, Chapter 5, Proposition 1.4], we can choose its generators to be reflections in Euclidean circles C_1, \dots, C_d such that:

(\star) For each i , the closure of the bounded component of $\widehat{\mathbb{C}} \setminus C_i$ does not contain any other C_j .

We will always assume that a chosen generating set for a Kleinian reflection group Γ satisfies Conditions (\star).

Definition 2.10. Let Γ be a Kleinian reflection group. The *domain of discontinuity* of Γ , denoted $\Omega(\Gamma)$, is the maximal open subset of $\widehat{\mathbb{C}}$ on which the elements of Γ form a normal family. The *limit set* of Γ , denoted by $\Lambda(\Gamma)$, is defined by $\Lambda(\Gamma) := \widehat{\mathbb{C}} \setminus \Omega(\Gamma)$.

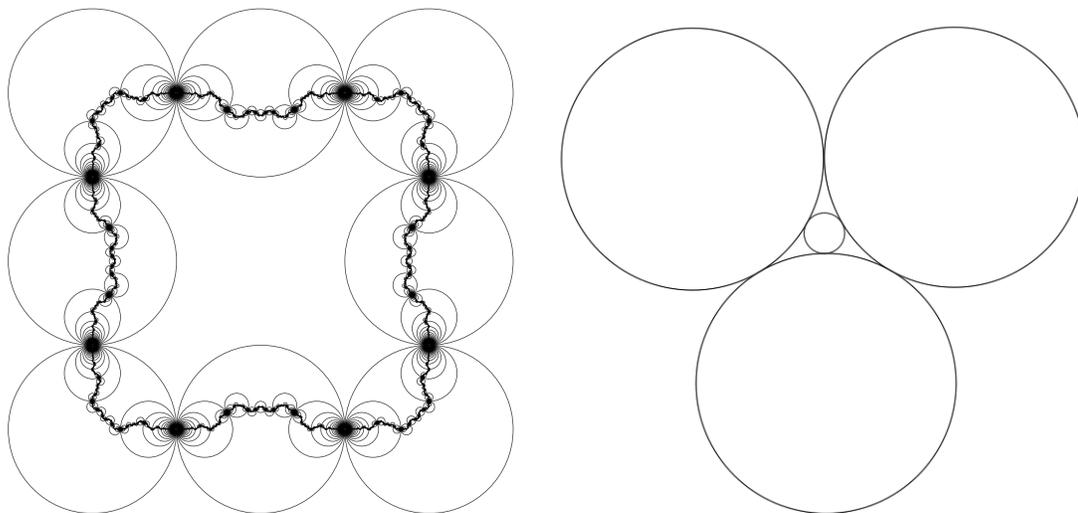


FIGURE 2. On the left is an interior necklace group, but the group generated by the circles pictured on the right violates condition (2) of Definition 2.11.

For a Euclidean circle C , the bounded complementary component of C will be called the *interior* of C , and will be denoted by $\text{int } C$.

Definition 2.11. Let Γ be a Kleinian reflection group. We say Γ is a *necklace group* (see Figure 2) if it can be generated by reflections in Euclidean circles C_1, \dots, C_d such that:

- (1) each circle C_i is tangent to C_{i+1} (with $i+1$ taken mod d),
- (2) the boundary of the unbounded component of $\widehat{\mathbb{C}} \setminus \cup_i C_i$ intersects each C_i , and
- (3) the circles C_i have pairwise disjoint interiors.

If, furthermore, C_{i-1} and C_{i+1} are the only circles to which any C_i is tangent, then Γ is an *interior necklace group*.

Remark 2.12. In Definition 2.11, Condition (2) ensures that each circle C_i is “seen” from ∞ - see Figure 2. When choosing a generating set for a necklace group, we always assume the generating set is chosen so as to satisfy Conditions (1)-(3), and the circles C_1, \dots, C_d are labelled clockwise around ∞ . We note that a necklace group Γ generated by reflections in d Euclidean circles is isomorphic to the free product of d copies of $\mathbb{Z}/2\mathbb{Z}$.

Notation 2.13. Given a necklace group Γ with generating set given by reflections in circles C_1, \dots, C_d , let

$$\mathcal{F}_\Gamma := \widehat{\mathbb{C}} \setminus \left(\bigcup_{i=1}^d (\text{int } C_i \cup \{C_j \cap C_i : j \neq i\}) \right).$$

Proposition 2.14. *Let Γ be a necklace group. Then \mathcal{F}_Γ is a fundamental domain for Γ .*

Proof. Let \mathcal{P}_Γ be the convex hyperbolic polyhedron (in \mathbb{H}^3) whose relative boundary in \mathbb{H}^3 is the union of the hyperplanes S_i (see Remark 2.9). Then, by [VS93, Part II, Chapter 5, Theorem 1.2], \mathcal{P}_Γ is a fundamental domain for the action of Γ on \mathbb{H}^3 . It now follows that $\mathcal{F}_\Gamma = \overline{\mathcal{P}_\Gamma} \cap \Omega(\Gamma)$ (where the closure is taken in $\Omega(\Gamma) \cup \mathbb{H}^3$) is a fundamental domain for the action of Γ on $\Omega(\Gamma)$ [Mar07, §3.5]. \square

It will be useful in our discussion to have a canonical interior necklace group to refer to:

Definition 2.15. Consider the Euclidean circles $\mathbf{C}_1, \dots, \mathbf{C}_d$ where \mathbf{C}_j intersects $|z| = 1$ at right-angles at the roots of unity $\exp(\frac{2\pi i \cdot (j-1)}{d})$, $\exp(\frac{2\pi i \cdot j}{d})$. Let ρ_j be the reflection map in the circle \mathbf{C}_j . By [VS93, Part II, Chapter 5, Theorem 1.2], this defines a necklace group

$$\mathbf{\Gamma}_d := \langle \rho_1, \dots, \rho_d : \rho_1^2 = \dots = \rho_d^2 = 1 \rangle,$$

that acts on the Riemann sphere.

Definition 2.16. Let Γ be a discrete subgroup of $\text{Aut}^\pm(\widehat{\mathbb{C}})$. An isomorphism

$$\xi : \mathbf{\Gamma}_d \rightarrow \Gamma$$

is said to be *weakly type-preserving*, or *w.t.p.*, if

- (1) $\xi(g)$ is orientation-preserving if and only if g is orientation-preserving, and
- (2) $\xi(g) \in \Gamma$ is a parabolic Möbius map for each parabolic Möbius map $g \in \mathbf{\Gamma}_d$.

In order to construct the *Bers slice* of the group $\mathbf{\Gamma}_d$ and describe its compactification, we need to define a representation space for $\mathbf{\Gamma}_d$. For necklace groups, the information encoded by a representation (defined below) is equivalent to the data given by a labeling of the underlying circle

packing. We will see in Section 3 that working with the space of representations (as opposed to the space of necklace groups without a labeling of the underlying circle packings) is crucial for the homeomorphism statement of Theorem A (compare Figure 5).

Definition 2.17. We define

$$\mathcal{D}(\mathbf{\Gamma}_d) := \{\xi : \mathbf{\Gamma}_d \rightarrow \Gamma \mid \Gamma \text{ is a discrete subgroup of } \text{Aut}^\pm(\widehat{\mathbb{C}}), \text{ and } \xi \text{ is a w.t.p. isomorphism}\}.$$

We endow $\mathcal{D}(\mathbf{\Gamma}_d)$ with the topology of *algebraic convergence*: we say that a sequence $(\xi_n)_{n=1}^\infty \subset \mathcal{D}(\mathbf{\Gamma}_d)$ converges to $\xi \in \mathcal{D}(\mathbf{\Gamma}_d)$ if $\xi_n(\rho_i) \rightarrow \xi(\rho_i)$ coefficient-wise (as $n \rightarrow \infty$) for $i = 1, \dots, d$.

Remark 2.18. Let $\xi \in \mathcal{D}(\mathbf{\Gamma}_d)$. Since for each $i \in \mathbb{Z}/d\mathbb{Z}$, the Möbius map $\rho_i \circ \rho_{i+1}$ is parabolic (this follows from the fact that each \mathbf{C}_i is tangent to \mathbf{C}_{i+1}), the w.t.p. condition implies that $\xi(\rho_i) \circ \xi(\rho_{i+1})$ is also parabolic. As each $\xi(\rho_i)$ is an anti-conformal involution, it follows that $\xi(\rho_i)$ is Möbius conjugate to the circular reflection $z \mapsto 1/\bar{z}$ or the antipodal map $z \mapsto -1/\bar{z}$. A straightforward computation shows that the composition of $-1/\bar{z}$ with either the reflection or the antipodal map with respect to any circle has two distinct fixed points in $\widehat{\mathbb{C}}$, and hence not parabolic. Therefore, it follows that no $\xi(\rho_i)$ is Möbius conjugate to the antipodal map $-1/\bar{z}$. Hence, each $\xi(\rho_i)$ must be the reflection in some Euclidean circle C_i . Thus, $\Gamma = \xi(\mathbf{\Gamma}_d)$ is generated by reflections in the circles C_1, \dots, C_d . The fact that $\xi(\rho_i) \circ \xi(\rho_{i+1})$ is parabolic now translates to the condition that each C_i is tangent to C_{i+1} (for $i \in \mathbb{Z}/d\mathbb{Z}$). However, new tangencies among the circles C_i may arise. Moreover, that ξ is an isomorphism rules out non-tangential intersection between circles C_i, C_j (indeed, a non-tangential intersection between C_i and C_j would introduce a new relation between $\xi(\rho_i)$ and $\xi(\rho_j)$, compare [VS93, Part II, Chapter 5, §1.1]). Therefore, $\Gamma = \xi(\mathbf{\Gamma}_d)$ is a Kleinian reflection group satisfying properties (1) and (3) of necklace groups.

Definition 2.19. Let τ be a conformal map defined in a neighborhood of ∞ with $\tau(\infty) = \infty$. We will say τ is *tangent to the identity at ∞* if $\tau'(\infty) = 1$. We will say τ is *hydrodynamically normalized* if

$$\tau(z) = z + O(1/z) \text{ as } z \rightarrow \infty.$$

Definition 2.20. Let $\text{Bel}_{\mathbf{\Gamma}_d}$ denote those Beltrami coefficients μ invariant under $\mathbf{\Gamma}_d$, satisfying $\mu = 0$ a.e. on \mathbb{D}^* . Let $\tau_\mu : \mathbb{C} \rightarrow \mathbb{C}$ denote the quasiconformal integrating map of μ , with the hydrodynamical normalization. The *Bers slice* of Γ is defined as

$$\beta(\mathbf{\Gamma}_d) := \{\xi \in \mathcal{D}(\mathbf{\Gamma}_d) \mid \xi(g) = \tau_\mu \circ g \circ \tau_\mu^{-1} \text{ for all } g \in \mathbf{\Gamma}_d, \text{ where } \mu \in \text{Bel}_{\mathbf{\Gamma}_d}\}.$$

Remark 2.21. There is a natural free $\text{PSL}_2(\mathbb{C})$ -action on $\mathcal{D}(\mathbf{\Gamma}_d)$ given by conjugation, and so it is natural to consider the space $\text{AH}(\mathbf{\Gamma}_d) := \mathcal{D}(\mathbf{\Gamma}_d)/\text{PSL}_2(\mathbb{C})$. The following definition of the Bers slice, where no normalization for τ_μ is specified, is more aligned with the classical Kleinian group literature:

$$(\star) \quad \{\xi \in \text{AH}(\mathbf{\Gamma}_d) \mid \xi(g) = \tau_\mu \circ g \circ \tau_\mu^{-1} \text{ for all } g \in \mathbf{\Gamma}_d, \text{ where } \mu \in \text{Bel}_{\mathbf{\Gamma}_d}\}.$$

Our Definition 2.20 of $\beta(\mathbf{\Gamma}_d)$ is simply a canonical choice of representative from each equivalence class of (\star) , and will be more appropriate for the present work.

Lemma 2.22. *Let $\mu \in \text{Bel}_{\mathbf{\Gamma}_d}$, and $\tau_\mu : \mathbb{C} \rightarrow \mathbb{C}$ an integrating map. Then $\tau_\mu \circ \mathbf{\Gamma}_d \circ \tau_\mu^{-1}$ is a necklace group.*

Proof. By definition, the maps $\tau_\mu \circ \rho_i \circ \tau_\mu^{-1}$ generate the group $\tau_\mu \circ \mathbf{\Gamma}_d \circ \tau_\mu^{-1}$. By invariance of μ , each $\tau_\mu \circ \rho_i \circ \tau_\mu^{-1}$ is an anti-conformal involution of $\widehat{\mathbb{C}}$, hence an anti-Möbius transformation. Since $\tau_\mu \circ \rho_i \circ \tau_\mu^{-1}$ fixes $\tau_\mu(C_i)$, and interchanges its two complementary components, it follows that $\tau_\mu(C_i)$ is a Euclidean circle and hence $\tau_\mu \circ \rho_i \circ \tau_\mu^{-1}$ is reflection in the circle $\tau_\mu(C_i)$. One readily verifies that the circles $\tau_\mu(C_i)$ satisfy the conditions of Definition 2.11, and so the result follows. \square

Proposition 2.23. *The Bers slice $\beta(\mathbf{\Gamma}_d)$ is pre-compact in $\mathcal{D}(\mathbf{\Gamma}_d)$, and for each $\xi \in \overline{\beta(\mathbf{\Gamma}_d)}$, the group $\xi(\mathbf{\Gamma}_d)$ is a necklace group.*

Proof. Let $(\xi_n)_{n=1}^\infty$ be a sequence in $\beta(\mathbf{\Gamma}_d)$, and $\tau_n : \mathbb{C} \rightarrow \mathbb{C}$ the associated quasiconformal maps as in Definition 2.20. Since each τ_n is conformal in \mathbb{D}^* and is hydrodynamically normalized, by a standard normal family result (see [CG93, Theorem 1.10] for instance) there exists a conformal map τ_∞ of \mathbb{D}^* such that $\tau_n \rightarrow \tau_\infty$ uniformly on compact subsets of \mathbb{D}^* , perhaps after passing to a subsequence which we reenumerate (τ_n) .

By Lemma 2.22, each $\tau_n(C_i)$ is a Euclidean circle which we denote by C_i^n . Since $\tau_n \rightarrow \tau_\infty$ uniformly on compact subsets of \mathbb{D}^* , $\tau_\infty(C_i \cap \mathbb{D}^*)$ must be a subarc of a Euclidean circle which we denote by C_i^∞ . Denote furthermore by ρ_i^n, ρ_i^∞ the reflections in the circles C_i^n, C_i^∞ (respectively), and by Γ_∞ the group generated by reflections in the circles $(C_i^\infty)_{i=1}^d$. Let $\xi_\infty : \mathbf{\Gamma}_d \rightarrow \Gamma_\infty$ be the homomorphism defined by $\xi_\infty(\rho_i) := \rho_i^\infty$. We see that $C_i^n \rightarrow C_i^\infty$ as $n \rightarrow \infty$ in the Hausdorff sense, whence it follows that $\rho_i^n \rightarrow \rho_i^\infty$. This proves algebraic convergence $(\xi_n) \rightarrow \xi_\infty$. Hausdorff convergence of $C_i^n \rightarrow C_i^\infty$ also implies that each C_i^∞ intersects tangentially with C_{i+1}^∞ , so that ξ_∞ is weakly type preserving. Similar considerations show that the circles C_i^∞ have pairwise disjoint interiors, and the boundary of the unbounded component of $\widehat{\mathbb{C}} \setminus \cup_i C_i^\infty$ intersects each C_i^∞ . In particular, there cannot be any non-tangential intersection among the circles C_i^∞ . It now follows that ξ_∞ is indeed an isomorphism, and $\Gamma_\infty = \xi_\infty(\mathbf{\Gamma}_d)$ is a necklace group. \square

Definition 2.24. We refer to $\overline{\beta(\mathbf{\Gamma}_d)} \subset \mathcal{D}(\mathbf{\Gamma}_d)$ as the *Bers compactification* of the Bers slice $\beta(\mathbf{\Gamma}_d)$. We refer to $\overline{\beta(\mathbf{\Gamma}_d)} \setminus \beta(\mathbf{\Gamma}_d)$ as the *Bers boundary*.

Remark 2.25. We will often identify $\xi \in \overline{\beta(\mathbf{\Gamma}_d)}$ with the group $\Gamma := \xi(\mathbf{\Gamma}_d)$, and simply write $\Gamma \in \overline{\beta(\mathbf{\Gamma}_d)}$, but always with the understanding of an associated representation $\xi : \mathbf{\Gamma}_d \rightarrow \Gamma$. Since ξ is completely determined by its action on the generators ρ_1, \dots, ρ_d of $\mathbf{\Gamma}_d$, this is equivalent to remembering the ‘labeled’ circle packing C_1, \dots, C_d , where $\xi(\rho_i)$ is reflection in the circle C_i , for $i = 1, \dots, d$.

Remark 2.26. The Apollonian gasket reflection group (see the right-hand side of Figure 2) is an example of a Kleinian reflection group in $\mathcal{D}(\mathbf{\Gamma}_d) \setminus \overline{\beta(\mathbf{\Gamma}_d)}$.

Notation 2.27. For $\Gamma \in \overline{\beta(\mathbf{\Gamma}_d)}$, we denote the component of $\Omega(\Gamma)$ containing ∞ by $\Omega_\infty(\Gamma)$.

Proposition 2.28. *Let $\Gamma \in \overline{\beta(\mathbf{\Gamma}_d)}$. Then the following hold true.*

- (1) $\Omega_\infty(\Gamma)$ is simply connected, and Γ -invariant.
- (2) $\partial\Omega_\infty(\Gamma) = \Lambda(\Gamma)$.
- (3) $\Lambda(\Gamma)$ is connected, and locally connected.
- (4) All bounded components of $\Omega(\Gamma)$ are Jordan domains.

Proof. 1) It is evident from the construction of the Bers compactification that for each $\Gamma \in \overline{\beta(\mathbf{\Gamma}_d)}$, there is a conformal map from \mathbb{D}^* onto $\Omega_\infty(\Gamma)$ that conjugates the action of $\mathbf{\Gamma}_d$ on \mathbb{D}^* to that of Γ on $\Omega_\infty(\Gamma)$. Hence, $\Omega_\infty(\Gamma)$ is simply connected, and invariant under Γ .

2) This follows from (1) and the fact that the boundary of an invariant component of the domain of discontinuity is the entire limit set.

3) Connectedness of $\Lambda(\Gamma)$ follows from (2) and that $\Omega_\infty(\Gamma)$ is simply connected. For local connectivity, first note that the index two Kleinian subgroup Γ^+ consisting of words of even length of Γ is geometrically finite. Then $\Lambda(\Gamma) = \Lambda(\Gamma^+)$, hence $\Lambda(\Gamma^+)$ is connected. Since Γ^+ is geometrically finite with a connected limit set, it now follows from [AM96] that $\Lambda(\Gamma^+) = \Lambda(\Gamma)$ is locally connected.

4) By (3), each component of $\Omega(\Gamma) \setminus \Omega_\infty(\Gamma)$ is simply connected with a locally connected boundary. That such a component \mathcal{U} is Jordan follows from the fact that $\partial\mathcal{U} \subset \Lambda(\Gamma) = \partial\Omega_\infty(\Gamma)$. \square

To a group $\Gamma \in \overline{\beta(\mathbf{\Gamma}_d)}$, we now associate a reflection map ρ_Γ that will play an important role in the present work.

Definition 2.29. Let $\Gamma \in \overline{\beta(\mathbf{\Gamma}_d)}$, generated by reflections $(r_i)_{i=1}^d$ in circles $(C_i)_{i=1}^d$. We define the associated *reflection map* ρ_Γ by:

$$\rho_\Gamma : \bigcup_{i=1}^d \overline{\text{int}(C_i)} \rightarrow \widehat{\mathbb{C}}$$

$$z \mapsto r_i(z) \text{ if } z \in \overline{\text{int}(C_i)}.$$

Definition 2.30. Let Γ be a Kleinian reflection group, and $f : D \rightarrow \widehat{\mathbb{C}}$ a mapping defined on a domain D . We say that Γ and f are *orbit-equivalent* if for any two points $z, w \in \widehat{\mathbb{C}}$, there exists $g \in \Gamma$ with $g(z) = w$ if and only if there exist non-negative integers n_1, n_2 such that $f^{\circ n_1}(z) = f^{\circ n_2}(w)$.

Proposition 2.31. Let $\Gamma \in \overline{\beta(\mathbf{\Gamma}_d)}$. The map ρ_Γ is orbit equivalent to Γ on $\widehat{\mathbb{C}}$.

Proof. Suppose $z, w \in \widehat{\mathbb{C}}$ are such that there exist $n_1, n_2 \in \mathbb{N}$ such that $\rho_\Gamma^{n_1}(z) = \rho_\Gamma^{n_2}(w)$. Since ρ_Γ acts by the generators r_i of the group Γ , it follows directly that there exists $g \in \Gamma$ with $g(z) = w$. Conversely, let $z, w \in \widehat{\mathbb{C}}$ be such that there exists $g \in \Gamma$ with $g(z) = w$. By definition, we have that $g = r_{s_1} r_{s_2} \cdots r_{s_n}$, for some $s_1, \dots, s_n \in \{1, \dots, d\}$. Suppose first that $n = 1$. Note that either z or w must belong to $\overline{\text{int} C_{s_1}}$. Since $r_{s_1}(z) = w$ implies $r_{s_1}(w) = z$, there is no loss of generality in assuming that $z \in \overline{\text{int} C_{s_1}}$. Now, the condition $r_{s_1}(z) = w$ can be written as $\rho_\Gamma(z) = w$. The case $n > 1$ now follows by induction. \square

Notation 2.32. For $\Gamma \in \overline{\beta(\mathbf{\Gamma}_d)}$, we will denote by $T^o(\Gamma)$ the union of all bounded components of the fundamental domain \mathcal{F}_Γ (see Proposition 2.14), and by $\Pi^o(\Gamma)$ the unique unbounded component of \mathcal{F}_Γ . We also set

$$T(\Gamma) := \overline{T^o(\Gamma)}, \text{ and } \Pi(\Gamma) := \overline{\Pi^o(\Gamma)}.$$

Remark 2.33. The set $T(\Gamma)$ should be thought of as the analogue of a droplet $T(\sigma_f)$ (this analogy will become transparent in Proposition 3.2). On the other hand, the notation $\Pi(\Gamma)$ is supposed to remind the readers that (the closure of) the unbounded component of \mathcal{F}_Γ is a ‘‘polygon.’’

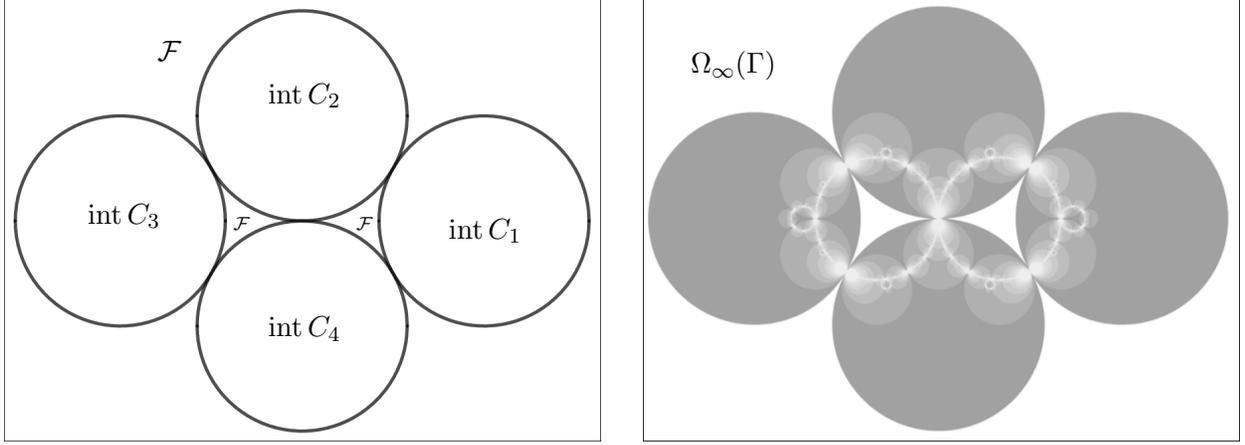


FIGURE 3. Left: The circles C_i generate a Kleinian reflection group $\Gamma \in \partial\beta(\mathbf{\Gamma}_4)$. The map ρ_Γ is defined piece-wise on the union of the closed disks $\overline{\text{int } C_i}$. The fundamental domain $\mathcal{F} = \mathcal{F}_\Gamma$ (for the action of Γ on $\Omega(\Gamma)$) is the complement of these open disks with the singular boundary points removed. The connected components of \mathcal{F} are marked. Right: The unbounded component of the domain of discontinuity $\Omega(\Gamma)$ is $\Omega_\infty(\Gamma)$. Every point in $\Omega(\Gamma)$ escapes to \mathcal{F} under iterates of ρ_Γ . The point of tangential intersection of C_2 and C_4 is the fixed point of an accidental parabolic of the index two Kleinian subgroup $\tilde{\Gamma}$.

Proposition 2.34. *Let $\Gamma \in \overline{\beta(\mathbf{\Gamma}_d)}$. Then:*

$$\Omega(\Gamma) = \bigcup_{n \geq 0} \rho_\Gamma^{-n}(\mathcal{F}_\Gamma), \text{ and } \Omega_\infty(\Gamma) = \bigcup_{n \geq 0} \rho_\Gamma^{-n}(\Pi^o(\Gamma)).$$

In particular, $\Lambda(\Gamma)$ is completely invariant under ρ_Γ .

Proof. This follows from Propositions 2.14, 2.28, 2.31. □

Remark 2.35. Let $\Gamma \in \overline{\beta(\mathbf{\Gamma}_d)}$. We now briefly describe the covering properties of $\rho_\Gamma : \Lambda(\Gamma) \rightarrow \Lambda(\Gamma)$. To this end, first note that

$$\Lambda(\Gamma) = \bigcup_{i=1}^d (\overline{\text{int } C_i} \cap \Lambda(\Gamma))$$

(see Figure 3). The interiors of these d “partition pieces” are disjoint, and ρ_Γ maps each of them injectively onto the union of the others. This produces a Markov partition for the degree d orientation-reversing covering map $\rho_\Gamma : \Lambda(\Gamma) \rightarrow \Lambda(\Gamma)$. In the particular case of the base group $\mathbf{\Gamma}_d$, the above discussion yields a Markov partition

$$\mathbb{T} = \bigcup_{j=1}^d \left[\exp\left(\frac{2\pi i(j-1)}{d}\right), \exp\left(\frac{2\pi i j}{d}\right) \right]$$

of the map $\rho_{\Gamma_d} : \mathbb{T} \rightarrow \mathbb{T}$. Note that the expanding map

$$\bar{z}^{d-1} : \mathbb{T} \rightarrow \mathbb{T},$$

or equivalently,

$$m_{-(d-1)} : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}, \theta \mapsto -(d-1)\theta$$

also admits the same Markov partition with the same transition matrix (identifying \mathbb{T} with \mathbb{R}/\mathbb{Z}). Following [LLMM18a, §3.2], one can define a homeomorphism

$$\mathcal{E}_{d-1} : \mathbb{T} \rightarrow \mathbb{T}$$

via the coding maps of $\rho_{\Gamma_d}|_{\mathbb{T}}$ and $\bar{z}^{d-1}|_{\mathbb{T}}$ such that \mathcal{E}_{d-1} maps 1 to 1, and conjugates ρ_{Γ_d} to \bar{z}^{d-1} (or $m_{-(d-1)}$). Since both ρ_{Γ_d} and \bar{z}^{d-1} commute with the complex conjugation map and \mathcal{E}_{d-1} fixes 1, one sees that \mathcal{E}_{d-1} commutes with the complex conjugation map as well.

The next result provides us with a model of the dynamics of ρ_{Γ} on the limit set $\Lambda(\Gamma)$ as a quotient of the action of ρ_{Γ_d} on the unit circle.

Proposition 2.36. *Let $\Gamma \in \overline{\beta(\Gamma_d)}$. There exists a conformal map $\phi_{\Gamma} : \mathbb{D}^* \rightarrow \Omega_{\infty}(\Gamma)$ such that*

$$(3) \quad \rho_{\Gamma_d}(z) = \phi_{\Gamma}^{-1} \circ \rho_{\Gamma} \circ \phi_{\Gamma}(z), \text{ for } z \in \mathbb{D}^* \setminus \text{int } \Pi(\Gamma_d).$$

The map ϕ_{Γ} extends continuously to a semi-conjugacy $\phi_{\Gamma} : \mathbb{T} \rightarrow \Lambda(\Gamma)$ between $\rho_{\Gamma_d}|_{\mathbb{T}}$ and $\rho_{\Gamma}|_{\Lambda(\Gamma)}$, and ϕ_{Γ} sends cusps of $\partial\Pi(\Gamma_d)$ to cusps of $\partial\Pi(\Gamma)$ with labels preserved.

Proof. Recall that Γ_d is generated by reflections ρ_i in circles $(\mathbf{C}_i)_{i=1}^d$. It follows from Propositions 2.14 and 2.28 that $\Pi^o(\Gamma_d)$ and $\Pi^o(\Gamma)$ are fundamental domains for the actions of Γ_d and Γ on \mathbb{D}^* and $\Omega_{\infty}(\Gamma)$, respectively. By the proofs of Lemma 2.22 and Proposition 2.23, there is a conformal mapping $\phi_{\Gamma} : \text{int } \Pi(\Gamma_d) \rightarrow \text{int } \Pi(\Gamma)$ whose extension to $\partial\Pi(\Gamma_d)$ is a label-preserving homeomorphism onto $\partial\Pi(\Gamma)$. Thus, by the Schwarz reflection principle, we may extend ϕ_{Γ} to a conformal mapping $\phi_{\Gamma} : \mathbb{D}^* \rightarrow \Omega_{\infty}(\Gamma)$ which satisfies (3) by construction.

Note that $\partial\Omega_{\infty}(\Gamma) = \Lambda(\Gamma)$. By local connectedness of $\Lambda(\Gamma)$ (see Proposition 2.28), the map ϕ_{Γ} extends continuously to a semi-conjugacy $\mathbb{T} \rightarrow \Lambda(\Gamma)$. \square

We will now introduce the notion of a *label-preserving homeomorphism*, which will play an important role in the proof of Theorem A.

Remark 2.37. For $f_0(z) := z - 1/(dz^d)$, label the non-zero critical points of f_0 as $\xi_1^{f_0}, \dots, \xi_{d+1}^{f_0}$ in counter-clockwise order with $\xi_1^{f_0} = e^{\frac{i\pi}{d+1}}$. Note that the critical points of f vary continuously depending on $f \in \Sigma_d^*$, and $f \in \Sigma_d^*$ can not have a double critical point on \mathbb{T} . Since Σ_d^* is connected by Proposition 4.19, there is a unique labeling $\xi_1^f, \dots, \xi_{d+1}^f$ of critical points of any $f \in \Sigma_d^*$ such that $f \mapsto \xi_i^f$ is continuous (for $i = 1, \dots, d+1$). This in turn determines a labeling of the cusps $\zeta_i^f := f(\xi_i^f)$ of $f(\mathbb{T})$ such that $f \mapsto \zeta_i^f$ is continuous ($i = 1, \dots, d+1$).

Similarly, label the cusps of $\partial T(\Gamma_d)$ as η_1, \dots, η_d in counter-clockwise order with $\eta_1 = 1$. This determines a labeling of cusps of $\partial T(\Gamma)$ for any $\Gamma \in \overline{\beta(\Gamma_d)}$ as the group Γ is the image under a representation of Γ_d .

Definition 2.38. Let $f \in \Sigma_d^*$ and $\Gamma \in \overline{\beta(\Gamma_{d+1})}$. We say that a homeomorphism $h : T(\Gamma) \rightarrow T(\sigma_f)$ is *label-preserving* if h maps cusps of $\partial T(\Gamma)$ to cusps of $\partial T(\sigma_f)$, and h preserves the labeling of cusps of $\partial T(\Gamma)$ and $\partial T(\sigma_f)$.

Similarly, for $f, f' \in \Sigma_d^*$ (respectively, for $\Gamma, \Gamma' \in \overline{\beta(\Gamma_{d+1})}$), a homeomorphism $h : T(\sigma_f) \rightarrow T(\sigma_{f'})$ (respectively, $h : T(\Gamma) \rightarrow T(\Gamma')$) is called *label-preserving* if h maps the boundary cusps to the boundary cusps preserving their labels.

We conclude this subsection with a discussion of the connection between the Bers slice of the reflection group Γ_d and a classical Teichmüller space. Let Γ_d^+ be the index two subgroup of Γ_d consisting of all Möbius maps in Γ_d . Then, Γ_d^+ is Fuchsian group (it preserves \mathbb{D} and \mathbb{D}^*). Using Proposition 2.14, it is seen that the top and bottom surfaces $S^+ := \mathbb{D}^*/\Gamma_d^+$ and $S^- := \mathbb{D}/\Gamma_d^+$ associated with the Fuchsian group Γ_d^+ are d times punctured spheres. Moreover, the anti-Möbius reflection ρ_i in the circle \mathbf{C}_i descends to anti-conformal involutions on S^\pm fixing all the punctures (the resulting involution is independent of $i \in \{1, \dots, d\}$). We will denote this involution on S^- by ι .

By definition, each $\xi \in \mathcal{D}(\Gamma_d)$ defines a discrete, faithful, w.t.p. representation of Γ_d^+ into $\mathrm{PSL}_2(\mathbb{C})$. If $\xi \in \beta(\Gamma_d)$, then ξ is induced by a quasiconformal map that is conformal on \mathbb{D}^* . Hence, such a representation of Γ_d^+ lies in the Bers slice of Γ_d^+ . Thus, $\beta(\Gamma_d)$ embeds into the Teichmüller space of a d times punctured sphere.

On the other hand, each $\xi \in \beta(\Gamma_d) \setminus \beta(\Gamma_d)$ induces a representation of Γ_d^+ that lies on the boundary of the Bers slice of the Fuchsian group Γ_d^+ . The index two Kleinian group Γ^+ of $\Gamma := \xi(\Gamma_d)$ is geometrically finite (a fundamental polyhedron for the action of Γ^+ on \mathbb{H}^3 is obtained by “doubling” a fundamental polyhedron for Γ , and hence it has finitely many sides). In fact, Γ^+ is a *cusplike group* that is obtained by pinching a special collection of simple closed curves on S^- . Indeed, since S^- is equipped with a natural involution ι , any Γ_d -invariant Beltrami coefficient on \mathbb{D} induces an ι -invariant Beltrami coefficient on S^- . Hence, the simple closed geodesics on S^- that can be pinched via quasiconformal deformations with Γ_d -invariant Beltrami coefficients are precisely the ones invariant under ι . Moreover, the ι -invariant simple closed geodesics on S^- bijectively correspond to pairs of non-tangential circles \mathbf{C}_i and \mathbf{C}_j ; more precisely, they are the projections to S^- of hyperbolic geodesics of \mathbb{D} with end-points at the two fixed points of the loxodromic Möbius map $\rho_i \circ \rho_j$. Hence, a group Γ^+ on the Bers boundary is obtained as a limit of a sequence of quasiFuchsian deformations of Γ_d^+ that pinch a disjoint union of ι -invariant simple, closed, essential geodesics on the bottom surface S^- without changing the (marked) conformal equivalence class of the top surface S^+ . If $\xi(\rho_i)$ is reflection in the circle C_i (for $i = 1, \dots, d$), then a point of intersection of some C_i and C_j with $j \neq i, i \pm 1 \pmod{d}$ corresponds to an *accidental parabolic* $\xi(\rho_i \circ \rho_j)$ for $\xi(\Gamma_d^+)$. Furthermore, the quotient

$$\mathcal{M}(\Gamma^+) := (\mathbb{H}^3 \cup \Omega(\Gamma^+)) / \Gamma^+$$

is an infinite volume 3-manifold whose conformal boundary $\partial\mathcal{M}(\Gamma^+) := \Omega(\Gamma^+) / \Gamma^+$ consists of finitely many punctured spheres.

2.3. Conformal Mating. In this Subsection we define the notion of conformal mating in Theorem A. Our definitions follow [PM12], to which we refer for a more extensive discussion of conformal mating.

Notation 2.39. For $\Gamma \in \overline{\beta(\Gamma_d)}$, recall $\Omega_\infty(\Gamma)$ denotes the unbounded component of $\Omega(\Gamma)$. We let $\mathcal{K}(\Gamma) := \mathbb{C} \setminus \Omega_\infty(\Gamma)$.

Remark 2.40. Let $w \mapsto p(w)$ be a monic, anti-holomorphic polynomial such that $\mathcal{J}(p)$ is connected and locally connected. Let $d := \deg(p)$, and denote by $\phi_p : \mathbb{D}^* \rightarrow \mathcal{B}_\infty(p)$ the Böttcher coordinate for p such that $\phi_p'(\infty) = 1$. We note that since $\partial\mathcal{K}(p) = \mathcal{J}(p)$ is locally connected by assumption, it follows that ϕ_p extends to a continuous semi-conjugacy between $z \mapsto \bar{z}^d|_{\mathbb{T}}$ and $p|_{\mathcal{J}(p)}$. Now let $\Gamma \in \overline{\beta(\Gamma_{d+1})}$. As was shown in Proposition 2.36, there is a natural continuous semi-conjugacy $\phi_\Gamma : \mathbb{T} \rightarrow \Lambda(\Gamma)$ between $\rho_{\Gamma_{d+1}}|_{\mathbb{T}}$ and $\rho_\Gamma|_{\Lambda(\Gamma)}$. Recall from Remark 2.35 that $\mathcal{E}_d : \mathbb{T} \rightarrow \mathbb{T}$ is a topological conjugacy between $\rho_{\Gamma_{d+1}}|_{\mathbb{T}}$ and $z \mapsto \bar{z}^d|_{\mathbb{T}}$.

Definition 2.41. Let notation be as in Remark 2.40. We define an equivalence relation \sim on $\mathcal{K}(\Gamma) \sqcup \mathcal{K}(p)$ by specifying \sim is generated by $\phi_\Gamma(t) \sim \phi_p(\overline{\mathcal{E}_d(t)})$ for all $t \in \mathbb{T}$.

Definition 2.42. Let $\Gamma \in \overline{\beta(\Gamma_{d+1})}$, p a monic, anti-holomorphic polynomial such that $\mathcal{J}(p)$ is connected and locally connected, and $f \in \Sigma_d^*$. We say that σ_f is a *conformal mating* of Γ with p if there exist continuous maps

$$\psi_p : \mathcal{K}(p) \rightarrow \widehat{\mathbb{C}} \setminus \mathcal{T}_\infty(\sigma_f) \text{ and } \psi_\Gamma : \mathcal{K}(\Gamma) \rightarrow \overline{\mathcal{T}_\infty(\sigma_f)},$$

conformal on $\text{int } \mathcal{K}(p)$, $\text{int } \mathcal{K}(\Gamma)$, respectively, such that

- (1) $\psi_p \circ p(w) = \sigma_f \circ \psi_p(w)$ for $w \in \mathcal{K}(p)$,
- (2) $\psi_\Gamma : T(\Gamma) \rightarrow T(\sigma_f)$ is label-preserving and $\psi_\Gamma \circ \rho_\Gamma(z) = \sigma_f \circ \psi_\Gamma(z)$ for $z \in \mathcal{K}(\Gamma) \setminus \text{int } T^o(\Gamma)$,
- (3) $\psi_\Gamma(z) = \psi_p(w)$ if and only if $z \sim w$ where \sim is as in Definition 2.41.

2.4. Convergence of Quadrilaterals. We conclude Section 2 by recalling a notion of convergence for quadrilaterals (see [LV73, §I.4.9]) which will be useful to us in the proof of Theorem A. We will usually denote a topological quadrilateral by Q , and its modulus by $M(Q)$.

Definition 2.43. The sequence of quadrilaterals Q_n (with a-sides a_i^n and b-sides b_i^n , $i = 1, 2$, $n \in \mathbb{N}$) converges to the quadrilateral Q (with a-sides a_i and b-sides b_i , $i = 1, 2$) if to every $\varepsilon > 0$ there corresponds an n_ε such that for $n \geq n_\varepsilon$, every point of a_i^n , b_i^n , $i = 1, 2$, and every interior point of Q_n has a spherical distance of at most ε from a_i , b_i , and Q , respectively.

Theorem 2.44. [LV73, §I.4.9] *If the sequence of quadrilaterals Q_n converges to a quadrilateral Q , then*

$$\lim_{n \rightarrow \infty} M(Q_n) = M(Q).$$

3. A HOMEOMORPHISM BETWEEN PARAMETER SPACES

The purpose of this Section is to define the mapping in Theorem A and prove that it is a homeomorphism. We will prove the conformal mating statement in Theorem A in Section 4. First we will need the following rigidity result.

Proposition 3.1. *Let Γ, Γ' be necklace groups. Suppose there exist homeomorphisms*

$$h_1 : T(\Gamma) \rightarrow T(\Gamma') \text{ and } h_2 : \Pi(\Gamma) \rightarrow \Pi(\Gamma')$$

which agree on cusps of $\partial T(\Gamma)$, and map cusps of $\partial T(\Gamma)$ to cusps of $\partial T(\Gamma')$. Suppose furthermore that h_1, h_2 are conformal on $T^\circ(\Gamma'), F^\circ(\Gamma')$, respectively. Then h_1, h_2 are restrictions of a common $M \in \text{Aut}(\mathbb{C})$ such that

$$\Gamma' = M \circ \Gamma \circ M^{-1}.$$

Proof. By iterated Schwarz reflection, we may extend the disjoint union of the maps h_1, h_2 to a conformal isomorphism of the ordinary sets $\Omega(\Gamma), \Omega(\Gamma')$. Since Γ, Γ' are geometrically finite, the conclusion then follows from [Tuk85, Theorem 4.2]. \square

Proposition 3.2. *Let $f \in \Sigma_d^*$. There exists a unique $\Gamma_f \in \overline{\beta(\mathbf{\Gamma}_{d+1})}$ such that there is a label-preserving homeomorphism*

$$h : T(\Gamma_f) \rightarrow T(\sigma_f)$$

with h conformal on $\text{int } T(\Gamma_f)$.

Proof of Existence. We first assume that $f(\mathbb{T})$ has no double points. Let

$$g : T(\mathbf{\Gamma}_{d+1}) \rightarrow T(\sigma_f)$$

be a label-preserving diffeomorphism such that

$$\|g_{\bar{z}}/g_z\|_{L^\infty(T(\mathbf{\Gamma}_{d+1}))} < 1.$$

Define a Beltrami coefficient μ_g by

$$\mu_g(u) := g_{\bar{z}}(u)/g_z(u) \text{ for } u \in T(\mathbf{\Gamma}_{d+1}),$$

and

$$\mu_g(u) := \begin{cases} \mu_g(r_i^{\circ n}(u)) & \text{if } u \in r_i^{-n}(T(\mathbf{\Gamma}_{d+1})) \text{ for } 1 \leq i \leq d+1 \text{ and } n \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Denote by $\tau_g : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ the integrating map of μ_g , normalized so that

$$\tau_g(z) = z + O(1/|z|) \text{ as } z \rightarrow \infty.$$

We claim that $\Gamma_f := \tau_g \circ \mathbf{\Gamma}_{d+1} \circ \tau_g^{-1}$ satisfies the conclusions of Proposition 3.2. Indeed,

$$\tau_g \circ \mathbf{\Gamma}_{d+1} \circ \tau_g^{-1} \in \beta(\mathbf{\Gamma}_{d+1})$$

since $\mu_g \equiv 0$ on \mathbb{D}^* . The map

$$h := g \circ \tau_g^{-1} : T(\Gamma_f) \rightarrow T(\sigma_f)$$

is conformal on $\text{int } T_{\Gamma_f}$ since τ_g is the integrating map for $g_{\bar{z}}/g_z$. Lastly, we see that h is label-preserving since τ_g^{-1} and g are both label-preserving by definition.

Next we consider the case that $f(\mathbb{T})$ has at least one double point. We claim the existence of $\Gamma \in \overline{\beta(\mathbf{\Gamma}_{d+1})}$ such that there is a label-preserving diffeomorphism $g : T(\Gamma) \rightarrow T(\sigma_f)$. Given the existence of such a Γ , the same quasiconformal deformation argument as above produces the desired group Γ_f and homeomorphism h .

The existence of such a Γ may be proven by pinching geodesics on the $(d+1)$ -times punctured sphere $\mathbb{D}/\mathbf{\Gamma}_d^+$ (where $\mathbf{\Gamma}_d^+$ is the index 2 Kleinian subgroup of $\mathbf{\Gamma}_{d+1}$ consisting of orientation-preserving automorphisms of \mathbb{C}), or adapting the techniques used in the proof of [LMM19, Theorem 4.11].

Alternatively, we may prove the existence of Γ by associating a planar vertex v_i , for $1 \leq i \leq d+1$, to each analytic arc connecting two cusps of $f(\mathbb{T})$, as in Figure 4. Connect two vertices v_i, v_j by an edge if and only if the corresponding analytic arcs have non-empty intersection. This defines a simplicial 2-complex K in the plane. K is a combinatorial closed disc, and hence [Ste05, Proposition 6.1] shows that there is a circle packing $(C'_i)_{i=1}^{d+1}$ of \mathbb{D} for K , with each C'_i tangent to $\partial\mathbb{D}$. Quasiconformally deforming this circle packing group so that there is a label-preserving conformal map to $\Pi(\mathbf{\Gamma}_{d+1})$ gives the desired $\Gamma \in \overline{\beta(\mathbf{\Gamma}_{d+1})}$ (up to Möbius conjugacy). \square

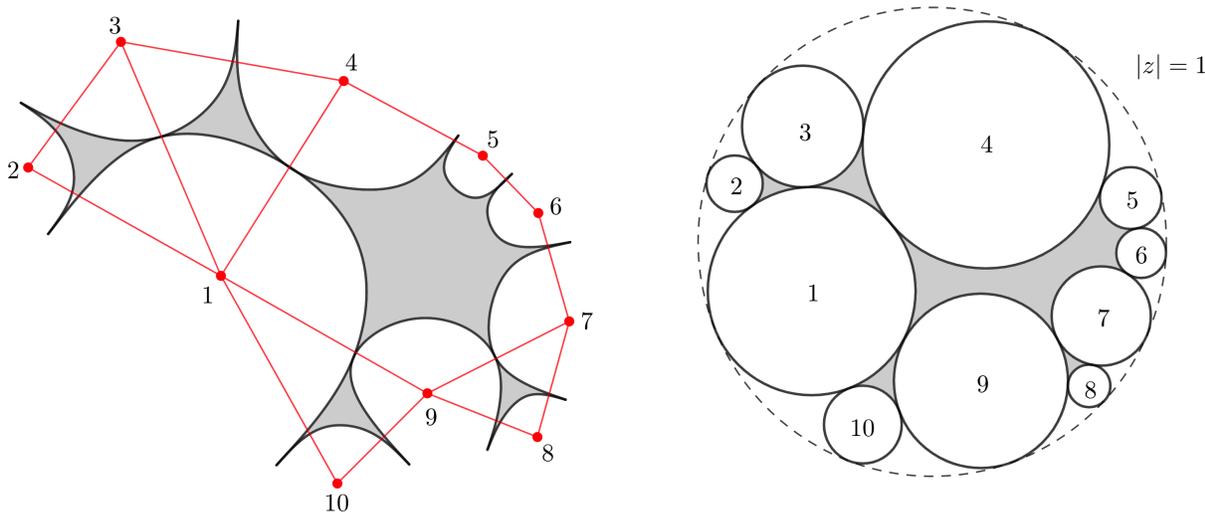


FIGURE 4. Illustrated is the procedure of associating a circle packing to an element of Σ_d^* .

Proof of Uniqueness. If Γ, Γ' both satisfy the conclusions of the Proposition, we may take h_1, h_2 as in Proposition 3.1, where $h_2(z) = z + O(1/z)$ as $z \rightarrow \infty$ since $\Gamma, \Gamma' \in \overline{\beta(\mathbf{\Gamma}_{d+1})}$. Thus as h_2 extends to an automorphism of \mathbb{C} by Proposition 3.1, it follows that $h_2 = \text{id}$, and hence $\Gamma = \Gamma'$. \square

Remark 3.3. The requirement that h be label-preserving is essential to the uniqueness statement in the conclusion of Proposition 3.2: see Figure 5.

Proposition 3.4. *The mapping*

$$(\star) \quad \begin{aligned} \Sigma_d^* &\rightarrow \overline{\beta(\mathbf{\Gamma}_{d+1})} \\ f &\mapsto \Gamma_f \end{aligned}$$

defined in Proposition 3.2 is a bijection.

Proof. We sketch a proof of surjectivity of (\star) . Suppose first that $\Gamma \in \overline{\beta(\mathbf{\Gamma}_{d+1})}$ is an interior necklace group, and let $f_0(z) := z - 1/(dz^d) \in \Sigma_d^*$. Pull back the standard conformal structure on $T^o(\Gamma)$ by a quasiconformal mapping $\mathfrak{h} : T(\sigma_{f_0}) \rightarrow T^o(\Gamma)$ which preserves vertices, spread this conformal

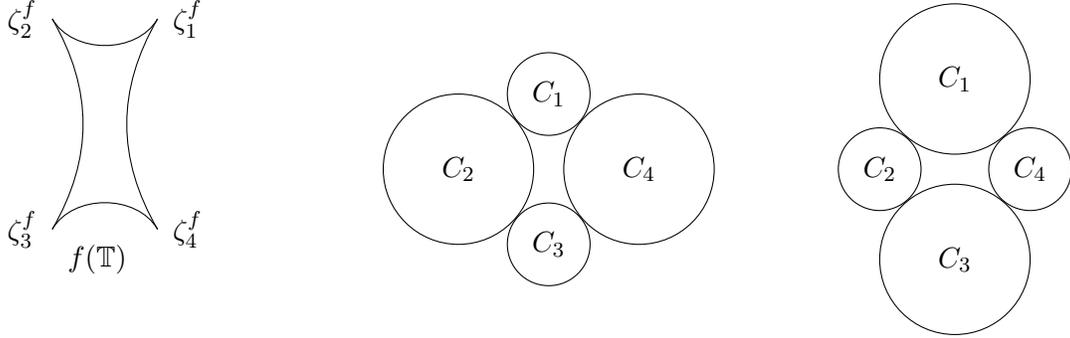


FIGURE 5. Let $f(z) := z + t/z - 1/(3z^3) \in \Sigma_3^*$ with $t > 0$. Then there are exactly two elements of $\overline{\beta(\mathbf{\Gamma}_4)}$ whose corresponding interior fundamental domains are conformally isomorphic (with cusps preserved) to $T(\sigma_f)$. However, only one of these conformal isomorphisms is label-preserving.

structure under the action of σ_f and extend elsewhere by the standard conformal structure, then straighten. This gives the desired element of Σ_d^* which maps to Γ (see the proof of [LMM19, Theorem 4.11] for details on quasiconformal deformations of f). If $\Gamma \in \overline{\beta(\mathbf{\Gamma}_{d+1})}$ is not an interior necklace group, Γ still satisfies Condition (2) of Definition 2.11 by Proposition 2.23, and so by [LMM19, Theorem 4.11] there exists $f \in \Sigma_d^*$ and a quasiconformal mapping $\mathfrak{h} : T(\sigma_f) \rightarrow T(\Gamma)$ preserving singularities, whence the above arguments apply.

We now show injectivity of (\star) . Let $f, f' \in \Sigma_d^*$ such that $\Gamma_f = \Gamma_{f'}$. Recall from Remark 2.5 that the Böttcher coordinates for $\sigma_f, \sigma_{f'}$ are both tangent to $z \mapsto wz$ at ∞ for the same w . Thus there is a conjugacy Ψ between $\sigma_f, \sigma_{f'}$ in a neighborhood of ∞ satisfying $\Psi'(\infty) = 1$. Since $\Gamma_f = \Gamma_{f'}$, there is a label-preserving conformal isomorphism of $T(\sigma_f) \rightarrow T(\sigma_{f'})$, which defines Ψ in a finite part of the plane (disjoint from the neighborhood of ∞ in which Ψ is a conjugacy). Since the 0-rays for $\sigma_f, \sigma_{f'}$ both land at a cusp with the same label (see Proposition 4.20), the definition of Ψ in the finite part of the plane and near ∞ can be connected along the 0-ray such that Ψ is a conjugacy along the 0-ray. The pullback argument of [LMM19, Theorem 5.1] now applies to show that Ψ is the restriction of a Möbius transformation M . Since $f, f' \in \Sigma_d^*$, the map M is multiplication by a $d + 1^{\text{st}}$ root of unity. Since $\Psi'(\infty) = 1$, it follows that $\Psi(z) \equiv z$. \square

We now wish to show that the mapping of Proposition 3.4 is in fact a homeomorphism, for which we first need the following lemma:

Lemma 3.5. *Let U, V be Jordan domains. Let $n \geq 4$, and suppose $u_1, \dots, u_n \in \partial U$ and $v_1, \dots, v_n \in \partial V$ are oriented positively with respect to U, V (respectively). Suppose furthermore that the quadrilaterals*

$$U(u_j, u_{j+1}, u_k, u_{k+1}), \quad V(v_j, v_{j+1}, v_k, v_{k+1})$$

have the same modulus for each j, k with $1 \leq j < j+2 \leq k \leq n-1$. Then there is a conformal map

$$f : U \rightarrow V \text{ such that } f(u_i) = v_i, \quad 1 \leq i \leq n.$$

Proof. Let $\Phi_U : U \rightarrow \mathbb{D}$, $\Phi_V : V \rightarrow \mathbb{D}$ be conformal maps such that

$$(4) \quad \Phi_U(u_j) = \Phi_V(v_j) \text{ for } 1 \leq j \leq 3.$$

Suppose by way of contradiction that

$$(5) \quad \Phi_U(u_4) \neq \Phi_V(v_4).$$

Since $U(u_1, u_2, u_3, u_4)$, $V(v_1, v_2, v_3, v_4)$ have the same modulus, it follows that there is a conformal map $g : U \rightarrow V$ with $g(u_j) = v_j$ for $1 \leq j \leq 4$. But then

$$\Phi_V \circ g \circ \Phi_U^{-1} : \mathbb{D} \rightarrow \mathbb{D}$$

is a Möbius transformation which fixes $\Phi_U(u_1)$, $\Phi_U(u_2)$, $\Phi_U(u_3) \in \partial\mathbb{D}$ by (4), but is not the identity by (5), and this is a contradiction. This shows that

$$\Phi_U(u_4) = \Phi_V(v_4),$$

and the same argument applied recursively shows that

$$\Phi_U(u_j) = \Phi_V(v_j), \text{ for } 4 \leq j \leq n.$$

The lemma follows by taking $f = \Phi_V^{-1} \circ \Phi_U$. □

Theorem 3.6. *The mapping*

$$\begin{aligned} \Sigma_d^* &\rightarrow \overline{\beta(\mathbf{\Gamma}_{d+1})} \\ f &\mapsto \Gamma_f \end{aligned}$$

defined in Proposition 3.2 is a homeomorphism.

Proof. Let $(f_n)_{n=1}^\infty \in \Sigma_d^*$, and suppose $f_n \rightarrow f_\infty \in \Sigma_d^*$. We abbreviate $\Gamma_n := \Gamma_{f_n}$, $\Gamma_\infty := \Gamma_{f_\infty}$. We want to show that

$$\Gamma_n \xrightarrow{n \rightarrow \infty} \Gamma_\infty \text{ in } \overline{\beta(\mathbf{\Gamma}_{d+1})}.$$

As $\overline{\beta(\mathbf{\Gamma}_{d+1})}$ is compact, we may assume, after passing to a subsequence, that Γ_n converges to some $\Gamma'_\infty \in \overline{\beta(\mathbf{\Gamma}_{d+1})}$.

We denote the critical values of f_n by $\zeta_1^n, \dots, \zeta_{d+1}^n$ (with the labeling chosen in Remark 2.37). For j, k with $1 \leq j < j+2 \leq k \leq d$, we consider the quadrilateral

$$Q_n := T(\sigma_{f_n})(\zeta_j^n, \zeta_{j+1}^n, \zeta_k^n, \zeta_{k+1}^n),$$

where we allow for the possibility that

$$(6) \quad \widehat{\zeta_j^n \zeta_{j+1}^n} \cap \widehat{\zeta_k^n \zeta_{k+1}^n} \neq \emptyset.$$

Note that the arcs in (6) may intersect in at most one point (see [LMM19, Proposition 4.8]), in which case we define

$$M(Q_n) := \infty.$$

Similarly, if

$$(7) \quad \widehat{\zeta_{j+1}^n \zeta_k^n} \cap \widehat{\zeta_{k+1}^n \zeta_j^n} \neq \emptyset, \text{ then } M(Q_n) := 0.$$

We note that only one of (6) or (7) may occur (see [LMM19, Proposition 4.8]).

We also consider the quadrilaterals

$$R_n := T(\Gamma_n)(h_n^{-1}(\zeta_j^n), h_n^{-1}(\zeta_{j+1}^n), h_n^{-1}(\zeta_k^n), h_n^{-1}(\zeta_{k+1}^n)),$$

where

$$h_n : R_n \rightarrow Q_n$$

is a label-preserving conformal isomorphism by Proposition 3.2, so that

$$(8) \quad M(Q_n) = M(R_n) \text{ for all } n.$$

Since $f_n \rightarrow f_\infty$ in Σ_d^* , it follows from Theorem 2.44 that

$$(9) \quad M(Q_n) \rightarrow M(Q_\infty) \text{ as } n \rightarrow \infty.$$

Now consider the quadrilateral

$$T(\Gamma'_\infty)(\eta_j^\infty, \eta_{j+1}^\infty, \eta_k^\infty, \eta_{k+1}^\infty),$$

where $\eta_j^\infty := \lim_n h_n^{-1}(\zeta_j^n)$ is a cusp of $T(\Gamma'_\infty)$. Since $\Gamma_n \rightarrow \Gamma'_\infty$ in $\overline{\beta(\mathbf{\Gamma}_{d+1})}$, it follows that

$$M(R_n) \rightarrow M(T(\Gamma'_\infty)(\eta_j^\infty, \eta_{j+1}^\infty, \eta_k^\infty, \eta_{k+1}^\infty)) \text{ as } n \rightarrow \infty.$$

Thus by (8) and (9),

$$M(Q_\infty) = M(T(\Gamma'_\infty)(\eta_j^\infty, \eta_{j+1}^\infty, \eta_k^\infty, \eta_{k+1}^\infty)).$$

As j, k are arbitrary, Lemma 3.5 applied to bounded components of $\mathbb{C} \setminus f_\infty(\mathbb{T})$ and $\text{int } T(\Gamma'_\infty)$ yields a label-preserving conformal isomorphism

$$T(\Gamma'_\infty) \rightarrow T(\sigma_{f_\infty}).$$

Thus by the uniqueness of Proposition 3.2, we have

$$\Gamma'_\infty = \Gamma_\infty,$$

as needed. We conclude that

$$\begin{aligned} \Sigma_d^* &\rightarrow \overline{\beta(\mathbf{\Gamma}_{d+1})} \\ f &\mapsto \Gamma_f \end{aligned}$$

is continuous, and the proof of continuity of the inverse is similar. \square

4. CONFORMAL MATINGS OF REFLECTION GROUPS AND POLYNOMIALS

The purpose of Section 4 is to prove the conformal mating statement of Theorem A. In Section 4.1 we will show that $\partial\mathcal{B}_\infty(\sigma_f)$ is locally connected for $f \in \Sigma_d^*$, whence in Section 4.2 we will show that $\mathcal{B}_\infty(\sigma_f)$ and $\mathcal{T}_\infty(\sigma_f)$ share a common boundary. Sections 4.3 and 4.4 study laminations of \mathbb{T} induced by σ_f and necklace groups Γ , whence it is shown in Section 4.5 that for $f \in \Sigma_d^*$, the laminations induced by σ_f and Γ_f are compatible. Finally, in Section 4.6, we deduce that σ_f is a conformal mating of Γ_f and $w \mapsto \bar{w}^d$ (see Definition 2.42).

4.1. Local Connectivity.

Lemma 4.1. *Let $f \in \Sigma_d^*$. Then*

$$|\bar{\partial}\sigma_f(z)| > 1, \forall z \in f(\mathbb{D}^*).$$

Proof. Let $\sigma := \sigma_f$. From Definition 2.3, we see that

$$(10) \quad \sigma(f(w)) = f(1/\bar{w}), \text{ for } |w| > 1.$$

Taking the $\bar{\partial}$ -derivative of (10) yields

$$(11) \quad \bar{\partial}\sigma(f(w)) \cdot \overline{f'(w)} = -\frac{1}{\bar{w}^2} \cdot f'\left(\frac{1}{\bar{w}}\right), \text{ for } |w| > 1.$$

By [LM14, Lemma 2.6], we also have that

$$(12) \quad f'\left(\frac{1}{\bar{w}}\right) = \bar{w}^{d+1} \overline{f'(w)}, \forall w \in \mathbb{C}.$$

Combining (11) and (12), we conclude that

$$|\bar{\partial}\sigma(f(w))| = |w|^{d-1} > 1 \text{ for } |w| > 1.$$

□

Proposition 4.2. *Let $f \in \Sigma_d^*$. Then $\partial\mathcal{B}_\infty(\sigma_f)$ is locally connected.*

Remark 4.3. Our proof follows the strategy taken in [DH85, Chapter 10].

Proof. Let $\sigma := \sigma_f$, $\phi_\sigma := \phi_{\sigma_f}$ be as in Remark 2.5, and $X := \mathcal{B}_\infty(\sigma) \cap \mathbb{C}$. Note that $\sigma : X \rightarrow X$ is a d :1 covering map. Define an equipotential curve

$$E(r) := \phi_\sigma(\{z \in \mathbb{C} : |z| = r\}).$$

We define, for $n \geq 1$, parametrizations $\gamma_n : \mathbb{T} \rightarrow E(2^{1/d^n})$ by:

$$\gamma_n(e^{2\pi i\theta}) := \phi_\sigma(2^{1/d^n} e^{2\pi i\theta}).$$

By (1), we have:

$$\sigma \circ \gamma_{n+1}(e^{2\pi i\theta}) = \gamma_n(e^{-2\pi i d\theta}).$$

We will show that the sequence $(\gamma_n)_{n=1}^\infty$ forms a Cauchy sequence in the complete metric space $C(\mathbb{T}, \mathbb{C})$. We will denote the length of a curve γ by $l(\gamma)$, and the lift of γ under σ by $\tilde{\gamma}$.

To this end, define $h : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ by

$$(13) \quad h(s) := \sup_{\substack{\gamma \in C(\mathbb{T}, X) \\ l(\gamma) \leq s}} \{l(\tilde{\gamma}) : \sigma(\tilde{\gamma}) = \gamma\}.$$

We claim that

$$(14) \quad h(s) < s \text{ and } h(ks) \leq kh(s), \forall s > 0 \text{ and } k \in \mathbb{N}.$$

Indeed, the first inequality of (14) follows from Lemma 4.1. The second inequality in (14) follows from the triangle inequality. It follows from (14) that $s - h(s) \rightarrow \infty$ as $s \rightarrow \infty$. Let $\ell := \text{dist}(\gamma_0, \gamma_1)$, and choose $L > \ell$ sufficiently large such that $L - h(L) > \ell$. One has:

$$\text{dist}(\gamma_2, \gamma_0) \leq \text{dist}(\gamma_2, \gamma_1) + \text{dist}(\gamma_1, \gamma_0) \leq h(\ell) + \ell \leq h(L) + \ell < L.$$

Similarly, an inductive procedure yields

$$\text{dist}(\gamma_n, \gamma_0) < L, \forall n \geq 0.$$

Observe that

$$\text{dist}(\gamma_n, \gamma_{n+p}) \leq h^{\circ n}(\text{dist}(\gamma_0, \gamma_p)) < h^{\circ n}(L).$$

The sequence $(h^{\circ n}(L))_{n=1}^{\infty} \in \mathbb{R}_{\geq 0}$ is decreasing and converges to a fixpoint of h , and this fixpoint must be 0 by (14). Thus $(\gamma_n)_{n=1}^{\infty}$ is a Cauchy sequence, and the limit is a continuous extension of

$$\phi_{\sigma} : \mathbb{D}^* \rightarrow \mathcal{B}_{\infty}(\sigma) \text{ to } \phi_{\sigma} : \mathbb{T} \rightarrow \partial\mathcal{B}_{\infty}(\sigma).$$

Local connectivity of $\partial\mathcal{B}_{\infty}(\sigma)$ follows from a theorem of Carathéodory. \square

4.2. The Limit Set is The Boundary of The Basin of Infinity. The goal of this Subsection is to prove the following:

Proposition 4.4. *Let $f \in \Sigma_d^*$. Then $\partial\mathcal{B}_{\infty}(\sigma_f) = \partial\mathcal{T}_{\infty}(\sigma_f)$.*

The proof of Proposition 4.4 will be carried out by way of several lemmas below. First we record the following definition:

Definition 4.5. Let $f \in \Sigma_d^*$. An *external ray* for σ_f is a curve

$$t \mapsto \phi_{\sigma_f}(te^{i\theta}), t \in (1, \infty)$$

for some $\theta \in [0, 2\pi)$, where ϕ_{σ_f} is the Böttcher coordinate of Remark 2.5. For $\theta \in [0, 2\pi)$, we refer to

$$\{\phi_{\sigma_f}(te^{i\theta}) : t \in (1, \infty)\}$$

as the θ -ray of σ_f .

Remark 4.6. By Proposition 4.2, each external ray of σ_f lands, in other words $\lim_{t \rightarrow 1^+} \phi_{\sigma_f}(te^{i\theta})$ exists for each $\theta \in [0, 2\pi)$.

Notation 4.7. Let $\Sigma_{d,k}^*$ denote the collection of those $f \in \Sigma_d^*$ such that $f(\mathbb{T})$ has exactly k double points.

For the remainder of this subsection we fix $f \in \Sigma_{d,k}^*$ and denote $\sigma := \sigma_f$. Let us first record the straightforward inclusion:

Lemma 4.8. $\partial\mathcal{B}_{\infty}(\sigma) \subset \partial\mathcal{T}_{\infty}(\sigma)$.

Proof. We note that $\mathcal{T}_{\infty}(\sigma)$ is open, whence the relation

$$(15) \quad \widehat{\mathcal{C}} = \mathcal{T}_{\infty}(\sigma) \sqcup \partial\mathcal{T}_{\infty}(\sigma) \sqcup \mathcal{B}_{\infty}(\sigma)$$

follows from the classical classification of periodic Fatou components and the observation that σ has only one singular value (at ∞). The Lemma follows from (15). \square

The proof of the opposite inclusion is a bit more involved, and we split the main arguments into a couple of lemmas.

Lemma 4.9. *The landing points of the fixed external rays of σ are singular points of $f(\mathbb{T})$.*

Proof. Note that the landing points of the fixed rays of σ are necessarily fixed points of σ on $\partial\mathcal{B}_\infty(\sigma)$. Since $\partial\mathcal{B}_\infty(\sigma) \subset \partial\mathcal{T}_\infty(\sigma)$ by Lemma 4.8, the result will follow if we can prove that the only fixed points of $\sigma|_{\partial\mathcal{T}_\infty(\sigma)}$ are the singular points of $f(\mathbb{T})$. This will be shown via the Lefschetz fixed-point formula.

As f has k double points, there are $k + 1$ forward-invariant components $\mathcal{U}_1, \dots, \mathcal{U}_{k+1}$ of $\mathcal{T}_\infty(\sigma)$, each containing a single component of $T^\circ(\sigma)$. Let $T_i := \mathcal{U}_i \cap T^\circ(\sigma)$, so that T_i is naturally a $(3 + j_i)$ -gon ($j_i \geq 0$) whose vertices are the $3 + j_i$ singularities of $f(\mathbb{T})$ lying on $\partial\mathcal{U}_i$. Each \mathcal{U}_i is a simply connected domain as it can be written as an increasing union of pullbacks (under σ) of T_i (see also [LLMM18a, Proposition 5.6]). Moreover, we can map each T_i conformally to a $(3 + j_i)$ -gon in \mathbb{D} whose edges are geodesics of \mathbb{D} . By iterated Schwarz reflection, one now obtains a Riemann map from \mathbb{D} onto \mathcal{U}_i . Using Lemma 4.1, one can mimic the proof of Proposition 4.2 to show that each $\partial\mathcal{U}_i$ is locally connected.

We now consider a quasiconformal homeomorphism $\chi_i : T^\circ(\mathbf{\Gamma}_{3+j_i}) \rightarrow T_i$ that sends the boundary cusps to the boundary cusps. Lifting χ_i by $\rho_{\mathbf{\Gamma}_{3+j_i}}$ and σ , we obtain a quasiconformal homeomorphism $\chi_i : \mathbb{D} \rightarrow \mathcal{U}_i$ that conjugates $\rho_{\mathbf{\Gamma}_{3+j_i}}$ to σ . Since $\partial\mathcal{U}_i$ is locally connected, χ_i extends continuously to the boundary, and yields a topological semi-conjugacy between $\rho_{\mathbf{\Gamma}_{3+j_i}}|_{\mathbb{T}}$ and $\sigma|_{\partial\mathcal{U}_i}$. It now follows from Remark 2.35 that

$$\widehat{\chi}_i := \chi_i \circ \mathcal{E}_{2+j_i}^{-1} : \mathbb{T} \rightarrow \partial\mathcal{U}_i$$

is a topological semi-conjugacy between $\bar{z}^{2+j_i}|_{\mathbb{T}}$ and $\sigma|_{\partial\mathcal{U}_i}$. Let $\widehat{\chi}_i : \overline{\mathbb{D}} \rightarrow \overline{\mathcal{U}_i}$ be an arbitrary continuous extension of $\widehat{\chi}_i|_{\mathbb{T}}$ such that $\widehat{\chi}_i$ maps \mathbb{D} homeomorphically onto \mathcal{U}_i .

We now (topologically) glue attracting basins into the domains \mathcal{U}_i :

$$\check{\sigma}(w) := \begin{cases} \sigma(w) & \text{on } \widehat{\mathbb{C}} \setminus \bigcup_{i=1}^{k+1} \mathcal{U}_i, \\ \widehat{\chi}_i \left(\overline{\widehat{\chi}_i^{-1}(w)}^{2+j_i} \right) & \text{on } \mathcal{U}_i, \text{ for } i = 1, \dots, k+1. \end{cases}$$

The map $\check{\sigma}$ is a degree d orientation-reversing branched cover of $\widehat{\mathbb{C}}$. We argue that each fixed point of $\check{\sigma}$ is either attracting or repelling. By Lemma 4.1 and construction of $\check{\sigma}$, this is the case for each fixed point in $\widehat{\mathbb{C}} \setminus \bigcup_i \partial\mathcal{U}_i$. We note that $\check{\sigma}$ has $k + 2$ attracting fixed points (one in each \mathcal{U}_i and one at ∞). Also by Lemma 4.1 and construction of $\check{\sigma}$, any fixed point of $\check{\sigma}$ on $\partial\mathcal{U}_i \setminus \{\text{singular values of } f(\mathbb{T})\}$ must be repelling. The singular values of $f(\mathbb{T})$ are fixed under $\check{\sigma}$ by construction. Such fixed points exhibit *parabolic* behavior under σ , and near such a fixed point, the complement of $\bigcup_{i=1}^{k+1} \mathcal{U}_i$ lies in the corresponding repelling petals (compare [LLMM18a, Propositions 6.10, 6.11]). Moreover, by construction of $\check{\sigma}$, the singular values of $f(\mathbb{T})$ are also repelling for $\check{\sigma}|_{\bigcup_{i=1}^{k+1} \mathcal{U}_i}$. Thus singular values of $f(\mathbb{T})$ are repelling fixed points of $\check{\sigma}$, and so each fixed point of $\check{\sigma}$ is either attracting or repelling.

By the Lefschetz fixed-point formula (see [LM14, Lemma 6.1]), we may thus conclude that $\check{\sigma}$ has $(d + 2k + 3)$ fixed points in \mathbb{S}^2 . We have already counted that $\check{\sigma}$ has $k + 2$ attracting fixed points, and $d + k + 1$ repelling fixed points at singular values of $f(\mathbb{T})$, so that we can conclude there are no other fixed points of $\check{\sigma}$. Since σ and $\check{\sigma}$ have the same fixed points on $\partial\mathcal{T}_\infty(\sigma)$, it follows that the singular points of $\partial\mathcal{T}(\sigma)$ are the only fixed points of σ on $\partial\mathcal{T}_\infty(\sigma)$. \square

Lemma 4.10. $\text{int } \overline{\mathcal{T}_\infty(\sigma)} = \mathcal{T}_\infty(\sigma)$. *In particular, each component of $\mathcal{T}_\infty(\sigma)$ is a Jordan domain.*

Proof. Let \mathcal{U} denote a component of $\mathcal{T}_\infty(\sigma)$. We first show that $\partial\mathcal{U} \subset \partial\mathcal{B}_\infty(\sigma)$. First assume \mathcal{U} is the forward-invariant component of $\mathcal{T}_\infty(\sigma)$ which contains the landing point p of the 0-ray of σ . As in the proof of Lemma 4.9, we note that $\sigma|_{\partial\mathcal{U}}$ is topologically semi-conjugate to $\rho_{\mathbb{R}_{2+j_i}}|_{\mathbb{T}}$. Thus the iterated pre-images of p under σ are dense in $\partial\mathcal{U}$. Since $p \in \partial\mathcal{B}_\infty(\sigma)$ by Lemma 4.9, and $\partial\mathcal{B}_\infty(\sigma)$ is completely invariant, it follows that $\partial\mathcal{U} \subset \partial\mathcal{B}_\infty(\sigma)$. A similar argument applies to show that the boundary of any forward-invariant component of $\mathcal{T}_\infty(\sigma)$ is contained in $\partial\mathcal{B}_\infty(\sigma)$. Lastly, any other component of $\mathcal{T}_\infty(\sigma)$ maps (under some iterate of σ) onto one of the invariant components of $\mathcal{T}_\infty(\sigma)$, so that $\partial\mathcal{U} \subset \partial\mathcal{B}_\infty(\sigma)$ for any component \mathcal{U} of $\mathcal{T}_\infty(\sigma)$.

Note that since $\mathcal{T}_\infty(\sigma)$ is open, we have $\mathcal{T}_\infty(\sigma) \subset \text{int } \overline{\mathcal{T}_\infty(\sigma)}$. Let us now pick a component W of $\text{int } \overline{\mathcal{T}_\infty(\sigma)}$. Since $\partial\mathcal{T}_\infty(\sigma)$ is nowhere dense in \mathbb{C} , it follows that W must intersect some component \mathcal{U} of $\mathcal{T}_\infty(\sigma)$. As W is a maximal open connected subset of $\overline{\mathcal{T}_\infty(\sigma)}$, it follows $\mathcal{U} \subset W$. However, if $\mathcal{U} \subsetneq W$, then $\partial\mathcal{U}$ must contain some point not belonging to $\partial\mathcal{T}_\infty(\sigma) = \partial\mathcal{B}_\infty(\sigma)$, and this contradicts what was shown in the previous paragraph. Thus $\mathcal{U} = W$, and so $\text{int } \overline{\mathcal{T}_\infty(\sigma)} \subset \mathcal{T}_\infty(\sigma)$. The conclusion of the lemma follows. \square

Proof of Proposition 4.4. Let

$$x \in \partial\mathcal{T}_\infty(\sigma) \subset \overline{\mathcal{T}_\infty(\sigma)} = \left(\text{int } \overline{\mathcal{T}_\infty(\sigma)} \right) \sqcup \overline{\partial\mathcal{T}_\infty(\sigma)}.$$

By Lemma 4.10 and the openness of $\mathcal{T}_\infty(\sigma)$, it follows that

$$x \in \overline{\partial\mathcal{T}_\infty(\sigma)} = \partial\mathcal{B}_\infty(\sigma).$$

Hence, $\partial\mathcal{T}_\infty(\sigma) \subset \partial\mathcal{B}_\infty(\sigma)$, and together with Lemma 4.8, this proves Proposition 4.4. \square

Corollary 4.11. *Let $f \in \Sigma_d^*$. Then*

$$\widehat{\mathbb{C}} = \mathcal{B}_\infty(\sigma_f) \sqcup \Lambda(\sigma_f) \sqcup \mathcal{T}_\infty(\sigma_f),$$

where $\Lambda(\sigma_f) = \partial\mathcal{B}_\infty(\sigma_f) = \partial\mathcal{T}_\infty(\sigma_f)$ is the limit set of σ_f .

Proof. This follows from (15), Proposition 4.4, and the definition of $\Lambda(\sigma_f)$ (see Subsection 2.1). \square

4.3. Lamination for The Limit Set $\Lambda(\sigma_f)$. In this Subsection, we study further the external rays of σ_f introduced already in Definition 4.5. Recall the Böttcher coordinate $\phi_{\sigma_f} : \mathbb{D}^* \rightarrow \mathcal{B}_\infty(\sigma_f)$ of Remark 2.5.

Remark 4.12. By Proposition 4.2, ϕ_{σ_f} extends continuously to a surjection

$$\phi_{\sigma_f} : \mathbb{T} \rightarrow \Lambda(\sigma_f) \text{ such that } \sigma_f \circ \phi_{\sigma_f}(u) = \phi_{\sigma_f}(\bar{u}^d).$$

We denote the set of all fibers of points of $\Lambda(\sigma_f)$ under the semi-conjugacy ϕ_{σ_f} by $\lambda(\sigma_f)$. Clearly, $\lambda(\sigma_f)$ defines an equivalence relation on \mathbb{T} . The description of $\lambda(\sigma_f)$ will be crucial to the proof of the conformal mating statement in Theorem A.

Remark 4.13. We will usually identify \mathbb{T} with \mathbb{R}/\mathbb{Z} and the map $z \mapsto \bar{z}^d$ on \mathbb{T} with the map

$$m_{-d} : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z},$$

defined by $m_{-d}(x) := -dx$.

Lemma 4.14. *Let $f \in \Sigma_d^*$. Then:*

- (1) Each cusp of $f(\mathbb{T})$ is the landing point of a unique external ray of $\mathcal{B}_\infty(\sigma_f)$, and the angle of this ray is fixed under m_{-d} .
- (2) Each double point of $f(\mathbb{T})$ is the landing point of exactly two external rays of $\mathcal{B}_\infty(\sigma_f)$, and the angles of the corresponding two rays form a 2-cycle under m_{-d} .

Proof. We abbreviate $\sigma := \sigma_f$ and $T := T(\sigma_f)$. Let ζ be a singular point of $f(\mathbb{T})$. Since $\zeta \in \partial\mathcal{T}_\infty(\sigma)$, Propositions 4.2 and 4.4 imply that ζ is the landing point of at least one external ray of σ .

Proof of (1). Assume ζ is a cusp of $f(\mathbb{T})$. We will show that ζ is not a cut-point of $\overline{\mathcal{T}_\infty(\sigma)}$. Let S be the connected component of $\overline{\mathcal{T}_\infty(\sigma)} \setminus \{\zeta\}$ which contains $T \setminus \{\zeta\}$. It follows from the covering properties of σ that

$$\bigcup_{k=0}^n \sigma^{-k}(T \setminus \{\zeta\})$$

is a connected subset of $\overline{\mathcal{T}_\infty(\sigma)} \setminus \{\zeta\}$, so that

$$\bigcup_{k=0}^n \sigma^{-k}(T \setminus \{\zeta\}) \subset S, \forall n \geq 0.$$

It follows that

$$\overline{\mathcal{T}_\infty(\sigma)} \subset \overline{\bigcup_{r=0}^{\infty} \sigma^{-r}(T \setminus \{\zeta\})} \subset \overline{S} \subset S \cup \{\zeta\}.$$

By our choice of S , we have that

$$\overline{\mathcal{T}_\infty(\sigma)} = S \cup \{\zeta\},$$

so that ζ is not a cut point of $\overline{\mathcal{T}_\infty(\sigma)}$. Hence, ζ is the landing point of exactly one external ray γ of $\mathcal{B}_\infty(\sigma)$. Since $\sigma(\zeta) = \zeta$, it follows that $\sigma(\gamma)$ also lands at ζ , whence $\sigma(\gamma) = \gamma$. In particular, the angle of γ is fixed under m_{-d} . \square

Proof of (2). Suppose now that ζ is a double point of $f(\mathbb{T})$. Let S_1, S_2 be the two components of $\overline{\mathcal{T}_\infty(\sigma)} \setminus \{\zeta\}$ such that $T \setminus \{\zeta\} \subset S_1 \cup S_2$. A similar argument as in the proof of (1) yields that

$$\overline{\mathcal{T}_\infty(\sigma)} = S_1 \cup S_2 \cup \{\zeta\}.$$

In particular, S_1, S_2 are the only components of $\overline{\mathcal{T}_\infty(\sigma)} \setminus \{\zeta\}$. Thus there are only two accesses to ζ from $\mathcal{B}_\infty(\sigma)$, and hence there are exactly two external rays landing at ζ . By (1), the $(d+1)$ fixed rays land at the $(d+1)$ distinct cusps on $f(\mathbb{T})$. Therefore, the angles of the two rays landing at ζ must be of period two, forming a 2-cycle under m_{-d} . \square

Notation 4.15. We denote by $\mathcal{A}^{\text{cusp}}$ the angles of external rays of σ_f landing at the cusps of $f(\mathbb{T})$, and by $\mathcal{A}^{\text{double}}$ the angles of external rays of σ_f landing at the double points of $f(\mathbb{T})$.

Remark 4.16. By Proposition 4.14, $\mathcal{A}^{\text{cusp}}$ is the set of angles fixed under m_{-d} . On the other hand, for $f \in \Sigma_{d,k}^*$, $\mathcal{A}^{\text{double}}$ consists of $2k$ angles of period two which we enumerate as $\{\alpha_1, \alpha'_1, \dots, \alpha_k, \alpha'_k\}$ where the rays at angles α_i, α'_i land at a common point.

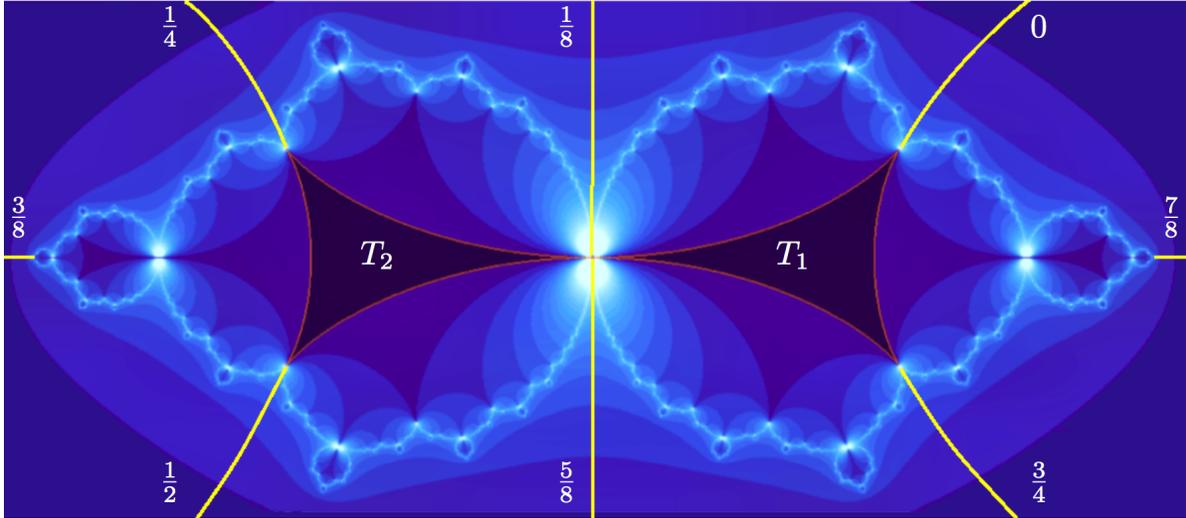


FIGURE 6. Shown is $\mathcal{T}_\infty(\sigma_f)$ for $f(z) := z - 2/(3z) - 1/(3z^3)$. Also pictured are several external rays for σ_f : here $\mathcal{A}^{\text{cusp}} = \{0, 1/4, 1/2, 3/4\}$, and $\mathcal{A}^{\text{double}} = \{1/8, 5/8\}$. This figure was made by Seung-Yeop Lee.

Remark 4.17. Let $f \in \Sigma_{d,k}^*$. The union of $T(\sigma_f)$ with the external rays of σ_f at angles $\mathcal{A}^{\text{cusp}}$ and $\mathcal{A}^{\text{double}}$ cut the limit set $\Lambda(\sigma)$ into $d + 2k + 1$ pieces that form a Markov partition for the dynamics $\sigma : \Lambda(\sigma) \rightarrow \Lambda(\sigma)$ (see Figure 6). Correspondingly, the angles in $\mathcal{A}^{\text{cusp}} \cup \mathcal{A}^{\text{double}}$ determine a Markov partition for $m_{-d} : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}$, and the elements of this Markov partition have diameter at most $1/d$.

Proposition 4.18. *Let $f \in \Sigma_{d,k}^*$, and $x \in \lambda(\sigma_f)$ be a non-trivial equivalence class. Then, the following hold true.*

- (1) $|x| = 2$, and $m_{-d}^{\circ n}(x) = \{\alpha_i, \alpha'_i\}$ for some $n \geq 0$ and $1 \leq i \leq k$.
- (2) If n_0 is the smallest non-negative integer with $m_{-d}^{\circ n_0}(x) = \{\alpha_i, \alpha'_i\}$, then x is contained in a connected component of $\mathbb{T} \setminus m_{-d}^{-(n_0-1)}(\mathcal{A}^{\text{double}} \cup \mathcal{A}^{\text{cusp}})$.

Proof of (1). We abbreviate $\sigma := \sigma_f$. If $x \in \lambda(\sigma_f)$ is non-trivial, x is a collection of ≥ 2 angles whose corresponding external rays for σ land at a cut-point w of $\Lambda(\sigma)$. Let θ, θ' be distinct angles in x . Suppose w is not a double point of $f(\mathbb{T})$. As w is a cut-point of $\Lambda(\sigma)$, it follows from Lemma 4.14 that w is not a cusp of $f(\mathbb{T})$.

Suppose by way of contradiction that no iterate of σ maps w to a double point of $f(\mathbb{T})$. Note that no iterate of σ can map w to a cusp of $f(\mathbb{T})$, as σ is a local homeomorphism on $\Lambda(\sigma) \setminus f(\mathbb{T})$ and cusps of $f(\mathbb{T})$ are not cut-points of $\Lambda(\sigma)$ by Lemma 4.14. Thus, w has a well-defined itinerary (or symbol sequence) with respect to the Markov partition of $\Lambda(\sigma)$ in Remark 4.17. This implies that the angles θ and θ' have the same (well-defined) itinerary with respect to the corresponding Markov partition of \mathbb{R}/\mathbb{Z} . However, this contradicts expansivity of the map $m_{-d} : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}$, as the distance between $m_{-d}^{\circ j}(\theta)$ and $m_{-d}^{\circ j}(\theta')$ must exceed $1/(d+1)$ for some $j \in \mathbb{N}$. This contradiction

proves that $\sigma^{\circ n}(w)$ is a double point of $f(\mathbb{T})$ for some $n \geq 0$. Let us choose the smallest n with this property, and call it n_0 .

By Lemma 4.14, there are exactly two rays $\{\alpha_i, \alpha'_i\}$ landing at the double point $\sigma^{\circ n_0}(w)$. As σ is a local homeomorphism on $\Lambda(\sigma) \setminus f(\mathbb{T})$, it follows that θ, θ' are the only two rays landing at w , and that $\sigma^{\circ n_0}$ maps the pair of rays at angles $\{\theta, \theta'\}$ to the pair of rays at angles $\{\alpha_i, \alpha'_i\}$. In other words, $x = \{\theta, \theta'\}$ and $m_{-d}^{\circ n_0}(x) = \{\alpha_i, \alpha'_i\}$. \square

Proof of (2). This follows from the landing patterns of the rays corresponding to the angles in $\mathcal{A}^{\text{double}} \cup \mathcal{A}^{\text{cusp}}$ and injectivity of σ on the interior of each piece of the Markov partition of $\Lambda(\sigma)$ defined in Remark 4.17. \square

We conclude this subsection with a proof of connectedness of Σ_d^* (which was used to define the labeling of the cusps on $f(\mathbb{T})$ in Remark 2.37), and a dynamical characterization of the cusp ζ_1^f as the landing point of the 0-ray of σ_f , for $f \in \Sigma_d^*$ (which was used in the injectivity step of the proof of Proposition 3.4).

Proposition 4.19. Σ_d^* is connected.

Proof. The main ideas of the proof are already present in [LMM19], so we only give a sketch.

Let $f_0(z) = z - 1/dz^d$. By [LMM19, Proposition 3.1], we have $f_0 \in \Sigma_d^*$ and $f_0(\mathbb{T})$ is a Jordan curve. We will denote the connected component of Σ_d^* containing f_0 by $\widetilde{\Sigma}_d^*$.

For a $(d+1)$ -st root of unity ω , the map M_ω is defined as $M_\omega(z) = \omega z$. The map M_ω induces a homeomorphism

$$(M_\omega)_* : \Sigma_d^* \rightarrow \Sigma_d^*, f \mapsto M_\omega \circ f \circ M_\omega^{-1}.$$

Since $(M_\omega)_*(f_0) = f_0$, it follows that $\widetilde{\Sigma}_d^*$ is invariant under $(M_\omega)_*$.

Let $f \in \Sigma_d^*$, and suppose that $f(\mathbb{T})$ has k double points, for some $0 \leq k \leq d-2$. Since $f_0(\mathbb{T})$ is a Jordan curve, we can think of $f_0(\mathbb{T})$ as a $(d+1)$ -gon with vertices at the cusp points. By repeated applications of [LMM19, Theorem 4.11] and quasiconformal deformation of Schwarz reflection maps, one can now “pinch” k suitably chosen pairs of non-adjacent sides of $f_0(\mathbb{T})$ producing some $\tilde{f} \in \Sigma_d^*$ such that there exists a homeomorphism $\mathfrak{h} : T(\sigma_{\tilde{f}}) \rightarrow T(\sigma_f)$ that is conformal on $\text{int} T(\sigma_{\tilde{f}})$. Note that the proof of [LMM19, Theorem 4.11] consists of two steps; namely, quasiconformally deforming Schwarz reflection maps and extracting limits of suitable sequences in Σ_d^* . Thanks to the parametric version of the Measurable Riemann Mapping Theorem and continuity of normalized Riemann maps, one can now conclude that $\tilde{f} \in \widetilde{\Sigma}_d^*$. Finally, due to the existence of a homeomorphism $\mathfrak{h} : T(\sigma_{\tilde{f}}) \rightarrow T(\sigma_f)$ that is conformal on $\text{int} T(\sigma_{\tilde{f}})$, the arguments of [LMM19, Theorem 5.1] apply mutatis mutandis to the current setting, and provide us with affine map A with $\sigma_f \equiv A \circ \sigma_{\tilde{f}} \circ A^{-1}$; i.e., $A(\tilde{f}(\mathbb{D}^*)) = f(\mathbb{D}^*)$. Arguing as in the injectivity step of [LMM19, Proposition 2.14], one now sees that $f = M_\omega \circ \tilde{f} \circ M_\omega^{-1} = (M_\omega)_*(\tilde{f})$, where ω is a $(d+1)$ -st root of unity. Since $\tilde{f} \in \widetilde{\Sigma}_d^*$ and $\widetilde{\Sigma}_d^*$ is invariant under $(M_\omega)_*$, it follows that $f \in \widetilde{\Sigma}_d^*$. Hence, $\Sigma_d^* = \widetilde{\Sigma}_d^*$; i.e., Σ_d^* is connected. \square

Recall from that the cusps on $f(\mathbb{T})$ were labeled as $\zeta_1^f, \dots, \zeta_{d+1}^f$ so that $f \mapsto \zeta_i^f$ is continuous.

Proposition 4.20. Let $f_0(z) := z - 1/(dz^d)$, $f \in \Sigma_d^*$ and $\omega_0 := e^{\frac{i\pi}{d+1}}$. Then:

- (1) The 0-ray of σ_{f_0} lands at the cusp $\zeta_1^{f_0} = (1 + 1/d)\omega_0$ of $f_0(\mathbb{T})$.
(2) The 0-ray of σ_f lands at the cusp point ζ_1^f of $f(\mathbb{T})$.

Proof. We abbreviate $\sigma := \sigma_{f_0}$. We will also employ our notation $\phi_\sigma : \mathbb{D}^* \rightarrow \mathcal{B}_\infty(\sigma_f)$ for the Böttcher coordinate for σ , where we recall the normalization $\phi'_\sigma(\infty) = d^{\frac{1}{d-1}}\omega_0$.

Proof of (1). We first note that $\zeta_1^{f_0} = f_0(\xi_1^{f_0}) = f_0(\omega_0) = (1 + 1/d)\omega_0$. A simple computation shows that

$$(16) \quad f(\omega_0 \cdot x) = \omega_0 \cdot \left(x + \frac{1}{dx^d} \right) \text{ for } x \in \mathbb{R}^+.$$

Next we note that

$$(17) \quad x + \frac{1}{dx^d} > 1 + \frac{1}{d} \text{ for } 0 < x < 1.$$

Let $\gamma := \{t\omega_0 : t > 1 + 1/d\}$. It follows from (16) and (17) that $\sigma(\gamma) \subset \gamma$. Moreover, the endpoints $(1 + 1/d)\omega_0, \infty$ of γ are fixed by σ , so since $|\bar{\partial}\sigma| > 1$ on γ by Lemma 4.1, it follows that $\gamma \subset \mathcal{B}_\infty(\sigma)$.

We claim it follows then that γ must be the 0-ray for σ . Indeed, suppose by way of contradiction that $te^{i\theta} \in \phi_\sigma^{-1}(\gamma)$ where $t > 1$ and $\theta \in (0, 2\pi)$, where we may assume θ is not a pre-image of 0 under m_{-d} . Then $t^{d^n}e^{i(-d)^n\theta} \in \phi_\sigma^{-1}(\gamma)$ for all $n > 1$. Thus there exists $\theta' \in (0, 2\pi)$ and a sequence $(z_n)_{n=1}^\infty \in \phi_\sigma^{-1}(\gamma)$ with $z_n \rightarrow \infty$ and $\arg(z_n) \rightarrow \theta'$ as $n \rightarrow \infty$. But then since $\arg(\phi'_\sigma(\infty)) = \arg(\omega_0)$, we have $\arg(\phi_\sigma(z_n)) \rightarrow \theta' + \arg(\omega_0)$ as $n \rightarrow \infty$. This is a contradiction since $\theta' + \arg(\omega_0) \not\equiv \arg(\omega_0) \pmod{2\pi}$, but $\arg(\phi_\sigma(z)) = \arg(\omega_0)$ for all $z \in \phi_\sigma^{-1}(\gamma)$. \square

Proof of (2). For $i \in \{1, \dots, d+1\}$, let us denote by X_i the set of $f \in \Sigma_d^*$ for which the 0-ray of σ_f lands at ζ_i^f .

We claim that each X_i is an open set. To this end, suppose that $f \in X_i$. It follows from the parabolic behavior of the cusps that the tail of the 0-ray of σ_f is contained in a repelling petal at ζ_i^f . In particular, we can assume that there exists some $r_0 > 0$ such that the part of the 0-ray of σ_f between potentials r_0/d and r_0 is contained in a sufficiently small repelling petal at ζ_i^f . Note that as cusps of $f(\mathbb{T})$ move continuously, so does a repelling petal at the cusp. It now follows from continuity of normalized Böttcher coordinates that for $f' \in \Sigma_d^*$ close to f , the part of the 0-ray of $\sigma_{f'}$ between potentials r_0/d and r_0 is contained in a repelling petal at $\zeta_i^{f'}$. Since a repelling petal is invariant under the inverse branch of $\sigma_{f'}$ fixing $\zeta_i^{f'}$, we conclude that for $f' \in \Sigma_d^*$ close to f , the tail of the 0-ray of $\sigma_{f'}$ is contained in a repelling petal at $\zeta_i^{f'}$. By Lemma 4.14, the 0-ray of $\sigma_{f'}$ must land at a cusp. Since a (sufficiently small) repelling petal has a unique cusp in its closure, it follows that the 0-ray of $\sigma_{f'}$ must land at $\zeta_i^{f'}$, for all $f' \in \Sigma_d^*$ close to f . This proves the claim.

Again, we have that $\Sigma_d^* = \sqcup_{i=1}^{d+1} X_i$ by Lemma 4.14. Now, connectedness of Σ_d^* (Proposition 4.19) and openness of each X_i together imply that $\Sigma_d^* = X_j$, for some $j \in \{1, \dots, d+1\}$. The result now follows from the fact that $f_0 \in X_1$. \square

4.4. **Lamination for The Limit Set $\Lambda(G)$.** Recall from Proposition 2.36 that for $\Gamma \in \overline{\beta(\mathbf{\Gamma}_d)}$, there exists a continuous semi-conjugacy $\phi_\Gamma : \mathbb{T} \rightarrow \Lambda(\Gamma)$ between $\rho_{\mathbf{\Gamma}_d}|_{\mathbb{T}}$ and $\rho_\Gamma|_{\Lambda(\Gamma)}$, and ϕ_Γ sends cusps of $\partial\Pi(\mathbf{\Gamma}_d)$ to cusps of $\partial\Pi(\Gamma)$ with labels preserved.

Remark 4.21. The fibers of the map $\phi_\Gamma : \mathbb{T} \rightarrow \Lambda(\Gamma)$ of Proposition 2.36 induce an equivalence relation on \mathbb{T} , and we will denote the set of all equivalence classes of this relation by $\lambda(\Gamma)$.

Adapting the arguments in the proof of Lemma 4.14, we have:

Lemma 4.22. *Let $\Gamma \in \overline{\beta(\mathbf{\Gamma}_d)}$. Then:*

- (1) *For any cusp η of $\partial T(\Gamma)$, we have $|\phi_\Gamma^{-1}(\eta)| = 1$, and $\rho_{\mathbf{\Gamma}_d}(\phi_\Gamma^{-1}(\eta)) = \phi_\Gamma^{-1}(\eta)$.*
- (2) *For each double point η of $\partial T(\Gamma)$, we have $|\phi_\Gamma^{-1}(\eta)| = 2$, and the elements of $\phi_\Gamma^{-1}(\eta)$ form a 2-cycle under $\rho_{\mathbf{\Gamma}_d}$.*

Remark 4.23. Let $\Gamma \in \overline{\beta(\mathbf{\Gamma}_d)}$. Consider the set of angles

$$\Theta := \{\phi_\Gamma^{-1}(\eta) : \eta \text{ is a cusp or a double point of } \partial T(\Gamma)\}.$$

These angles cut \mathbb{T} into finitely many pieces that form a Markov partition for $\rho_{\mathbf{\Gamma}_d} : \mathbb{T} \rightarrow \mathbb{T}$. Analogously, the union of the cusps and double points of $\partial T(\Gamma)$ determines a Markov partition for $\rho_\Gamma : \Lambda(\Gamma) \rightarrow \Lambda(\Gamma)$.

Using the Markov partition of Remark 4.23, the proof of Lemma 4.18 may be adapted to show the following:

Proposition 4.24. *Let $\Gamma \in \overline{\beta(\mathbf{\Gamma}_d)}$, and $x \in \lambda(\Gamma)$ be a non-trivial equivalence class. Then:*

- (1) *$|x| = 2$, and there is a double point η of $\partial T(\Gamma)$ such that $\rho_{\mathbf{\Gamma}_d}^{\circ n}(x) = \phi_\Gamma^{-1}(\eta)$ for some $n \geq 0$.*
- (2) *If n_0 is the smallest non-negative integer with the above property, then x is contained in a connected component of $\mathbb{T} \setminus \rho_{\mathbf{\Gamma}_d}^{-(n_0-1)}(\Theta)$.*

Remark 4.25. For $f \in \Sigma_{d,k}^*$ with $k \geq 1$, the index two Kleinian subgroup Γ_f^+ of Γ_f has k accidental parabolics. These accidental parabolics correspond to a collection of k simple, closed, essential geodesics on $S^- = \mathbb{D}/\mathbf{\Gamma}_d^+$ that can be pinched to obtain Γ_f^+ . These geodesics lift by $\mathbf{\Gamma}_d$ to the universal cover \mathbb{D} giving rise to a geodesic lamination of \mathbb{D} [Mar07, §3.9]. By [MS13] (also compare [Mar07, p. 266]), the quotient of \mathbb{T} by identifying the endpoints of the leaves of this lamination produces a topological model of the limit set $\Lambda(\Gamma_f^+) = \Lambda(\Gamma_f)$. Therefore, up to rotation by a $(d+1)$ -st root of unity, the set of equivalence classes of this geodesic lamination is equal to $\lambda(\Gamma_f)$. Moreover, the continuous map $\phi_{\Gamma_f} : \mathbb{T} \rightarrow \Lambda(\Gamma_f)$ is a *Cannon-Thurston* map for Γ_f (see [MS13, §2.2] for a discussion of Cannon-Thurston maps).

4.5. **Relating The Laminations of Schwarz and Kleinian Limit Sets.** Given $f \in \Sigma_d^*$, we discussed the lamination of \mathbb{T} induced by σ_f in Subsection 4.3, and the lamination of \mathbb{T} induced by Γ_f in Subsection 4.4. The purpose of Subsection 4.5 is to relate these two laminations.

Proposition 4.26. *Let $f \in \Sigma_d^*$. Then the homeomorphism $\mathcal{E}_d : \mathbb{T} \rightarrow \mathbb{T}$ descends to a homeomorphism*

$$\mathcal{E}_d : \mathbb{T}/\lambda(\Gamma_f) \rightarrow \mathbb{T}/\lambda(\sigma_f).$$

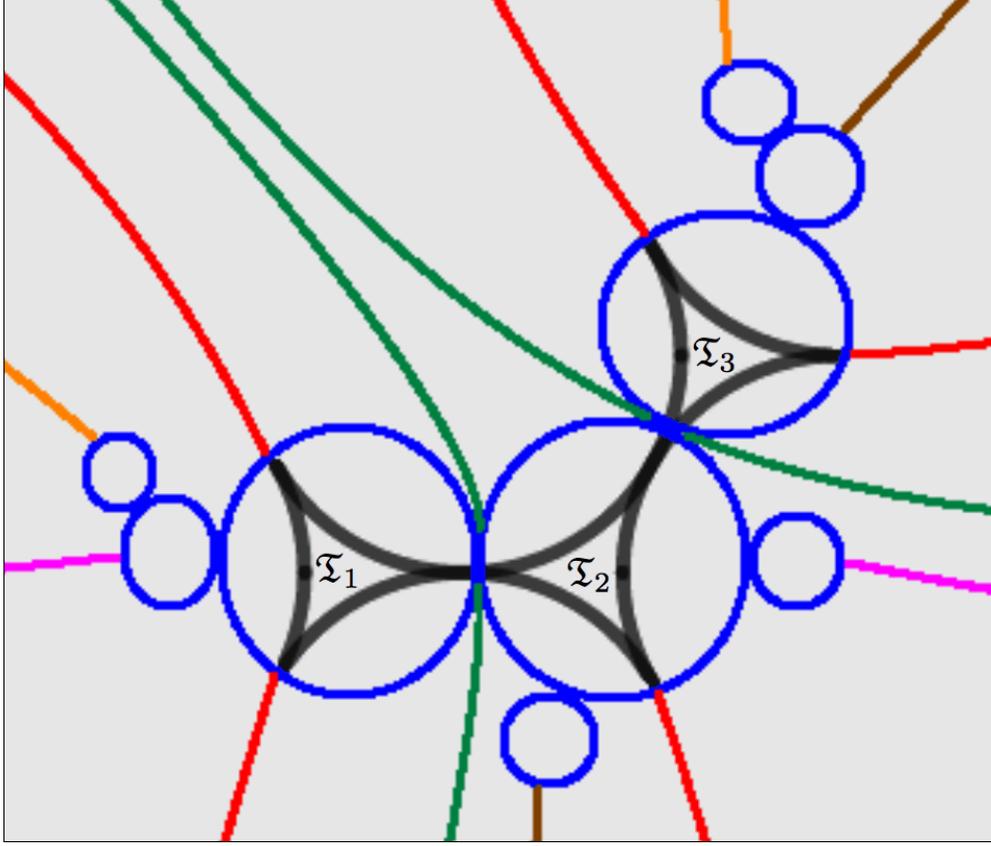


FIGURE 7. A cartoon of the rays of period 1 and 2 landing on $\Lambda(\sigma)$. Each cusp is the landing point of a unique fixed ray (in red), and each double point is the landing point of exactly two rays of period two (in green). The other rays of period two land at non-cut points of $\Lambda(\sigma)$. These rays are colored such that the two rays of the same color form a 2-cycle. That the same pattern holds for the limit set $\Lambda(\Gamma)$ is the crux of the proof of Proposition 4.26.

Proof. We let $0 \leq k \leq d - 2$ and fix $f \in \Sigma_{d,k}^*$. We abbreviate $\sigma := \sigma_f$, $\Gamma := \Gamma_f$. We denote by ϕ_σ the Böttcher coordinate for σ , and ϕ_Γ the map of Proposition 2.36. Recall that the homeomorphism

$$h : T(\Gamma) \rightarrow T(\sigma)$$

of Proposition 3.2 is label-preserving. By Proposition 4.18, $\lambda(\sigma)$ is generated by

$$\{\phi_\sigma^{-1}(\zeta) : \zeta \text{ is a double point of } f(\mathbb{T})\}.$$

Moreover, ζ is a double point of $f(\mathbb{T})$ if and only if $h^{-1}(\zeta)$ is a double point of $\partial T(\Gamma)$. Thus, by Proposition 4.24, $\lambda(\Gamma)$ is generated by

$$\{\phi_\Gamma^{-1}(h^{-1}(\zeta)) : \zeta \text{ is a double point of } f(\mathbb{T})\}.$$

Thus it will suffice to show that

$$(\star) \quad \mathcal{E}_d^{-1}(\phi_\sigma^{-1}(\zeta)) = \phi_\Gamma^{-1}(h^{-1}(\zeta)) \text{ for each double point } \zeta \text{ of } f(\mathbb{T}).$$

Let ζ be a double point of $f(\mathbb{T})$. By Lemma 4.14, $\phi_\sigma^{-1}(\zeta)$ is a 2-cycle for m_{-d} on \mathbb{T} . Similarly, by Lemma 4.22, $\phi_\Gamma^{-1}(h^{-1}(\zeta))$ is a 2-cycle for $\rho_{\Gamma_{d+1}}$. Note that the maps m_{-d} and $\rho_{\Gamma_{d+1}}$ both have the same fixed points on $\mathbb{T} \cong \mathbb{R}/\mathbb{Z}$ which we label counter-clockwise as $\theta_1, \dots, \theta_{d+1}$ with $\theta_1 = 0$. From the Markov property, there is a simple description of all 2-cycles of m_{-d} on \mathbb{T} : there is exactly one 2-cycle $\{x, m_{-d}(x)\}$ in each pair of non-adjacent intervals (θ_i, θ_{i+1}) , (θ_j, θ_{j+1}) with $x \in (\theta_i, \theta_{i+1})$, $m_{-d}(x) \in (\theta_j, \theta_{j+1})$. The same description holds for all 2-cycles of $\rho_{\Gamma_{d+1}}$, and by definition via the Markov-property, the map \mathcal{E}_d^{-1} sends the 2-cycle of m_{-d} in (θ_i, θ_{i+1}) , (θ_j, θ_{j+1}) to the 2-cycle of $\rho_{\Gamma_{d+1}}$ in (θ_i, θ_{i+1}) , (θ_j, θ_{j+1}) .

Now observe that by Proposition 4.20 and the label-preserving statement in Proposition 2.36, we have the relation:

$$(18) \quad h^{-1}(\phi_\sigma(1)) = \phi_\Gamma(1).$$

For $1 \leq i \leq d+1$, $\phi_\sigma(\theta_i)$ is a cusp of $f(\mathbb{T})$ by Lemma 4.14, and $\phi_\Gamma(\theta_i)$ is a cusp of $\partial T(\Gamma)$ by Proposition 2.36. Since h is label-preserving, it then follows from (18) that:

$$(19) \quad h^{-1}(\phi_\sigma(\theta_i)) = \phi_\Gamma(\theta_i) \text{ for } 1 \leq i \leq d+1.$$

Thus it follows from the mapping properties of h that

$$\zeta \in \phi_\sigma(\theta_i, \theta_{i+1}) \cap \phi_\sigma(\theta_j, \theta_{j+1}) \text{ if and only if } h^{-1}(\zeta) \in \phi_\Gamma(\theta_i, \theta_{i+1}) \cap \phi_\Gamma(\theta_j, \theta_{j+1}).$$

Hence, the 2-cycle $\phi_\sigma^{-1}(\zeta)$ for m_{-d} lies in (θ_i, θ_{i+1}) , (θ_j, θ_{j+1}) if and only if the 2-cycle $\phi_\Gamma^{-1}(h^{-1}(\zeta))$ for $\rho_{\Gamma_{d+1}}$ lies in (θ_i, θ_{i+1}) , (θ_j, θ_{j+1}) . By the definition of the homeomorphism \mathcal{E}_d via the Markov-partitions for m_{-d} and $\rho_{\Gamma_{d+1}}$, it follows then that $\mathcal{E}_d^{-1}(\phi_\sigma^{-1}(\zeta)) = \phi_\Gamma^{-1}(h^{-1}(\zeta))$, as needed. \square

Remark 4.27. Let notation be as in Proposition 4.26, and denote by ϕ_{σ_f} the Böttcher coordinate of σ_f , and ϕ_{Γ_f} the map of Proposition 2.36. It follows from Proposition 4.26 that

$$\phi_{\sigma_f} \circ \mathcal{E}_d \circ \phi_{\Gamma_f}^{-1} : \Lambda(\Gamma_f) \rightarrow \Lambda(\sigma_f)$$

is well defined, and indeed a topological conjugacy (see Figure 9).

4.6. Proof of Conformal Mating. With Proposition 4.26 in hand, we can finally prove the conformal mating statement of Theorem A. We follow Definition 2.42 of conformal mating.

Proof of Theorem A. The map

$$\begin{aligned} \Sigma_d^* &\rightarrow \overline{\beta(\mathbf{\Gamma}_{d+1})} \\ f &\mapsto \Gamma_f \end{aligned}$$

was already defined and proven to be a homeomorphism in Section 3. The uniqueness statement of Theorem A is evident since if $\Gamma \in \overline{\beta(\mathbf{\Gamma}_{d+1})}$ is such that σ_f is a conformal mating of Γ and $w \mapsto \overline{w^d}$, then Condition (2) of Definition 2.42 and the uniqueness statement in Proposition 3.2 imply that $\Gamma = \Gamma_f$. Thus it only remains to show that σ_f is indeed a conformal mating of Γ_f and $w \mapsto \overline{w^d}$. Fix $f \in \Sigma_d^*$. We will abbreviate $\sigma := \sigma_f$ and $\Gamma := \Gamma_f$.

Recall from Remark 2.5 the Böttcher coordinate

$$\phi_\sigma : \mathbb{D}^* \rightarrow \mathcal{B}_\infty(\sigma) \text{ satisfying } \phi_\sigma^{-1} \circ \sigma \circ \phi_\sigma(u) = \bar{u}^d, \forall u \in \mathbb{D}^*.$$

By Corollary 4.11, we have the relation

$$(20) \quad \widehat{\mathcal{C}} = \mathcal{B}_\infty(\sigma) \sqcup \Lambda(\sigma) \sqcup \mathcal{T}_\infty(\sigma).$$

Thus $\widehat{\mathcal{C}} \setminus \mathcal{T}_\infty(\sigma) = \mathcal{B}_\infty(\sigma) \sqcup \Lambda(\sigma)$. By Proposition 4.2, $\Lambda(\sigma) = \partial\mathcal{B}_\infty(\sigma)$ is locally connected, so that ϕ_σ extends as a semi-conjugacy $\mathbb{T} \rightarrow \Lambda(\sigma)$. Thus taking $p(w) := \bar{w}^d$ (so that $\mathcal{K}(p) = \overline{\mathbb{D}}$), and $\psi_p(w) := \phi_\sigma(1/w)$ for $w \in \overline{\mathbb{D}}$, it is evident that ψ_p is conformal in $\text{int } \mathcal{K}(p) = \mathbb{D}$ and satisfies Condition (1) of Definition 2.42.

Let $h : T(\Gamma) \rightarrow T(\sigma)$ be the mapping of Proposition 3.2 applied to f . Define $\psi_\Gamma(z) := h(z)$ for $z \in T^\circ(\Gamma)$. Note that ψ_Γ is label-preserving by Proposition 3.2. Lifting ψ_Γ by ρ_Γ and σ , we extend ψ_Γ to a conformal map

$$\psi_\Gamma : \bigcup_{n=0}^{\infty} \rho_\Gamma^{-n}(T^\circ(\Gamma)) \rightarrow \bigcup_{n=0}^{\infty} \sigma^{-n}(T^\circ(\sigma)).$$

Recall our notation $\mathcal{K}(\Gamma) := \mathbb{C} \setminus \Omega_\infty(\Gamma)$. Then $\text{int } \mathcal{K}(\Gamma)$ is the union of all bounded components of $\Omega(\Gamma)$, and we have

$$(21) \quad \mathcal{K}(\Gamma) = \text{int } (\mathcal{K}(\Gamma)) \sqcup \Lambda(\Gamma) \text{ and } \overline{\mathcal{T}_\infty(\sigma)} = \mathcal{T}_\infty(\sigma) \sqcup \Lambda(\sigma).$$

By Proposition 2.34 and Definition 2.3, we have

$$\bigcup_{n=0}^{\infty} \rho_\Gamma^{-n}(T^\circ(\Gamma)) = \text{int } \mathcal{K}(\Gamma) \text{ and } \bigcup_{n=0}^{\infty} \sigma^{-n}(T^\circ(\sigma)) = \mathcal{T}_\infty(\sigma).$$

Thus $\psi_\Gamma : \text{int } \mathcal{K}(\Gamma) \rightarrow \mathcal{T}_\infty(\sigma)$ is conformal. Moreover, by the definition of ψ_Γ via lifting, we have

$$(22) \quad \psi_\Gamma \circ \rho_\Gamma(z) = \sigma_f \circ \psi_\Gamma(z) \text{ for } z \in \text{int } \mathcal{K}(\Gamma) \setminus \text{int } T^\circ(\Gamma).$$

Thus in order to conclude that Condition (2) of Definition 2.42 holds, by (21) it only remains to show that ψ_Γ extends to a semi-conjugacy $\Lambda(\Gamma) \rightarrow \Lambda(\sigma)$. We will show that in fact ψ_Γ extends as a topological conjugacy.

Let $\phi_\Gamma : \mathbb{D}^* \rightarrow \Omega_\infty(\Gamma)$ denote the conformal map of Proposition 2.36. As observed in Remark 4.27, Proposition 4.26 implies that the map

$$\phi_\sigma \circ \mathcal{E}_d \circ \phi_\Gamma^{-1} : \Lambda(\Gamma) \rightarrow \Lambda(\sigma)$$

is a well-defined homeomorphism, so that we only need to show that $\phi_\sigma \circ \mathcal{E}_d \circ \phi_\Gamma^{-1}$ is an extension of $\psi_\Gamma : \text{int } \mathcal{K}(\Gamma) \rightarrow \mathcal{T}_\infty(\sigma_f)$. Note that by construction and the normalization in Remark 2.5, ψ_Γ and $\phi_\sigma \circ \mathcal{E}_d \circ \phi_\Gamma^{-1}$ agree on the cusps of $\partial T(\Gamma)$. One may then verify via the definition of ψ_Γ (by lifting ρ_Γ and σ) and \mathcal{E}_d (in Remark 2.35) that ψ_Γ and $\phi_\sigma \circ \mathcal{E}_d \circ \phi_\Gamma^{-1}$ agree on all preimages of cusps of $\partial T(\Gamma)$. As these preimages form a dense subset of $\Lambda(\Gamma)$, it follows that $\phi_\sigma \circ \mathcal{E}_d \circ \phi_\Gamma^{-1}$ is the desired homeomorphic extension of ψ_Γ .

It remains only to show Condition (3) of Definition 2.42. Let $t \in \mathbb{T}$ and consider $\phi_\Gamma(t) \in \Lambda(\Gamma)$, $\phi_p \circ \overline{\mathcal{E}_d(t)} \in \mathcal{J}(p)$, where we note $\phi_p \equiv \text{id}$. We readily compute that

$$\psi_p(\phi_p \circ \overline{\mathcal{E}_d(t)}) = \phi_\sigma(1/\overline{\mathcal{E}_d(t)}) = \phi_\sigma(\mathcal{E}_d(t)) = \phi_\sigma \circ \mathcal{E}_d \circ \phi_\Gamma^{-1}(\phi_\Gamma(t)) = \psi_\Gamma(\phi_\Gamma(t)).$$

Thus for $z \in \mathcal{K}(\Gamma)$, $w \in \mathcal{K}(p)$, we see that $z \sim w \implies \psi_\Gamma(z) = \psi_p(w)$.

If, conversely, $\psi_\Gamma(z) = \psi_p(w)$, we must have firstly that $\psi_\Gamma(z) = \psi_p(w) \in \Lambda(\sigma)$. Thus $z \in \Lambda(\Gamma)$, and $w \in \mathbb{T}$. Recalling $\psi_\Gamma = \phi_\sigma \circ \mathcal{E}_d \circ \phi_\Gamma^{-1}$ on $\Lambda(\Gamma)$ and $\psi_p(w) = \phi_\sigma(\bar{w})$ for $w \in \mathbb{T}$, we see that

$$(23) \quad \phi_\sigma \circ \mathcal{E}_d \circ \phi_\Gamma^{-1}(z) = \phi_\sigma(\bar{w}).$$

As already noted, $\psi_\Gamma = \phi_\sigma \circ \mathcal{E}_d \circ \phi_\Gamma^{-1} : \Lambda(\Gamma) \mapsto \Lambda(\sigma)$ is a homeomorphism, so that we deduce from (23) that $z = \phi_\Gamma \circ \mathcal{E}_d^{-1}(\bar{w})$. Letting $t = \mathcal{E}_d^{-1}(\bar{w})$, we see that $\phi_\Gamma(t) = z$ and $\phi_p \circ \overline{\mathcal{E}_d(t)} = w$, so that by Definition 2.41, $z \sim w$ as needed. \square

5. SULLIVAN'S DICTIONARY

Definition 5.1. An *abstract angled tree* is a triple $(\mathcal{T}, \deg, \angle)$, where:

- (1) \mathcal{T} is a tree,
- (2) $\deg : V(\mathcal{T}) \rightarrow \mathbb{N}$ is a function with $\deg(v) \geq 2$ for each vertex v of \mathcal{T} ,
- (3) $\text{valence}(v) \leq 1 + \deg(v)$ for each vertex v of \mathcal{T} , and
- (4) \angle is a skew-symmetric, non-degenerate, additive function defined on pairs of edges incident at a common vertex, and takes values in $\frac{2\pi}{1+\deg(v)}\mathbb{Z}/2\pi\mathbb{Z}$.

Remark 5.2. If $(\mathcal{T}, \deg, \angle)$ is an abstract angled tree, the positive integer

$$d := 1 + \sum_{v \in V(\mathcal{T})} (\deg(v) - 1)$$

is called the *total degree* of the angled tree. Two angled trees are said to be isomorphic if there is a tree isomorphism between them that preserves the functions \deg and \angle .

Example 5.3. To any $f \in \Sigma_{d,k}^*$, we will associate an abstract angled tree $\mathcal{T}(f)$ with $k+1$ vertices as follows. Denote by T_1, \dots, T_{k+1} the components of $T^o(\sigma_f)$. Let $j_i \geq 0$ be such that the boundary of T_i has $3 + j_i$ cusps. Assign a vertex v_i to each component T_i , and connect two vertices v_i, v_j by an edge if and only if T_i, T_j share a common boundary point. We define the \deg function by:

$$\begin{aligned} \deg : V(\mathcal{T}(f)) &\rightarrow \mathbb{N} \\ v_i &\mapsto 2 + j_i. \end{aligned}$$

It remains to define the \angle function for two edges e, e' meeting at a vertex v_i . Suppose e, e' correspond to two cusps $\zeta, \zeta' \in \partial T_i$, and denote by γ_i the component of $\partial T_i \setminus \{\zeta, \zeta'\}$ which, when traversed counter-clockwise, is oriented positively with respect to T_i . Then

$$\angle(e, e') := \frac{2\pi}{3 + j_i} \cdot (1 + \#\{\text{cusps of } \partial T_i \text{ on the curve } \gamma_i\}).$$

We leave it to the reader to verify that $(\mathcal{T}(f), \deg, \angle)$ satisfies Definition 5.1 of an abstract angled tree. Note that if $f \in \Sigma_{d,d-2}^*$, then the tree $(\mathcal{T}(f), \deg, \angle)$ is simply a *bi-angled tree* in the language of [LMM19, §2.5].

Proposition 5.4. *For each $f \in \Sigma_{d,k}^*$, there exists an anti-polynomial p_f of degree d such that:*

- (1) p_f has a total of $k+1$ distinct critical points in \mathbb{C} ,
- (2) Each critical point of p_f is fixed by p_f , and

(3) *The angled Hubbard tree of p_f is isomorphic to $(\mathcal{T}(f), \deg, \angle)$.*

Proof. We continue to use the notation introduced in Example 5.3. One readily verifies that:

$$-(2 + j_i) \left(\frac{2\pi n}{3 + j_i} \right) = \frac{2\pi n}{3 + j_i} \pmod{2\pi} \text{ for } i = 1, \dots, k + 1.$$

Thus, $\text{id} : \mathcal{T}(f) \rightarrow \mathcal{T}(f)$ is an orientation-reversing angled tree map (see [LMM19, §2.7]) of degree

$$1 + \sum_{i=1}^{k+1} (\deg(v_i) - 1) = 1 + \sum_{i=1}^{k+1} (1 + j_i) = d.$$

Moreover, since all vertices of \mathcal{T} are critical and fixed under id , it follows that all vertices are of *Fatou type* (again, see [LMM19, §2.7]). Hence, the realization theorem [Poi13, Theorem 5.1] applied to the orientation-reversing angled tree map id yields a postcritically finite anti-polynomial p_f of degree d such that the angled Hubbard tree of p_f is isomorphic to $(\mathcal{T}(f), \deg, \angle)$. That p_f satisfies (1), (2) follows since the Hubbard tree of p_f is isomorphic to $(\mathcal{T}(f), \deg, \angle)$. \square

Proposition 5.5. *Let $f \in \Sigma_{d,k}^*$, and p_f as in Proposition 5.4. Denote by U_1, \dots, U_{k+1} the immediate attracting basins of the fixed critical points of p_f . Then*

$$U := \bigcup_{i=1}^{k+1} \overline{U_i}$$

is connected. Moreover, p_f has exactly $2k + d + 2$ fixed points in \mathbb{C} , of which:

- (1) $k + 1$ are critical points,
- (2) k are cut-points of U and belong to $\mathcal{J}(p_f)$, and
- (3) $d + 1$ are not cut-points of U and belong to $\mathcal{J}(p_f)$.

Proof. We abbreviate $p := p_f$. Enumerate the critical points of p by c_1, \dots, c_{k+1} . Since

$$(24) \quad p(c_i) = c_i \text{ for } 1 \leq i \leq k + 1,$$

p can not have any indifferent fixed point. It also follows from (24) that $(c_i)_{i=1}^{k+1}$ are the only attracting fixed points of p , since any basin of attraction of p must contain a critical value. It then follows from the Lefschetz fixed point formula (see [LM14, Lemma 6.1]) that p has a total of $2k + d + 2$ fixed points in \mathbb{C} , of which $k + d + 1$ are repelling and thus belong to $\mathcal{J}(p)$.

Note that $p|_{U_i}$ is conformally conjugate to $\bar{z}^{2+j_i}|_{\mathbb{D}}$. As p is hyperbolic, ∂U_i is locally connected (see [Mil06, Lemma 19.3]), and so this conjugacy extends to the boundary. Thus, $p|_{\partial U_i}$ has $3 + j_i$ fixed points for each i . We claim that:

- (\star) a repelling fixed point can lie on the boundary of at most two $(U_i)_{i=1}^{k+1}$, and
- ($\star\star$) there is at least one repelling fixed point which is on the boundary of precisely one $(U_i)_{i=1}^{k+1}$.

Statement (\star) follows from the fact that the basins of attraction are invariant under p and that p is an orientation-reversing homeomorphism in a neighborhood of a repelling fixed point. Statement ($\star\star$) follows from fullness of the filled Julia set $\mathcal{K}(p)$. Thus if we suppose, by way of contradiction,

that, say $\overline{U_1}$ is disjoint from $\cup_{i=2}^{k+1} \overline{U_i}$, a counting argument yields that p has at least

$$\underbrace{4 + j_1}_{\text{fixed points in } \overline{U_1}} + \underbrace{k}_{\text{critical fixed points in } U_i, i>1} + \underbrace{\sum_{i>1} (3 + j_i)}_{\text{fixed points in } \partial U_i, i>1} - \underbrace{(k-1)}_{\text{shared fixed points}} = 2k + d + 3$$

fixed points in \mathbb{C} , which is a contradiction. Thus U is connected.

An elementary argument using fullness of $\mathcal{K}(p)$ shows that two $\overline{U_i}, \overline{U_j}$ can intersect in at most one point, and that an intersection point of $\overline{U_i}, \overline{U_j}$ must be a fixed point of p (see [LMM19, Proposition 6.2]). Thus by (\star) above and connectedness of U , it follows that U has at least k cut-points which are in $\mathcal{J}(f)$. Futhermore, by fullness of $\mathcal{K}(p)$, U can not have more than k cut-points. \square

Notation 5.6. Let p_f be as in Propositions 5.4, 5.5. We denote by $\text{Rep}(p_f)$ the set of repelling fixed points of p_f , and by $\text{Cut}(p_f)$ the cut points of U .

Remark 5.7. Consider a Böttcher coordinate $\phi_p : \mathbb{D}^* \rightarrow \mathcal{B}_\infty(p)$ for $p = p_f$ as in Proposition 5.4. Note that $\mathcal{J}(p)$ is connected as each finite critical point of p is fixed. Thus since p is hyperbolic, it follows that $\mathcal{J}(p)$ is locally connected [Mil06, Theorem 19.2]. Hence ϕ_p extends continuously to a surjection $\phi_p : \mathbb{T} \rightarrow \mathcal{J}(p)$ which semi-conjugates $m_{-d}|_{\mathbb{R}/\mathbb{Z}}$ to $p|_{\mathcal{J}(p)}$, and all external rays of $\mathcal{B}_\infty(p)$ land. The fibers of $\phi_p|_{\mathbb{T}}$ induce an equivalence relation on \mathbb{T} which we denote by $\lambda(p)$. Note that $\lambda(p)$ depends on a normalization of the Böttcher coordinate.

Lemma 5.8. *Let $f \in \Sigma_d^*$, p_f as in Proposition 5.4, and ϕ_p any Böttcher coordinate for p_f . Then:*

- (1) *Each $\beta \in \text{Rep}(p_f) \setminus \text{Cut}(p_f)$ is the landing point of a unique external ray. The angle of this external ray is fixed by m_{-d} .*
- (2) *Each $\beta \in \text{Rep}(p_f) \cap \text{Cut}(p_f)$ is the landing point of exactly two external rays. The angles of these two rays form a 2-cycle under m_{-d} .*

Proof. We abbreviate $p := p_f$, and continue to use the notation of Propositions 5.4, 5.5. By Remark 5.7, each $\beta \in \text{Rep}(p)$ is the landing point of at least one external ray.

Proof of (1): Let $\beta \in \text{Rep}(p_f) \setminus \text{Cut}(p_f)$. Denote by S the component of $\mathcal{K}(p) \setminus \{\beta\}$ containing $U \setminus \{\beta\}$. It follows from the covering properties of p that $\cup_{r=0}^n p^{-r}(U \setminus \{\beta\})$ is connected for each $n \geq 0$. Thus

$$\bigcup_{r=0}^n p^{-r}(U \setminus \{\beta\}) \subseteq S \text{ for all } n \geq 0.$$

Observe that

$$(25) \quad \text{int } \mathcal{K}(p) = \bigcup_{r=0}^{\infty} p^{-r} \left(\bigcup_{i=1}^{k+1} U_i \right).$$

We then have

$$(26) \quad \mathcal{K}(p) = \overline{\text{int } \mathcal{K}(p)} \subseteq \overline{\bigcup_{r=0}^{\infty} p^{-r}(U \setminus \{\beta\})} \subseteq \overline{S} = S \cup \{\beta\},$$

where the first equality in (26) follows from [Mil06, Corollary 4.12], and the proceeding \subseteq relation follows from (25). By definition of S , we have $\mathcal{K}(p) = S \cup \{\beta\}$, so that β is not a cut point of $\mathcal{K}(p)$. Thus β is the landing point for exactly one external ray. Since β is fixed, it follows that the angle of the external ray landing at β is fixed under m_{-d} .

Proof of (2): Let $\beta \in \text{Rep}(p_f) \cap \text{Cut}(p_f)$. Let S_1, S_2 be the two components of $\mathcal{K}(p) \setminus \{\beta\}$ such that $U \setminus \{\beta\} \subseteq S_1 \cup S_2$. A similar argument as for (1) shows that S_1 and S_2 are the only components of $\mathcal{K}(p) \setminus \{\beta\}$. Thus there are only two accesses to β in $\mathcal{B}_\infty(p)$, and hence exactly two external rays landing at β . Since the $d+1$ fixed external rays land at the $d+1$ points of $\text{Rep}(p_f) \setminus \text{Cut}(p_f)$ by (1), it follows that the external rays landing at β must have period 2, and hence form a 2-cycle under m_{-d} . \square

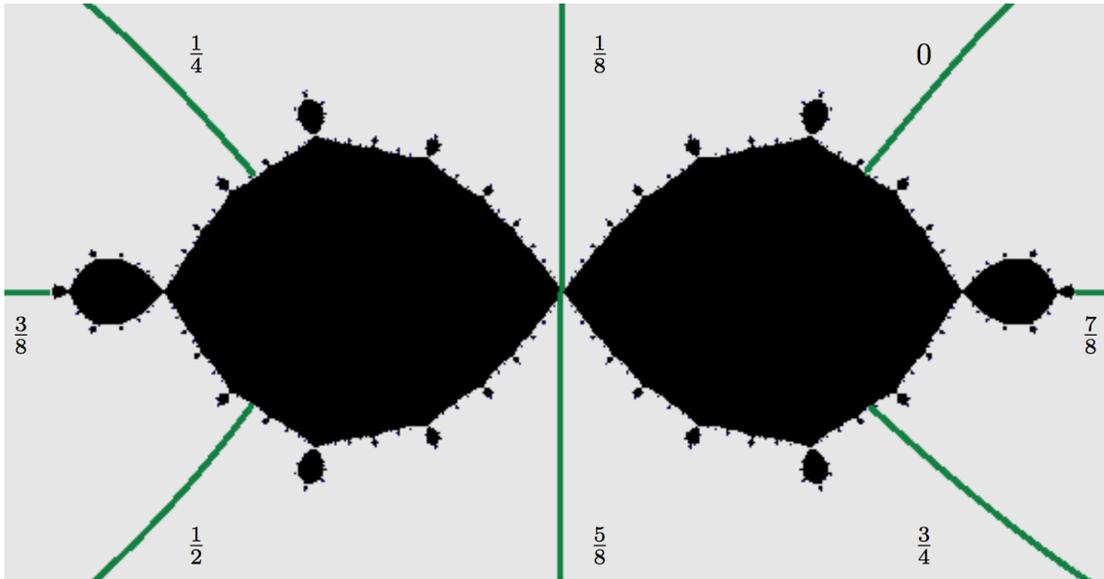


FIGURE 8. Illustrated is the dynamical plane of p_f , where $f(z) := z + 2/(3z) - 1/(3z^3)$. For this particular f , an explicit formula is known for p_f : $p_f(z) = \bar{z}^3 - \frac{3i}{2}\bar{z}$ (the figure displayed is a $\pi/4$ -rotate of the actual dynamical plane). Also shown are all external rays of p_f of period 1 and 2 (with angles indicated). The idea of the proof of Proposition 5.9 is to show that the external rays of σ_f landing at the double points of $f(\mathbb{T})$ have the same landing pattern as for those rays landing at the cut-points of the immediate basins of attraction for p_f .

Proposition 5.9. *Let $f \in \Sigma_d^*$, and p_f as in Proposition 5.4. There is a normalization of the Böttcher coordinate for p_f such that $\lambda(p_f) = \lambda(\sigma_f)$.*

Remark 5.10. The idea of the proof is similar to that of Proposition 4.26, for which we refer to Figure 8.

Proof. Let k be such that $f \in \Sigma_{d,k}^*$. We abbreviate $\sigma := \sigma_f$, $p := p_f$. Consider the isomorphism of the angled Hubbard tree of p with the abstract angled tree of f as defined in Example 5.3. Thus there is, first of all, a bijection between the attracting basins U_1, \dots, U_{k+1} of p and the components T_1, \dots, T_{k+1} of $T^o(\sigma)$. We ensure the labeling is such that U_i is mapped to T_i . Since the deg function is preserved, the number of singular points on each ∂T_i is equal to the number of fixed points of $p|_{\partial U_i}$. Moreover, since the \angle function is preserved, for each $1 \leq i \leq k+1$ there is a bijection

$$\chi_i : \partial U_i \cap \text{Rep}(p) \rightarrow \partial T_i \cap \text{Sing}(f(\mathbb{T}))$$

satisfying:

- (1) For $1 \leq j \leq k+1$, one has $\chi_i(\beta) \in \partial T_i \cap \partial T_j$ if and only if $\beta \in \partial U_i \cap \partial U_j$;
- (2) $(\beta_1, \beta_2, \beta_3)$ is oriented positively with respect to U_i if and only if $(\chi_i(\beta_1), \chi_i(\beta_2), \chi_i(\beta_3))$ is oriented positively with respect to T_i .

By (1), the map $\chi : \text{Rep}(p) \rightarrow \text{Sing}(f(\mathbb{T}))$ defined piecewise as χ_i on each $\text{Rep}(p) \cap \partial U_i$ is well-defined, whence it follows that χ is a bijection.

Denote by ϕ_p, ϕ_σ the Böttcher coordinates for p, σ , respectively. We normalize ϕ_p so that

$$\chi \circ \phi_p(1) = \phi_\sigma(1).$$

Recall that the cusps of ∂T_i are the landing points of the fixed rays in $\partial \mathcal{B}_\infty(\sigma)$ by Lemma 4.14, and the points $\text{Rep}(p) \setminus \text{Cut}(p)$ are the landing points of the fixed rays in $\partial \mathcal{B}_\infty(p)$ by Lemma 5.8. The fixed rays of $\partial \mathcal{B}_\infty(\sigma)$ and $\partial \mathcal{B}_\infty(p)$ have the same angles, and we enumerate them $\theta_1, \dots, \theta_{d+1}$ where $\theta_1 := 0$. There is a 2-cycle (under m_{-d}) on \mathbb{T} in each pair of non-adjacent intervals (θ_i, θ_{i+1}) , (θ_j, θ_{j+1}) , and this constitutes all 2-cycles of m_{-d} .

By Lemma 5.8, for each $\beta \in \text{Rep}(p) \cap \text{Cut}(p)$, the set $\phi_p^{-1}(\beta)$ is a 2-cycle on \mathbb{T} . The 2-cycle $\phi_p^{-1}(\beta)$ is in the pair of intervals (θ_i, θ_{i+1}) , (θ_j, θ_{j+1}) if and only if β lies on both $\phi_p((\theta_i, \theta_{i+1}))$ and $\phi_p((\theta_j, \theta_{j+1}))$. Similarly, for each double point ζ of $f(\mathbb{T})$, the set $\phi_\sigma^{-1}(\zeta)$ is a 2-cycle on \mathbb{T} by Lemma 4.14. And moreover, the 2-cycle $\phi_\sigma^{-1}(\zeta)$ is in the pair of intervals (θ_i, θ_{i+1}) , (θ_j, θ_{j+1}) if and only if ζ lies on both $\phi_\sigma((\theta_i, \theta_{i+1}))$ and $\phi_\sigma((\theta_j, \theta_{j+1}))$. Thus, by the definition of χ , for $\beta \in \text{Rep}(p) \cap \text{Cut}(p)$, the 2-cycle $\phi_p^{-1}(\beta)$ is in the pair of intervals (θ_i, θ_{i+1}) , (θ_j, θ_{j+1}) if and only if $\phi_\sigma^{-1}(\chi(\beta))$ is in the same pair of intervals. As there is only one 2-cycle in any such pair, it follows that $\phi_p^{-1}(\beta) = \phi_\sigma^{-1}(\chi(\beta))$.

Recall that by Proposition 4.18, the pairs $\phi_\sigma^{-1}(\zeta)$ over all double points ζ of $f(\mathbb{T})$ generate $\lambda(\sigma)$. A completely analogous proof to that of Proposition 4.18 shows $\lambda(p)$ is generated by pairs $\phi_p^{-1}(\beta)$ where β ranges over $\text{Rep}(p) \cap \text{Cut}(p)$. Thus since χ is a bijection and $\phi_p^{-1}(\beta) = \phi_\sigma^{-1}(\chi(\beta))$ for all $\beta \in \text{Rep}(p) \cap \text{Cut}(p)$, it follows that $\lambda(\sigma) = \lambda(p)$. \square

Remark 5.11. Let notation be as in Proposition 5.9, and denote by $\phi_{p_f}, \phi_{\sigma_f}$ the Böttcher coordinates of p_f, σ_f (respectively) with ϕ_{p_f} normalized as in Proposition 5.9. It follows from Proposition 5.9 that

$$\phi_{p_f} \circ \phi_{\sigma_f}^{-1} : \Lambda(\sigma_f) \rightarrow \mathcal{J}(p_f)$$

is well-defined, and indeed a topological conjugacy (see Figure 9).

We note that Theorem B follows immediately from:

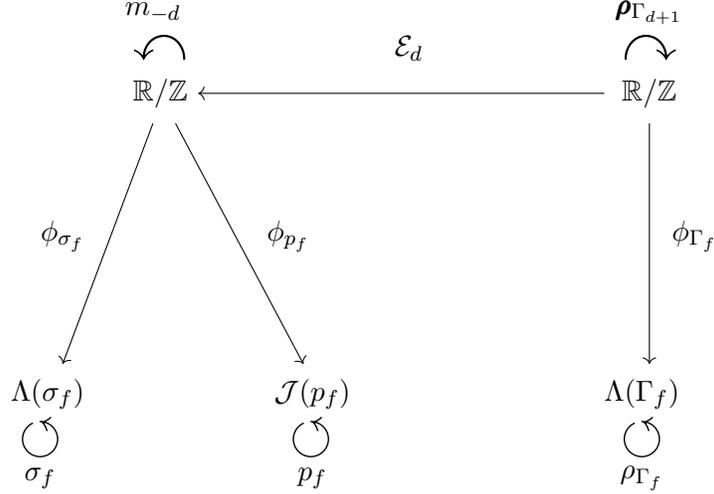


FIGURE 9. Various topological conjugacies.

Theorem C. Let $f \in \Sigma_d^*$. Denote by σ_f , Γ_f , p_f the Schwarz reflection map, Kleinian reflection group, and critically fixed anti-polynomial determined by Definition 2.3, Theorem A, and Proposition 5.4, respectively. Then the dynamical systems

$$\begin{aligned} \sigma_f &: \Lambda(\sigma_f) \rightarrow \Lambda(\sigma_f), \\ \rho_{\Gamma_f} &: \Lambda(\Gamma_f) \rightarrow \Lambda(\Gamma_f), \\ p_f &: \mathcal{J}(p_f) \rightarrow \mathcal{J}(p_f) \end{aligned}$$

are pairwise topologically conjugate.

Proof of Theorem C. That $\sigma_f|_{\Lambda(\sigma_f)}$ and $p_f|_{\mathcal{J}(p_f)}$ are topologically conjugate is a consequence of Proposition 5.9 as explained in Remark 5.11. That $\sigma_f|_{\Lambda(\sigma_f)}$ and $\rho_{\Gamma_f}|_{\Lambda(\Gamma_f)}$ are topologically conjugate follows from Proposition 4.26, as explained in Remark 4.27. \square

Remark 5.12. In the spirit of [LLMM19, Theorem 7.2], it is natural to ask whether $\Lambda(\sigma_f)$, $\Lambda(\Gamma_f)$, $\mathcal{J}(p_f)$ can be distinguished by their quasisymmetry groups.

Remark 5.13. In light of Proposition 5.9, we can conjugate p_f by an affine map to assume that p_f is monic, centered, and $\lambda(p_f) = \lambda(\sigma_f)$, where $\lambda(p_f)$ is determined by the Böttcher coordinate of p_f that is tangent to the identity at ∞ . In fact, p_f becomes unique with such normalization. Moreover, it directly follows from the proof of Proposition 4.26 and Remark 4.25 that the circle homeomorphism \mathcal{E}_d transports the geodesic lamination that produces a topological model for $\Lambda(\Gamma_f)$ to the lamination that produces a topological model for $\mathcal{J}(p_f)$.

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