Anomalous diffusion due to long-range velocity fluctuations in the absence of a mean flow

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The dispersion of a tracer in a heterogeneous medium in which the tracer's velocity has zero mean and a covariance that decays as $x^{-\gamma}$ with distance x is studied using nonlocal advection–diffusion theory. If the velocity covariance decays slowly, $\gamma \leqslant 2$, the tracer's dispersive motion is non-Fickian even at long times after its release. Under these circumstances, it is not possible to predict the dispersion by assuming that the tracer samples the velocity fluctuations primarily by molecular diffusion, even if the fluctuations are weak. Instead, we develop a self-consistent theory in which the tracer samples each velocity fluctuation by the motion resulting from the other fluctuations. It is shown that in cases of anomalous diffusion, the tracer's mean-square displacement grows faster than linearly with time—as $t^{4/(2+\gamma)}$ for $0<\gamma<2$ and as $t(\ln t)^{1/2}$ for $\gamma \equiv 2$ as $t \to \infty$.

I. INTRODUCTION

The transport of heat or mass from particles to a flowing fluid stream has been a problem of long-standing interest to Professor Acrivos and his students.[1–6] The coupling between advection of the fluid and molecular diffusion (or conductivity) results in a variety of different physical processes, each with a different dependence on the Peclet number—the nondimensional parameter measuring the relative importance of advection to diffusion. His work has been instrumental in unraveling these different processes, in determining the proper scalings, and in establishing a foundation upon which to base further studies. Our work over the last several years on dispersion in porous media has its roots firmly planted in the rich soil laid down by Professor Acrivos and his students and colleagues. We wish to dedicate not only the present contribution, but also our previous work to Professor Acrivos on the occasion of his 60th birthday, which is commemorated in this issue.

The diffusion of tracers or passive scalars in disordered media is a subject of considerable practical and theoretical interest. Problems ranging from the motion of electrons in semiconductors to gel formation by polymeric reaction to dispersion of pollutants in the environment can all be cast into this form. Current interest has focused on situations in which the diffusive motion is anomalous, that is, the mean-square displacement of the tracer does not grow linearly with time at long times. Although anomalous diffusion is not a new problem,[7,8] recent work has demonstrated the underlying unity of the physical processes, and a variety of mechanisms for anomalous behavior have been identified and the corresponding temporal scalings derived.[9–14]

In a previous paper[9] we studied the dispersion arising from flow through heterogeneous porous media with long-range permeability variations. The long-range permeability fluctuations induce equally long-range velocity fluctuations which lead to the anomalous behavior. For pressure driven flow through porous media, the mean velocity is necessarily nonzero, since without a mean pressure gradient driving the mean flow there would be no motion and therefore no velocity fluctuations. In addition to providing a bias to the average motion, the mean flow also provides the primary mechanism by which a tracer samples the fluctuating velocity field. (This is true whether or not the amplitude of the fluctuations is small and whether or not molecular diffusion is present.) This one-dimensional "path" for sampling the fluctuations leads to anomalous diffusion with the mean-square displacement growing as $t^{2-\gamma}$, when $\gamma < 1$. Here, γ measures the rate of decay of the covariance of the velocity fluctuations with separation $\mathbf{x}$, i.e., the covariance is proportional to $x^{-\gamma}$ as $x \to \infty$. For $\gamma > 1$ the diffusive process is normal (mean-square displacement grows linearly with time), and in the crossover case $\gamma \equiv 1$, there is a $t \ln t$ behavior.[9]

In the present paper we focus on the situation in which the mean velocity is zero. We have already noted that this case is not relevant to pressure driven flows in porous media. There are other physical processes, however, in which long-range velocity fluctuations in the absence of a mean flow arise, and the analysis presented below may thus be applicable to these situations. In general, the situation considered in this paper falls into the class of so-called random-field problems in the "viscous" or "high-damping" limit—the velocity of the tracer is proportional to the random forcing (random force field), rather than the acceleration of the tracer being proportional to the forcing. Our primary interest in this problem is not, however, based on any particular application. Rather, interest stems from the different way in which the tracer samples the fluctuations, which leads to different anomalous scalings.

When the mean velocity is zero, the tracer samples the random velocity field either by molecular diffusion or by the motion induced by the random fluctuations themselves. It will be seen below that in the anomalous regime the sampling

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induced by the fluctuations dominates and the motion of the tracer must now be determined self-consistently—the rate at which fluctuations are visited is determined by, and determines the spreading of, the tracer. When $\gamma > 2$, the sampling "path" is a random Brownian walk with fractal dimension 2, and the diffusion is Fickian. However, the mean-square displacement is anomalous for $\gamma \leqslant 2$, growing as $t^{4/(2+\gamma)}$ as $t \to \infty$ for $\gamma < 2$ and as $t(\ln t)^{1/2}$ for $\gamma \equiv 2$.

Bouchaud *et al.*[12] have given a simple qualitative argument leading to a criterion for the existence of anomalous diffusion in cases of zero mean flow. They also employed a group renormalization argument to predict the scaling of the mean-square displacement with time in the limit in which the deviation from normal diffusive behavior is small. In Sec. II, we shall confirm the results of the Bouchaud *et al.*[12] using a continuum nonlocal advection–diffusion theory,[9,15,16] which suggests a simple interpretation of the requisite renormalization. In addition, we shall obtain the mean-square displacement in cases of *very* anomalous diffusion. In Sec. III we shall offer some simple physical arguments to explain the anomalous behavior obtained in this case and in the case of nonzero mean flow.

II. ANALYSIS

We consider the dispersion of a tracer whose velocity $\mathbf{u}$ is a stationary random function of position $\mathbf{x}$ in three-dimensional space. The tracer concentration c satisfies the advection–diffusion equation

$$\frac{\partial c}{\partial t} + \nabla\cdot(\mathbf{u}c - D\nabla c) = s, \tag{1}$$

where ∇ is the nabla operator, D is the tracer's molecular diffusivity, which is assumed constant, and s is the source of the tracer. We shall consider the spread of a tracer initially released at the origin, i.e., $s = \delta(\mathbf{x})\delta(t)$.

We consider random, isotropic, incompressible velocity fields, i.e., $\nabla\cdot\mathbf{u} = 0$, and $\langle\mathbf{uu}\rangle = \frac{1}{3}\mathbf{I}\langle u^2\rangle$. Here, $\langle\ \rangle$ indicates an unconditional ensemble average over all possible realizations of the random velocity field. Generalizations to other velocity fields are, of course, possible. The only effect of anisotropy in the velocity field is to make the nonlocal diffusivity in (11) an anisotropic tensor. (We note, however, that unusual results occur in the singular case in which the velocity is unidirectional and varies in a second, normal, direction.[14] Here, molecular diffusion is required to sample the velocity fluctuations, and the structure of the sampling "path" is different than that discussed below.)

The most important consequence of a compressible velocity field is the fact that it presents the possibility that the equilibrium tracer distribution c_{eq}—the distribution in the absence of any average tracer concentration gradients, i.e., $\nabla\langle c\rangle = 0$—may be nonuniform. This implies that the average velocity of the tracer at equilibrium $\mathbf{V} \equiv \langle\mathbf{u}c_{\text{eq}}\rangle/\langle c_{\text{eq}}\rangle$ is not necessarily the same as the medium's average velocity $\mathbf{U} \equiv \langle\mathbf{u}\rangle$.[15,16] In a compressible random flow field, it would be necessary to ensure that $\mathbf{V} = 0$ in order that the medium was not sampled by an average advective motion.

For simplicity we consider the random velocity field to be composed of independent velocity fluctuations of the form

$$\nabla^2\langle\mathbf{u}(\mathbf{x}|\mathbf{r}_1;a,\mathbf{p})\rangle_1 = U\mathbf{p}\cdot(\mathbf{I}\nabla^2 - \nabla\nabla)e^{-a^{-1}|\mathbf{x}-\mathbf{r}_1|}, \tag{2}$$

where $\langle\ \rangle_1$ denotes the conditional ensemble average with one velocity fluctuation fixed. This is the form of the velocity that would result from a local source of momentum $U\mathbf{p}e^{-a^{-1}|\mathbf{x}-\mathbf{r}_1|}$ in a porous medium described by Darcy's law, and it provides a convenient mathematical form for the subsequent analysis. The fluctuation is centered at $\mathbf{r}_1$, has a magnitude U (taken to be a constant), a direction $\mathbf{p}$, and a radial extent a. The number density $N(a,\mathbf{p})$ of fluctuations is specified to be independent of the position $\mathbf{r}_1$, so that we have a statistically homogeneous medium. We further require all possible directions of $\mathbf{p}$ to be equally likely, i.e.,

$$N(a,\mathbf{p}) = \widetilde{N}(a)\Omega(\mathbf{p}), \tag{3}$$

where $\Omega(\mathbf{p}) = 1/4\pi$, so that the velocity field is isotropic. This distribution of $\mathbf{p}$ also ensures that the mean velocity is zero, $\langle\mathbf{u}\rangle = 0$. The particular choice of (2) for the velocity fluctuations does not affect the qualitative results to be obtained below, as long as $\langle\mathbf{u}\rangle_1$ decays exponentially as $a^{-1}|\mathbf{x}-\mathbf{r}_1| \to \infty$.

To examine the anomalous regime, we take the number density to decay algebraically with fluctuation size a as $a \to \infty$, i.e.,

$$\widetilde{N}(a) \sim k_1 a^{-4-\gamma} \quad \text{as } a \to \infty. \tag{4}$$

If the number density decays faster than algebraically, then a classical diffusion process results as $t \to \infty$.

The covariance of the velocity field, a central element in assessing the dispersion, is given by

$$\langle\mathbf{u}(\mathbf{x})\mathbf{u}(\mathbf{0})\rangle = \int da \int d\mathbf{r}_1 \int d\mathbf{p}\, N(a,\mathbf{p})\langle\mathbf{u}(\mathbf{x}|\mathbf{r}_1;a,\mathbf{p})\rangle_1 \times \langle\mathbf{u}(\mathbf{0}|\mathbf{r}_1;a,\mathbf{p})\rangle_1. \tag{5}$$

The product of the conditional averages in (5) follows from the assumed statistical independence of the fluctuations. The magnitude of the covariance decays with radial separation x according to

$$\langle\mathbf{u}(\mathbf{x})\mathbf{u}(\mathbf{0})\rangle = \mathbf{I}k_2 x^{-\gamma}, \tag{6}$$

where $k_2 = 2\pi/27U^2k_1(3 + 4\gamma + \gamma^2)\Gamma(\gamma)$. In obtaining (6) we have used the convolution theorem and the Fourier transform of $\langle\mathbf{u}\rangle$, which is given by

$$\langle\hat{\mathbf{u}}\rangle_1 = [8\pi a^3/(4\pi^2a^2\xi^2 + 1)^2]U\mathbf{p}\cdot(\mathbf{I} - \tilde{\boldsymbol{\xi}}\tilde{\boldsymbol{\xi}}). \tag{7}$$

In (7), $\hat{}$ denotes the Fourier transform, $\boldsymbol{\xi}$ is the transform variable corresponding to $\mathbf{x} - \mathbf{r}_1$, and $\tilde{\boldsymbol{\xi}} = \boldsymbol{\xi}/\xi$. We have adopted Lighthill's[17] convention for Fourier transforms, using $e^{-2\pi i\mathbf{x}\cdot\boldsymbol{\xi}}$.

The unconditionally averaged mass conservation equation may be written without approximation as

$$\frac{\partial\langle c\rangle}{\partial t} - D\nabla^2\langle c\rangle + \nabla\cdot\langle\mathbf{u}c\rangle = \frac{\partial\langle c\rangle}{\partial t} - D\nabla^2\langle c\rangle + \nabla\cdot\int d\mathbf{r}_1\int da\int d\mathbf{p}\, N(a,\mathbf{p}) \times\langle\mathbf{u}(\mathbf{x}-\mathbf{r}_1)\rangle_1\langle c'(\mathbf{x}-\mathbf{r}_1)\rangle_1 = \langle s\rangle, \tag{8}$$

where $c' \equiv c - \langle c \rangle$. In writing the final equality in (8), we have used the fact that the velocity fluctuations are uncorrelated. An equation for the concentration disturbance induced by a velocity fluctuation centered at $\mathbf{r}_1$, obtained by subtracting the unconditionally from the conditionally averaged mass conservation equation, is

$$\frac{\partial \langle c' \rangle_1}{\partial t} - D\nabla^2 \langle c' \rangle_1 + \nabla\cdot(\langle \mathbf{u}c' \rangle_1 - \langle \mathbf{u}c' \rangle) = -\langle \mathbf{u} \rangle_1 \cdot \nabla \langle c \rangle, \tag{9}$$

where we have used the fact that $\langle \mathbf{u} \rangle = 0$ and have stipulated that there be no fluctuation in the source of the tracer, i.e., $\langle s' \rangle_1 = 0$.

Writing P for the Green's function for the left-hand side of (9), the bulk mass conservation equation (8) may be written as

$$\frac{\partial \langle c \rangle}{\partial t} - \nabla\cdot\int d\mathbf{x}_1 \int_{-\infty}^{t} \mathbf{D}(\mathbf{x}-\mathbf{x}_1, t-t_1) \cdot\nabla_1 \langle c(\mathbf{x}_1,t_1) \rangle = \langle s \rangle, \tag{10}$$

where the nonlocal dispersivity is given by

$$\mathbf{D}(\mathbf{x}-\mathbf{x}_1,t-t_1) = \mathbf{I}D\delta(\mathbf{x}-\mathbf{x}_1)\delta(t-t_1) + \int d\mathbf{r}_1 \int da \int d\mathbf{p}\, N(a,\mathbf{p}) \times \langle \mathbf{u}(\mathbf{x}-\mathbf{r}_1) \rangle_1 P(\mathbf{x}-\mathbf{x}_1,t-t_1) \langle \mathbf{u}(\mathbf{x}_1-\mathbf{r}_1) \rangle_1. \tag{11}$$

Solving (10) for the average concentration, after taking the Fourier and Laplace transforms in space and time, simply gives

$$\langle \hat{c} \rangle = \langle \hat{s} \rangle / (\sigma + 4\pi^2 \boldsymbol{\xi}\cdot\hat{\mathbf{D}}\cdot\boldsymbol{\xi}), \tag{12}$$

where σ is the Laplace transform variable corresponding to t. In general, the space-time evolution of the bulk concentration field may be obtained from (12) along with (11).

The greatest difficulty in using (12) and (11) is to determine the Green's function P for the concentration disturbance equation (9). Here P is also the conditional transition probability—the probability of a tracer transiting from $\mathbf{x}_1$ at t_1 to $\mathbf{x}$ at t, given a fluctuation centered at $\mathbf{r}_1$. Using the statistical independence of the velocity fluctuations, the concentration disturbance equation may be written in the form

$$\frac{\partial \langle c' \rangle_1}{\partial t} - D\nabla^2 \langle c' \rangle_1 + \nabla\cdot\langle \mathbf{u} \rangle_1 \langle c' \rangle_1 + \nabla\cdot\int d\mathbf{r}_2 \int da_2 \int d\mathbf{p}_2\, N(a_2,\mathbf{p}_2) \langle \mathbf{u}(\mathbf{x}-\mathbf{r}_2) \rangle_1 \times \langle c''(\mathbf{x}|\mathbf{r}_1,\mathbf{r}_2) \rangle_2 = -\langle \mathbf{u} \rangle_1 \cdot \nabla \langle c \rangle, \tag{13}$$

where

$$c'' \equiv c - \langle c \rangle - \langle c'(\mathbf{x}|\mathbf{r}_1) \rangle_1 - \langle c'(\mathbf{x}|\mathbf{r}_2) \rangle_1.$$

Noting that the integral over $N(a_2,\mathbf{p}_2)$ in (13) has the same physical significance and mathematical form as that occurring in the bulk conservation equation (8), we may write (13) as

$$\frac{\partial \langle c' \rangle_1}{\partial t} + \nabla\cdot\langle \mathbf{u} \rangle_1 \langle c' \rangle_1 - \nabla\cdot\int d\mathbf{x}_1 \int_{-\infty}^{t} dt_1 \times \mathbf{D}(\mathbf{x}-\mathbf{x}_1,t-t_1)\cdot\nabla_1 \langle c'(\mathbf{x}_1,t_1|\mathbf{r}_1) \rangle_1 = -\langle \mathbf{u} \rangle_1 \cdot \nabla \langle c \rangle, \tag{14}$$

where $\mathbf{D}$ is the same nonlocal dispersivity tensor as appears in (10).

If the fixed fluctuation has a small effect on the concentration disturbance so that the second term on the left-hand side of (14) may be neglected compared to the third, then the transport of the tracer is dominated by the dispersive motion induced by all of the velocity fluctuations, not by an advective motion associated with the fluctuation fixed at $\mathbf{r}_1$, i.e., the relevant Peclet number (one that contains the sought-after dispersivity) is small. In this case, we may approximate the Fourier–Laplace transform of the Green's function or transition probability $\hat{P}$ of (14) by

$$\hat{P} = 1/(\sigma + 4\pi^2 \boldsymbol{\xi}\cdot\hat{\mathbf{D}}\cdot\boldsymbol{\xi}). \tag{15}$$

Equation (15) contains the renormalization required to treat dispersion in a random velocity field with zero mean. If the mean velocity were nonzero, the denominator of (15) would contain a term $i\boldsymbol{\xi}\cdot\langle \mathbf{u} \rangle$, indicating that the tracer sampled the random velocity field (at least in part) by translating with the mean velocity. When the mean velocity is zero, the tracer can sample the random velocity field only by molecular diffusion and the motion induced by the random field itself. It will be seen below that in cases of anomalous diffusion sampling by the random field dominates. The "self-consistent" approximation embodied in (15), in which the tracer samples each velocity fluctuation by dispersing according to the bulk dispersion operator, is valid when the tracer is influenced by many fluctuations, each one individually having a small effect on the tracer's motion, i.e., $\nabla\cdot\langle \mathbf{u} \rangle_1 \langle c' \rangle_1$ is small in (14).

The transform of the nonlocal diffusivity (11) obtained by using the convolution theorem is

$$\hat{\mathbf{D}}(\boldsymbol{\xi},\sigma) = D\mathbf{I} + \int da \int d\mathbf{p}\, N(a,\mathbf{p}) \int d\boldsymbol{\xi}' \langle \hat{\mathbf{u}}(-\boldsymbol{\xi};a,\mathbf{p}) \rangle_1 \times \hat{P}(\boldsymbol{\xi}+\boldsymbol{\xi}',\sigma;a,\mathbf{p}) \langle \hat{\mathbf{u}}(\boldsymbol{\xi};a,\mathbf{p}) \rangle_1. \tag{16}$$

Inserting (7) and (15) in (16) gives

$$\hat{\mathbf{D}}(\boldsymbol{\xi},\sigma) = D\mathbf{I} + U^2 \int da \int d\mathbf{p}\, N(a,\mathbf{p}) \int d\boldsymbol{\xi}' \times \frac{[8\pi a^3 \mathbf{p}\cdot(\mathbf{I} - \tilde{\boldsymbol{\xi}}'\tilde{\boldsymbol{\xi}}')]^2}{(4\pi^2 a^2 \xi'^2 + 1)^4 [\sigma + 4\pi^2(\boldsymbol{\xi}+\boldsymbol{\xi}')\cdot\hat{\mathbf{D}}\cdot(\boldsymbol{\xi}+\boldsymbol{\xi}')]}. \tag{17}$$

Classical Fickian diffusion with the mean-square displacement growing linearly with time results if the transformed nonlocal diffusivity (17) approaches a constant as $\xi \to 0$ and $\sigma \to 0$, i.e., if

$$\mathbf{D}_L \equiv \hat{\mathbf{D}}(0,0) = d\,\mathbf{I} = \mathbf{I}\left(D + \frac{10\pi}{9}\frac{U^2}{d}\int da\, \tilde{N}(a) a^5\right) \tag{18}$$

is finite. Here, $\mathbf{D}_L$ indicates the local, Fickian diffusivity. Fickian diffusion occurs in the long-time limit if the fluctuation number density $\tilde{N}(a)$ decays faster than a^{-6} as $a \to \infty$ so that the integral in (18) converges. This corresponds to a requirement that the velocity covariance decay faster than x^{-2} as $x \to \infty$, cf. (6), i.e., $\gamma > 2$.

The criterion for normal diffusion in the absence of a mean flow then is that an *area* integral of the velocity covariance exists. This is in contrast to the requirement that a *linear* integral of the velocity covariance exists in order to obtain normal diffusion in the presence of a mean flow.[9] A physical explanation of this distinction is that in the presence of a mean flow the tracer samples space by means of a one-dimensional path—a streamline of the mean flow. On the other hand, in the absence of a mean flow, the tracer samples space through a random walk of fractal dimension 2. Thus, in order for the tracer velocity to become sufficiently uncorrelated with its previous values to yield normal diffusion, we require the existence of a one-dimensional integral of the velocity covariance in the presence of a mean flow and the existence of a two-dimensional integral in the absence of a mean flow.

In the case of anomalous diffusion, $\gamma \leqslant 2$, the concentration response to an impulse source is non-Gaussian for all time and may be obtained from (12) with (17). In our earlier work,[9] we presented the non-Gaussian profiles that result for anomalous diffusion in the presence of a mean flow. Here, however, we shall seek only the behavior of the mean-square displacement of the tracer introduced at $\mathbf{x}=0$ at time $t=0$, i.e., $\langle s\rangle = \delta(\mathbf{x})\delta(t)$ or $\langle\hat{s}\rangle = 1$.

The moments of the average concentration field may be obtained by taking derivatives of the transform of the concentration (12) with respect to $\boldsymbol{\xi}$ and evaluating them at $\boldsymbol{\xi}=0$. Thus the zeroth moment is $H(t)$, where H is the Heaviside step function; the first moment is zero; and the second moment is

$$\int d\mathbf{x}\,\mathbf{x}\mathbf{x}\langle c\rangle = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{2\,\mathrm{sym}\,\widehat{\mathbf{D}}(\xi=0,\sigma)}{\sigma^2}\, e^{\sigma t}\, d\sigma, \tag{19}$$

where $\mathrm{sym}\,\widehat{\mathbf{D}} = \frac{1}{2}(\widehat{\mathbf{D}} + \widehat{\mathbf{D}}^\dagger)$ and $\dagger$ indicates the transpose. In determining the above moments, we have assumed that $\widehat{\mathbf{D}}(\xi=0,\sigma)$ is finite, and that $\nabla_\xi \widehat{\mathbf{D}} \sim O(\xi)$ and $\nabla_\xi\nabla_\xi \widehat{\mathbf{D}} \sim O(\xi^2)$ as $\xi \to 0$. From (17) it is apparent that this is true as long as $\widetilde{N}(a)$ decays faster than a^{-4} as $a\to\infty$, or, equivalently, as long as the velocity covariance decays as $x\to\infty$, i.e., $\gamma > 0$.

Here, $\widehat{\mathbf{D}}(\xi=0,\sigma)$ is independent of all vector and tensor quantities and is therefore isotropic. Substituting the definition

$$\widehat{\mathbf{D}}(\xi=0,\sigma) = \mathbf{I}d(\sigma) \tag{20}$$

into (17), we obtain the following implicit equation for d:

$$d(\sigma) = D + \frac{1}{3}U^2\int da \int d\mathbf{p}\, N(a,\mathbf{p}) \times \int d\boldsymbol{\xi}' \frac{[8\pi a^3 \mathbf{p}\cdot(\mathbf{I} - \tilde{\boldsymbol{\xi}}'\tilde{\boldsymbol{\xi}}')]^2 : \mathbf{I}}{(4\pi a^2 \xi'^2 + 1)^4(\sigma + 4\pi^2\xi'^2 d)}. \tag{21}$$

In writing (21) we have used the fact that, in the long-time limit of interest here, the dominant contribution to the integral over wavenumbers ξ' comes from the asymptotically small values of ξ'. Thus we have replaced $\widehat{\mathbf{D}}(\boldsymbol{\xi}+\boldsymbol{\xi}',\sigma)$ in the denominator of (17) with its small ξ' limit $\mathbf{I}d$. Performing the integration in $\mathbf{p}$, and introducing the new variables $\eta = 2\pi a\xi'$ and $\beta = a\sigma^{1/2}/d^{1/2}$, d becomes

$$d = D + \tfrac{64}{9}U^2 b d^{-\gamma/2}\sigma^{(\gamma-2)/2}, \quad 0<\gamma<2, \tag{22}$$

where

$$b = k_1\int_0^\infty d\beta\,\beta^{1-\gamma}\int_0^\infty \frac{\eta^2\, d\eta}{(\eta^2+1)^4(\eta^2+\beta^2)}, \tag{23a}$$

$$b = \frac{1}{24}k_1\Gamma\left(\frac{2-\gamma}{2}\right)\Gamma\left(\frac{\gamma}{2}\right)\Gamma\left(\frac{3-\gamma}{2}\right)\Gamma\left(\frac{5+\gamma}{2}\right). \tag{23b}$$

When the diffusion is anomalous, $\gamma<2$, the second term on the right-hand side of (22) is large compared to the first in the limit of long times (small σ), indicating, as expected, that the spread of the tracer by the long-range velocity fluctuations is more efficient than by molecular diffusion. Thus we can solve (22) for d neglecting molecular diffusion to give, in the long-time limit,

$$d = [(64U^2/9)b]^{2/(\gamma+2)}\sigma^{(\gamma-2)/(\gamma+2)}. \tag{24}$$

Finally, substituting (24) and (20) into (19) and performing the inverse Laplace transform, the leading behavior of the mean-square displacement at long times is found to be

$$\lim_{t\to\infty}\int d\mathbf{x}\,\mathbf{x}\mathbf{x}\langle c\rangle = \mathbf{I}2\left(\frac{64}{9}U^2 b\right)^{2/(\gamma+2)} \times \frac{1}{\Gamma[(6+\gamma)/(2+\gamma)]}\, t^{4/(\gamma+2)}, \quad 0<\gamma<2. \tag{25}$$

According to (25), the growth of the mean-square displacement in cases of anomalous diffusion, $0<\gamma<2$, is intermediate between the linear growth of a normal diffusive behavior, and the quadratic growth of a purely convective spread. By means of group renormalization arguments, Bouchaud *et al.*[12] predicted that the mean-square displacement grows as $t^{1+\epsilon/4}$, where $\epsilon \equiv 2-\gamma$ is a measure of the extent of the anomalous nature of the diffusion and the condition $\epsilon \ll 1$ is required for their analysis. Our expression (25) agrees with Bouchaud *et al.*[12] in the limit $\gamma\to 2$ ($\epsilon\to 0$), but (25) is not restricted to small ϵ.

In the "crossover" case of $\gamma=2$, the expression (24) for d becomes, to leading order in small σ,

$$d = [(5\pi/9)k_1U^2]^{1/2}[\ln(1/\sigma)]^{1/2}, \quad \gamma\equiv 2. \tag{26}$$

The mean-square displacement becomes in lieu of (25)

$$\lim_{t\to\infty}\int d\mathbf{x}\,\mathbf{x}\mathbf{x}\langle c\rangle = \mathbf{I}2\left(\frac{5\pi}{9}k_1U^2\right)^{1/2} t(\ln t)^{1/2}, \quad \gamma\equiv 2, \tag{27}$$

which also agrees with the expression of Bouchaud *et al.*[12] for $\epsilon\equiv 0$.

III. INTERPRETATION

Bouchaud *et al.*[18] have presented a simple heuristic argument based on a discrete description of the random medium to predict the same criterion, $\gamma\leqslant 2$, for anomalous diffusion as that derived here. We shall now demonstrate that similar arguments may be used to rationalize the results obtained in Ref. 9 and here for the scaling of the mean-square displacement with time when anomalous diffusion occurs in the presence and absence of a mean flow, respectively.

One way of achieving a velocity fluctuation covariance that decays as $x^{-\gamma}$ is to have an $O(R^{3-\gamma})$ number of sites with the same velocity in a spherical volume of radius R. (Here, the radius is nondimensionalized by the linear size of a "site.") Thus there are R^{γ} independent values of the velocity in a radius R. The idea here is to think of the velocity correlation function as a measure of the number of sites within a region surrounding the origin with the same value of the velocity as at the origin. The number of sites with the same velocity in a region of size R is the integral of the correlation function over this volume, i.e., $R^{3-\gamma}$ for a covariance that decays as $x^{-\gamma}$.

If a tracer samples the random field primarily by a mean advective flow, then it traverses a volume of radius t in a time t, which is nondimensionalized by the ratio of the site size to the magnitude of the mean velocity. Thus the tracer experiences t^{γ} independent values of the fluctuating velocity, encountering each independent value $t^{1-\gamma}$ times (so that the total number of encounters is, of course, t). Hence the tracer's displacement relative to the mean is

$$x - \langle x \rangle_T = t^{1-\gamma} \sum_{i=1}^{t^{\gamma}} \Delta s_i, \qquad (28)$$

where the sum is over the relative displacements Δs_i resulting from each encounter with a different value of the random velocity field. Since the t^{γ} values of the velocity fluctuation are independent, the Δs_i are independent, and the central limit theorem applies to the sum of the Δs_i in (28). Thus we find the mean-square displacement to be

$$\langle (x - \langle x \rangle_T)^2 \rangle_T \sim t^{2-\gamma}, \quad 0<\gamma<1, \quad \langle \mathbf{u} \rangle \neq 0, \qquad (29)$$

which is the result we derived in Ref. 9. Here, $\langle\ \rangle_T$ refers to a "tracer" ensemble average defined by $\langle x^2 \rangle_T \equiv \int d\mathbf{x}\, xx \langle c \rangle$.

In the absence of a mean advective flow, the tracer samples space by means of the velocity fluctuations themselves. Thus in a time t a volume of radius $R \sim t^{\alpha}$ is sampled, where t^{α} is the rate of growth of the root-mean-square displacement of the tracer. Here, t is nondimensionalized by the ratio of the site size to a characteristic magnitude of the fluctuating velocity field. Hence the tracer samples $t^{\alpha\gamma}$ independent values of the randomly fluctuating velocity field, encountering each value $t^{1-\alpha\gamma}$ times. The tracer's displacement is then given by

$$x = t^{1-\alpha\gamma} \sum_{i=1}^{t^{\alpha\gamma}} \Delta s_i. \qquad (30)$$

Again, the central limit theorem may be applied to the sum of the independent variables Δs_i, and the mean-square displacement is

$$\langle x^2 \rangle_T \sim t^{2-\alpha\gamma}. \qquad (31)$$

We have already expressed the growth of the root-mean-square tracer displacement as t^{α}. Thus we require $2\alpha = 2 - \alpha\gamma$, or $\alpha = 2/(2+\gamma)$, giving for the mean-square displacement

$$\langle x^2 \rangle_T \sim t^{4/(2+\gamma)}, \quad 0<\gamma<2, \quad \langle \mathbf{u} \rangle \equiv 0, \qquad (32)$$

which is the result (25) that we derived from the nonlocal diffusion theory in Sec. II. In the crossover cases $\gamma \equiv 1$, $\langle \mathbf{u} \rangle \neq 0$ and $\gamma \equiv 2$, $\langle \mathbf{u} \rangle \equiv 0$, the scalings change, with the mean-square displacements given by $t \ln t$, $\langle \mathbf{u} \rangle \neq 0$, and $t(\ln t)^{1/2}$, $\langle \mathbf{u} \rangle \equiv 0$.

IV. CONCLUSIONS

We have applied the nonlocal advection–diffusion theory developed in Refs. 9, 15, and 16 to derive the mean-square displacement of a tracer in a randomly fluctuating flow field in the absence of a mean flow. A self-consistent renormalization of the equation for the transport of the tracer, reflecting the fact that the tracer samples the random field by a nonlocal effective diffusivity resulting from the random field itself, is required to predict the dispersive behavior. This renormalization arises quite naturally from the nonlocal theory. We have also shown that there are simple heuristic arguments, based on an identification that the central limit theorem may still be applied to anomalous diffusion processes if properly reformulated, to predict the anomalous scalings derived by our nonlocal theory.

The situation treated in this paper represents the simplest case of anomalous diffusion in the absence of a mean flow, that is, it is concerned with the motion of a single tracer particle. Another relevant quantity is the relative spreading of *two* tracer particles initially released near each other. This case is considerably more complex than that treated here, requiring, at the minimum, information on the correlations of the velocity gradient tensor. Furthermore, in the anomalous regime, there is no reason to suspect that the relative or pair dispersion approaches the single dispersion behavior at long times, as would be the case in a normal diffusion process. The analysis given here is nonetheless a "first step" in the more general and complete problem of anomalous diffusion of an initially released cloud of particles, which may be more relevant to the dilution of a contaminant.

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