Symmetry-protected sign problem and magic in quantum phases of matter

Tyler D. Ellison,[1,2,*] Kohtaro Kato,[3,4] Zi-Wen Liu,[2] and Timothy H. Hsieh[2,†]

[1] *University of Washington, Seattle, WA 98195, USA*
[2] *Perimeter Institute for Theoretical Physics, Waterloo, Ontario N2L 2Y5, Canada*
[3] *Institute for Quantum Information and Matter, California Institute of Technology, Pasadena, CA 91125, USA*
[4] *Center for Quantum Information and Quantum Biology, Institute for Open and Transdisciplinary Research Initiatives, Osaka University, Osaka 560-8531, Japan*

(Dated: August 12, 2021)

We introduce the concepts of a symmetry-protected sign problem and symmetry-protected magic, defined by the inability of symmetric finite-depth quantum circuits to transform a state into a non-negative real wave function and a stabilizer state, respectively. We show that certain symmetry-protected topological (SPT) phases have these properties, as a result of their anomalous symmetry action at a boundary. For example, one-dimensional $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPT states (e.g. cluster state) have a symmetry-protected sign problem, and two-dimensional $\mathbb{Z}_2$ SPT states (e.g. Levin-Gu state) have both a symmetry-protected sign problem and magic. We also comment on the relation of a symmetry-protected sign problem to the computational wire property of one-dimensional SPT states and the greater implications of our results for measurement based quantum computing.

CONTENTS

* E-mail: ellisont@uw.edu
† E-mail: thsieh@pitp.ca

I. INTRODUCTION

The concept of entanglement is an important tool for diagnosing the complexity of quantum states and has led to a deeper understanding of quantum phases of matter and quantum phase transitions. However, entanglement by itself does not fully capture the quantum complexity of a state. Moreover, some quantum states can be efficiently simulated by classical systems, despite the presence of entanglement. This motivates using diagnostics beyond entanglement to assess the quantum complexity of many-body states and to further inform us of the quantum information structures intrinsic to phases of matter. In this work, we focus on two means for evaluating the complexity of a state: (i) its 'magic' and (ii) its sign structure.

The magic of a state is an assessment of its inability to be efficiently described within the stabilizer formalism. The stabilizer formalism is based on a restricted set of operations - the stabilizer operations, and ultimately is incapable of simulating arbitrary quantum systems [1]. The stabilizer operations can nonetheless be promoted to a universal set of operations when used in conjunction with special ancillary states, known as magic states [2]. Similar to entanglement, the magic in a magic state can be formally understood as a quantum informational resource [3] that serves as a measure of the complexity of a state.

The sign structure of a state, on the other hand, refers to the complex probability amplitudes of a wave function [4, 5]. Complex probability amplitudes are responsible for inherently non-classical phenomena, such as quantum interference, and for this reason, characterize the quantum nature of the state. Indeed, if the amplitudes happen to be all non-negative, then there is a classical probability distribution whose square root gives the amplitudes of the wave function. The sign structure of a state is, of

course, basis dependent, so to make a meaningful assessment of the complexity of the state, we consider the sign structure modulo local basis changes. Following Ref. [5], we say the wave function has a sign problem if the amplitudes cannot be made non-negative by any local basis transformation.

While the magic in a many-body state and the notion of a sign problem are promising metrics for the quantum complexity of states, they are notoriously challenging to study analytically and numerically, though substantial progress has been made [3, 5–18]. We therefore propose a simplification by imposing symmetry constraints. In particular, we introduce symmetry-protected magic and a symmetry-protected sign problem. These simplified diagnostics of the complexity of a state allow us to make analytical statements about the structure of quantum information in quantum phases of matter.

More specifically, we consider symmetry-protected topological (SPT) phases of matter, whose properties can be characterized by short-range entangled (SRE) states. Despite the short-range entanglement, SPT phases are responsible for a rich set of quantum phenomena including the helical edge modes at the boundary of topological insulators [19, 20] and symmetry-protected degeneracies useful for measurement-based quantum computing [21–27]. It is therefore valuable to have a complete understanding of the quantum information structures of SPT phases to be able to both simulate their novel behaviors and harness their resources for quantum computing.

In this work, we contribute to the understanding of the quantum complexity of SPT states, by showing that certain SPT states have symmetry-protected magic and that some possess a symmetry-protected sign problem. The symmetry-protected magic implies that the SPT states cannot be simulated efficiently by stabilizer operations, or put differently, they have magic that cannot be removed by making local symmetry-preserving changes to the state. Whereas, the symmetry-protected sign problem informs us about the sign structure of SPT states and poses an obstruction to finding a non-negative representation through local symmetry-respecting basis changes. We note that this constitutes the first analytic proof of a sign problem at the level of the wave function.

Structure of the paper:

Our main application of symmetry-protected magic and a symmetry-protected sign problem are to SPT states. Therefore, we begin by defining SPT states and SPT phases in Section II A. For convenience, we work with a definition of SPT phases phrased in terms of finite-depth quantum circuits. Then, in Section II B, we describe a characteristic feature of SPT phases – the symmetry acts anomalously near a boundary. In the following section, Section II C, we discuss how the effects of the anomalous symmetry action can be detected using a strange correlator. To illustrate these concepts on a concrete example, we apply them the 1D cluster state in Section II D.

We then move on to assess the complexity of SPT states, starting with symmetry-protected magic in Section III. We first review the stabilizer formalism in Section III A before defining symmetry-protected magic in Section III B. Subsequently, in Section III C, we use the anomalous boundary symmetry action, to show that a subset of SPT phases (the group cohomology SPT phases in spatial dimensions $D \geq 2$) have symmetry-protected magic.

Next, we turn to the symmetry-protected sign problem in Section IV. In Section IV A, we give a precise definition for a symmetry-protected sign problem, and then, in Section IV B, we argue that SPT states in dimensions $D \leq 2$ have a symmetry-protected sign problem relative to local bases where the symmetry is diagonal. The argument relies on the expected "strange correlations" in SPT states. We also provide a second argument in Section IV B based on the incompatibility between the computational wire property of one-dimensional SPT phases and bounds on measurement-induced entanglement in non-negative wave functions [5].

We conclude by commenting on relations to previous work and by proposing future directions for studying the quantum complexity of topological phases of matter. We also state a number of conjectures, and in particular, we conjecture that states belonging to the double semion phase have magic that is robust to arbitrary unitary local operations.

II. PRIMER ON SPT PHASES

To begin, we define SPT phases in terms of the circuit complexity of states, following Ref. [28]. We then describe a characteristic property of (non-trivial) SPT phases in Section II B: the symmetry acts on the system in an anomalous fashion in the presence of a boundary. In certain cases, the effects of the anomalous symmetry action can be detected using strange correlators, which we define in Section II C. In Section II D, we illustrate the concepts of the anomalous boundary symmetry action and strange correlators with an example of a well-known SPT state - the 1D cluster state.

A. Definition of SPT phases

We define SPT states and SPT phases using finite-depth quantum circuits (FDQCs). Recall that a FDQC is any unitary operator that can be written in the form:

$$\mathcal{U} = \prod_{\ell=1}^{d} \Big(\prod_{j_\ell} U_{j_\ell} \Big). \tag{1}$$

Here, the first product runs over layers, up to a depth d, and j_ℓ indexes unitary operators U_{j_ℓ} in the layer ℓ. The unitary operators U_{j_ℓ}, referred to as gates, are taken to

be local[1] and to have non-overlapping supports within a given layer. We note that the circuit is "finite-depth", if the depth d is both finite and constant in the system size.

To define SPT states in D dimensions, we consider Hilbert spaces of the form:

$$\mathcal{H} = \bigotimes_{i\in\Lambda} \mathcal{H}_i, \tag{2}$$

where i labels sites on a lattice Λ embedded in a D dimensional manifold without boundary, and each site hosts a finite-dimensional Hilbert space $\mathcal{H}_i$. For SPT phases protected by a G symmetry, we assume the G symmetry is represented by an *onsite* representation.[2] That is, every g in G is represented by an operator:

$$u(g) = \bigotimes_{i\in\Lambda} u_i(g), \tag{3}$$

with each $u_i(g)$ forming a linear representation of G on $\mathcal{H}_i$. With this, an SPT state is any state that satisfies the following three conditions:

- **Short-range entangled:** It can be prepared from a product state by a finite-depth quantum circuit.
- **Symmetric:** It is invariant under the onsite representation of the G symmetry.
- **SPT parent Hamiltonian:** It is the unique ground state of a symmetric local gapped Hamiltonian.

The SPT states are then organized into SPT phases by imposing an equivalence relation. Two SPT states are equivalent, or belong to the same phase, if one can be constructed from the other by a FDQC composed of symmetric gates – with the possible use of ancillary lower-dimensional SPT states. We say an SPT state is trivial if it belongs to the same equivalence class as a product state, whereas a non-trivial SPT state has entanglement that cannot be removed by making symmetry preserving local changes to the state. In other words, a non-trivial SPT state cannot be disentangled by applying a FDQC with symmetric gates.

B. Anomalous symmetry action at a boundary

Having defined SPT phases, an important question is: what properties characterize an SPT phase? For non-trivial SPT phases, the symmetry action near a boundary is anomalous – there is an obstruction to finding an effective boundary symmetry action that is onsite.[3] Here, we give a heuristic description of the effective boundary symmetry action and refer to Ref. [29] for more details. In Appendix A, we outline an argument that the obstruction indeed gives a well-defined quantized invariant of the SPT phase.

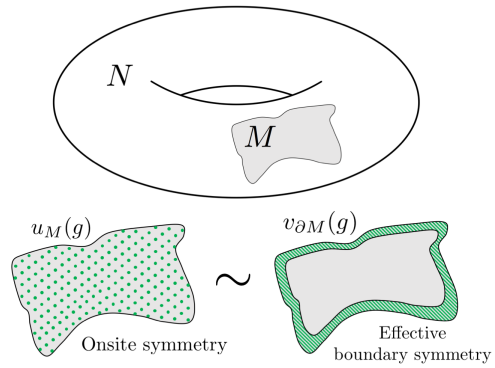


FIG. 1. To determine the SPT phase associated to a given SPT state, we compute an effective boundary symmetry action. This is done by truncating a corresponding SPT Hamiltonian defined on a closed manifold N to a submanifold M with boundary. In the low energy Hilbert space of the truncated Hamiltonian, the onsite symmetry action $u_M(g)$ (green dots) on M is equivalent to an effective boundary symmetry action $v_{\partial M}(g)$ (striped green) supported near the boundary of M. The symbol "$\sim$" denotes that $u_M(g)$ and $v_{\partial M}(g)$ are only required to be equivalent in the low energy Hilbert space. We use the effective boundary symmetry action to show that certain SPT states have symmetry-protected magic in Section III C.

To describe the effective boundary symmetry action, we consider a choice of SPT state along with a parent SPT Hamiltonian on a manifold N without boundary. We call the energy gap between the ground state and the first excited state Δ. We then imagine truncating the Hamiltonian to a submanifold M with boundary by removing any terms whose support includes sites outside of M (Fig. 1).[4] Furthermore, we use the tensor product structure to restrict the Hilbert space and onsite symmetry to M.

After restricting to M, we expect the spectrum of the truncated Hamiltonian to look qualitatively different – states now possibly lie within the energy window Δ. We assume that the low energy states are similar to the ground state of the un-truncated Hamiltonian in

[1] Throughout the text, by local, we mean that the support of the operator can be contained in a ball of fixed finite diameter.

[2] Note that, unless otherwise stated, we take the symmetry to be a unitary finite Abelian 0-form symmetry.

[3] Moreover, the effective boundary symmetry action cannot be made onsite through a combination of taking the tensor product with the effective boundary symmetry action of lower-dimensional SPT phases and conjugation by a FDQC.

[4] We assume M is large compared with the Lieb-Robinson length of a FDQC that prepares the ground state of the Hamiltonian on N.

regions far from the boundary.[5] Hence, the low energy states correspond to either excitations localized near the boundary or degenerate ground states.

We then define the boundary Hilbert space as the Hilbert space spanned by the states within the bulk gap Δ. With this, the effective boundary symmetry action is any unitary linear representation of the G symmetry in the boundary Hilbert space, such that (i) its support is localized[6] near the boundary of M and (ii) its action agrees with the global symmetry on states within the boundary Hilbert space (Fig. 1).

While the symmetry on M is onsite, the effective boundary symmetry action may be non-onsite. Ref. [29] showed that the so-called group cohomology SPT phases [32] exhibit an obstruction to an onsite effective boundary symmetry action, which is captured by group cohomology. In D-dimensions with a G symmetry, the obstruction corresponds to an element of $H^{D+1}(G, U(1))$, the $(D+1)^{\text{th}}$ group cohomology of G with coefficients in $U(1)$. It is believed that $H^{D+1}(G, U(1))$ gives a complete classification of (bosonic) SPT phases protected by unitary symmetries in dimensions $D < 4$ [33–35]. In the remainder of the text, we focus our attention on the group cohomology SPT phases.

C. Strange correlator

The anomalous symmetry action at a boundary, in the previous section, enforces long-range entanglement in states describing non-trivial SPT phases on a manifold with boundary.[7] This has been shown carefully in one and two spatial dimensions using a tensor network approach [36] and is believed to hold in higher dimensions. We emphasize that on a manifold *without* boundary, the states in an SPT phase are short-range entangled by definition – the long-range entanglement only appears explicitly when a boundary to a trivial SPT phase is exposed.

One tool that has been developed to probe the long-range entanglement of SPT phases in the presence of a boundary is the strange correlator [38–41]. The strange correlator takes the general form:

$$\frac{\langle \Omega | \mathcal{O}_i \mathcal{O}_j | \psi_{\text{SPT}} \rangle}{\langle \Omega | \psi_{\text{SPT}} \rangle}, \tag{4}$$

where $\langle \Omega |$ is a symmetric product state, $|\psi_{\text{SPT}}\rangle$ is an SPT state on a manifold *without* boundary, and $\mathcal{O}_i$ and $\mathcal{O}_j$ are operators localized near the sites i and j. More specifically, $\mathcal{O}_i$ and $\mathcal{O}_j$ correspond to a strange order parameter, given by a collection of unitary local operators $\{\mathcal{O}_k\}_{k\in\Lambda}$ such that each $\mathcal{O}_k$ has a non-trivial definite charge under the symmetry. That is, for a finite Abelian symmetry G and any $g \in G$, $\mathcal{O}_k$ satisfies:

$$u(g)\mathcal{O}_k u(g)^\dagger = e^{i\kappa(g)}\mathcal{O}_k, \tag{5}$$

where $e^{i\kappa(g)}$ forms a non-trivial one dimensional representation of G.

The general expectation is that, for a non-trivial SPT state in either one or two dimensions, there exists a strange order parameter such that the strange correlator in Eq. (4) has a power law decay or is constant as the separation between i and j goes to infinity. This is based on numerous examples as well as physical intuition from a tensor network representation of $\langle \Omega | \psi_{\text{SPT}} \rangle$.

Given a tensor network representation of the D-dimensional SPT state $|\psi_{\text{SPT}}\rangle$, we can interpret the overlap $\langle \Omega | \psi_{\text{SPT}} \rangle$ as a partition function for a $(D-1)$-dimensional system, as pictured in Fig. 2. The $(D-1)$-dimensional system is invariant under an anomalous symmetry, similar to the anomalous boundary symmetry action of an SPT phase. This can be seen by acting with the symmetry restricted to a subregion M with a boundary. Ref. [37] argued that the symmetry action on M can be replaced with an effective symmetry action on the virtual bonds of the tensor network along the boundary of M (see Fig. 2).

For a non-trivial SPT state, the effective symmetry action on the virtual bonds is anomalous, and hence, the $(D-1)$-dimensional partition function $\langle \Omega | \psi_{\text{SPT}} \rangle$ is invariant under an anomalous symmetry action. This implies that $\langle \Omega | \psi_{\text{SPT}} \rangle$ should be thought of as a partition function for a long-range entangled state, and the strange correlator probes the correlations in this state. Therefore, the strange correlator measures correlations similar to those that arise on the boundary of a state in an SPT phase. (See also Refs. [38] and [39] for a physical interpretation of the strange correlator.)

The use of strange correlators can be rigorously justified for 1D SPT states using the notion of string-order parameters. We illustrate this for the cluster state assuming the working definition of SPT phases, given in Section II A. We claim that the argument can be generalized straightforwardly to other 1D SPT states.

D. Example: cluster state

To make the discussion more concrete, we describe the cluster state, an example of a non-trivial 1D SPT state with a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry. The cluster state is defined on a 1D lattice with $2N$ qubits and periodic boundary conditions. We denote the Pauli X and Pauli Z operator

[5] More precisely, the reduced density matrices agree on regions sufficiently far from the boundary. This is the TQO-2 assumption in Refs. [30] and [31].

[6] In particular, we assume the effective boundary symmetry action is supported on M and within a fixed distance from the boundary of M.

[7] We note that, by an SPT phase on a manifold with boundary, we have in mind a collection of ground states, where the parent Hamiltonians are truncated SPT Hamiltonians with an arbitrary local symmetric Hamiltonian supported near the boundary.

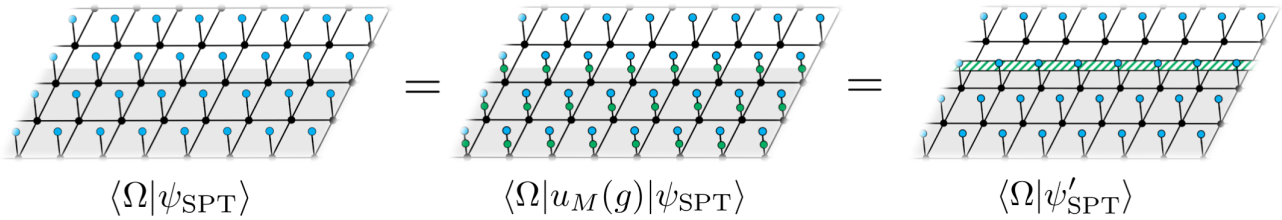


FIG. 2. The overlap between $\langle\Omega|$ (light blue tensors) and $|\psi_{\rm SPT}\rangle$ (black tensors) takes the form of a Euclidean partition function for a $(D-1)$-dimensional system, for which, a spacetime configuration corresponds to a set of fixed indices on the virtual bonds. The $(D-1)$-dimensional system defined by $\langle\Omega|\psi_{\rm SPT}\rangle$ is invariant under an anomalous symmetry if $|\psi_{\rm SPT}\rangle$ belongs to a non-trivial SPT phase. This can be argued by first using the symmetry of $\langle\Omega|$ to insert the symmetry action $u_M(g)$ (green circles) restricted to a region M (shaded gray). By the arguments in Ref. [37], $u_M(g)$ applied to $|\psi_{\rm SPT}\rangle$ is equivalent to inserting a certain tensor network operator (striped green) along the virtual bonds on the boundary of M. If $|\psi_{\rm SPT}\rangle$ is a non-trivial SPT state, then the effective symmetry action on the virtual bonds is anomalous, and the $(D-1)$-dimensional system has an anomalous symmetry. $|\psi'_{\rm SPT}\rangle$ denotes the state with the tensor network operator applied on the virtual bonds. This motivates the use of strange correlators to prove a symmetry-protected sign problem in certain SPT states (see Section IV B).

at the site i by X_i and Z_i, respectively. The onsite $\mathbb{Z}_2\times\mathbb{Z}_2$ symmetry is then generated by the operators:

$$u((g,1)) \equiv \prod_j X_{2j}, \quad u((1,g)) \equiv \prod_j X_{2j+1}, \tag{6}$$

where we have labeled the elements of $\mathbb{Z}_2\times\mathbb{Z}_2$ as:

$$\mathbb{Z}_2\times\mathbb{Z}_2 = \{(1,1),(g,1),(1,g),(g,g)\}. \tag{7}$$

The cluster state can be prepared from a product state by the FDQC $\mathcal{U}_{\rm CS}$ given as:

$$\mathcal{U}_{\rm CS} \equiv \prod_{\langle i,i+1\rangle} CZ_{i(i+1)}. \tag{8}$$

Here, the product is over pairs of neighboring sites, and the control-Z operator $CZ_{i(i+1)}$ is the two qubit operator whose action on an arbitrary computational basis state $|a\rangle_i|b\rangle_{i+1}$ is:

$$CZ_{i(i+1)}|a\rangle_i|b\rangle_{i+1} = (-1)^{ab}|a\rangle_i|b\rangle_{i+1}, \quad a,b\in\{0,1\}. \tag{9}$$

Explicitly, the cluster state is:

$$|\psi_{\rm CS}\rangle \equiv \mathcal{U}_{\rm CS}|+\ldots+\rangle, \tag{10}$$

where $|+\ldots+\rangle$ is the simultaneous $+1$ eigenstate of all Pauli X operators.

A parent Hamiltonian for the cluster state is:

$$H_{\rm CS} \equiv \mathcal{U}_{\rm CS}\left(-\sum_i X_i\right)\mathcal{U}_{\rm CS}^\dagger = -\sum_{i=1}^{2N} Z_{i-1}X_iZ_{i+1}. \tag{11}$$

$H_{\rm CS}$ is gapped and has a unique ground state given that it has the same spectrum as the paramagnet Hamiltonian: $-\sum_i X_i$. Further, it can be checked that each term of $H_{\rm CS}$ is symmetric. Therefore, $H_{\rm CS}$ is an SPT Hamiltonian.

To see that the ground state is in a non-trivial SPT phase, we introduce a boundary and study the effective symmetry action near the boundary, as described below.

Anomalous boundary symmetry action:

In dimension $D=1$, SPT phases with a G symmetry are classified by $H^2(G,U(1))$ [42, 43], where the elements of $H^2(G,U(1))$ correspond to projective representations of G. We compute an effective boundary symmetry action for the cluster state model and show that it forms a projective representation of $\mathbb{Z}_2\times\mathbb{Z}_2$. This is the non-trivial element of $H^2(\mathbb{Z}_2\times\mathbb{Z}_2,U(1))=\mathbb{Z}_2$.

To start, we truncate the Hamiltonian $H_{\rm CS}$ in Eq. (11) to a lattice with $2M$ sites and open boundary conditions. This gives us the Hamiltonian $H_{\rm CS}^M$:

$$H_{\rm CS}^M \equiv -\sum_{i=2}^{2M-1} Z_{i-1}X_iZ_{i+1}. \tag{12}$$

$H_{\rm CS}^M$ has a 4-fold degenerate ground state subspace, which follows from the fact that we have removed the terms associated to the sites $i=1$ and $i=2M$. The degenerate ground state subspace of $H_{\rm CS}^M$ defines the boundary Hilbert space.

We now derive an effective boundary symmetry action. The states in the boundary Hilbert space are $+1$ eigenstates of the terms in $H_{\rm CS}^M$, since the terms are mutually commuting and un-frustrated. Therefore, in the boundary Hilbert space, we have:

$$Z_{i-1}X_iZ_{i+1} \sim 1, \quad \forall i\in\{2,\ldots,2M-1\}, \tag{13}$$

where $\sim$ emphasizes that this holds in the boundary Hilbert space. Using the relation in Eq. (13), the generators of the $\mathbb{Z}_2\times\mathbb{Z}_2$ symmetry can be written in the boundary Hilbert space as:

$$u((g,1)) \sim Z_1(Z_{2M-1}X_{2M}), \quad u((1,g)) \sim (X_1Z_2)Z_{2M}. \tag{14}$$

We define the right-hand side of the equations in Eq. (14) as the operators:

$$v\big((g,1)\big) \equiv Z_1(Z_{2M-1}X_{2M}), \quad v\big((1,g)\big) \equiv (X_1Z_2)Z_{2M}. \tag{15}$$

These define a $\mathbb{Z}_2 \times \mathbb{Z}_2$ effective boundary symmetry action, since they form a unitary linear representation of $\mathbb{Z}_2 \times \mathbb{Z}_2$, are localized near the boundary, and, by definition, agree with the global symmetry action in the boundary Hilbert space.

The effective boundary symmetry action generated by the operators in Eq. (15) is not onsite – i.e., it is not in the form of a tensor product of linear representations at each site (as defined in Section II A). Instead, the action at endpoints $i = 1$ and $i = 2M$ are independently projective representations of $\mathbb{Z}_2 \times \mathbb{Z}_2$. For example, at the endpoint $i = 1$, we have:

$$v_{i=1}\big((g,1)\big) \equiv Z_1, \quad v_{i=1}\big((1,g)\big) \equiv X_1Z_2. \tag{16}$$

These give a projective representation, as can be seen by the commutation relations between $v_{i=1}\big((g,1)\big)$ and $v_{i=1}\big((1,g)\big)$:

$$v_{i=1}\big((g,1)\big)v_{i=1}\big((1,g)\big) = -v_{i=1}\big((1,g)\big)v_{i=1}\big((g,1)\big). \tag{17}$$

Thus, the effective boundary symmetry action is anomalous.[8] In Appendix A, we show that the anomalous symmetry action implies that the cluster state cannot be disentangled by a FDQC composed of symmetric gates.

Strange correlator:

For the cluster state, we can use the exactly solvable Hamiltonian H_{CS} to identify a suitable strange order parameter. To see this, we consider the product of Hamiltonian terms:

$$\prod_{k=i}^{j-1} Z_{2k}X_{2k+1}Z_{2k+2} = (\prod_{k=i}^{j-1} X_{2k+1})Z_{2i}Z_{2j}. \tag{18}$$

This gives us the identity:

$$\begin{aligned}\langle + \ldots + | \psi_{\mathrm{CS}} \rangle &= \langle + \ldots + | (\prod_{k=i}^{j-1} X_{2k+1})Z_{2i}Z_{2j} | \psi_{\mathrm{CS}} \rangle \\ &= \langle + \ldots + | Z_{2i}Z_{2j} | \psi_{\mathrm{CS}} \rangle,\end{aligned} \tag{19}$$

where the first equality comes from the fact that $|\psi_{\mathrm{CS}}\rangle$ is a $+1$ eigenstate of each term of H_{CS}, and the second equality uses that $\langle + \ldots + |$ is a $+1$ eigenstate of every Pauli X operator. From Eq. (19), we have:

$$\frac{\langle + \ldots + | Z_{2i}Z_{2j} | \psi_{\mathrm{CS}} \rangle}{\langle + \ldots + | \psi_{\mathrm{CS}} \rangle} = 1, \tag{20}$$

for any choice of i and j. By redefining the sites so that the pair of sites $2k-1$ and $2k$ form a single site labeled by k, we can use the set $\{Z_{2k}\}$ as a strange order parameter. Moreover, we have found a strange order parameter for $|\psi_{\mathrm{CS}}\rangle$ such that the strange correlator is constant in the separation of $2i$ and $2j$, according to Eq. (20).

As for other states in the same phase as $|\psi_{\mathrm{CS}}\rangle$, we can use the operator in Eq. (18) to identify a strange order parameter for which the strange correlator is constant in the limit $|i-j| \to \infty$. For example, let $|\psi'_{\mathrm{CS}}\rangle$ be the state prepared from $|\psi_{\mathrm{CS}}\rangle$ by the FDQC $\mathcal{U}_{\mathrm{sym}}$ composed of symmetric gates:

$$|\psi'_{\mathrm{CS}}\rangle \equiv \mathcal{U}_{\mathrm{sym}} |\psi_{\mathrm{CS}}\rangle. \tag{21}$$

$|\psi'_{\mathrm{CS}}\rangle$ is invariant under the operator:

$$\mathcal{U}_{\mathrm{sym}} \left[(\prod_{k=i}^{j-1} X_{2k+1}) Z_{2i}Z_{2j} \right] \mathcal{U}^{\dagger}_{\mathrm{sym}}. \tag{22}$$

Since $\mathcal{U}_{\mathrm{sym}}$ is built from symmetric gates, the operator in Eq. (22) is equal to:

$$(\prod_{k=i}^{j-1} X_{2k+1}) \mathcal{O}_i \mathcal{O}_j, \tag{23}$$

for some unitary local charged operators $\mathcal{O}_i$ and $\mathcal{O}_j$. Following Eqs. (19) and (20), we can define a strange order parameter from the collection of $\mathcal{O}_i$ and $\mathcal{O}_j$ for varying endpoints i and j. Thus, every state in the SPT phase (as defined in Section II A) admits a strange order parameter with a constant strange correlator.

We note that the operators in Eqs. (19) and (22) are the more familiar string-order parameters that characterize 1D SPT phases. These naturally lead to strange order parameters with a constant strange correlation.

III. SYMMETRY-PROTECTED MAGIC

In this section, we introduce symmetry-protected magic and demonstrate that it is a feature of a large class of SPT states. To start, we review the stabilizer formalism and describe how it can be simulated efficiently on a classical computer. The stabilizer formalism is insufficient for universal quantum computing, but that leads us to the concept of magic, which is a resource that can be used to help overcome the limitations of the stabilizer formalism. We then define the notion of symmetry-protected magic and use it to assess the magic in SPT states. In particular, we show that SPT states belonging to group cohomology phases in $D \geq 2$ dimensions have symmetry-protected magic.

[8] Importantly, the projective representations at the endpoints cannot be made into linear representations by conjugating by a FDQC.

A. Review of the stabilizer formalism

The stabilizer formalism has been instrumental to our understanding of the complexity of quantum phases of matter, as it often provides simple, workable examples. In this section, we give a brief review of the stabilizer formalism to ensure the text is self-contained. We refer to Refs. [3, 44–47] for more thorough reviews.

To make the discussion general, we first define a generalization of Pauli operators to q-dimensional Hilbert spaces, i.e., to qudits. The computational basis states, in this case, are labeled by $j \in \mathbb{Z}_q$, and the usual Pauli Z and Pauli X operators are generalized to:

$$Z \equiv \sum_{j\in\mathbb{Z}_q} e^{\frac{2\pi i}{q}j}|j\rangle\langle j|, \quad X \equiv \sum_{j\in\mathbb{Z}_q} |j+1\rangle\langle j|. \tag{24}$$

If q is odd, then the set of Pauli operators on a qudit is generated by products of Z and X, and if q is even, the Pauli operators are generated by Z, X, and the phase i.[9] For systems of more than one qudit, we call a tensor product of Pauli operators a Pauli string. Furthermore, we say a Pauli string is Z-type or X-type if, up to a phase, it consists of only products of Z operators or products of X operators, respectively.

With this, we introduce Clifford unitaries and stabilizer states. A Clifford unitary is any unitary operator that maps Pauli strings to Pauli strings by conjugation. Explicitly, for any Pauli string P, a Clifford unitary U satisfies:

$$UPU^\dagger = Q, \tag{25}$$

for some Pauli string Q. A stabilizer state is then any state that can be generated by applying a Clifford unitary to the computational basis state $|0\ldots0\rangle$. Here, $|0\ldots0\rangle$ is the simultaneous $+1$ eigenstate of every Pauli Z operator. Thus, by definition, a stabilizer state $|\psi_\mathcal{S}\rangle$ can always be written as:

$$|\psi_\mathcal{S}\rangle = U|0\ldots0\rangle, \tag{26}$$

for some Clifford unitary U.

At this point, one can define a computational scheme based on applying Clifford unitaries to stabilizer states and making measurements of Pauli strings. However, this restricted set of operations - the stabilizer operations - can be efficiently simulated by a classical computer. This is the statement of the Gottesman–Knill theorem [1] and a consequence of the fact that a stabilizer state can be fully characterized by a stabilizer group, as described below.

Indeed, for every stabilizer state $|\psi_\mathcal{S}\rangle$, we can find a group $\mathcal{G}$ of mutually commuting Pauli strings such that $|\psi_\mathcal{S}\rangle$ is the unique state satisfying:

$$S|\psi_\mathcal{S}\rangle = |\psi_\mathcal{S}\rangle, \quad \forall S \in \mathcal{G}. \tag{27}$$

We refer to the elements in $\mathcal{G}$ as stabilizers and call the group $\mathcal{G}$ a stabilizer group. We say the stabilizer group $\mathcal{G}$ "stabilizes" or "fixes" $|\psi_\mathcal{S}\rangle$ to mean that $|\psi_\mathcal{S}\rangle$ is in the simultaneous $+1$ eigenspace of all of the stabilizers. More generally, any group of mutually commuting Pauli strings is a stabilizer group, although it may not fix a unique stabilizer state.

To find a suitable stabilizer group for $|\psi_\mathcal{S}\rangle$, we first consider the state $|0\ldots0\rangle$. $|0\ldots0\rangle$ is uniquely stabilized by the stabilizer group $\mathcal{G}_0$ generated by a Pauli Z operator for each site:

$$\mathcal{G}_0 \equiv \langle Z_i : i \in \text{sites}\rangle. \tag{28}$$

From this, we can identify a stabilizer group $\mathcal{G}$ that uniquely fixes $|\psi_\mathcal{S}\rangle$. To do so, we conjugate the elements of $\mathcal{G}_0$ by a choice of Clifford unitary U that prepares $|\psi_\mathcal{S}\rangle$ from $|0\ldots0\rangle$, as in Eq. (26):

$$\mathcal{G} \equiv \langle UZ_iU^\dagger : i \in \text{sites}\rangle. \tag{29}$$

By construction, the stabilizer group $\mathcal{G}$ in Eq. (29) is generated by N Pauli strings, where N is the number of sites. The state can therefore be efficiently specified by a stabilizer group, and moreover, the effects of evolution by a Clifford unitary and measurements of Pauli strings can be determined by appropriately modifying the stabilizer group. We see that the stabilizer operations are no more powerful than a classical computer, and additional ingredients are needed to promote it to a universal set of operations.

Before describing how the stabilizer formalism can be supplemented to achieve universal quantum computation, we remark that the generators of a stabilizer group can be used to build a stabilizer Hamiltonian. More specifically, given a stabilizer group $\mathcal{G}$ that uniquely stabilizes a state $|\psi_\mathcal{S}\rangle$, we can construct a Hamiltonian:

$$H_\mathcal{S} \equiv -\sum_{S\in\mathcal{S}} S + \text{h.c.}, \tag{30}$$

where $\mathcal{S}$ denotes a set of stabilizers that generate $\mathcal{G}$. The unique ground state of $H_\mathcal{S}$ is $|\psi_\mathcal{S}\rangle$, since it is a $+1$ eigenstate of each $S \in \mathcal{S}$ and is uniquely fixed by $\mathcal{G}$. We note that $H_\mathcal{S}$ might not be local.

B. Definition of symmetry-protected magic

Stabilizer operations, reviewed in the previous section, can be simulated efficiently on a classical computer, but the full power of quantum computation can be recovered by supplementing the stabilizer operations with ancillary non-stabilizer states. In fact, any non-stabilizer (pure) state can be used as ancillary states to promote the stabilizer operations to a universal set of operations. In this

[9] The group generated by the Pauli operators is commonly called the Heisenberg-Weyl group.

context, the non-stabilizer states are referred to as magic states. In a precise sense, the magic of a state (or the "non-stabilizerness") can be treated as a resource, similar to viewing entanglement as a resource. Consequently, resource-theoretical tools have been developed to quantify the amount of magic in a state (see e.g., Refs. [3, 48–51]), however few analytical statements have been made about magic in many-body systems. To make progress in this direction, we define the following coarse measure of the magic to help understand the large-scale structure of magic in a many-body state:

Definition 1 (Long-range magic) *A state $|\psi\rangle$ has long-range magic, if, for any finite-depth quantum circuit $\mathcal{U}$, the state $\mathcal{U}|\psi\rangle$ is a magic state.*

In other words, a state with long-range magic has magic that cannot be removed by any FDQC. In this sense, the state can serve as a robust source of magic. We would like to point out that concepts similar to long-range magic have been recently introduced in Ref. [13] in the context of conformal field theories. We hope to comment on long-range magic in future work, but in the present text we focus on a restricted notion of long-range magic.

In particular, we consider magic that cannot be removed by any FDQC composed of *symmetric* gates. We say such a state has symmetry-protected magic. This is defined more precisely as:

Definition 2 (Symmetry-protected magic) *A state $|\psi\rangle$ has symmetry-protected magic, if, for any finite-depth quantum circuit $\mathcal{U}_{sym}$ composed of symmetric gates, the state $\mathcal{U}_{sym}|\psi\rangle$ is a magic state.*

As a proof of concept, we show that certain SPT states have symmetry-protected magic.

C. Symmetry-protected magic in SPT states

Our main objective in this section is to show that SPT states, in particular those belonging to a non-trivial group cohomology SPT phase in dimensions $D \geq 2$, have symmetry-protected magic. This includes, for example, the $\mathbb{Z}_2$ SPT model introduced in Ref. [52]. The argument relies on the anomalous boundary symmetry action characteristic of non-trivial group cohomology SPT phases (see Section II B). After proving the proposition below, we comment on SPT states that fall outside of our argument – these correspond to SPT phases that can indeed be described efficiently by the stabilizer formalism.

Proposition 1 *An SPT state has symmetry-protected magic, if it belongs to a non-trivial group cohomology phase in $D \geq 2$ dimensions and is protected by a G symmetry represented by tensor products of Pauli operators.*

Proof of Proposition 1. We defined SPT phases as collections of SRE states that are equivalent under FDQCs composed of symmetric gates. Therefore, if an SPT state has symmetry-protected magic, it implies that every state in the SPT phase must be a magic state. To prove the proposition, it is then sufficient to show that there are no stabilizer states belonging to non-trivial group cohomology phases in $D \geq 2$ dimensions with a G symmetry represented by Pauli strings.

With this, we proceed by deriving a contradiction. We assume that there is a stabilizer state $|\psi_{\mathcal{S}}\rangle$ belonging to a non-trivial group cohomology phase in $D \geq 2$ dimensions protected by a G symmetry represented by a Pauli string $P(g)$ for every $g \in G$. We argue that this is in conflict with the anomalous boundary symmetry action expected in the non-trivial SPT phase.

The first step is to find a local symmetric stabilizer Hamiltonian whose unique ground state is $|\psi_{\mathcal{S}}\rangle$. Since $|\psi_{\mathcal{S}}\rangle$ is an SPT state, it has a local parent Hamiltonian (albeit possibly non-stabilizer), and it is invariant under the G symmetry, i.e., $P(g)|\psi_{\mathcal{S}}\rangle = |\psi_{\mathcal{S}}\rangle$, for all $g \in G$. In Appendix B, we show that this, in fact, implies that there exists a local symmetric stabilizer Hamiltonian $H_{\mathcal{S}}$ whose unique ground state is $|\psi_{\mathcal{S}}\rangle$ and which commutes with the G symmetry (see Lemma 2).

We can now determine the SPT phase by using $H_{\mathcal{S}}$ to compute the anomalous symmetry action at the boundary (analogous to the calculation of the anomalous boundary symmetry action for the cluster state in Section II D). For this purpose, we introduce a boundary by truncating the Hamiltonian $H_{\mathcal{S}}$ to a region M with boundary. We define the truncated Hamiltonian $H_{\mathcal{S}}^M$ by removing any term whose support is not entirely contained within M.[10] The global symmetry action $P(g)$ can also be restricted to M using that $P(g)$ is a tensor product of linear representations $P_i(g)$ of G:

$$P(g) \equiv \prod_i P_i(g) \longrightarrow P_M(g) \equiv \prod_{i \in M} P_i(g). \tag{31}$$

The truncated Hamiltonian has a G symmetry represented by the operators $P_M(g)$, given above.

Similar to $H_{\mathcal{S}}$, the truncated Hamiltonian is a sum of symmetric commuting stabilizer terms, but unlike $H_{\mathcal{S}}$, it will generically have a ground state degeneracy, and the degenerate ground state subspace forms the boundary Hilbert space. The anomalous behavior of the symmetry, characteristic of the SPT phase, is revealed by the effective symmetry action on the boundary Hilbert space. We recall from Section II B that the effective boundary symmetry action is any operator localized near the boundary

[10] To avoid pathologies, we require that M is large compared to the size of the supports of the terms in $H_{\mathcal{S}}$. More precisely, we require $d_M \gg d_S$, where d_M is the diameter of the largest ball inscribing M, and d_S is the minimum diameter such that the support of each stabilizer term fits within a ball of diameter d_S.

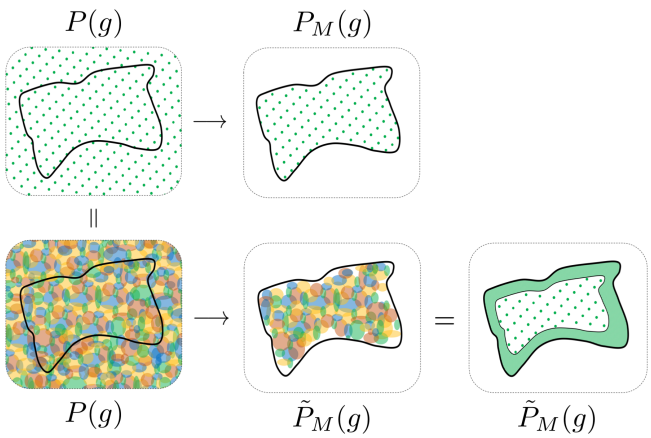


FIG. 3. We determine the effective boundary symmetry action from $H_{\mathcal{S}}$ by first observing that the global symmetry $P(g)$, for any $g \in G$, can be expressed as a product of terms in $H_{\mathcal{S}}$ (the supports of the stabilizer terms are depicted with colored ovals). The global symmetry action can be restricted to a submanifold M (outlined in black) in two ways. Restricting to M by using the tensor product structure of $P(g)$ results in $P_M(g)$, while restricting to M using the product of stabilizer terms gives $\tilde{P}_M(g)$. $\tilde{P}_M(g)$ acts like the onsite symmetry away from the boundary of M.

of M, whose action is equivalent to the symmetry action of $P_M(g)$, within the boundary Hilbert space.

The effective symmetry action at the boundary can be computed by first observing that the global symmetry action $P(g)$ can be written as a product of the stabilizer terms of the un-truncated Hamiltonian $H_{\mathcal{S}}$. This is because, $H_{\mathcal{S}}$ commutes with $P(g)$ and has a unique ground state. Therefore, $P(g)$ is contained in the stabilizer group generated by the terms of $H_{\mathcal{S}}$ (see Lemma 1 in Appendix B). Consequently, $P(g)$ can be written as:

$$P(g) = \prod_{S_j \in \mathcal{S}_{P(g)}} S_j, \tag{32}$$

where $\mathcal{S}_{P(g)}$ is defined as the set of terms in $H_{\mathcal{S}}$ whose product is $P(g)$. By using the expression for $P(g)$ in Eq. (32), $P(g)$ can instead be truncated to M by retaining only the stabilizers S_j whose support $\mathrm{supp}(S_j)$ is entirely contained in M (see Fig. 3):

$$\tilde{P}_M(g) \equiv \prod_{\substack{S_j \in \mathcal{S}_{P(g)} \\ \mathrm{supp}(S_j) \subset M}} S_j. \tag{33}$$

We note that $\tilde{P}_M(g)$ is a product of terms in $H_{\mathcal{S}}^M$ and as such, acts as the identity on the boundary Hilbert space. This implies, in particular, that $P_M(g)$ is equivalent to $P_M(g)\tilde{P}_M^\dagger(g)$ in the boundary Hilbert space. Further, by comparing Eqs. (32) and (33), we see that the action of $\tilde{P}_M(g)$ is equivalent to that of $P(g)$ on sites in M greater than a fixed distance from the boundary of M. Therefore, the support of $P_M(g)\tilde{P}_M^\dagger(g)$ is contained in M and localized near the boundary of M. As a result, we can take the effective boundary symmetry action to be:

$$\mathcal{P}(g) \equiv P_M(g)\tilde{P}_M^\dagger(g). \tag{34}$$

The characteristic group cohomology class of the SPT order can be deduced from the effective boundary symmetry action using the methods of Ref. [29], where the group cohomology class manifests as an obstruction to realizing the effective boundary symmetry action onsite (as a tensor product of linear representations at each site). From the definition in Eq. (34), we see that $\mathcal{P}(g)$ can be written as a tensor product of Pauli operators:

$$\mathcal{P}(g) = \prod_k \mathcal{P}_k(g), \tag{35}$$

with the product over sites k in M close to the boundary of M. While $\mathcal{P}(g)$ forms a linear representation of G in the boundary Hilbert space, the operators $\mathcal{P}_k(g)$ might only satisfy the group laws projectively. In dimensions $D \geq 2$ this does not pose an obstruction to an onsite representation of the effective boundary symmetry action. The algorithm in Ref. [29] shows that the effective boundary symmetry action in Eq. (35) corresponds to the trivial element of $H^{D+1}(G, U(1))$ (if $D \geq 2$).

To motivate this conclusion, we argue that any projective representations formed by $\mathcal{P}_k(g)$ can be resolved by modifying $H_{\mathcal{S}}$ with decoupled 1D SPT Hamiltonians acting on ancillary qudits. Importantly, the 1D SPT Hamiltonians do not change the $D \geq 2$ SPT phase described by $H_{\mathcal{S}}$. Moreover, the 1D SPT Hamiltonians can always be chosen so that their projective effective boundary symmetry actions (see Section II D, for example) compensate for the projective representations formed by the $\mathcal{P}_k(g)$. Then by locally redefining the sites, the projective representation from $\mathcal{P}_k(g)$ and the effective boundary symmetry of the 1D SPT phases form a linear representation on the composite site.

Therefore, the effective boundary symmetry action in Eq. (35) is non-anomalous, and by the universality of the anomalous boundary symmetry action (see Appendix A), $|\psi_{\mathcal{S}}\rangle$ cannot be a member of a non-trivial group cohomology SPT phase in $D \geq 2$. This contradicts the initial assumption and implies that there are no stabilizer states in non-trivial group cohomology SPT phases in $D \geq 2$ dimensions protected by a symmetry represented by Pauli strings. Thus, the non-trivial SPT states described in the proposition have symmetry-protected magic. □

While we have shown that a large class of SPT states have symmetry-protected magic, there are notable examples of SPT states without symmetry-protected magic. For example, the cluster state, described in Section II D, is a stabilizer state. In this case, the anomalous boundary symmetry action corresponds to projective representations, and there is no obstruction to forming projective representations with Pauli strings. As another example,

the ground state of the 2D CZX-model in Ref. [36] is a stabilizer state, but the onsite symmetry is not represented by a product of Pauli operators.

There are also well-known examples of stabilizer states in non-trivial SPT phases protected by subsystem symmetries (e.g. the 2D cluster state) [53, 54] or protected by higher-form symmetries (e.g. the 3D cluster state) [21, 55–57]. In our argument, we assumed that the protecting symmetry is a 0-form symmetry, i.e., it is supported on a codimension-0 manifold. The assessment of the anomalous nature of the effective boundary symmetry action was specific to 0-form SPT phases. We expect that the proposition can be generalized by accounting for the anomalies associated to subsystem SPT phases and higher-form SPT phases, as described in Refs. [53] and [58]. Evidently, in some cases, the anomalous boundary symmetry action of these SPT phases can be described by Pauli operators.

We have qualified that Proposition 1 applies to SPT phases classified by group cohomology, but our results may be more general. There are, in fact, known SPT phases in dimensions $D \geq 3$ that are outside of the group cohomology classification – aptly named the beyond cohomology phases [34, 59]. In dimension $D = 3$, there is a beyond cohomology phase protected by time-reversal symmetry that admits a stabilizer representation [60]. However, this SPT phase falls outside of the purview of our argument, since the symmetry is anti-unitary and is not represented by Pauli strings. On the other hand, in $D = 4$, there is a beyond cohomology SPT phase protected by a unitary $\mathbb{Z}_2$ symmetry represented by a Pauli string [61], and we expect the proof of Proposition 1 applies in this case. Indeed, it was recently argued that the effective boundary symmetry action of the SPT phase corresponds to a non-trivial 3D quantum cellular automaton [61, 62]. The operator in Eq. (35) is certainly a trivial quantum cellular automaton.

IV. SYMMETRY-PROTECTED SIGN PROBLEM

The sign problem is a notorious obstacle in efficiently simulating many-body quantum systems using Monte Carlo methods. Often, the sign problem refers to a difficulty in writing the partition function of a quantum system as a classical partition function with non-negative Boltzmann weights. Here, however, our focus is on a sign problem that manifests in the sign structure of a quantum state [5], i.e., in the complex amplitudes of a wave function. Therefore, to get started, we state the sign problem of interest. Then, we define a symmetry-constrained variation of the sign problem, which we call the symmetry-protected sign problem. We illustrate this concept by showing that a subset of SPT states exhibits a symmetry-protected sign problem.

A. Definition of symmetry-protected sign problem

Complex probability amplitudes are a key feature of quantum states and are essential for describing non-classical phenomena such as quantum interference. For this reason, a non-negative wave function can be regarded as more classical, and indeed, the amplitudes of a non-negative wave function correspond to (the square root of) a classical probability distribution. Whether a state has non-negative amplitudes, however, is basis dependent, i.e., it may be possible to remove a complex sign structure by making a local basis change. This motivates defining the following sign problem at the level of probability amplitudes:

Definition 3 (Sign problem) *A state $|\psi\rangle$ has a sign problem relative to a basis $\{|\alpha\rangle\}$, if, for any finite-depth quantum circuit $\mathcal{U}$, at least one amplitude of the state $\mathcal{U}|\psi\rangle$ in the basis $\{|\alpha\rangle\}$ is outside of the set $\mathbb{R}_{\geq 0}$.*

It is natural to take the basis $\{|\alpha\rangle\}$ to be the computational basis and to interpret the FDQC $\mathcal{U}$ as a local basis change – then, a state has a sign problem if there is no local basis in which the amplitudes of the state are all non-negative. We make the basis $\{|\alpha\rangle\}$ explicit here to more readily generalize to a symmetry-protected sign problem below.

It remains an open question as to whether any many-body system exhibits a sign problem in the sense above. We note that, in Refs. [16–18], it is shown that certain topological phases of matter have an obstruction to finding a parent Hamiltonian that is stoquastic - i.e., where the off-diagonal matrix elements of the Hamiltonian are all non-positive [63, 64]. While a stoquastic parent Hamiltonian is sufficient to guarantee that the ground state is non-negative, it is not necessary. Nonetheless, it is natural to conjecture that these same phases of matter exhibit a sign problem related to the sign structure of a ground state wave function.

Notably, SPT states do not have a sign problem. This is because SPT states can be disentangled into a product state by applying a FDQC. However, we consider a symmetry-protected variant of the sign problem, and show that certain SPT states indeed exhibit a *symmetry-protected* sign problem, defined as:

Definition 4 (Symmetry-protected sign problem) *A state $|\psi\rangle$ has a symmetry-protected sign problem relative to a basis $\{|\alpha\rangle\}$, if, for any finite-depth quantum circuit $\mathcal{U}_{sym}$ composed of symmetric gates, at least one amplitude of the state $\mathcal{U}_{sym}|\psi\rangle$ in the basis $\{|\alpha\rangle\}$ is outside of the set $\mathbb{R}_{\geq 0}$.*

In other words, relative to the reference basis $\{|\alpha\rangle\}$, there are no symmetry-preserving local basis changes that make the wave function non-negative. With this simplification of the sign problem to symmetry-preserving basis changes, we are able to show that particular SPT phases have a symmetry-protected sign problem.

B. Symmetry-protected sign problem for SPT states

In this section, we argue that SPT states in dimensions $D \leq 2$ have a symmetry-protected sign problem in the symmetry-charge basis, where a symmetry-charge basis is a basis of product states in which the symmetry is diagonal. Our proof relies on strange correlators, defined in Section II C, and before stating the main proposition, we would like to emphasize:

Remark: Every non-trivial 1D SPT state, as defined in Section II A, has a strange order parameter with constant strange correlations. This is a consequence of the existence of string order parameters, described in Section II D. As for higher dimensional SPT phases, it is conjectured that every non-trivial SPT state in 2D has a strange order parameter that has power law decaying or constant strange correlations (see the argument in Section II C).

Proposition 2 *Let $|\psi_{SPT}\rangle$ be a state belonging to an SPT phase such that, for every state in the SPT phase, there exists a strange order parameter with power law decaying or constant strange correlations. Then $|\psi_{SPT}\rangle$ has a symmetry-protected sign problem relative to product state bases in which the symmetry is represented by products of diagonal Pauli operators.*

Proof of Proposition 2. Without loss of generality, we assume the symmetry is represented by X-type Pauli strings, which are diagonal in the X-basis.

We derive a contradiction by assuming, contrary to the proposition, that $|\psi_{\text{SPT}}\rangle$ *does not* have a symmetry-protected sign problem relative to the X-basis. This means that there is some FDQC $\mathcal{U}_{\text{sym}}$ composed of symmetric gates such that all of the amplitudes of $\mathcal{U}_{\text{sym}}|\psi_{\text{SPT}}\rangle$ are non-negative in the X-basis. We denote $\mathcal{U}_{\text{sym}}|\psi_{\text{SPT}}\rangle$ by $|\varphi_{\text{SPT}}\rangle$, and note that, according to the definition of SPT phases, it belongs to the same SPT phase as $|\psi_{\text{SPT}}\rangle$.

By assumption, there exists a strange order parameter $\{\mathcal{O}_k\}$ for which the strange correlator:

$$\frac{\langle + \ldots + | \mathcal{O}_i \mathcal{O}_j | \varphi_{\text{SPT}} \rangle}{\langle + \ldots + | \varphi_{\text{SPT}} \rangle} \tag{36}$$

decays according to a power law or is constant in the separation of i and j.[11] Here, $|+\ldots+\rangle$ is the symmetric product state with a $+1$ eigenvalue under every Pauli X operator.[12] In fact, as shown in Appendix C, the strange order parameter can always be chosen to be a set of Pauli Z strings $\{Q_k^Z\}$. In what follows, we argue that the existence of this strange order parameter is in conflict with the non-negative amplitudes of the SPT state $|\varphi_{\text{SPT}}\rangle$.

To see this, we consider $\{Q_k^Z\}$ as a spontaneous-symmetry breaking order parameter for $|\varphi_{\text{SPT}}\rangle$:[13]

$$\langle \varphi_{\text{SPT}} | Q_i^Z Q_j^Z | \varphi_{\text{SPT}} \rangle. \tag{37}$$

We evaluate the correlator by expanding $\langle \varphi_{\text{SPT}}|$ in the X-basis. We label an X-basis state by x and denote the non-negative amplitudes of $\langle \varphi_{\text{SPT}}|$ by $\sqrt{p(x)}$, where $p(x) \in \mathbb{R}_{\geq 0}$. $\langle \varphi_{\text{SPT}}|$ can then be written as:

$$\langle \varphi_{\text{SPT}}| = \sum_x \sqrt{p(x)} \langle x|. \tag{38}$$

Inserting this into Eq. (37) gives us:

$$\begin{aligned} \sum_x \sqrt{p(x)} \langle x | Q_i^Z Q_j^Z | \varphi_{\text{SPT}} \rangle = \\ \sqrt{p(+\ldots+)} \langle + \ldots + | Q_i^Z Q_j^Z | \varphi_{\text{SPT}} \rangle \\ + \sum_{x \neq +\ldots+} \sqrt{p(x)} \langle x | Q_i^Z Q_j^Z | \varphi_{\text{SPT}} \rangle. \end{aligned} \tag{39}$$

The first term on the right-hand side of Eq. (39) is proportional to the strange correlator with strange order parameter $\{Q_k^Z\}$. Consequently, it decays slowly with the separation of i and j, i.e., it decays according to a power law or is constant in $|i-j|$. The second term on the right-hand side of Eq. (39) is non-negative, because the amplitudes of $|\varphi_{\text{SPT}}\rangle$ are non-negative and Q_i^Z, Q_j^Z preserve non-negativity (in the X-basis).

From Eq. (39), we see that $|\varphi_{\text{SPT}}\rangle$ has long-range order. This contradicts the assumption that $|\varphi_{\text{SPT}}\rangle$ is an SPT state with short-range entanglement. Therefore, $|\psi_{\text{SPT}}\rangle$ must have a symmetry-protected sign problem relative to the X-basis, where the symmetry is represented by products of diagonal Pauli operators. □

For a concrete application of Proposition 2, we can consider the cluster state, discussed in Section II D. The $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry of the cluster state is represented by Pauli X operators, and therefore, the cluster state has a symmetry-protected sign problem relative to the X-basis. This is to say, in the symmetry-charge basis, at least one amplitude of the cluster state is outside of $\mathbb{R}_{\geq 0}$, and moreover, there are no symmetry-preserving local basis changes from the X-basis that make all of the amplitudes of the cluster state non-negative.

[11] Note that if $|+\ldots+\rangle$ and $|\varphi_{\text{SPT}}\rangle$ are orthogonal, we can always construct a state $|\varphi'_{\text{SPT}}\rangle$ from $|\varphi_{\text{SPT}}\rangle$ by applying pairs of Pauli Z operators, such that $|\varphi'_{\text{SPT}}\rangle$ has a non-zero overlap with $|+\ldots+\rangle$, still belongs to the same phase as $|\psi_{\text{SPT}}\rangle$, and has non-negative amplitudes in the X-basis. (Pauli Z operators are positivity preserving in the X-basis.) Our argument can then be applied to $|\varphi'_{\text{SPT}}\rangle$ instead of $|\varphi_{\text{SPT}}\rangle$.

[12] Despite the notation $|+\ldots+\rangle$, the argument holds for systems of qudits.

[13] Note that Q_i^Z and Q_j^Z are charged, so the following matrix elements vanish: $\langle + \ldots + | Q_i^Z | \varphi_{\text{SPT}} \rangle = \langle + \ldots + | Q_j^Z | \varphi_{\text{SPT}} \rangle = 0$.

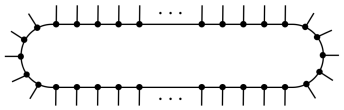
We note that, although the cluster state has a symmetry-protected sign problem relative to the X-basis, there still exists product state bases in which the amplitudes of the cluster state are non-negative. In particular, the amplitudes of the cluster state are non-negative if the Z-basis is used on even sites and the X-basis is used on odd sites.[14] Proposition 2 does not apply in this case, because, in this basis, the symmetry is not diagonal, and it cannot be mapped to the X-basis by symmetry-preserving local transformations.[15]

To highlight the quantum nature of states with a symmetry-protected sign problem, we would like to describe an alternative proof of Proposition 2 that applies to 1D SPT states. The idea is to make use of the quantum wire property, where, for certain 1D SPT states, measurements in the symmetry-charge basis can be used to generate long-range entanglement [22, 65, 66]. We argue that, if a state can serve as a quantum wire and it is non-negative in the symmetry-charge basis, then it contradicts the results of Ref. [5], in which a bound is set on the entanglement created by making measurements of a non-negative state. More formally, we show the following:

Proposition 2′ *Let $|\psi_{SPT}\rangle$ be a state belonging to a 1D SPT phase protected by an Abelian symmetry and corresponding to a maximally non-commutative cohomology class.*[16] *Then $|\psi_{SPT}\rangle$ has a symmetry-protected sign problem relative to product state bases in which the symmetry is represented by products of diagonal unitaries.*

Remark: Here, we make the technical assumption that $|\psi_{\rm SPT}\rangle$ belongs to an SPT phase labeled by a maximally non-commutative cohomology class to guarantee that $|\psi_{\rm SPT}\rangle$ exhibits the quantum wire property [65]. We note that we are also working under the assumption that $|\psi_{\rm SPT}\rangle$ is defined on a lattice with periodic boundary conditions and can be prepared exactly from a product state by a FDQC, as established in Section II A.

Proof of Proposition 2′. First, we use the results of Ref. [65] to show that long-range entanglement can be generated from measurements of $|\psi_{\rm SPT}\rangle$. To see this, we consider a matrix product state (MPS) representation of $|\psi_{\rm SPT}\rangle$:

$|\psi_{\rm SPT}\rangle =$  . (40)

[14] In fact, it can be checked that the Hamiltonian is stoquastic in this basis.

[15] Specifically, the bases are related by applying Hadamard gates on the even sites. Hadamard gates do not commute with the symmetry formed by products of Pauli X operators.

[16] A cohomology class $[\omega] \in H^2(G, U(1))$ is maximally non-commutative if for every element $g \in G$ other than the identity, there exists an $h \in G$ such that $\omega(g,h) \neq \omega(h,g)$.

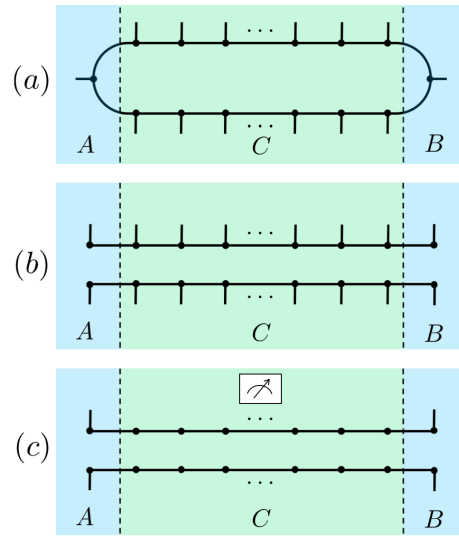
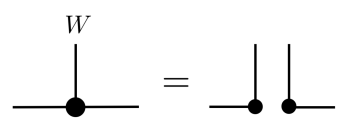


FIG. 4. (a) We partition the coarse grained MPS into regions A, B, and C by choosing super-sites A and B. (b) Applying the isometry $W_A \otimes W_B$ to $|\psi_{\rm SPT}\rangle$ splits it into two unentangled MPS: $|\psi^1_{\rm SPT}\rangle$ and $|\psi^2_{\rm SPT}\rangle$. (c) The measurement on the super-sites in C fixes the physical indices in the region C according to the measurement outcome $|x\rangle$ and leaves us with the state $|\psi^x_{AB}\rangle$ on $A \cup B$. $|\psi^x_{AB}\rangle$ is entangled between A and B through the virtual bonds, as described in Ref. [65].

We then coarse grain the lattice by combining a constant number of neighboring sites into super-sites, such that, for each local tensor of the coarse grained MPS, there exists an isometry W that graphically satisfies:



. (41)

Here, W is an isometry that maps from the d-dimensional physical Hilbert space to a pair of Hilbert spaces of dimension χ_L and χ_R, where χ_L and χ_R are the dimensions of the left and right virtual Hilbert spaces, respectively. Importantly, W disentangles the states in the left virtual Hilbert space from the states in the right virtual Hilbert space. Heuristically, W_A at the super-site A can be interpreted as first acting with a unitary operator supported on A that locally disentangles $|\psi_{\rm SPT}\rangle$ and then subsequently removing the unentangled degrees of freedom.

The next step is to choose well-separated super-sites A and B and make measurements in the symmetry-charge basis on the complement of $A \cup B$, denoted by C (Fig. 4). We claim that measurements in the symmetry-charge basis on C generate entanglement between A and B lower bounded by a value that is independent of the separation of A and B. To show this, we apply the isometry $W_A \otimes W_B$ to $|\psi_{\rm SPT}\rangle$, with W_A and W_B defined as in Eq. (41). The isometry splits $|\psi_{\rm SPT}\rangle$ into two independent MPS, as shown in Fig. 4:

$$W_A \otimes W_B |\psi_{\rm SPT}\rangle = |\psi^1_{\rm SPT}\rangle \otimes |\psi^2_{\rm SPT}\rangle. \qquad (42)$$

Note that, after applying the isometry $W_A \otimes W_B$, the degrees of freedom at A and B correspond to the virtual bonds of the MPS, up to local positive diagonal operators (see Cororally 3.12 of Ref. [67]). This is important given that the results of Ref. [65] show that measurements of an SPT state create entanglement at the level of the virtual bonds.

We now measure the sites in C in the symmetry-charge basis and with probability p_x obtain the product state $|x\rangle$ on C. We define $|\psi^x_{AB}\rangle$ to be the state on $A \cup B$ given by fixing the degrees of freedom of $|\psi^1_{\mathrm{SPT}}\rangle \otimes |\psi^2_{\mathrm{SPT}}\rangle$ on C according to the product state $|x\rangle$. By the assumption that $|\psi_{\mathrm{SPT}}\rangle$ belongs to an SPT phase corresponding to a maximally non-commutative cohomology class, Theorem 1 of Ref. [65] tells us that $|\psi^x_{AB}\rangle$ can be written in the form:

$$\begin{aligned} |\psi^x_{AB}\rangle = &\left(|\varphi^1_{\mathrm{junk}}\rangle \otimes U^{x,1}_B |\Phi^1_{\max}\rangle\right) \\ &\otimes \left(|\varphi^2_{\mathrm{junk}}\rangle \otimes U^{x,2}_B |\Phi^2_{\max}\rangle\right). \end{aligned} \tag{43}$$

Here, $|\varphi^1_{\mathrm{junk}}\rangle$,$|\varphi^2_{\mathrm{junk}}\rangle$ are unimportant states that depend on the details of $|\psi_{\mathrm{SPT}}\rangle$, $|\Phi^1_{\max}\rangle$,$|\Phi^2_{\max}\rangle$ are $\sqrt{|G|}$-dimensional maximally entangled states between A and B, and $U^{x,1}_B$,$U^{x,2}_B$ are some unitary operators supported only on B. Note that the first and second lines of Eq. (43) correspond to independent contributions from $|\psi^1_{\mathrm{SPT}}\rangle$ and $|\psi^2_{\mathrm{SPT}}\rangle$. For any measurement outcome $|x\rangle$, the entanglement entropy of $|\psi^x_{AB}\rangle$ between A and B is therefore bounded below as:

$$S\left(\rho^x_A\right) \geq 2\log_2 \sqrt{|G|}, \tag{44}$$

with ρ^x_A denoting the reduced density matrix of $|\psi^x_{AB}\rangle$ on A. Since $W_A \otimes W_B$ has no affect on the entanglement generated between A and B, we see that making measurements of $|\psi_{\mathrm{SPT}}\rangle$ on C induces entanglement between A and B with a constant lower bound, as claimed.

On the other hand, Proposition 4.1 of Ref. [5] implies that, if $|\psi_{\mathrm{SPT}}\rangle$ is non-negative in the symmetry-charge basis, then the average entanglement entropy after the measurements is bounded from above as:

$$\sum_x p_x S\left(\rho^x_A\right) \leq f(L), \tag{45}$$

where L is the distance between the super-sites A and B, and $f(L)$ is a function that decays rapidly to zero (faster than any polynomial). For a sufficiently large L, the bound in Eq. (45) conflicts with Eq. (44).

Therefore, $|\psi_{\mathrm{SPT}}\rangle$ cannot be non-negative in the symmetry-charge basis. Furthermore, the argument applies to any state constructed from $|\psi_{\mathrm{SPT}}\rangle$ by a FDQC composed of symmetric gates, since the quantum wire property is shared by states in the same phase. We can thus conclude that $|\psi_{\mathrm{SPT}}\rangle$ has a symmetry-protected sign problem relative to the symmetry-charge basis. □

V. DISCUSSION

We have introduced the concepts of symmetry-protected magic and a symmetry-protected sign problem to facilitate the study of many-body magic and the sign structure of wave functions. We have applied these concepts to SPT states to assess their quantum complexity. Using the universal properties of non-trivial group cohomology phases in $D \geq 2$ dimensions, we showed that the corresponding SPT states have symmetry-protected magic, assuming the symmetry is represented by products of Pauli operators. This implies that there is no stabilizer state representative in these SPT phases. We also argued that SPT states in dimensions $D \leq 2$ have a symmetry-protected sign problem in bases where the symmetry is diagonal. Consequently, in this basis, there is an obstruction to a description of the SPT phase by a non-negative wave function.

By imposing symmetry constraints, we were able to make analytic statements about the complexity of quantum states, including the first verification of a sign problem at the level of probability amplitudes. We note that a restriction to symmetric systems has also been beneficial for studying the No Low-energy Trivial States conjecture in Ref. [68]. We anticipate that, moving forwards, symmetry constraints will be a valuable tool for addressing outstanding quantum information problems.

We would like to emphasize that our assessment of the symmetry-protected magic in SPT phases is independent of the dimensions of the qudits. This is noteworthy given that magic is more easily quantified and better understood in systems of qudits with odd dimensions (or sometimes just odd prime dimensions), thanks to the discrete Wigner formalism [69, 70]. The associated discrete Wigner function maps states to quasi-probabilities, and for systems of odd dimensional qudits, the negative quasi-probabilities can be used to define a measure of the amount of magic in the state [3].

In the case that the qudits are odd dimensional, symmetry-protected magic can be interpreted as a sign problem, which manifests through the quasi-probability distribution of the discrete Wigner function (known as the discrete Hudson's theorem) [70]. Symmetry-protected magic says that the signs in the quasi-probability distribution cannot be removed by making symmetry preserving unitary local changes to the state. We point out that this sign problem has appeared in the simulation of random quantum circuits as in Ref. [71].

Our work therefore deals with two different notions of a sign problem – one relates to the quasi-probability distribution of a discrete Wigner function, while the other corresponds to the complex probability amplitudes of a state. It should be noted that these are distinct from the usual notion of a sign problem related to the "stoquasticity" of a Hamiltonian and discussed in the context of simulating quantum systems by Monte Carlo methods. However, a sign problem at the level of the amplitudes of a wave function, in fact, implies that any gapped parent

Hamiltonian suffers from a sign problem in the stoquastic sense (see Ref. [5] and Appendix A of Ref. [18]).

To conclude, we would like to further comment on related work, make a few conjectures, and discuss some promising directions for future work.

Symmetry-protected magic:

Proposition 1 shows that stabilizer operations are insufficient for simulating certain SPT states. It is important to note that this also implies that those SPT states can be used as a source of magic for quantum computing. Moreover, our results show that magic is present in every state belonging to group cohomology SPT phases in dimensions $D \geq 2$ (protected by a symmetry represented by Pauli strings). It would be interesting if a quantized universal property of these SPT phases - say, their responses to probing with symmetry defects - could be exploited to reliably produce a standard magic state (e.g. a CCZ state), independent of the microscopic details of the systems. We also speculate that there is a series of adaptive measurements that produces a standard magic state on a large length scale, similar to how a series of local measurements of 1D SPT states can create entanglement between distant sites [66].

It is also interesting to consider the implications of our work for the use of group cohomology SPT states as resources for measurement based quantum computing (MBQC). Remarkable progress has been made in identifying computationally universal phases of matter protected by subsystem symmetries [23, 24], but much remains to be understood about the MBQC utility of SPT phases with global (0-form) symmetries. In Refs. [72] and [73], it was recognized that a particular "fixed point" wave function in a $2D$ $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ group cohomology SPT phase harbors magic. Furthermore, it was shown that the state can be used as a resource for universal MBQC using only Pauli measurements. It is natural to wonder whether the entire phase can be used for universal MBQC with Pauli measurements. Our results are notably consistent with this conjecture and suggest that other group cohomology SPT states may be able to serve as universal resources just as well. For a related discussion on the quantification of magic, see Ref. [15].

According to Proposition 1, certain SPT states must be non-stabilizer and, as such, cannot be prepared from $|0\ldots0\rangle$ by a Clifford unitary. Inspired by Ref. [74], we speculate that the higher levels of the Clifford hierarchy may also be useful for understanding the complexity of SPT states. The first level of the hierarchy is the set of Pauli strings $\mathcal{C}_1$, and the higher levels of the hierarchy are obtained recursively as:

$$\mathcal{C}_{D+1} \equiv \left\{U : UPU^\dagger \in \mathcal{C}_D, \forall P \in \mathcal{C}_1\right\}. \tag{46}$$

The Clifford unitaries form the second level of the Clifford hierarchy $\mathcal{C}_2$. We conjecture that SPT states belonging to non-trivial group cohomology SPT phases in D dimensions (protected by symmetries represented by Pauli strings) cannot be prepared from $|0\ldots0\rangle$ by any FDQC in the Dth level of the hierarchy. Ref. [74] showed that this is the case for a specific FDQC that prepares a fine tuned representative SPT state.

In this text, we focused on the magic in quantum phases characterized by SRE states, but an important avenue for future work is to study magic in systems with long-range entanglement, such as in conformal field theories (CFTs) and intrinsic topological orders. Refs. [13] and [14] have made the first steps in numerically studying the emergence of magic at a critical point, and Ref. [13] conjectured that CFTs generically have magic at large length scales, detectable by correlations.

Regarding the magic inherent in topologically ordered phases, Ref. [75] provided a classification of systems in 2D that can be described by a local stabilizer Hamiltonian, assuming the stabilizer Hamiltonian is translationally invariant and defined on a Hilbert space built of prime dimensional qudits. These results place important restrictions on the phases of matter that admit a representation by a stabilizer state. However, more work is needed to lift the assumptions and better understand long-range magic (Definition 1) in phases with intrinsic topological order. We conjecture that the states in the double semion phase, for example, have long-range magic, and we look forward to commenting further on this conjecture in upcoming work.

Symmetry-protected sign problem:

We argued that non-trivial SPT states in dimensions $D \leq 2$ have a symmetry-protected sign problem relative to the symmetry-charge basis, where the symmetry is diagonal. It is unclear whether these SPT states have a symmetry-protected sign problem relative to other bases, as our techniques are specialized for the symmetry-charge basis. For instance, does the cluster state have a symmetry-protected sign problem relative to the Z-basis? A complete characterization of the symmetry-protected sign problems might lead to new tools useful for tackling the sign problem in the absence of symmetry constraints. New techniques are also needed to study the symmetry-protected sign problem in SPT states in dimensions $D \geq 3$, since the strange correlations may no longer be a reliable way to diagnose the SPT order.

In Section III C, we also argued that the quantum wire property of non-trivial 1D SPT states is incompatible with a non-negative wave function in the symmetry-charge basis. This suggests a potential operational consequence of a symmetry-protected sign problem. In particular, for non-trivial 1D SPT states, entanglement can be generated between any two regions by making measurements on the complement in the symmetry-charge basis. We speculate that, more generally, a symmetry-protected sign problem relative to a basis $|\{\alpha\}\rangle$ implies that measurements in the $|\{\alpha\}\rangle$ basis can be used to create entanglement between distant regions. In any event, further work is needed to build off of the results of Ref. [5]

and to fully understand the connection between the sign structure of a quantum state and its localizable entanglement [76].

Acknowledgements – We would like to acknowledge Tomotaka Kuwahara for helpful conversations related to Proposition 2, and we thank Sergey Bravyi for insights that led us to Lemma 2. TDE thanks Kyle Kawagoe, Alex Kubica, and Beni Yoshida for clarifying discussions, and he is especially grateful to Lukasz Fidkowski and Nathanan Tantivasadakarn for providing feedback on the manuscript. TDE is also appreciative of the hospitality of Perimeter Institute, where this work was initiated. ZWL is supported by Perimeter Institute for Theoretical Physics. Perimeter Institute is supported in part by the Government of Canada through the Department of Innovation, Science and Economic Development Canada and by the Province of Ontario through the Ministry of Economic Development, Job Creation and Trade. KK is supported by MEXT Quantum Leap Flagship Program (MEXT Q-LEAP) Grant Number JPMXS0120319794. TH acknowledges support from the Natural Sciences and Engineering Research Council of Canada (NSERC) through a Discovery Grant.

Appendix A: Universality of the anomalous symmetry action

In Section II B, we stated that group cohomology SPT phases can be characterized by anomalies – i.e., obstructions to finding an effective boundary symmetry action that is onsite.[17] The calculation of the anomaly, as described in Section II B, (seemingly) depends on the following choices: (i) a representative SPT state, (ii) a parent SPT Hamiltonian for the SPT state, and (iii) an effective boundary symmetry action derived from the parent Hamiltonian. Following Appendix C of Ref. [29], we sketch an argument below that the computation of the anomaly ultimately does not depend on these choices. In other words, we argue that the anomaly is well-defined as an invariant of the SPT phase. The strategy is to study the anomaly at the interface between two different possible choices of (i), (ii), and (iii). To simplify the discussion, we assume that the parent Hamiltonians are un-frustrated. (This is sufficient for our purposes in the main text.) We expect that the argument can be generalized to show that the anomaly is well-defined for any choice of parent SPT Hamiltonian.

We consider two states $|\psi_1\rangle$ and $|\psi_2\rangle$ belonging to the same D-dimensional SPT phase along with a choice of corresponding SPT Hamiltonians H_1 and H_2. For concreteness, we take $|\psi_1\rangle$ and $|\psi_2\rangle$ to be defined on a D-sphere S^D. Since $|\psi_1\rangle$ and $|\psi_2\rangle$ are in the same phase, there exists a FDQC $\mathcal{U}_{\text{sym}}$ composed of symmetric gates such that:

$$\mathcal{U}_{\text{sym}}|\psi_2\rangle = |\psi_1\rangle. \tag{A1}$$

We can then construct the SPT Hamiltonian $\tilde{H}_2$, defined as:

$$\tilde{H}_2 \equiv \mathcal{U}_{\text{sym}} H_2 \mathcal{U}_{\text{sym}}^\dagger, \tag{A2}$$

which has the unique ground state $|\psi_1\rangle$.

Now, we combine terms from H_1 and $\tilde{H}_2$ to form a new SPT Hamiltonian H on S^D whose ground state is also $|\psi_1\rangle$. Roughly speaking, H is comprised of the terms in H_1 on the left half of the sphere and the terms in $\tilde{H}_2$ on the right half. More specifically, we divide the D-sphere into two overlapping regions L_+ and R_-, as shown in Fig. 5. L_+ and R_- cover the D-sphere, and their intersection is a thickened $(D-1)$-sphere. We take the "width" of the intersection to be large compared with the Lieb-Robinson length of some (arbitrary) FDQC that prepares $|\psi_1\rangle$. To construct H, we truncate H_1 to L_+, to define $H_1^{L_+}$, and we truncate $\tilde{H}_2$ to R_-, to define $\tilde{H}_2^{R_-}$. The SPT Hamiltonian H is then:

$$H \equiv H_1^{L_+} + \tilde{H}_2^{R_-}, \tag{A3}$$

with the unique ground state $|\psi_1\rangle$. The fact that $|\psi_1\rangle$ is the ground state follows from the assumption that H_1 and H_2 are un-frustrated.

Next, we study the possible anomaly at the interface between H_1 and $\tilde{H}_2$. In particular, we introduce a boundary by truncating H to the region $L \cup R$, where L and R are defined as:

$$L \equiv L_+ - L_+ \cap R_-, \quad R \equiv R_- - L_+ \cap R_-, \tag{A4}$$

as depicted in Fig. 5. In other words, we remove any term in H that is supported (in part) on the overlap between L_+ and R_-. We are left with the truncated Hamiltonian $H^{L\cup R}$:

$$H^{L\cup R} \equiv H_1^L + \tilde{H}_2^R, \tag{A5}$$

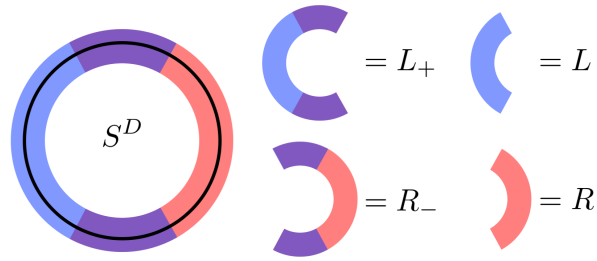


FIG. 5. $|\psi_1\rangle$ and $|\psi_2\rangle$ are defined on a D-sphere S^D. We partition S^D into overlapping regions L_+ (union of blue and purple) and R_- (union of red and purple). L_+ contains the subregion L (blue), and R_- contains the subregion R (red).

[17] More specifically, the obstructions are to finding an effective boundary symmetry action that is onsite up to taking tensor products with effective actions of lower-dimensional SPT phases and by conjugating the effective symmetry action with a FDQC.

where H_1^L is the truncation of H_1 to L and $\tilde{H}_2^R$ is the truncation of $\tilde{H}_2$ to R. Importantly, we consider $H^{L\cup R}$ as a Hamiltonian on the full Hilbert space of S^D.

The boundary Hilbert space of $H^{L\cup R}$ can be decomposed into a tensor product of the following three Hilbert spaces: (i) the low-energy Hilbert space of H_1^L on L, (ii) the full Hilbert space on the intersection $L_+ \cap R_-$, and (iii) the low-energy Hilbert space of $\tilde{H}_2^R$ on R. Accordingly, we can construct an effective symmetry action near the boundary of $L \cup R$ by multiplying an effective action on L, an onsite symmetry on $L_+ \cap R_-$, and an effective action on R. More explicitly, we can choose the effective boundary symmetry action representing $g \in G$ to be $v^L(g)$ on L and $v^R(g)$ on R, so that an effective boundary symmetry action $v(g)$ on $L \cup R$ is:

$$v(g) \equiv v^L(g) \left(\prod_{i \in L_+ \cap R_-} u_i(g) \right) v^R(g). \tag{A6}$$

We note that the effective boundary symmetry action in Eq. (A6) is localized near $L_+ \cap R_-$.

The algorithm defined in Ref. [29] can now be applied to $v(g)$ to identify potential obstructions to making $v(g)$ onsite. The obstruction corresponds to an element $[\omega] \in H^{D+1}(G, U(1))$, and one can show that it can be divided into a contribution $[\omega^L] \in H^{D+1}(G, U(1))$ from $v^L(g)$ and a contribution $[\omega^R] \in H^{D+1}(G, U(1))$ from $v^R(g)$, so that:

$$[\omega] = [\omega^L] \cdot [\omega^R]. \tag{A7}$$

The last step is to argue that $[\omega]$ calculated from $v(g)$ using the procedure in Ref. [29] must correspond to the trivial class in $H^{D+1}(G, U(1))$. Therefore, regardless of the choices made in determining $v^L(g)$ and $v^R(g)$, we have $[\omega^L] = [\omega^R]^{-1}$. This constraint implies that the anomaly is well-defined, since $v^L(g)$ and $v^R(g)$ can be chosen independently. For simplicity, we show that $[\omega]$ is the trivial class for only the 1D case. We note that the 2D case can be found in Appendix C of Ref. [29], and we fully expect that the argument can be generalized straightforwardly to higher dimensions.

To show that $[\omega]$ must be trivial in the 1D case, we consider the state $|\psi_1\rangle$. Since $|\psi_1\rangle$ belongs to the boundary Hilbert space of $H^{L\cup R}$, the symmetry action $u(g)$, for any $g \in G$, can be replaced by $v(g)$ when acting on $|\psi_1\rangle$. Therefore, we have the equality:

$$v(g)|\psi_1\rangle = |\psi_1\rangle, \quad \forall g \in G. \tag{A8}$$

In 1D, the support of $v(g)$ can be partitioned into two connected components, which we label as A and B. Consequently, $v(g)$ can be split[18] into an operator $v_A(g)$ supported on A and $v_B(g)$ supported on B. From Eq. (A8), we have:

$$v_A(g) v_B(g)|\psi_1\rangle = |\psi_1\rangle, \quad \forall g \in G. \tag{A9}$$

Furthermore, we can always define $v_A(g)$ and $v_B(g)$ so that:

$$v_A(g)|\psi_1\rangle = |\psi_1\rangle, \quad \forall g \in G. \tag{A10}$$

It follows that $v_A(g)$ forms a linear representation of G on $|\psi_1\rangle$:

$$v_A(g) v_A(h)|\psi_1\rangle = |\psi_1\rangle = v_A(gh)|\psi_1\rangle, \quad \forall g, h \in G. \tag{A11}$$

Since $v_A(g)$ forms a trivial projective representation (i.e., a linear representation), the associated element of $H^2(G, U(1))$ must be the trivial class.

Cluster state example:

Using the ideas above, we argue that the cluster state belongs to a non-trivial SPT phase. In particular, the projective representation satisfied by the effective boundary symmetry action poses an obstruction to finding a FDQC with symmetric gates that can disentangle the cluster state. We show this by deriving a contradiction.

Suppose that $|\psi_{\text{CS}}\rangle$ can be disentangled by a FDQC $\mathcal{U}_{\text{sym}}$ composed of symmetric gates:

$$\mathcal{U}_{\text{sym}}|\psi_{\text{CS}}\rangle = |+\ldots+\rangle. \tag{A12}$$

Then the Hamiltonian $\tilde{H}_{\text{CS}} \equiv \mathcal{U}_{\text{sym}} H_{\text{CS}} \mathcal{U}_{\text{sym}}^\dagger$ has the unique product state ground state $|+\ldots+\rangle$. Further, we can identify an effective boundary symmetry action for $\tilde{H}_{\text{CS}}$ by conjugating the effective action computed using H_{CS} [copied from Eq. (15)]:

$$\begin{aligned} v\big((g,1)\big) &= Z_1(Z_{2M-1}X_{2M}), \\ v\big((1,g)\big) &= (X_1 Z_2) Z_{2M}, \end{aligned} \tag{A13}$$

by the FDQC $\mathcal{U}_{\text{sym}}$:

$$\begin{aligned} \tilde{v}\big((g,1)\big) &\equiv \mathcal{U}_{\text{sym}} v\big((g,1)\big) \mathcal{U}_{\text{sym}}^\dagger, \\ \tilde{v}\big((1,g)\big) &\equiv \mathcal{U}_{\text{sym}} v\big((1,g)\big) \mathcal{U}_{\text{sym}}^\dagger. \end{aligned} \tag{A14}$$

Similar to the effective action in Eq. (A13), when $\tilde{v}\big((g,1)\big)$ and $\tilde{v}\big((1,g)\big)$ are restricted to a region near either the endpoint 1 or $2M$, they form a projective representation of $\mathbb{Z}_2 \times \mathbb{Z}_2$, corresponding to the non-trivial element of $H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1))$.

We compare $\tilde{H}_{\text{CS}}$ to the paramagnet SPT Hamiltonian H_0:

$$H_0 \equiv -\sum_i X_i, \tag{A15}$$

which also has $|+\ldots+\rangle$ as its unique ground state. An effective boundary symmetry action computed with respect to H_0 is given by:

$$\begin{aligned} v^0\big((g,1)\big) &\equiv X_2 X_{2M} \\ v^0\big((1,g)\big) &\equiv X_1 X_{2M-1}. \end{aligned} \tag{A16}$$

[18] The operator can be split unambiguously up to a g dependent phase.

The restrictions of $v^0((g,1))$ and $v^0((1,g))$ to an endpoint forms a linear representation of $\mathbb{Z}_2 \times \mathbb{Z}_2$, corresponding to the trivial element of $H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1))$.

We see that the quantized invariant for the SPT phase containing $|+\ldots+\rangle$ computed using $\tilde{H}_{\rm CS}$ differs from the quantized invariant computed using H_0. This contradicts the fact the anomaly is well-defined. Therefore, the cluster state $|\psi_{\rm CS}\rangle$ cannot be disentangled using a FDQC composed of symmetric gates.

Appendix B: Existence of a local stabilizer parent Hamiltonian

In the proof of Proposition 1, we claimed that a stabilizer SPT state admits a local symmetric stabilizer Hamiltonian. Here, we justify this claim. We start by proving a lemma regarding the stabilizer group $\mathcal{G}$, defined in Eq. (29).

Lemma 1 *Let $\mathcal{G}$ be the stabilizer group defined as:*

$$\mathcal{G} \equiv \langle UZ_iU^\dagger : i \in \textit{sites}\rangle, \tag{B1}$$

which uniquely fixes the stabilizer state $|\psi_{\mathcal{S}}\rangle$ constructed from $|0\ldots0\rangle$ by the Clifford unitary U. Then, if a Pauli string P satisfies one of the following, P is contained in $\mathcal{G}$:

(i) P commutes with every element of $\mathcal{G}$

(ii) P stabilizes $|\psi_{\mathcal{S}}\rangle$, i.e., $P|\psi_{\mathcal{S}}\rangle = |\psi_{\mathcal{S}}\rangle$.

Proof of Lemma 1: We make use of the fact that $\mathcal{G}$ can be constructed by conjugating the generators of the stabilizer group $\mathcal{G}_0$ by the Clifford unitary U, where $\mathcal{G}_0$ is:

$$\mathcal{G}_0 \equiv \langle Z_i : i \in \text{sites}\rangle. \tag{B2}$$

If the Pauli string P satisfies (i), then $U^\dagger PU$ is a Pauli string that commutes with every element of $\mathcal{G}_0$. Any Pauli string that commutes with $\mathcal{G}_0$ must be a product of Pauli Z operators. This implies that P is a product of Pauli strings of the form $UZ_iU^\dagger$. Hence, P belongs to $\mathcal{G}$.

If the Pauli string P satisfies (ii), then $U^\dagger PU$ is a Pauli string that leaves $|0\ldots0\rangle$ invariant: $U^\dagger PU|0\ldots0\rangle = |0\ldots0\rangle$. The only Pauli strings that leave $|0\ldots0\rangle$ invariant are products of Pauli Z operators. Therefore, $U^\dagger PU$ is a product of Pauli Z operators, and P is a product of elements in $\mathcal{G}$. □

Now, we can prove the following statement about the existence of a local symmetric stabilizer Hamiltonian.

Lemma 2 *Let $|\psi_{\mathcal{S}}\rangle$ be a stabilizer state which is a unique ground state of a geometrically local Hamiltonian*[19] *H_{loc}. Then, there exists a local stabilizer Hamiltonian $H_{\mathcal{S}}$ such that $|\psi_{\mathcal{S}}\rangle$ is the unique ground state of $H_{\mathcal{S}}$. Furthermore, if $|\psi_{\mathcal{S}}\rangle$ is invariant under a Pauli string P, i.e., $P|\psi_{\mathcal{S}}\rangle = |\psi_{\mathcal{S}}\rangle$, then $H_{\mathcal{S}}$ commutes with P.*

Proof of Lemma 2. Since $|\psi_{\mathcal{S}}\rangle$ is a stabilizer state, there is a stabilizer group $\mathcal{G}$ that uniquely fixes $|\psi_{\mathcal{S}}\rangle$, such as the stabilizer group in Lemma 1. We claim that the generators of $\mathcal{G}$ can always be chosen to be geometrically local. To see this, we imagine minimizing the largest support of the generators over all possible choices for generators of $\mathcal{G}$. We let d_S denote the minimum length such that each stabilizer term can be contained in a ball of diameter d_S. We argue that d_S is constant, i.e., independent of the system size.

If there exists a generator S that is not contained in a constant size ball, then there is a stabilizer state $|\phi_{\mathcal{S}}\rangle$ that has a $+1$ eigenvalue for all the generators except S, for which the eigenvalue is some root of unity (not equal to $+1$). $|\psi_{\mathcal{S}}\rangle$ and $|\phi_{\mathcal{S}}\rangle$ are orthogonal, and yet they have the same reduced density matrices on any constant-size region. The latter follows from the fact that S is not contained in any constant-size region and that the reduced density matrices of stabilizer states on a region M depend solely on the stabilizer group elements whose support is contained in M [77, 78]. (Note that if there is another generator T such that ST is supported in a constant-sized region M, then we can use ST as a generator instead of S. This contradicts the assumption that we have minimized the maximum support of the generators.)

Since $|\psi_{\mathcal{S}}\rangle$ and $|\phi_{\mathcal{S}}\rangle$ have the same reduced density matrices on constant-sized regions, $|\phi_{\mathcal{S}}\rangle$ gives another ground state of $H_{\rm loc}$. This conflicts with the assumption that $H_{\rm loc}$ has a unique ground state. Thus, the support of S must be contained in a constant-size ball, and d_S can be chosen independent of the system size. Therefore, it is possible to find a set of generators for $\mathcal{G}$, which are geometrically local. We define $H_{\mathcal{S}}$ to be the negative sum of the local generators (with their Hermitian conjugates). $|\psi_{\mathcal{S}}\rangle$ is the unique ground state of $H_{\mathcal{S}}$, since the terms of $H_{\mathcal{S}}$ span $\mathcal{G}$.

Lastly, if $|\psi_{\mathcal{S}}\rangle$ is invariant under a Pauli operator P, then P is an element of $\mathcal{G}$, assuming $\mathcal{G}$ is the stabilizer group in Lemma 1. This follows directly from Lemma 1. Since $\mathcal{G}$ is a commuting group, P commutes with $H_{\mathcal{S}}$. □

Appendix C: Simplified strange order parameter

In this appendix, we show that the strange order parameters used in the proof of Proposition 2 can always be chosen to be Z-type Pauli strings.

Lemma 3 *Let $\{\mathcal{O}_k\}$ be a strange order parameter such that the strange correlator:*

$$\frac{\langle +\ldots+|\mathcal{O}_i\mathcal{O}_j|\varphi_{SPT}\rangle}{\langle +\ldots+|\varphi_{SPT}\rangle}, \tag{C1}$$

[19] We recall that a geometrically local Hamiltonian is a sum of terms such that the support of each term can be contained within a ball of fixed finite diameter.

decays slowly, i.e., it decays as a power law or is constant in the separation of i and j. Then, there exists a strange order parameter formed by Z-type Pauli string operators $\{Q_k^Z\}$ that decays slowly in $|i-j|$.

Proof of Lemma 3: The strange order parameter $\{\mathcal{O}_k\}$ can be simplified by commuting the Pauli X operators in $\mathcal{O}_i$ and $\mathcal{O}_j$ to act on $\langle +\ldots+|$ in Eq. (C1). This leaves us with:

$$\frac{\langle +\ldots+|\mathcal{O}_i\mathcal{O}_j|\varphi_{\rm SPT}\rangle}{\langle +\ldots+|\varphi_{\rm SPT}\rangle} = \frac{\langle +\ldots+|\mathcal{Z}_i\mathcal{Z}_j|\varphi_{\rm SPT}\rangle}{\langle +\ldots+|\varphi_{\rm SPT}\rangle}, \quad \text{(C2)}$$

where $\mathcal{Z}_i$ and $\mathcal{Z}_j$ are charged operators generated by sums of Z-type Pauli strings. To be explicit, $\mathcal{Z}_i$ and $\mathcal{Z}_j$ can be written as:

$$\mathcal{Z}_i = \sum_{P^Z} C^i_{P^Z} P^Z, \quad \mathcal{Z}_j = \sum_{P^Z} C^j_{P^Z} P^Z, \quad \text{(C3)}$$

where the sums are over all Z-type Pauli strings P^Z, and $C^i_{P^Z}$ and $C^j_{P^Z}$ are some complex valued coefficients. Note that due to the locality of the $\mathcal{O}_i$ and $\mathcal{O}_j$, $\mathcal{Z}_i$ and $\mathcal{Z}_j$ are localized near i and j, respectively, and consequently, there are only finitely many non-zero coefficients $C^i_{P^Z}$ and $C^j_{P^Z}$. Furthermore, the unitarity of $\mathcal{O}_i$ and $\mathcal{O}_j$, implies $|C^i_{P^Z}|, |C^j_{P^Z}| \leq 1$ for every P^Z. We also note that neither $\mathcal{Z}_i$ nor $\mathcal{Z}_j$ is the identity, since $\mathcal{O}_i$ and $\mathcal{O}_j$ are charged.

With this, we can expand the right-hand side of Eq. (C2) as:

$$\sum_{P_i^Z}\sum_{P_j^Z} C^i_{P_i^Z} C^j_{P_j^Z} \frac{\langle +\ldots+|P_i^Z P_j^Z|\varphi_{\rm SPT}\rangle}{\langle +\ldots+|\varphi_{\rm SPT}\rangle}. \quad \text{(C4)}$$

Given that the strange correlator in Eq. (C1) decays slowly in $|i-j|$, the expression above also decays slowly in $|i-j|$. Furthermore, since there are finitely many non-zero coefficients $C^i_{P_i^Z}, C^j_{P_j^Z}$, all of which are bounded from above, there must be a choice of P_i^Z and P_j^Z such that the expression:

$$\frac{\langle +\ldots+|P_i^Z P_j^Z|\varphi_{\rm SPT}\rangle}{\langle +\ldots+|\varphi_{\rm SPT}\rangle} \quad \text{(C5)}$$

decays slowly with the separation of i and j. Let us denote this choice of P_i^Z and P_j^Z by Q_i^Z and Q_j^Z. The set $\{Q_k^Z\}$ forms a strange order parameter for $|\varphi_{\rm SPT}\rangle$ with strange correlations that decay slowly in the separation of i and j. □

[1] Daniel Gottesman, "The Heisenberg Representation of Quantum Computers," Group22: Proceedings of the XXII International Colloquium on Group Theoretical Methods in Physics, eds. S. P. Corney, R. Delbourgo, and P. D. Jarvis, pp. 32-43 (Cambridge, MA, International Press, 1999).

[2] Sergey Bravyi and Alexei Kitaev, "Universal quantum computation with ideal clifford gates and noisy ancillas," Phys. Rev. A **71**, 022316 (2005).

[3] Victor Veitch, S A Hamed Mousavian, Daniel Gottesman, and Joseph Emerson, "The resource theory of stabilizer quantum computation," New Journal of Physics **16**, 013009 (2014).

[4] Tarun Grover and Matthew P. A. Fisher, "Entanglement and the sign structure of quantum states," Phys. Rev. A **92**, 042308 (2015).

[5] M. B. Hastings, "How quantum are non-negative wavefunctions?" Journal of Mathematical Physics **57**, 015210 (2016).

[6] Giacomo Torlai, Juan Carrasquilla, Matthew T. Fishman, Roger G. Melko, and Matthew P. A. Fisher, "Wave-function positivization via automatic differentiation," Phys. Rev. Research **2**, 032060 (2020).

[7] Matthias Troyer and Uwe-Jens Wiese, "Computational complexity and fundamental limitations to fermionic quantum monte carlo simulations," Phys. Rev. Lett. **94**, 170201 (2005).

[8] Milad Marvian, Daniel A. Lidar, and Itay Hen, "On the computational complexity of curing non-stoquastic hamiltonians," Nature Communications **10**, 1571 (2019).

[9] Joel Klassen, Milad Marvian, Stephen Piddock, Marios Ioannou, Itay Hen, and Barbara Terhal, "Hardness and ease of curing the sign problem for two-local qubit hamiltonians," (2020), arXiv:1906.08800 [quant-ph].

[10] Joel Klassen and Barbara M. Terhal, "Two-local qubit Hamiltonians: when are they stoquastic?" Quantum **3**, 139 (2019).

[11] Ryan Levy and Bryan K. Clark, "Mitigating the sign problem through basis rotations," (2019), arXiv:1907.02076 [cond-mat.str-el].

[12] Dominik Hangleiter, Ingo Roth, Daniel Nagaj, and Jens Eisert, "Easing the monte carlo sign problem," Science Advances **6** (2020), 10.1126/sciadv.abb8341.

[13] Christopher David White, ChunJun Cao, and Brian Swingle, "Conformal field theories are magical," arXiv e-prints , arXiv:2007.01303 (2020), arXiv:2007.01303 [quant-ph].

[14] S Sarkar, C Mukhopadhyay, and A Bayat, "Characterization of an operational quantum resource in a critical many-body system," New Journal of Physics **22**, 083077 (2020).

[15] Zi-Wen. Liu and Andreas Winter, "Many-body quantum magic," to appear.

[16] Zohar Ringel and Dmitry L. Kovrizhin, "Quantized gravitational responses, the sign problem, and quantum complexity," Science Advances **3** (2017), 10.1126/sciadv.1701758.

[17] Omri Golan, Adam Smith, and Zohar Ringel, "Intrinsic sign problem in fermionic and bosonic chiral topological matter," Phys. Rev. Research **2**, 043032 (2020).

[18] Adam Smith, Omri Golan, and Zohar Ringel, "Intrinsic sign problems in topological quantum field theories," Phys. Rev. Research **2**, 033515 (2020).

[19] C. L. Kane and E. J. Mele, "Z_2 topological order and the quantum spin hall effect," Phys. Rev. Lett. **95**, 146802 (2005).

[20] C. L. Kane and E. J. Mele, "Quantum spin hall effect in graphene," Phys. Rev. Lett. **95**, 226801 (2005).

[21] Robert Raussendorf and Hans J. Briegel, "A one-way quantum computer," Phys. Rev. Lett. **86**, 5188–5191 (2001).

[22] Robert Raussendorf, Dong-Sheng Wang, Abhishodh Prakash, Tzu-Chieh Wei, and David T. Stephen, "Symmetry-protected topological phases with uniform computational power in one dimension," Phys. Rev. A **96**, 012302 (2017).

[23] Robert Raussendorf, Cihan Okay, Dong-Sheng Wang, David T. Stephen, and Hendrik Poulsen Nautrup, "Computationally universal phase of quantum matter," Phys. Rev. Lett. **122**, 090501 (2019).

[24] David T. Stephen, Hendrik Poulsen Nautrup, Juani Bermejo-Vega, Jens Eisert, and Robert Raussendorf, "Subsystem symmetries, quantum cellular automata, and computational phases of quantum matter," Quantum **3**, 142 (2019).

[25] Austin K. Daniel, Rafael N. Alexander, and Akimasa Miyake, "Computational universality of symmetry-protected topologically ordered cluster phases on 2D Archimedean lattices," Quantum **4**, 228 (2020).

[26] Trithep Devakul and Dominic J. Williamson, "Universal quantum computation using fractal symmetry-protected cluster phases," Phys. Rev. A **98**, 022332 (2018).

[27] Akimasa Miyake, "Quantum computation on the edge of a symmetry-protected topological order," Phys. Rev. Lett. **105**, 040501 (2010).

[28] Xie Chen, Zheng-Cheng Gu, and Xiao-Gang Wen, "Local unitary transformation, long-range quantum entanglement, wave function renormalization, and topological order," Phys. Rev. B **82**, 155138 (2010), arXiv:1004.3835 [cond-mat.str-el].

[29] Dominic V. Else and Chetan Nayak, "Classifying symmetry-protected topological phases through the anomalous action of the symmetry on the edge," Phys. Rev. B **90**, 235137 (2014).

[30] Sergey Bravyi, Matthew B Hastings, and Spyridon Michalakis, "Topological quantum order: stability under local perturbations," Journal of mathematical physics **51**, 093512 (2010).

[31] Sergey Bravyi and Matthew B Hastings, "A short proof of stability of topological order under local perturbations," Communications in mathematical physics **307**, 609 (2011).

[32] Xie Chen, Zheng-Cheng Gu, Zheng-Xin Liu, and Xiao-Gang Wen, "Symmetry protected topological orders and the group cohomology of their symmetry group," Phys. Rev. B **87**, 155114 (2013).

[33] Davide Gaiotto and Theo Johnson-Freyd, "Symmetry protected topological phases and generalized cohomology," Journal of High Energy Physics **2019**, 7 (2019).

[34] Anton Kapustin, "Symmetry Protected Topological Phases, Anomalies, and Cobordisms: Beyond Group Cohomology," arXiv e-prints , arXiv:1403.1467 (2014), arXiv:1403.1467 [cond-mat.str-el].

[35] Kazuya Yonekura, "On the cobordism classification of symmetry protected topological phases," Communications in Mathematical Physics **368**, 1121–1173 (2019).

[36] Xie Chen, Zheng-Xin Liu, and Xiao-Gang Wen, "Two-dimensional symmetry-protected topological orders and their protected gapless edge excitations," Phys. Rev. B **84**, 235141 (2011).

[37] Dominic J. Williamson, Nick Bultinck, Michael Mariën, Mehmet B. Şahinoğlu, Jutho Haegeman, and Frank Verstraete, "Matrix product operators for symmetry-protected topological phases: Gauging and edge theories," Phys. Rev. B **94**, 205150 (2016).

[38] R. Shankar and Ashvin Vishwanath, "Equality of bulk wave functions and edge correlations in some topological superconductors: A spacetime derivation," Phys. Rev. Lett. **107**, 106803 (2011).

[39] Yi-Zhuang You, Zhen Bi, Alex Rasmussen, Kevin Slagle, and Cenke Xu, "Wave function and strange correlator of short-range entangled states," Phys. Rev. Lett. **112**, 247202 (2014).

[40] Robijn Vanhove, Matthias Bal, Dominic J. Williamson, Nick Bultinck, Jutho Haegeman, and Frank Verstraete, "Mapping topological to conformal field theories through strange correlators," Phys. Rev. Lett. **121**, 177203 (2018).

[41] Nick Bultinck, Robijn Vanhove, Jutho Haegeman, and Frank Verstraete, "Global anomaly detection in two-dimensional symmetry-protected topological phases," Phys. Rev. Lett. **120**, 156601 (2018).

[42] Xie Chen, Zheng-Cheng Gu, and Xiao-Gang Wen, "Classification of gapped symmetric phases in one-dimensional spin systems," Phys. Rev. B **83**, 035107 (2011).

[43] Lukasz Fidkowski and Alexei Kitaev, "Topological phases of fermions in one dimension," Phys. Rev. B **83**, 075103 (2011).

[44] Michael A. Nielsen and Isaac L. Chuang, *Quantum Computation and Quantum Information: 10th Anniversary Edition* (Cambridge University Press, 2010).

[45] Daniel Gottesman, *Stabilizer codes and quantum error correction*, Ph.D. thesis, California Institute of Technology (1997).

[46] Héctor J. García, Igor L. Markov, and Andrew W. Cross, "On the geometry of stabilizer states," Quantum Info. Comput. **14**, 683–720 (2014).

[47] Victor Veitch, Christopher Ferrie, David Gross, and Joseph Emerson, "Negative quasi-probability as a resource for quantum computation," New Journal of Physics **14**, 113011 (2012).

[48] Mark Howard and Earl Campbell, "Application of a resource theory for magic states to fault-tolerant quantum computing," Phys. Rev. Lett. **118**, 090501 (2017).

[49] Sergey Bravyi, Dan Browne, Padraic Calpin, Earl Campbell, David Gosset, and Mark Howard, "Simulation of quantum circuits by low-rank stabilizer decompositions," Quantum **3**, 181 (2019).

[50] Zi-Wen Liu, Kaifeng Bu, and Ryuji Takagi, "One-shot operational quantum resource theory," Phys. Rev. Lett. **123**, 020401 (2019).

[51] Xin Wang, Mark M. Wilde, and Yuan Su, "Efficiently computable bounds for magic state distillation," Phys. Rev. Lett. **124**, 090505 (2020).

[52] Michael Levin and Zheng-Cheng Gu, "Braiding statistics approach to symmetry-protected topological phases," Phys. Rev. B **86**, 115109 (2012).

[53] Trithep Devakul, Dominic J. Williamson, and Yizhi You,

"Classification of subsystem symmetry-protected topological phases," Phys. Rev. B **98**, 235121 (2018).

[54] Yizhi You, Trithep Devakul, F. J. Burnell, and S. L. Sondhi, "Subsystem symmetry protected topological order," Phys. Rev. B **98**, 035112 (2018).

[55] Beni Yoshida, "Topological phases with generalized global symmetries," Phys. Rev. B **93**, 155131 (2016).

[56] Robert Raussendorf, Sergey Bravyi, and Jim Harrington, "Long-range quantum entanglement in noisy cluster states," Phys. Rev. A **71**, 062313 (2005).

[57] Yu-An Chen, Tyler D. Ellison, and Nathanan Tantivasadakarn, "Disentangling supercohomology symmetry-protected topological phases in three spatial dimensions," arXiv e-prints , arXiv:2008.05652 (2020), arXiv:2008.05652 [cond-mat.str-el].

[58] Lokman Tsui and Xiao-Gang Wen, "Lattice models that realize $\mathbb{Z}_n$-1 symmetry-protected topological states for even n," Phys. Rev. B **101**, 035101 (2020).

[59] Juven C. Wang, Zheng-Cheng Gu, and Xiao-Gang Wen, "Field-theory representation of gauge-gravity symmetry-protected topological invariants, group cohomology, and beyond," Phys. Rev. Lett. **114**, 031601 (2015).

[60] F. J. Burnell, Xie Chen, Lukasz Fidkowski, and Ashvin Vishwanath, "Exactly soluble model of a three-dimensional symmetry-protected topological phase of bosons with surface topological order," Phys. Rev. B **90**, 245122 (2014).

[61] Lukasz Fidkowski, Jeongwan Haah, and Matthew B. Hastings, "Exactly solvable model for a $4+1$D beyond-cohomology symmetry-protected topological phase," Phys. Rev. B **101**, 155124 (2020).

[62] Jeongwan Haah, Lukasz Fidkowski, and Matthew B. Hastings, "Nontrivial Quantum Cellular Automata in Higher Dimensions," arXiv e-prints , arXiv:1812.01625 (2018), arXiv:1812.01625 [quant-ph].

[63] Sergey Bravyi, David P. Divincenzo, Roberto Oliveira, and Barbara M. Terhal, "The complexity of stoquastic local hamiltonian problems," Quantum Info. Comput. **8**, 361–385 (2008).

[64] Lalit Gupta and Itay Hen, "Elucidating the interplay between non-stoquasticity and the sign problem," Advanced Quantum Technologies **3**, 1900108 (2020).

[65] Dominic V. Else, Ilai Schwarz, Stephen D. Bartlett, and Andrew C. Doherty, "Symmetry-protected phases for measurement-based quantum computation," Phys. Rev. Lett. **108**, 240505 (2012).

[66] Iman Marvian, "Symmetry-protected topological entanglement," Phys. Rev. B **95**, 045111 (2017).

[67] J.I. Cirac, D. Pérez-García, N. Schuch, and F. Verstraete, "Matrix product density operators: Renormalization fixed points and boundary theories," Annals of Physics **378**, 100–149 (2017).

[68] Sergey Bravyi, Alexander Kliesch, Robert Koenig, and Eugene Tang, "Obstacles to State Preparation and Variational Optimization from Symmetry Protection," arXiv e-prints , arXiv:1910.08980 (2019), arXiv:1910.08980 [quant-ph].

[69] William K Wootters, "A wigner-function formulation of finite-state quantum mechanics," Annals of Physics **176**, 1 – 21 (1987).

[70] D. Gross, "Hudson's theorem for finite-dimensional quantum systems," Journal of Mathematical Physics **47**, 122107 (2006).

[71] Hakop Pashayan, Joel J. Wallman, and Stephen D. Bartlett, "Estimating outcome probabilities of quantum circuits using quasiprobabilities," Phys. Rev. Lett. **115**, 070501 (2015).

[72] Jacob Miller and Akimasa Miyake, "Hierarchy of universal entanglement in 2D measurement-based quantum computation," npj Quantum Information **2**, 16036 (2016).

[73] Jacob Miller and Akimasa Miyake, "Latent computational complexity of symmetry-protected topological order with fractional symmetry," Phys. Rev. Lett. **120**, 170503 (2018).

[74] Beni Yoshida, "Gapped boundaries, group cohomology and fault-tolerant logical gates," Annals of Physics **377**, 387 – 413 (2017).

[75] Jeongwan Haah, "Classification of translation invariant topological Pauli stabilizer codes for prime dimensional qudits on two-dimensional lattices," arXiv e-prints , arXiv:1812.11193 (2018), arXiv:1812.11193 [quant-ph].

[76] M. Popp, F. Verstraete, M. A. Martín-Delgado, and J. I. Cirac, "Localizable entanglement," Phys. Rev. A **71**, 042306 (2005).

[77] David Fattal, Toby S. Cubitt, Yoshihisa Yamamoto, Sergey Bravyi, and Isaac L. Chuang, "Entanglement in the stabilizer formalism," arXiv e-prints , quant-ph/0406168 (2004), arXiv:quant-ph/0406168 [quant-ph].

[78] Noah Linden, Frantisek Matus, Mary Beth Ruskai, and Andreas Winter, "The Quantum Entropy Cone of Stabiliser States," in *8th Conference on the Theory of Quantum Computation, Communication and Cryptography (TQC 2013)*, Leibniz International Proceedings in Informatics (LIPIcs), Vol. 22, edited by Simone Severini and Fernando Brandao (Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, Dagstuhl, Germany, 2013) pp. 270–284.