

# EULER PRODUCT ASYMPTOTICS FOR DIRICHLET $L$ -FUNCTIONS

IKUYA KANEKO

**ABSTRACT.** Via the work of Ramanujan, we establish the asymptotic behaviour of partial Euler products for Dirichlet  $L$ -functions under the Generalised Riemann Hypothesis (GRH). Understanding the behaviour of Euler products on the critical line is called the Deep Riemann Hypothesis (DRH). This work manifests the relation between GRH and DRH.

## 1. INTRODUCTION

**1.1. Overview and motivation.** This work is motivated by the beautiful work of Ramanujan on asymptotics for the partial Euler product of the Riemann zeta function  $\zeta(s)$ . We handle the family of Dirichlet  $L$ -functions

$$L(s, \chi) = \sum_{n=1}^{\infty} \chi(n)n^{-s} = \prod_p (1 - \chi(p)p^{-s})^{-1} \quad \text{with } \Re(s) > 1,$$

and aim at proving the asymptotic behaviour of partial Euler products

$$\prod_{p \leq x} (1 - \chi(p)p^{-s})^{-1} \tag{1.1}$$

in the critical strip  $0 < \Re(s) < 1$  with recourse to the Generalised Riemann Hypothesis (GRH) for this family. In 1984, Mertens [6] conceived of the partial Euler products for  $\zeta(s)$  and  $L(s, \chi_4)$  at  $s = 1$ , where  $\chi_4$  is the primitive character modulo 4. The  $\sqrt{2}$  phenomenon occurs at the central point  $s = 1/2$ , which was observed by Conrad [2].

For technical convenience, let  $\chi$  modulo  $q$  be a primitive character throughout this article. Let  $\varphi(q)$  be Euler's totient function and let  $\Lambda(n)$  be the von Mangoldt function. We then define the allied counting functions

$$\vartheta(x; q, a) = \sum_{\substack{p \leq x \\ p \equiv a \pmod{q}}} \log p \quad \text{and} \quad \psi(x; a, q) = \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} \Lambda(n).$$

To understand the behaviour of (1.1) as  $x$  tends to infinity, we shall follow an idea of Ramanujan [9] and utilise his technique. A feature of his method is to use an accurate version of the explicit formula. This was created in the process of studying the maximal order of the divisor function by introducing highly composite numbers. The aim of this article is to generalise the formula due to Ramanujan to the context of Dirichlet  $L$ -functions.

**Theorem 1.1.** *Let  $\chi$  be a primitive Dirichlet character modulo  $q$ . Write  $s = \sigma + it$  with  $\Re(s) > 0$  and*

$$S_s(x, \chi) = -\frac{s}{\varphi(q)} \sum_{a \pmod{q}}^* \chi(a) \sum_{\psi \pmod{q}} \bar{\psi}(a) \sum_{L(\rho, \psi)=0} \frac{x^{\rho-s}}{\rho(\rho-s)}.$$

*If GRH for a Dirichlet  $L$ -function associated to  $\chi$  is assumed, we then have for  $q \leq \sqrt{x}/(\log x)^{2+\epsilon}$  that*

**Case I.**  $0 < \Re(s) < 1/2$ :

$$\begin{aligned} \prod_{p \leq x} (1 - \chi(p)p^{-s})^{-1} = \exp \left( \frac{1}{\varphi(q)} \sum_{a \pmod{q}}^* \chi(a) \text{Li}((\varphi(q)\vartheta(x; q, a))^{1-s}) - \frac{1}{\varphi(q)} \text{Li}(x^{1-\varphi(q)s}) - \dots \right. \\ \left. - \frac{1}{n} \text{Li}(x^{1-n}) + \frac{(2s-1+\delta_{\chi^2=1})x^{1/2-s}}{(1-2s)\log x} - \frac{S_s(x, \chi)}{\log x} + O\left(\frac{x^{1/2-\sigma}}{(\log x)^2}\right) \right) \end{aligned}$$

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with  $n$  the largest multiple of  $\varphi(q)$  not exceeding  $[1 + 1/2\sigma]$ ,

**Case II.**  $\Re(s) = 1/2$ :

$$\begin{aligned} & \prod_{p \leq x} (1 - \chi(p)p^{-s})^{-1} \\ &= L(s, \chi) \exp \left( \frac{1}{\varphi(q)} \sum_{a \pmod{q}}^* \chi(a) \text{Li}((\varphi(q)\vartheta(x; q, a))^{1-s}) + \frac{x^{1/2-s} + S_s(x, \chi)}{\log x} + O\left(\frac{1}{(\log x)^2}\right) \right) \\ & \quad \times \begin{cases} \sqrt{2} & \text{if } s = 1/2 \text{ and } \chi^2 = 1, \\ \exp\left(\delta_{\chi^2=1} \frac{x^{1-2s}(2x^{s-1/2} - 1)}{2(2s-1)\log x}\right) & \text{otherwise,} \end{cases} \end{aligned}$$

**Case III.**  $\Re(s) > 1/2$ :

$$\begin{aligned} \prod_{p \leq x} (1 - \chi(p)p^{-s})^{-1} &= L(s, \chi) \exp \left( \frac{1}{\varphi(q)} \sum_{a \pmod{q}}^* \chi(a) \text{Li}((\varphi(q)\vartheta(x; q, a))^{1-s}) \right. \\ & \quad \left. + \frac{(2s-1 + \delta_{\chi^2=1})x^{1/2-s}}{(2s-1)\log x} + \frac{S_s(x, \chi)}{\log x} + O\left(\frac{x^{1/2-\sigma}}{(\log x)^2}\right) \right). \end{aligned}$$

Disregarding the exponential multipliers in Theorem 1.1, it is conjectured for  $\chi \neq 1$  that

$$\lim_{x \rightarrow \infty} \prod_{p \leq x} (1 - \chi(p)p^{-s})^{-1} = L(s, \chi) \quad (1.2)$$

on the half-plane  $\Re(s) > 1/2$ . Conrad [2] has shown that GRH is equivalent to (1.2). As a strengthened version of (1.2), DRH asserts the following in the case of Dirichlet  $L$ -functions.

**Conjecture 1.2** (DRH for Dirichlet  $L$ -functions). *If  $\chi \neq 1$  and a complex number  $s$  is on the critical line  $\Re(s) = 1/2$  with  $m$  the order of vanishing of  $L(s, \chi)$ , we have*

$$\lim_{x \rightarrow \infty} \left( (\log x)^m \prod_{p \leq x} (1 - \chi(p)p^{-s})^{-1} \right) = \frac{L^{(m)}(s, \chi)}{e^{m\gamma} m!} \times \begin{cases} \sqrt{2} & \text{if } s = 1/2 \text{ and } \chi^2 = 1, \\ 1 & \text{otherwise.} \end{cases} \quad (1.3)$$

The two statements that the limit on the left-hand side of (1.3) exists for *some*  $s$  on  $\Re(s) = 1/2$  and that it exists for *every*  $s$  on  $\Re(s) = 1/2$  are equivalent. Moreover, the conjecture (1.3) is known to be equivalent to

$$\vartheta(x; q, a) - \frac{x}{\varphi(q)} = o(\sqrt{x} \log x). \quad (1.4)$$

This bound is better than what one can reach under GRH. Case II of Theorem 1.1 shows that DRH holds when the error term in the Prime Number Theorem in arithmetic progressions is bounded as in (1.4).

Conrad [2] considered partial Euler products for various  $L$ -functions along their critical line and demystified the  $\sqrt{2}$  phenomenon. He found the equivalence between the Euler product asymptotics and the estimate  $\psi_L(x) := \sum_{N \mathfrak{p}^k \leq x} (\alpha_{\mathfrak{p},1}^k + \cdots + \alpha_{\mathfrak{p},d}^k) \log N \mathfrak{p} = o(\sqrt{x} \log x)$ , which is stronger than GRH. Given an elliptic curve  $E/\mathbb{Q}$  with  $N$  the conductor, Kuo–Murty [5] established the equivalence between the Birch and Swinnerton-Dyer conjecture and the bound  $\sum_{n \leq x} \tilde{c}_n = o(x)$ . Here  $\tilde{c}_n$  signifies that

$$\tilde{c}_n = \begin{cases} \frac{\alpha_p^k + \beta_p^k}{k} & n = p^k \text{ for } p \nmid N, \\ 0 & \text{otherwise,} \end{cases}$$

with  $\alpha_p$  and  $\beta_p$  the Frobenius eigenvalues at  $p$ . Akatsuka [1] has studied DRH for the Riemann zeta function. With a simple pole of  $\zeta(s)$  in mind, DRH is equivalent to the estimate  $\vartheta(x) - x = o(\sqrt{x} \log x)$ .

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## 2. PRELIMINARIES

**2.1. The work of Ramanujan.** Ramanujan [9] extended, beyond the boundary, the result of Mertens to  $s > 0$  in the process of obtaining the maximal order of the divisor function. His formula then asserts that if the Riemann Hypothesis (RH) for  $\zeta(s)$  is assumed, we have

$$\prod_{p \leq x} (1 - p^{-s})^{-1} = -\zeta(s) \exp \left( \text{Li}(\vartheta(x)^{1-s}) + \frac{2sx^{\frac{1}{2}-s}}{(2s-1)\log x} + \frac{S_s(x)}{\log x} + O\left(\frac{x^{\frac{1}{2}-s}}{(\log x)^2}\right) \right)$$

in  $1/2 < s < 1$ , where  $\vartheta(x) = \sum_{p \leq x} \log p$  is the Chebyshev function and

$$S_s(x) = -s \sum_{\zeta(\rho)=0} \frac{x^{\rho-s}}{\rho(\rho-s)}.$$

His method is contained in the article [7] entitled ‘Highly Composite Numbers’ and the rest [9] was published in 1997. There are two manuscripts by him (handwritten by Watson) on sums involving primes. These are found on pages 228–232 in [8]. His original manuscripts are stored in the library at Trinity College, Cambridge.

**2.2. Prime Number Theorem in arithmetic progressions.** Let  $\chi$  be a primitive Dirichlet character modulo  $q$ . Then a Dirichlet  $L$ -function  $L(s, \chi)$  satisfies the functional equation

$$\Lambda(s, \chi) := \left(\frac{q}{\pi}\right)^{s/2} \Gamma\left(\frac{s+\nu}{2}\right) L(s, \chi) = \epsilon(\chi) \Lambda(1-s, \bar{\chi}),$$

where  $\nu = (1 - \chi(-1))/2$  and  $\epsilon(\chi) = i^{-\nu} q^{-1/2} \tau(\chi)$  with the Gauss sum  $\tau(\chi)$ . The completed  $L$ -function  $\Lambda(s, \chi)$  has a meromorphic continuation to  $\mathbb{C}$  and is entire if  $\chi \neq 1$ . Let  $\pi(x; q, a)$  be the number of primes  $p$  up to  $x$  belonging to the arithmetic progression  $a \pmod{q}$ . In particular, we set  $\pi(x) := \pi(x; 1, 1)$  and this abbreviation also applies to other functions below. The Prime Number Theorem in arithmetic progressions shows that in any residue class  $a \pmod{q}$ , the primes are equidistributed amongst the plausible arithmetic progressions modulo  $q$ :

$$\pi(x; q, a) \sim \frac{\pi(x)}{\varphi(q)}$$

as  $x \rightarrow \infty$  whenever  $(a, q) = 1$  and  $q \geq 1$ . An important question here is the uniformity in  $q$ , which is relevant to the distribution of zeroes of  $L(s, \chi)$ . Moreover, the Siegel–Walfisz theorem [10, 11] asserts that

$$\pi(x; q, a) = \frac{1}{\varphi(q)} \text{Li}(x) + O(x \exp(-c\sqrt{\log x})) \quad (2.1)$$

for any  $q \leq (\log x)^A$  where  $c$  and the implicit constant depend on  $A$  alone (not effectively computable if  $A \geq 2$ ). Nonetheless, it is beneficial to weaken the restriction on  $q$  for applications. The assumption of GRH yields (2.1) in a much wider regime  $q \leq \sqrt{x}/(\log x)^{2+\epsilon}$ . Since we assume GRH throughout this article, such a restriction on  $q$  adheres to our discussion. It is conjectured that the following asymptotic is available:

$$\pi(x; q, a) = \frac{1}{\varphi(q)} \text{Li}(x) + O(x^{1/2+\epsilon})$$

uniformly for  $q \leq x^{1/2-\epsilon}$ . For notational convenience, we introduce

$$E(x; q, a) = \frac{x}{\varphi(q)} - \vartheta(x; q, a).$$

It is well known that the bound  $|E(x; q, a)| \ll \sqrt{x}(\log x)^2$  is tantamount to GRH.

**2.3. Summation formulæ.** Following Ramanujan, we start with considering the partial summation that if  $\Phi'(x)$  is a continuous function, then

$$\Phi(p_1) \log p_1 + \Phi(p_2) \log p_2 + \cdots + \Phi(p_n) \log p_n = \Phi(x) \vartheta(x; q, a) - \int_{p_1}^x \Phi'(t) \vartheta(t; q, a) dt,$$

where  $p_1 < p_2 < \dots < p_n$  is an ascending sequence of consecutive primes of the form  $mq + a$  and  $p_1$  (resp.  $p_n$ ) stands for the smallest (resp. largest) prime below  $x$  of such a form. Integrating by parts gives

$$\Phi(x)\vartheta(x; q, a) - \int_{p_1}^x \Phi'(t)\vartheta(t; q, a)dt = \text{const} + \frac{1}{\varphi(q)} \int_{p_1}^x \Phi(t)dt - \Phi(x)E(x; q, a) + \int_{p_1}^x \Phi'(t)E(t; q, a)dt$$

where ‘const’ depends on  $a, q$  and  $\Phi$ . In what follows, we assume GRH for Dirichlet  $L$ -functions, which allows us to work in the regime  $q \leq \sqrt{x}/(\log x)^{2+\epsilon}$ . Taylor’s theorem then yields that

$$\int_{p_1}^{\varphi(q)\vartheta(x; q, a)} \Phi(t)dt = \int_{p_1}^x \Phi(t)dt - \Phi(x)\varphi(q)E(x; q, a) + \frac{1}{2}(\varphi(q)E(x; q, a))^2\Phi'(x + O(\sqrt{x}(\log x)^2)).$$

Gathering together these formulæ, one sees that

$$\begin{aligned} & \Phi(p_1) \log p_1 + \Phi(p_2) \log p_2 + \dots + \Phi(p_n) \log p_n \\ &= \text{const} + \frac{1}{\varphi(q)} \int_{p_1}^{\varphi(q)\vartheta(x; q, a)} \Phi(t)dt + \int_{p_1}^x \Phi'(t)E(t; q, a)dt - \frac{1}{2}(\varphi(q)E(x; q, a))^2\Phi'(x + O(\sqrt{x}(\log x)^2)), \end{aligned} \quad (2.2)$$

### 3. PROOF OF THEOREM 1.1

In this section, we establish Theorem 1.1.

**3.1. Partial Euler products for Dirichlet  $L$ -functions.** Let  $\chi$  be a primitive character modulo  $q$ . We exploit the summation formula (2.2)  $\varphi(q)$  times. For our purpose, we assume that  $\Phi(x) = \chi(a)/(x^s - \chi(a))$  for each  $a$  with  $(a, q) = 1$ . We also assume for the sake of simplicity that  $\Re(s) > 0$  throughout this article. Hence one derives

$$\begin{aligned} & \frac{\chi(2) \log 2}{2^s - \chi(2)} + \frac{\chi(3) \log 3}{3^s - \chi(3)} + \frac{\chi(5) \log 5}{5^s - \chi(5)} + \dots + \frac{\chi(p) \log p}{p^s - \chi(p)} \\ &= \text{const} + \frac{1}{\varphi(q)} \sum_{a \pmod{q}}^* \int_{p_1}^{\varphi(q)\vartheta(x; q, a)} \frac{\chi(a)dt}{t^s - \chi(a)} \\ & \quad - s \sum_{a \pmod{q}}^* \chi(a) \int_{p_1}^x \frac{E(t; q, a)dt}{t^{1-s}(t^s - \chi(a))^2} + O(x^{-s}(\log x)^4), \end{aligned} \quad (3.1)$$

where  $p$  signifies the largest prime below  $x$ . Moreover, the explicit formula (cf. [3]) renders that

$$\varphi(q)E(t; q, a) = \delta_2(q, a)\sqrt{t} + \delta_3(q, a)\sqrt[3]{t} + \sum_{\psi \pmod{q}} \bar{\psi}(a) \sum_{L(\rho, \psi)=0} \frac{t^\rho}{\rho} - \sum_{\psi \pmod{q}} \bar{\psi}(a) \sum_{L(\rho, \psi)=0} \frac{t^{\rho/2}}{\rho} + O(t^{1/5}), \quad (3.2)$$

where

$$\delta_m(q, a) = \#\{x \pmod{q} : x^m \equiv a \pmod{q}\},$$

the outer sums over  $\psi$  range over all Dirichlet characters modulo  $q$  and the inner sums are over nontrivial zeroes of  $L(s, \psi)$ . Note that the number of primitive Dirichlet characters modulo  $q$  is given by  $\sum_{d|q} \mu(d)\phi(r)$ . Using the Chinese Remainder Theorem, the function  $\delta_2(q, a)$  is multiplicative if  $q = 2^{n_2}3^{n_3}5^{n_5} \dots p_1^{n_{p_1}}$ . To be accurate, counting the solutions to  $x^2 \equiv a \pmod{q}$ , we infer for  $a$  with  $(a/q) = 1$  that

$$\delta_2(q, a) = \prod_p \delta(p^{n_p}, a) = 2^{\omega(q)-1} \times \begin{cases} 4 & \text{if } n_2 \geq 3, \\ 2 & \text{if } n_2 = 2, \\ 1 & \text{if } n_2 = 1, \\ 2 & \text{otherwise,} \end{cases}$$

where  $\omega(q)$  is the number of different prime factors of  $q$ . Then it turns out that the contribution from the fourth term on the right hand side of (3.2) is bounded as

$$\sum_{\psi \pmod{q}} \bar{\psi}(a) \sum_{L(\rho, \psi)=0} \frac{1}{\rho} \int_{p_1}^x \frac{x^{s+\rho/2-1} dx}{(x^s - \chi(a))^2} \ll \left| \sum_{\psi \pmod{q}} \sum_{L(\rho, \psi)=0} \frac{x^{\rho/2-s}}{\rho(\rho/2-s)} \right| \ll x^{1/4-\sigma},$$

Since  $(x^s - \chi(a))^{-2} = x^{-2s} + O(x^{-3\sigma})$ , the contribution of the third term on the right hand side of (3.2) becomes

$$\begin{aligned} \sum_{a \pmod{q}}^* \chi(a) \sum_{\psi \pmod{q}} \bar{\psi}(a) \sum_{L(\rho, \psi)=0} \frac{1}{\rho} \int_{p_1}^x \frac{x^{s+\rho-1} dx}{(x^s - \chi(a))^2} \\ = \text{const} + \sum_{a \pmod{q}}^* \chi(a) \sum_{\psi \pmod{q}} \bar{\psi}(a) \sum_{L(\rho, \psi)=0} \frac{x^{\rho-s}}{\rho(\rho-s)} + O(x^{1/2-2s}). \end{aligned}$$

Hence we recall the definition

$$S_s(x, \chi) = -\frac{s}{\varphi(q)} \sum_{a \pmod{q}}^* \chi(a) \sum_{\psi \pmod{q}} \bar{\psi}(a) \sum_{L(\rho, \psi)=0} \frac{x^{\rho-s}}{\rho(\rho-s)}$$

to arrive at the expression

$$\begin{aligned} \frac{\chi(2) \log 2}{2^s - \chi(2)} + \frac{\chi(3) \log 3}{3^s - \chi(3)} + \cdots + \frac{\chi(p) \log p}{p^s - \chi(p)} \\ = \text{const} + \frac{1}{\varphi(q)} \sum_{a \pmod{q}}^* \int_{p_1}^{\varphi(q) \vartheta(x; q, a)} \frac{\chi(a) dt}{t^s - \chi(a)} - \sum_{a \pmod{q}}^* s \int_{p_1}^x \chi(a) \frac{\delta_2(q, a) \sqrt{t} + \delta_3(q, a) \sqrt[3]{t}}{t^{1-s} (t^s - \chi(a))^2} dt \\ + S_s(x, \chi) + O(x^{1/2-2s} + x^{1/4-s}). \quad (3.3) \end{aligned}$$

**3.2. Completion of the proof.** One can establish the following lemma:

**Lemma 3.1.** *Let  $\chi$  be a Dirichlet character modulo  $q$  and let  $m \geq 1$  be a positive integer. We then have*

$$\sum_{a \pmod{q}}^* \chi(a) \delta_m(q, a) = \begin{cases} \varphi(q) & \text{if } \chi^m = 1, \\ 0 & \text{otherwise,} \end{cases}$$

*Proof.* The proof relies on the identity

$$\sum_{a \pmod{q}}^* \chi(a) \delta_m(q, a) = \sum_{a \pmod{q}}^* \chi(a)^m$$

to which we can apply the classical orthogonality relation.  $\square$

We use Lemma 3.1 after expanding the integrands of the two integrals on the right hand side of (3.3) respectively. Upon truncating unnecessary terms, the sum of the contributions from these integrals equals

$$\begin{aligned} \text{const} + \frac{1}{\varphi(q)} \sum_{a \pmod{q}}^* \left( \frac{\chi(a)}{1-s} (\varphi(q) \vartheta(x; q, a))^{1-s} + \frac{\chi(a)^2}{1-2s} (\varphi(q) \vartheta(x; q, a))^{1-2s} \right) \\ + \frac{x^{1-\varphi(q)s}}{1-\varphi(q)s} + \cdots + \frac{x^{1-ns}}{1-ns} - \delta_{\chi^2=1} \frac{2sx^{1/2-s}}{1-2s} - \delta_{\chi^3=1} \frac{3sx^{1/3-s}}{1-3s} - \delta_{\chi^2=1} \frac{4sx^{1/2-2s}}{1-4s} + O(x^{1/2-2s}), \end{aligned}$$

where  $n$  is the largest multiple of  $\varphi(q)$  not exceeding  $[2 + 1/2s]$ . It therefore follows that

$$\begin{aligned} \frac{\chi(2) \log 2}{2^s - \chi(2)} + \frac{\chi(3) \log 3}{3^s - \chi(3)} + \cdots + \frac{\chi(p) \log p}{p^s - \chi(p)} \\ = -\frac{L'(s, \chi)}{L(s, \chi)} + \frac{1}{\varphi(q)} \sum_{a \pmod{q}}^* \left( \frac{\chi(a)}{1-s} (\varphi(q) \vartheta(x; q, a))^{1-s} + \frac{\chi(a)^2}{1-2s} (\varphi(q) \vartheta(x; q, a))^{1-2s} \right) + \frac{x^{1-\varphi(q)s}}{1-\varphi(q)s} + \cdots \\ + \frac{x^{1-ns}}{1-ns} - \delta_{\chi^2=1} \frac{2sx^{1/2-s}}{1-2s} - \delta_{\chi^3=1} \frac{3sx^{1/3-s}}{1-3s} - \delta_{\chi^2=1} \frac{4sx^{1/2-2s}}{1-4s} + S_s(x, \chi) + O(x^{1/2-2\sigma} + x^{1/4-\sigma}). \quad (3.4) \end{aligned}$$

We note that  $\text{const} = -L'(s, \chi)/L(s, \chi)$  is justified by the work of Conrad [2]. If we consider the Euler products at  $s = 1, 1/2, 1/3$  and  $1/4$ , we must take the limit of the right hand side of (3.4). In order to finish our proof, one should replace  $s = \sigma + it$  with  $u + it$  and integrate the asymptotic formula (3.4) in  $u$  once from  $\infty$  to  $\sigma$ , obtaining

$$\begin{aligned} \log \prod_{p \leq x} (1 - \chi(p)p^{-s}) &= -\log L(s, \chi) + \frac{1}{\varphi(q)} \sum_{a \pmod{q}}^* \chi(a) \text{Li}((\varphi(q)\vartheta(x; q, a))^{1-s}) \\ &\quad - \delta_{\chi^2=1} \frac{1}{2} \text{Li}(x^{1-2s}) + \delta_{\chi^2=1} \frac{1}{2} \text{Li}(x^{1/2-s}) - \frac{1}{\varphi(q)} \text{Li}(x^{1-\varphi(q)s}) - \dots \\ &\quad - \frac{1}{n} \text{Li}(x^{1-ns}) - \frac{x^{1/2-s} + S_s(x, \chi)}{\log x} + O\left(\frac{x^{1/2-\sigma}}{(\log x)^2}\right), \end{aligned} \quad (3.5)$$

where we have used that

$$\int_{\infty}^{\sigma} \frac{x^{a+b(u+it)}}{a+b(u+it)} du = \frac{1}{b} \text{Li}(x^{a+bs}), \quad \int_{\infty}^{\sigma} S_{u+it}(x, \chi) du = -\frac{S_s(x, \chi)}{\log x} + O\left(\frac{x^{1/2-\sigma}}{(\log x)^2}\right).$$

Exponentiating (3.5) and classifying our resulting formula into the three cases  $0 < \Re(s) < 1/2$ ,  $\Re(s) = 1/2$  and  $\Re(s) > 1/2$ , we can deduce the desired formula. This concludes the proof of Theorem 1.1.  $\square$

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DEPARTMENT OF MATHEMATICS, CALIFORNIA INSTITUTE OF TECHNOLOGY, 1200 E CALIFORNIA BLVD, PASADENA, CA 91125, USA

Email address: ikuyak@icloud.com

URL: <https://sites.google.com/view/ikuyakaneko/>