

Accurately Incorporating a Reduced-Speed-of-Light in Magnetohydrodynamic-Particle-in-Cell Simulations

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ABSTRACT

A reduced-speed-of-light (RSOL) approximation is usually adopted in magnetohydrodynamic (MHD)-particle-in-cell (PIC) simulations, in which relativistic cosmic ray (CR) particles moving at nearly the speed of light c and background (non-relativistic) plasma are evolved concurrently. With a RSOL, some “in-code” speed-of-light \tilde{c} is set to much lower values than the true c , allowing simulations to take larger timesteps (which are restricted by the Courant condition given the large CR speeds). However, due to the absence of a well-formulated RSOL implementation in the literature for MHD-PIC simulations, the CR properties in simulations (e.g. CR energy or momentum density, gyro radius) vary artificially with respect to each other and with respect to the converged ($\tilde{c} \rightarrow c$) solutions with different implementations of a RSOL, breaking the correspondence between simulation results and physical reality. Here, we derive a new formulation of the MHD-PIC equations with a RSOL, and show that (1) it guarantees all steady-state properties of the CR distribution function and background plasma/MHD quantities are *independent* of the RSOL \tilde{c} even for $\tilde{c} \ll c$, (2) ensures that the simulation can *simultaneously* represent the real physical values of CR number, mass, momentum, and energy density, (3) retains the correct physical meaning of various terms like the electric field, and (4) ensures the numerical timestep for CRs can always be safely increased by a factor $\sim c/\tilde{c}$. This should enable greater self-consistency and reduced CPU cost in simulations of CR-MHD interactions.

Key words: cosmic rays — plasmas — methods: numerical — MHD — galaxies: evolution — ISM: structure

1 INTRODUCTION

Cosmic rays (CRs) are relativistic charged particles which can exchange energy and momentum with the surrounding medium, and thus can potentially play a significant role in the kinematic and thermal evolution of astrophysical fluids, as well as being interesting in their own right as probes of high-energy astro-particle physics. CRs are expected to be in equipartition with the turbulent and magnetic energy density in the ISM (Boulares & Cox 1990), and have been shown to be influential at galactic and extragalactic scales in a number of recent numerical studies, with implications for galaxy formation (Chan et al. 2019; Su et al. 2019; Hopkins et al. 2020c; Buck et al. 2020), galactic outflows/winds (Ruszkowski et al. 2017; Farber et al. 2018; Hopkins et al. 2020a), cosmic inflows and virial shocks (Ji et al. 2021) and the phase structure of the circumgalactic medium (Salem et al. 2016; Butsky et al. 2020; Ji et al. 2020; Wang et al. 2020). In simulations of CRs on ISM or star formation or galaxy scales, as well as classical models of Galactic CR transport which are compared to Solar system CR experiments (Strong & Moskalenko 2001; Jóhannesson et al. 2016; Evoli et al. 2017), a fluid or Fokker-Planck type approximation for CRs is necessarily adopted, usually with some simplified assumptions for the effective “diffusion coefficient” or “streaming speed” (or some combination thereof). The studies above, and others comparing different assumptions for these CR “transport coefficients” (Butsky & Quinn 2018; Hopkins et al. 2020b) have shown that the effects of CRs and their interactions with the background plasma, let alone Solar system observables, are highly dependent on these coefficients. Therefore, it is critical to obtain a better understanding of CR transport physics and CR “feedback” effects (coupling to the gas via magnetic fields) at a microscopic level.

In recent years, hybrid magnetohydrodynamic (MHD)-particle-in-cell (PIC) simulations have been developed to investigate the interactions between CRs and background plasma, where CRs are evolved kinetically with the PIC treatment while the ionized gas is treated a MHD fluid (Bai et al. 2015; Mignone et al. 2018). This is possible because, for many astrophysical applications including all those above, the background plasma (1) is non-relativistic (bulk

speeds $u \ll c$), (2) is sufficiently well-ionized that ideal MHD is a good approximation, and (3) has gyro radii which are vastly smaller than the CRs (e.g. for typical ISM protons and electrons, the gyro radii are factors $\sim 10^5 - 10^6$ times smaller than the \sim GeV CRs carrying most of the CR energy density). Compared with previous pure PIC simulations where both CRs and ions (and sometimes electrons as well) are simulated as particles (e.g., Spitkovsky 2005; Gargaté & Spitkovsky 2011; Kunz et al. 2014), the hybrid MHD-PIC method can simulate CR kinetic effects and feedbacks to gas by resolving the CR gyro-radius (\sim AU for GeV CRs), without needing to explicitly resolve the extremely tiny background plasma ion inertial length of $\sim 10^{-6}$ AU or even smaller background electron skin depth of $\sim 10^{-8}$ AU. This method has been proven capable of capturing the CR streaming instability (Kulsrud & Pearce 1969; Skilling 1971; Bai et al. 2019) and accounting for the ion-neutral damping effects (Kulsrud & Cesarsky 1971; Zweibel & Shull 1982; Plotnikov et al. 2021; Bambic et al. 2021). Therefore, the MHD-PIC simulations have a unique advantage in following the evolution of CRs and their feedback/coupling to gas in many astrophysical contexts on much greater scales and over much longer durations than pure PIC simulations.

Even with this approximation, however, one difficulty in simulating CRs is that CRs are relativistic with a velocity close to the speed of light c , far greater than other characteristic MHD velocities, which severely limits the simulation timestep (for any standard Courant-type condition) and thus makes simulations computationally prohibitive. To circumvent this difficulty, a reduced-speed-of-light (RSOL) is usually adopted by assigning a RSOL value \tilde{c} smaller than c , but still much larger than other characteristic MHD velocities (Bai et al. 2015; Mignone et al. 2018). In other contexts, such as radiation-hydrodynamics (RHD), application of an RSOL is commonly used to allow larger timesteps for the same reason (photons moving “across the grid” at c require very small numerical timesteps). But in the RHD case, considerable effort has gone into carefully formulating a version of the RHD equations that ensures results are not systematically biased if the RSOL $\tilde{c} \ll c$. In fact most current RHD formulations (and some applications in other fields;

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see e.g. Hopkins et al. 2021) ensure that steady-state solutions are independent of \tilde{c} and that the “true” ($\tilde{c} = c$) values of the photon energy and flux/momentum density can always be recovered (see e.g. Kimm & Cen 2013; Rosdahl et al. 2013; Skinner & Ostriker 2013, and references therein). But there does not yet exist a similar formulation of a RSOL for CR MHD-PIC simulations, which inevitably causes some physical inconsistencies. For instance, since the expressions of the CR energy density, momentum density and mass/number density contain factors of \tilde{c}^2 , \tilde{c}^1 and \tilde{c}^0 respectively, simply replacing c with \tilde{c} for any arbitrary choice of $\tilde{c} \neq c$ makes it impossible to *simultaneously* match the CR energy density, mass/number density, and momentum density to their correct physical values (those with $\tilde{c} = c$). The “back-reaction” forces from CRs to gas thus can be incorrectly calculated due to the CR mass/energy/momentum density mismatch. Moreover, the CR gyro-radius also varies with the choices of \tilde{c} , which leads to inconsistency in related length scales, e.g., scales of CR scattering, in simulations with RSOL adopted. There is no guarantee, even, in some previous CR-MHD-PIC implementations, that the RSOL universally ensures that the timestep can be increased by a factor $\sim c/\tilde{c}$ (which is the goal of introducing it in the first place), nor that steady-state solutions are independent of \tilde{c} .

These challenges motivate us to develop a new formulation of the MHD-PIC equations with a RSOL, which resolves all of the issues above. The paper is organized as follows. §2 presents the derivation of the new RSOL formulation. We compare to previous applications of a RSOL in MHD-PIC simulations in §3, discuss the numerical implementation in §4, and finally conclude in §5.

2 DERIVATION

Begin by considering the general Vlasov equation for a population of CRs:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + \mathbf{F} \cdot \nabla_{\mathbf{p}} f = \frac{\partial f}{\partial t} \Big|_{\text{coll}} \quad (1)$$

where $\nabla_{\mathbf{x}, \mathbf{p}}$ represent the gradients in position and momentum space, respectively, \mathbf{F} is the force, f is the distribution function, the term on the right-hand-side represents collisional processes (which we will neglect for now, and return to later), t is time, and this is defined in the lab (simulation) frame. For Lorentz forces $\mathbf{F} \equiv q [\mathbf{E} + (\mathbf{v}/c) \times \mathbf{B}]$, where q is the CR charge, \mathbf{v} the velocity, \mathbf{B} the magnetic field (\mathbf{E} electric field), and c the *true* speed of light. We expect Lorentz forces to dominate in the regimes of interest, but note that our derivation is robust to the form of \mathbf{F} .

Following the standard practice for radiation-hydrodynamics and newly-developed methods which evolve the Fokker-Planck equation for a population of CRs (Skinner & Ostriker 2013; Hopkins et al. 2021), we multiply this equation by $1/c$ but then ad-hoc replace the value $1/c$ *only* in front of the time derivative $\partial_t f$ with a different value \tilde{c} , to introduce the RSOL.

$$\frac{1}{\tilde{c}} \frac{\partial f}{\partial t} + \frac{\mathbf{v}}{c} \cdot \nabla_{\mathbf{x}} f + \frac{\mathbf{F}}{c} \cdot \nabla_{\mathbf{p}} f = \frac{1}{c} \frac{\partial f}{\partial t} \Big|_{\text{coll}} \quad (2)$$

This is equivalent to taking $\partial_t f^{\text{rsol}} = (\tilde{c}/c) \partial_t f^{\text{true}}$ – i.e. the time variation of f is systematically slowed by a factor of \tilde{c}/c , equivalent to a rescaling of time “as seen by” the CRs. This is what allows us, fundamentally, to take larger timesteps by a factor c/\tilde{c} , as the time variation is slower. But this also ensures that one still recovers *exactly* the correct steady-state solutions ($\partial_t f \rightarrow 0$) for the distribution function f , independent of the choice of \tilde{c} .

Now we can turn this into the equation for a “single” CR group. More accurately for a MHD-PIC method which Monte-Carlo samples the dynamics of a CR population, we assume that f is a sum of δ -functions each representing a group or “packet” of CRs (which are

represented in-code by some “super-particles” or other tracer field):

$$f(\mathbf{x}, \mathbf{p}, q, s, \dots) \equiv \sum_j N_j \delta(\mathbf{x} - \langle \mathbf{x} \rangle_j, \mathbf{p} - \langle \mathbf{p} \rangle_j, q - \langle q \rangle_j, s - s_j, \dots) \quad (3)$$

where s denotes the species (with individual CR mass m_s), etc.. Inserting this into Eq. (2) and integrating over $d^3 \mathbf{x} d^3 \mathbf{p} dq ds \dots$ times $\mathbf{x}, \mathbf{p}, q, s, \dots$, we obtain the evolution equation for each “packet.”

Trivially (ignoring collisions), we have $d_t N_j = 0$, $d_t \langle q \rangle_j = 0$, $d_t s_j = 0$, $d_t \langle m_s \rangle_j = 0$, where $d_t \psi_j = d\psi_j/dt = \dot{\psi}_j$ represents the Lagrangian time derivative with each packet of some quantity ψ , i.e. CR number, and individual charge, species, mass are conserved along the trajectories of individual CRs (since we have neglected spallation and other catastrophic processes).

For \mathbf{x} , we obtain: $\tilde{c}^{-1} \partial_t \langle \mathbf{x} \rangle_j = c^{-1} \langle \mathbf{v} \rangle_j$, or $\partial_t \langle \mathbf{x} \rangle_j = (\tilde{c}/c) \langle \mathbf{v} \rangle_j$. This makes it clear how certain terms should be interpreted. Here and in all the equations above, \mathbf{v} and $\langle \mathbf{v} \rangle_j$ represent the *true* CR speed, which is an intrinsic property of the CR particles. However, $\partial_t \langle \mathbf{x} \rangle_j$ is the *effective* CR advection speed across the grid (which we can define as $\mathbf{v}_j^{\text{eff}}$), which is reduced by our introduction of $\tilde{c} < c$. If $|\langle \mathbf{v} \rangle_j| = c$, then trivially $|\partial_t \langle \mathbf{x} \rangle_j| = \tilde{c}$, so it functions like a “reduced speed of light” (RSOL) as desired. The momentum equation is (using various properties of δ functions and cross-products to simplify the algebra):

$$\frac{1}{\tilde{c}} d_t \langle \mathbf{p} \rangle_j = \frac{1}{c} \langle \mathbf{F} \rangle_j = \frac{\langle q \rangle_j}{c} \left[\mathbf{E} + \frac{\langle \mathbf{v} \rangle_j}{c} \times \mathbf{B} \right] \quad (4)$$

where $\mathbf{E} = -(\mathbf{u}/c) \times \mathbf{B}$ to leading order here (with \mathbf{u} the fluid velocity, and ρ the fluid density), and in the last equality we have inserted the assumption that \mathbf{F} comes primarily from Lorentz forces, but our expression $d_t \langle \mathbf{p} \rangle_j = (\tilde{c}/c) \langle \mathbf{F} \rangle_j$ is completely general and allows for the introduction of any other forces (e.g. gravity).

We can make this more clear by defining the following for notation purposes. Take quantities such as the Lorentz factors $\beta \equiv |\boldsymbol{\beta}|$ and γ , and likewise the energy E and momentum \mathbf{p} of individual CRs to have their *true* values, so they are intrinsic properties of each CR particle. Then we have:

$$\mathbf{v}_j^{\text{eff}} \equiv \partial_t \langle \mathbf{x} \rangle_j = \frac{\tilde{c}}{c} \langle \mathbf{v} \rangle_j = \tilde{c} \langle \boldsymbol{\beta} \rangle_j \quad (5)$$

$$\langle \boldsymbol{\beta} \rangle_j \equiv \frac{\langle \mathbf{v} \rangle_j}{c} = \frac{\mathbf{v}_j^{\text{eff}}}{\tilde{c}} \quad (6)$$

$$\langle \gamma \rangle_j \equiv \frac{1}{\sqrt{1 - \langle \boldsymbol{\beta} \rangle_j \cdot \langle \boldsymbol{\beta} \rangle_j}} \quad (7)$$

$$\langle E \rangle_j \equiv \langle \gamma \rangle_j \langle m_s \rangle_j c^2 \quad (8)$$

$$\langle \mathbf{p} \rangle_j \equiv \langle \gamma \rangle_j \langle \boldsymbol{\beta} \rangle_j \langle m_s \rangle_j c = \langle \gamma \rangle_j \langle m_s \rangle_j \langle \mathbf{v} \rangle_j = \frac{c}{\tilde{c}} \langle \gamma \rangle_j \langle m_s \rangle_j \mathbf{v}_j^{\text{eff}} \quad (9)$$

By definition, then the *total* CR number, mass, momentum, and energy represented by a given “packet” are: N_j , $M_j \equiv N_j \langle m_s \rangle_j$, $\mathcal{E}_j \equiv N_j \langle E \rangle_j$, $\mathbf{P}_j \equiv N_j \langle \mathbf{p} \rangle_j$. We can then re-write the momentum equation for the CR “packet” as:

$$\frac{d}{dt} (\langle \gamma \rangle_j \langle \boldsymbol{\beta} \rangle_j) \equiv \left(\frac{\tilde{c}}{c} \right) \left(\frac{\langle q \rangle_j}{\langle m_s \rangle_j} \right) [\mathbf{E} + \boldsymbol{\beta}_j \times \mathbf{B}] = \frac{1}{\tilde{c}} \frac{d}{dt} (\langle \gamma \rangle_j \mathbf{v}_j^{\text{eff}}) \quad (10)$$

So this is exactly the “true” acceleration, reduced by \tilde{c}/c . We now have a momentum equation which can evolve $\mathbf{v}_j^{\text{eff}}$, the effective advection speed of a CR “packet” across the grid, entirely written in terms of background MHD quantities (which retain exactly their usual meaning) and well-defined properties of the “packet.”

We can derive the “back-reaction” force on the gas by going back to the original Vlasov equation, calculate the force or change

in momentum for $\tilde{c} = c$, integrate over the distribution function, and apply the equal-and-opposite change to gas, as in Hopkins et al. (2021). Or, because we are assuming tight coupling between non-relativistic ions and magnetic fields, we could also calculate a current effect and use Ampere’s law, to get to the force on the gas, as in Zweibel (2017); Thomas & Pfrommer (2019), and obtain the identical answer. The important thing is that just like the radiation case, the “normal” value of c is what appears here, which ensures that the force on the gas/magnetic fields has its correct behavior (identical to $\tilde{c} = c$) when the CR distribution function is in steady state. This gives:

$$\begin{aligned} \frac{\partial(\rho \mathbf{u})}{\partial t} &= - \int d^3 \mathbf{p} f \mathbf{F} = - \sum_j n_j \mathbf{F}_j = - \sum_j n_j \langle q \rangle_j [\mathbf{E} + \boldsymbol{\beta}_j \times \mathbf{B}] \\ &= - \sum_j \frac{1}{V_j} \frac{c}{\tilde{c}} d_t \mathbf{P}_j = - \sum_j n_j \frac{c}{\tilde{c}} d_t \langle \mathbf{p} \rangle_j \\ &= - \sum_j \frac{M_j}{V_j} \left(\frac{c}{\tilde{c}} \right)^2 \frac{d}{dt} \left(\langle \gamma \rangle_j \mathbf{v}_j^{\text{eff}} \right) \end{aligned} \quad (11)$$

where $n_j \equiv N_j/V_j$ is the local CR number density with V_j some effective differential volume assigned to the “packet.”

With these insights, we can also incorporate collisional effects. Returning to Eq. (2), ignoring the terms in \mathbf{F} , we simply have $d_t \psi_j = (\tilde{c}/c) d_t \psi_j|_{\text{coll}}$ for some ψ and relevant collisional effect (e.g. catastrophic losses). So for example if catastrophic losses remove CRs at a “true” rate $\dot{n}_j = -\langle \sigma v \rangle_j n_n n_j$ (where n_n is the local nucleon number density), the total number/mass/momentum/energy of a “packet” is reduced accordingly as $d_t \psi_j \rightarrow -(\tilde{c}/c) \langle \sigma v \rangle_j n_n \psi_j$ for $\psi_j = (N_j, M_j, \mathcal{E}_j, \mathbf{P}_j)$.

Upon examination, a few points are clear. First, as intended, the manner in which \tilde{c} enters is equivalent to a rescaling of time as seen by the CRs, so it guarantees that we can always increase the timestep by c/\tilde{c} . This means that the gyro-frequency is reduced in the lab frame (necessarily) by the same factor (as $\Omega_g^{\text{eff}} = \Omega_g (\tilde{c}/c)$), so the actual *effective* time it requires to complete a gyro orbit is longer. However, this means that for a given true \mathbf{v}_j or equivalently CR energy or γ_j , the gyro *radius* on the grid is entirely independent of \tilde{c} . We can trivially confirm either if we think about the RSOL as a rescaling in time, or simply examine the “effective” transport speeds across the grid, that the Courant criteria for CR “advection” (gyro orbits) become e.g. $\Delta t < C/\Omega_g^{\text{eff}} = (c/\tilde{c}) \Delta t (\tilde{c} = c)$; likewise for advection “through a cell” (which restricts both CR and MHD timesteps), $\Delta t < C \Delta x/v^{\text{eff}} = (c/\tilde{c}) \Delta t (\tilde{c} = c)$. Just like with radiation-hydrodynamics, the “conserved momentum” here upon MHD-CR exchange is $\rho \mathbf{u} + (c/\tilde{c}) \mathbf{P}$ (where $\mathbf{P} \equiv \sum_j \mathbf{P}_j$), not $\rho \mathbf{u} + \mathbf{P}$ (which is only true if $\tilde{c} = c$), but this is purely a matter of definitions. From the form of the Vlasov equation and force on gas written in terms of the distribution function f , it is also immediately obvious that so long as the *distribution function is in steady-state*, i.e. $\partial_t f \rightarrow 0$, then we are guaranteed to recover the correct steady-state solutions for the distribution function, i.e. the correct steady state distribution of CR velocities, phase and pitch angles, momenta/Lorentz factors, etc, as well as the correct force on the gas. We are also guaranteed that processes such as scattering occur on the correct length scales: e.g. scattering rates (or e.g. collisional/catastrophic loss rates) will be slowed by a factor \tilde{c}/c , but so will transport speeds, so by the time a CR has traveled a given *distance* ℓ down a magnetic field line, it will have an identical integrated scattering/catastrophic loss probability, independent of \tilde{c} .

3 COMPARISON TO OTHER MHD-PIC RSOL METHODS

While “pure PIC” methods are quite mature, MHD-PIC methods for applications such as CR dynamics have only been widely-developed

relatively recently. In other previous implementations, for example Bai et al. (2015); Mignone et al. (2018); Bai et al. (2019), the authors simply treat the RSOL \tilde{c} as if the “true” SOL c were reduced for the CRs specifically, such that c specifically no longer has any meaning in their equations. But this immediately leads to several conceptual difficulties and inconsistencies, including: (1) it is impossible to simultaneously reproduce the true CR energy and/or momentum and/or number densities, regardless of whatever limit the system is in, since for example in their formulation $e_j \equiv d\mathcal{E}_j/d^3 \mathbf{x} = n_j \gamma_j \langle m_s \rangle_j \tilde{c}^2$, so either n_j or e_j must be incorrect (and likewise for momentum density); (2) the transition between relativistic and ultra-relativistic behavior occurs at the incorrect CR energy/momentum; (3) this leads to a momentum equation (in our notation) $d_t (\langle \gamma \rangle_j \mathbf{v}_j^{\text{eff}}) \rightarrow (\langle q \rangle_j / \langle m_s \rangle_j c) (c \mathbf{E} + \mathbf{v}_j^{\text{eff}} \times \mathbf{B})$, which is different from Eq. (10) by $(c/\tilde{c})^2$ in the electric-field term and $(c/\tilde{c})^1$ in the magnetic-field term, breaking the physical correspondence between the two unless one redefines the electric field \mathbf{E} to be different from that assumed in the MHD derivation; (4) likewise if one rescales this momentum equation to keep $(q/m_s c)$ fixed, then this leads not to a rescaling of Ω with \tilde{c} , but gives the same Ω (requiring the same timestep as simulations with $\tilde{c} = c$, for gyro orbits), instead with a smaller gyro radius $\tilde{c} \Omega$; (5) there is no guarantee that when the distribution function is in steady-state (even for the simplest homogeneous gas and CR configuration), the distribution function and/or the net back-reaction force on the gas will actually be correct, compared to their value with $\tilde{c} = c$. As a result, in these approaches, one obtains certain results that vary systematically with \tilde{c} and must be extrapolated (e.g. by running a large suite of simulations with different \tilde{c}) to estimate their correct $\tilde{c} \rightarrow c$ values. In contrast, our approach allows for formal convergence of steady-state properties with $\tilde{c} \ll c$.

4 NUMERICAL IMPLEMENTATION & TIMESTEP CONSTRAINTS

The form of the RSOL proposed here can be immediately and trivially implemented in any MHD-PIC code, as it amounts to a trivial renormalization of the time-derivative terms which would be computed anyways. We have explored existing implementations in the code GIZMO in Ji (2021), but note it immediately applies to any other MHD-PIC implementation, for example both the “full- f ” and “ δf ” methods in Bai et al. (2015, 2019). In these methods the MHD (non-relativistic ion/neutral/electron) dynamics are solved in some fluid limit with whatever solver is most useful, additional physics (e.g. radiation transport or dust as in Ji 2021) can be added as well, and then the CRs are integrated on this background with a “super-particle” approach. In this, one simply identifies each super-particle with a “packet” above, and immediately obtain their evolution equations. The only difference between our approach and one with no RSOL is the insertion of appropriate factors of \tilde{c}/c in the code.

With this formulation, we see also that when initializing the system, one should simply initialize the correct N_j such that e.g. $N \equiv \sum N_j = M_{\text{cr}}^{\text{total}} / \langle m_s \rangle$, where $\langle m_s \rangle$ is the true individual CR mass. Or equivalently $N = E_{\text{cr}}^{\text{total}} / \langle E \rangle_j = E_{\text{cr}}^{\text{total}} / \langle \gamma \rangle_j \langle m_s \rangle_j c^2$. This ensures that the correct total CR energy and momentum densities ϵ and \mathbf{P} are also initialized – these are all automatically consistent with one another. Inconsistency only arises if we incorrectly identify $\langle \gamma \rangle_j \langle m_s \rangle_j \mathbf{v}_j^{\text{eff}}$ as the individual CR momentum, but this is not correct because the relation between $\mathbf{v}_j^{\text{eff}}$ and the momentum as it appears in conservation and dynamics equations is changed by the introduction of the RSOL. As detailed above, quantities like \mathbf{p} , \mathbf{v} , γ , $\boldsymbol{\beta}$, E , etc. all retain their usual intrinsic physical meaning (that they would have with $\tilde{c} = c$), but the actual advection velocity $\partial_t \langle \mathbf{x} \rangle_j$ of a

“super-particle” is v_j^{eff} , and the CR momentum equation is modified with the appropriate \tilde{c}/c factors.

The result of this is that all timestep constraints for the CRs also rescale by c/\tilde{c} , as all the time-derivative terms for CRs rescale with this. Obviously, one requires $\Delta t_j < C \Omega_j^{-1} = C \langle r_g \rangle_j / \tilde{c} = (c/\tilde{c}) C (\langle r_g \rangle_j / c)$, where $\langle r_g \rangle_j$ is the gyro radius, and C is a Courant factor, or more generally $\Delta t_j < C |v_j^{\text{eff}}| / |d_t v_j^{\text{eff}}|$, and if there are some catastrophic losses, similarly $\Delta t_j < C N_j / |\dot{N}_j| \sim (c/\tilde{c}) C / ((\sigma v)_j n_n)$. And for the Courant condition of CRs propagating through the gas with some cell size Δx , we have $\Delta t_j < C \Delta x / |\partial_t \mathbf{x}|_j = C \Delta x / |v_j^{\text{eff}}| = (c/\tilde{c}) C \Delta x / |\langle \mathbf{v} \rangle_j|$.

Finally, just like any other RSOL implementation, convergence to correct solutions requires choosing a sufficiently large value of \tilde{c} compared to other (e.g. MHD) signal speeds in the problem. Although formally steady-state solutions are independent of \tilde{c} for all \tilde{c} with this method, if one chose \tilde{c} too small compared to e.g. some rms turbulent velocity of the MHD fluid, $\tilde{c} \ll |\delta u|$, then the plasma would artificially “outpace” the CRs and the system will not actually be able to converge to a steady-state distribution function f in the first place.

5 CONCLUDING REMARKS

In this paper, we propose a new, well-defined formulation of a RSOL which can be applied to MHD-PIC simulations. The major advantages of the new formulation are that: (1) The timestep can be safely increased by c/\tilde{c} as intended (regardless of if the timestep restriction from gyro orbits, advection over the grid, grid response to CRs, or collisional terms dominates). But meanwhile (2) the key properties of CRs in numerical simulations, such as energy density, momentum density, mass/number density and CR gyro-radius, are *independent* of the choices of \tilde{c} , and (3) they can *simultaneously* reproduce their true, physical values (those one would have with $\tilde{c} = c$). And perhaps most importantly (4) solutions for CR bulk properties, distribution function, and their interactions with/forces on the magnetic fields and background fluid are independent of \tilde{c} as well (even if $\tilde{c} \ll c$) so long as the CR distribution function f approaches a local steady-state. These advantages naturally recommend the formulation here as a standard and straightforward implementation for MHD-PIC simulations whenever a RSOL is required.

DATA AVAILABILITY STATEMENT

There are no new data associated with this article.

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