Supplemental Material: Fermionic Chern insulator from twisted light with linear polarization

Utso Bhattacharya,1, 2 Swati Chaudhary,2 Tobias Grass,1 Allan S. Johnson,1 Simon Wall,1,3 and Maciej Lewenstein1,4

1ICFO-Institut de Ciencies Fotoniques, The Barcelona Institute of Science and Technology, Av. Carl Friedrich Gauss 3 08860 Castelldefels (Barcelona), Spain
2Institute of Quantum Information and Matter and Department of Physics, California Institute of Technology, Pasadena, CA 91125, USA
3Department of Physics and Astronomy, Aarhus University, Ny Munkegade 120, 8000 Aarhus C, Denmark
4ICREA, Pg. Lluís Companys 23, 08010 Barcelona, Spain

We describe the method to obtain effective Floquet Hamiltonians in the driven system, and provide the definition of the local markers of Chern number used to topologically characterize the phases.

General form of Floquet Hamiltonian

Under periodic boundary conditions, the graphene system is translationally invariant, and for \( \Delta_0 = 0 \), it has a gapless spectrum at some isolated points in the quasi-momentum space; the quasi-momentum being a good quantum number due to translational invariance. Such points are called Dirac points (DP) due to their massless relativistic dispersion at very low energies. The Chern number at half filling in a gapless graphene system is ill-defined. A gap is necessary for the system to become a topological Chern insulator. But, these gapless points are protected by three symmetries: chiral (same mass of the two sublattices), time-reversal symmetry (TRS), and crystal symmetry \( C_3 \). Hence, to open a gap, one can place the monolayer on a substrate such as hexagonal Silicon Carbide or hexagonal Boron Nitride, which introduces an energy difference between the sublattices [1], described by the Semonoff mass term [2]. This breaks the sublattice or chiral symmetry but preserves TRS. However, this results in a gap that is topologically trivial as evidenced by the zero value of the topological Chern invariant calculated over the filled bands of the system. Therefore, to realize a topologically non-trivial insulator, TRS must be broken, which can be achieved by driving the system with either a circularly polarised (CP) light or light with orbital angular momentum (OAM) as we show here.

To describe such a periodically driven scenario, it is necessary to study the time independent Floquet Hamiltonian of the system defined at (time periodic) strobscopic times. The general form of the Floquet Hamiltonian which has been used through out this work is given by,

\[
H = \sum_{n=-N}^{N} (H_0 \otimes |n\rangle \langle n| + n\omega I \otimes |n\rangle \langle n|) + H_m \otimes |n\rangle \langle n + m| + H_m^\dagger \otimes |n\rangle \langle n - m|)
\]

where \( H_0 \) is the undriven Hamiltonian for graphene, and \( |n\rangle \) denotes the photon degree of freedom [3]. The elements \( H_m \) are Fourier coefficients of the Hamiltonian \( H_m = \int_0^T dt H(t) e^{-im\omega t} \).

In matrix notation, the Floquet Hamiltonian reads

\[
\begin{pmatrix}
H_0 + N\omega & H_1 & \cdots & H_{2N} \\
H_1^\dagger & \ddots & \ddots & \vdots \\
\vdots & \ddots & H_0 & \ddots \\
H_{2N}^\dagger & \cdots & H_1^\dagger & H_0 - N\omega
\end{pmatrix}
\]

(2)

By numerical diagonalization we have obtained the spectrum and eigenstates of the system by truncating the Floquet Hamiltonian beyond \( N = 1 \) sector. We have verified that increasing \( N \) does not result in any significant change in the LMCN or the spectra.

The graphene Hamiltonian \( H_0 \) consists of nearest-neighbor hopping on a lattice as shown in Fig.1, and, potentially, a substrate-induced mass term between the two sublattices. The vectors connecting one site to its three nearest neighbors are given by:

\[
d_1 = a(0, 1), \quad d_2 = a \left( \cos \left( \frac{\pi}{6} \right), -\sin \left( \frac{\pi}{6} \right) \right), \quad d_3 = a \left( -\cos \left( \frac{\pi}{6} \right), -\sin \left( \frac{\pi}{6} \right) \right),
\]

where \( a \) is the distance between nearest neighbor atoms on a honeycomb lattice.

The periodic driving of the graphene lattice, either with circularly polarized Gaussian light or with linearly polarized
OAM light, is incorporated via the Peierls’ substitution:

\[
t_{r,r_j} = e^{i t_0 A \cdot dr_{r,r_j}} = e^{i \left( f_0^a A \cdot dr - f_0^b A \cdot dr \right)}
\]

where \( A \) is the vector potential of the light used to irradiate the sample.

**Effective Floquet Hamiltonian for CP drive**

For the CP drive, the vector potential reads \( A = A_0 \cos \omega t \hat{x} + A_0 \sin \omega t \hat{y} \) and thus Peierls’ phase takes the following form

\[
e^{i \left( f_0^a A \cdot dr - f_0^b A \cdot dr \right)} = e^{i \alpha \sin(\omega t + \beta)}
\]

where \( \beta \) depends on \( r_i \) and \( r_j \). Now, we express the above expression using Jacobi-Anger expansion:

\[
e^{i \alpha \sin(\omega t + \beta)} = \sum_{n} J_n(\alpha) e^{i \beta} e^{i n \omega t}.
\]

This allows us to find the Fourier components \( H_n \) which occur in Eq.(2). Specifically, the real space \( H_{\pm 1} \) are:

\[
H_1 = t_0 \sum_{r_i,j=1,2,3} J_1(A) e^{i \phi_{ij}} \left( b_{r_i+d, a_{r_i}}^\dagger - a_{r_i}^\dagger b_{r_i+d} \right)
\]

and

\[
H_{-1} = t_0 \sum_{r_i,j=1,2,3} J_{-1}(A) e^{-i \phi_{ij}} \left( b_{r_i+d, a_{r_i}}^\dagger - a_{r_i}^\dagger b_{r_i+d} \right)
\]

where \( \phi_1 = \pi, \phi_2 = \frac{\pi}{3}, \phi_3 = -\frac{\pi}{3} \), and \( A = \frac{eF a}{\omega} \). The \( \phi \) phase is obtained via a Peierls’ substitution of the vector potential. It can be shown that the commutation

\[
[H_1, H_{-1}] = t_0^2 (J_1(A))^2 \sum_{r_i, d_k-d_j=a_1, a_2, a_3} 2i \sin(\phi_k - \phi_j) \left( b_{r_i+d_k, a_{r_i}}^\dagger - a_{r_i}^\dagger b_{r_i+d_k} \right) + \text{h.c}
\]

where \( a_3 = -a_1 - a_2 \) and \( a_1, a_2 \) are two lattice vectors for the given honeycomb lattice (see Fig.3). The hopping becomes complex, but remains independent from position, as \( \phi_1 - \phi_2 = \phi_2 - \phi_3 = \phi_3 - \phi_1 = \frac{2\pi}{3} \). Thus, to lowest order (i.e. in the high-frequency regime), the effective Floquet Hamiltonian becomes identical to the static Haldane model with \( t_2 \propto t_0^2 E_\omega \).

**Effective Floquet Hamiltonian for OAM drive**

In the case of OAM drive, the vector potential \( A(x, y) = (A_x(r) e^{i \phi_x e^{i \omega t}} + A_y(r) e^{-i \phi_y e^{-i \omega t}}) \hat{x} \), and after Peierls’ phase substitution, we get \( H_{\pm 1} \):

\[
H_1 = t_0 \sum_{r_i,j=1,2,3} J_1(A_{ij}) e^{i \phi_{ij}} \left( b_{r_i+d_j, a_{r_i}}^\dagger - a_{r_i}^\dagger b_{r_i+d_j} \right)
\]

with

\[
H_{-1} = t_0 \sum_{r_i,j=1,2,3} J_{-1}(A_{ij}) e^{-i \phi_{ij}} \left( b_{r_i+d_j, a_{r_i}}^\dagger - a_{r_i}^\dagger b_{r_i+d_j} \right)
\]

where

\[
A_{ij} = \frac{e}{\omega} \int_0^{r_i} e^{i \beta \vec{r} \cdot \vec{E}} \cdot d\vec{r} - \int_0^{r_i} e^{i \beta \vec{r} \cdot \vec{E}} \cdot d\vec{r}
\]

and

\[
\phi_{ij} = \text{Arg} \left( \int_0^{r_i} e^{i \beta \vec{r} \cdot \vec{E}} \cdot d\vec{r} - \int_0^{r_i} e^{i \beta \vec{r} \cdot \vec{E}} \cdot d\vec{r} \right).
\]

The first contribution \( H^{(1)} \) comes out to be

\[
H^{(1)} = [H_1, H_{-1}] = t_0^2 \sum_{r_i, d_k-d_j=a_1, a_2, a_3} i J_1(A_{ik}) J_1(A_{ij}) \left( \sin(\phi_{ik} - \phi_j) b_{r_i+d_k, a_{r_i}}^\dagger - a_{r_i}^\dagger b_{r_i+d_k} \right) + \text{h.c}
\]

where \( r_{i+\alpha} = r_i + a_\alpha \). Again, in the high-frequency regime the effective OAM Hamiltonian is given by

\[
H_{\text{eff}} = H_0 + H^{(1)} + \mathcal{O} \left( \frac{1}{\omega^2} \right).
\]

As for CP driven case discussed above, \( H^{(1)} \) introduces complex hopping between next-nearest neighbor sites, so this analysis reveals the formal resemblance of the two cases. However, in the OAM driven case, the phase and amplitude of the next-nearest neighbor hopping depends on position in a complicated manner. This impedes an analytical solution, in particular as periodic boundary conditions cannot be applied. For this reason, information about the topology of the system has been obtained from the local marker of the Chern number \([4, 5]\) which we explicitly define in next section.

In order to evaluate the validity of the high-frequency expansion we compare the spectrum of the full Floquet Hamiltonian \( H_F \) from Eq. (2) (truncated at \( N = 1 \)), and \( H_{\text{new}} \) in Fig. 2. This confirms the validity of \( H_{\text{new}} \) for driving frequency \( \omega \approx 25t_0 \).

**Local Chern number**

To identify topological properties of the system we have used local Chern numbers according to the recipe discussed in Refs. [4, 5]. The local Chern numbers are defined as

\[
C(r_i) = \frac{2\pi}{A_c} \langle r_i | [P_x P, P_y P] | r_i \rangle
\]
where $P$ is the projection on states with $E > E_F$ given by:

$$P = \sum_{E > E_F} |\psi_\lambda\rangle \langle \psi_\lambda|$$

with $x = \sum_{\mathbf{r}_i} x_i |\mathbf{r}_i\rangle \langle \mathbf{r}_i|$. 

### Edge Conductance

In order to verify the topological features of Floquet Hamiltonian, we investigate the edge conductance arising from gapless edge modes. We calculate DC edge conductance by using the Floquet scattering matrix approach [6] for a rectangular sample. This scattering matrix was obtained in KWANT for two static leads and we used Floquet representation of the Hamiltonian to account for the photon degree of freedom in the driven sample. We show some plots of edge conductance for the setup for (a) CP light (see Fig. 3) and (b) OAM light (see Fig. 4). In both the cases we see perfectly quantised unit edge conductance in the topological phase and zero in the trivial phase. The jump in the edge conductance across the phase transition exactly matches the point where the in gap states appear in the plots of the energy spectrum. We also show in Fig. 5 that upon increasing the OAM of light from $l = +1$ to $l = +2$, the edge conductance is again quantized to a unit value and shows the existence of a topological phase illustrating that any finite OAM is sufficient to prepare the system in a Chern phase.

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* utso.bhattacharya@icfo.eu


FIG. 4. Edge conductance (top most) and spectra for open boundary conditions for sublattice energy $\Delta_0 = 0.05 t_0$ (blue) and $\Delta_0 = 0.1 t_0$ (orange) around zero energy for the full Floquet Hamiltonian truncated at $n = 2$ for $l = 1$ OAM drive for frequency $\omega = 4t_0$. 
FIG. 5. Edge conductance (top most) and spectra for open boundary conditions around zero energy for the full Floquet hamiltonian truncated at $n = 2$ for OAM drive for frequency $\omega = 4t_0$ and OAM $l = +2$. 