

CIRCULAR GROUPS, PLANAR GROUPS, AND THE EULER CLASS

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ABSTRACT. We study groups of C^1 orientation-preserving homeomorphisms of the plane, and pursue analogies between such groups and circularly orderable groups. We show that every such group with a bounded orbit is circularly orderable, and show that certain generalized braid groups are circularly orderable.

We also show that the Euler class of C^∞ diffeomorphisms of the plane is an *unbounded* class, and that any closed surface group of genus > 1 admits a C^∞ action with arbitrary Euler class. On the other hand, we show that $\mathbb{Z} \oplus \mathbb{Z}$ actions satisfy a homological rigidity property: every orientation-preserving C^1 action of $\mathbb{Z} \oplus \mathbb{Z}$ on the plane has trivial Euler class.

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1. INTRODUCTION

We are motivated by the following question: what kinds of countable groups G act on the plane? And for a given group G known to act faithfully, what is the best possible analytic quality for a faithful action?

This is a very general problem, and it makes sense to narrow focus in order to draw useful conclusions. Groups can be sifted through many different kinds of strainers: finitely presented, hyperbolic, amenable, property T, residually finite, etc. Here we propose that “acts on a circle” or “acts on a line” is an interesting sieve to apply to groups G which act on the plane.

The theory of group actions on 1-dimensional manifolds is rich and profound, and has many subtle connections with algebra, logic, analysis, topology, ergodic theory, etc. One would hope that some of the depth of this theory would carry across to the study of group actions on 2-dimensional manifolds.

The most straightforward way to establish a connection between groups which act in 1 and 2 dimensions is to study when the groups acting in either case are abstractly isomorphic. Therefore we study subgroups $G < \text{Homeo}^+(\mathbb{R}^2)$, and ask under what general conditions are they isomorphic (as abstract groups) to subgroups of $\text{Homeo}^+(S^1)$.

One reason to compare the groups $\text{Homeo}^+(S^1)$ and $\text{Homeo}^+(\mathbb{R}^2)$ comes from their cohomology as discrete groups. A basic theorem of Mather and Thurston says that the cohomology of both of these groups, thought of as discrete groups, is equal, and is equal to $\mathbb{Z}[e]$ where $[e]$ in dimension 2 is free, and is the *Euler class*. Thus at a classical algebraic topological level, these groups are not easily distinguished, and we should not be surprised if many subgroups of $\text{Homeo}^+(\mathbb{R}^2)$ can be naturally made to act faithfully on a circle. We establish that countable C^1 groups of homeomorphisms of the plane which satisfy a certain dynamical condition — that they have a bounded orbit — are all isomorphic to subgroups of $\text{Homeo}^+(S^1)$.

On the other hand, the *bounded cohomology* of these groups is dramatically different. The classical Milnor–Wood inequality says that the Euler class on $\text{Homeo}^+(S^1)$ is a *bounded class*. By contrast, the Euler class on $\text{Homeo}^+(\mathbb{R}^2)$ is *unbounded*. This was known to be true for C^0 homeomorphisms; in this paper we establish that it is also true for C^∞ homeomorphisms. However, a surprising rigidity phenomenon manifests itself: for C^1 actions of $\mathbb{Z} \oplus \mathbb{Z}$ we show that the Euler class must always vanish, which would be implied by boundedness. This is surprising for two reasons: firstly, because we show that the Euler class can take on any value for C^∞ actions of higher genus surface groups, and secondly because the Euler class can take on any value for C^0 actions of $\mathbb{Z} \oplus \mathbb{Z}$. It would be very interesting to understand the full range of this homological rigidity.

We now turn to a more precise statement of results.

1.1. Statement of results. Section §2 contains background material on left orderable and circularly orderable groups. This material is all standard, and is collected there for the convenience of the reader. The main results established there are that a countable group is *left orderable* iff it admits an injective homomorphism to $\text{Homeo}^+(\mathbb{R})$, and is *circularly orderable* iff it admits an injective homomorphism to $\text{Homeo}^+(S^1)$. Thus, group actions on 1–manifolds can be characterized in purely algebraic terms. The expert may feel free to skip §2 and move on to §3.

Section §3 concerns C^1 subgroups $G < \text{Homeo}^+(\mathbb{R}^2)$ with *bounded orbits*. Our first main result is a generalization of a theorem of Dehornoy about orderability of the usual braid groups.

For C a compact, totally disconnected subset of the open unit disk D , we use the notation B_C to denote the group of homotopy classes of homeomorphisms of $\overline{D} \setminus C$ to itself which are fixed on the boundary, and B'_C to denote the group of homotopy classes of (orientation–preserving) homeomorphisms which might or might not be fixed on the boundary. Informally, B_C is the “braid group” of C . In particular, if C consists of n isolated points, B_C is the usual braid group on n strands.

Theorem A. *Let C be a compact, totally disconnected subset of the open unit disk D . Then B'_C is circularly orderable, and B_C is left orderable.*

Using this theorem and the Thurston stability theorem, we show the following:

Theorem B. *Let G be a group of orientation preserving C^1 homeomorphisms of \mathbb{R}^2 with a bounded orbit. Then G is circularly orderable.*

Section §4 concerns the Euler class for planar actions. As intimated above, we show that the Euler class can take on any value for C^∞ actions of higher genus surface groups:

Theorem C. *For each integer i , there is a C^∞ action*

$$\rho_i : \pi_1(S) \rightarrow \text{Diffeo}^+(\mathbb{R}^2)$$

where S denotes the closed surface of genus 2, satisfying

$$\rho_i^*([e])([S]) = i$$

In particular, the Euler class $[e] \in H^2(\text{Diffeo}^+(\mathbb{R}^2); \mathbb{Z})$ is unbounded.

This answers a question of Bestvina.

Using this result, we are able to construct examples of finitely generated torsion free groups of orientation-preserving homeomorphisms of \mathbb{R}^2 which are not circularly orderable, thereby answering a question of Farb.

It might seem from this theorem that there are no homological constraints on group actions on \mathbb{R}^2 , but in fact for C^1 actions, we show the following:

Theorem D. *Let $\rho : \mathbb{Z} \oplus \mathbb{Z} \rightarrow \text{Homeo}^+(\mathbb{R}^2)$ be a C^1 action. Then the Euler class $\rho^*([e]) \in H^2(\mathbb{Z} \oplus \mathbb{Z}; \mathbb{Z})$ is zero.*

It should be emphasized that this is *not* a purely local theorem, but uses in an essential way Brouwer's famous theorem on fixed-point free orientation-preserving homeomorphisms of \mathbb{R}^2 .

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I would especially like to single out Étienne Ghys for thanks, for reading an earlier version of this paper and providing me with copious comments, observations, references, and counterexamples to some naive conjectures.

1.3. Dedication. This paper is dedicated to Andrew Casson, on the occasion of his sixtieth birthday. Happy birthday Andrew!

2. LEFT-ORDERABLE GROUPS AND CIRCULAR GROUPS

In this section we define left-orderable and circularly-orderable groups, and present some of their elementary properties. None of the material in this section is new, but perhaps the exposition will be useful to the reader. Details and references can be found in [24], [31], [11], [16] and [15], as well as other papers mentioned in the text as appropriate.

2.1. Left-invariant orders.

Definition 2.1.1. Let G be a group. A *left invariant order* on G is a total order $<$ such that, for all α, β, γ in G ,

$$\alpha < \beta \text{ iff } \gamma\alpha < \gamma\beta$$

A group which admits a left invariant order is said to be *left orderable*.

We may sometimes abbreviate “left orderable” to LO. Note that a left orderable group may admit many distinct left invariant orders. For instance, the group \mathbb{Z} admits exactly two left invariant orders.

The following lemma gives a characterization of left orderable groups:

Lemma 2.1.2. *A group G admits a left invariant order iff there is a disjoint partition of $G = P \cup N \cup \text{Id}$ such that $P \cdot P \subset P$ and $P^{-1} = N$.*

Proof. If G admits a left invariant order, set $P = \{g \in G : g > \text{Id}\}$. Conversely, given a partition of G into P, N, Id with the properties above, we can define a left-invariant order by setting $h < g$ iff $h^{-1}g \in P$. \square

Notice that lemma 2.1.2 implies that any nontrivial LO group is *infinite*, and *torsion free*.

LO is a *local* property. We make this explicit in the next two lemmas. First we show that if a group fails to be left orderable, this fact can be verified by examining a *finite* subset of the multiplication table for the group, and applying the criterion of lemma 2.1.2.

Lemma 2.1.3. *A group G is not left orderable iff there is some finite symmetric subset $S = S^{-1}$ of G with the property that for every disjoint partition $S \setminus \text{Id} = P_S \cup N_S$, one of the following two properties holds:*

- (1) $P_S \cap P_S^{-1} \neq \emptyset$ or $N_S \cap N_S^{-1} \neq \emptyset$
- (2) $(P_S \cdot P_S) \cap N_S \neq \emptyset$ or $(N_S \cdot N_S) \cap P_S \neq \emptyset$

Proof. Firstly, it is clear that the existence of such a subset contradicts lemma 2.1.2. So it suffices to show the converse.

The set of partitions of $G \setminus \text{Id}$ into disjoint sets P, N is just $2^{G \setminus \text{Id}}$, which is compact with the product topology by Tychonoff’s theorem. By abuse of notation, if $\pi \in 2^{G \setminus \text{Id}}$ and $g \in G \setminus \text{Id}$, we write $\pi(g) = P$ or $\pi(g) = N$ depending on whether the element g is put into the set P or N under the partition corresponding to π .

For every element $\alpha \in G \setminus \text{Id}$, define A_α to be the *open* subset of $2^{G \setminus \text{Id}}$ for which $\pi(\alpha) = \pi(\alpha^{-1})$. For every pair of elements $\alpha, \beta \in G \setminus \text{Id}$ with $\alpha \neq \beta^{-1}$, define $B_{\alpha, \beta}$ to be the *open* subset of $2^{G \setminus \text{Id}}$ for which $\pi(\alpha) = \pi(\beta)$ but $\pi(\alpha) \neq \pi(\alpha\beta)$.

Now, if G is not LO, then by lemma 2.1.2, every partition $\pi \in 2^{G \setminus \text{Id}}$ is contained in some A_α or $B_{\alpha, \beta}$. That is, the sets $A_\alpha, B_{\alpha, \beta}$ are an open cover of $2^{G \setminus \text{Id}}$. By compactness, there is some *finite* subcover. Let S denote the set of indices of the sets A, B, \dots appearing in this finite subcover, together with their inverses. Then S satisfies the statement of the lemma. \square

Remark 2.1.4. An equivalent statement of this lemma is that for a group G which is not LO, there is a finite subset $S = \{g_1, \dots, g_n\} \subset G \setminus \text{Id}$ with $S \cap S^{-1} = \emptyset$ such that for all choices of signs $e_i \in \pm 1$, the semigroup generated by the $g_i^{e_i}$ contains Id .

To see this, observe that a choice of sign $e_i \in \pm 1$ amounts to a choice of partition of $S \cup S^{-1}$ into P_S and N_S . Then if G is not LO, the semigroup of positive products of the P_S must intersect the semigroup of positive products of the N_S ; that is, $p = n$ for p in the semigroup generated by P_S and n in the semigroup generated by N_S . But this implies n^{-1} is in the semigroup generated by P_S , and therefore so too is the product $n^{-1}p = \text{Id}$.

Remark 2.1.5. Given a finite symmetric subset S of G and a multiplication table for G , one can check by hand whether the set S satisfies the hypotheses of lemma 2.1.3. It follows that if G is a group for which there is an algorithm to solve the word problem, then if G is not left orderable, one can certify that G is not left orderable by a finite combinatorial certificate.

The next lemma follows easily from lemma 2.1.3:

Lemma 2.1.6. *A group G is left orderable iff every finitely generated subgroup is left orderable.*

Proof. A left ordering on G restricts to a left ordering on any finitely generated subgroup $H < G$.

Conversely, suppose G is not left orderable. By lemma 2.1.3 we can find a finite set S satisfying the hypotheses of that lemma. Let H be the group generated by S . There is a restriction map

$$\text{res} : 2^{G \setminus \text{Id}} \rightarrow 2^{H \setminus \text{Id}}$$

which is surjective, and continuous with respect to the product topologies. It follows that the union of the sets $\text{res}(A_\alpha), \text{res}(B_{\alpha,\beta})$ with $\alpha, \beta \in H$ is an open cover of $2^{H \setminus \text{Id}}$, and therefore H is not left orderable. \square

We now study homomorphisms between LO groups.

Definition 2.1.7. Let S and T be totally ordered sets. A map $\phi : S \rightarrow T$ is *monotone* if for every pair $s_1, s_2 \in S$ with $s_1 > s_2$, either $\phi(s_1) > \phi(s_2)$ or $\phi(s_1) = \phi(s_2)$.

Let G and H be left orderable groups, and choose a left invariant order on each of them. A homomorphism $\phi : G \rightarrow H$ is *monotone* if it is monotone as a map or totally ordered sets.

LO behaves well under short exact sequences:

Lemma 2.1.8. *Suppose K, H are left orderable groups, and suppose we have a short exact sequence*

$$0 \rightarrow K \rightarrow G \rightarrow H \rightarrow 0$$

Then for every left invariant order on K and H , the group G admits a left invariant order compatible with that of K , such that the surjective homomorphism to H is monotone.

Proof. Let $\phi : G \rightarrow H$ be the homomorphism implicit in the short exact sequence. The order on G is uniquely determined by the properties that it is required to satisfy:

- (1) If $\phi(g_1) \neq \phi(g_2)$ then $g_1 > g_2$ in G iff $\phi(g_1) > \phi(g_2)$ in H
- (2) If $\phi(g_1) = \phi(g_2)$ then $g_2^{-1}g_1 \in K$, so $g_1 > g_2$ in G iff $g_2^{-1}g_1 > \text{Id}$ in K

This defines a total order on G and is left-invariant, as required. \square

Definition 2.1.9. A group G is *locally LO-surjective* if every finitely generated subgroup H admits a surjective homomorphism $\phi_H : H \rightarrow L_H$ to an infinite LO group L_H .

A group G is *locally indicable* if every finitely generated subgroup H admits a surjective homomorphism to \mathbb{Z} . In particular, a locally indicable group is locally LO-surjective, though the converse is not true.

The following theorem is proved in [4]. We give a sketch of a proof.

Theorem 2.1.10 (Burns–Hale). *Suppose G is locally LO–surjective. Then G is LO.*

Proof. Suppose G is locally LO–surjective but not LO. Then by remark 2.1.4, there is some finite subset $\{g_1, \dots, g_n\} \subset G \setminus \text{Id}$ such that, for all choices of signs $e_i \in \pm 1$, the semigroup of positive products of the elements $g_i^{e_i}$ contains Id. Choose a set of such g_i such that n is smallest possible (obviously, $n \geq 2$). Let $G' = \langle g_1, \dots, g_n \rangle$. Then G' is finitely generated. Moreover, by construction G' is not left orderable. Since G is locally LO–surjective, G' admits a surjective homomorphism to an infinite LO group

$$\varphi : G' \rightarrow H$$

with kernel K . Since G' is not LO, at least one g_i is in K . Since H is nontrivial and φ is surjective, at least one g_j is not in K .

Reorder the indices of the g_i so that $g_1, \dots, g_k \notin K$ and $g_{k+1}, \dots, g_n \in K$. Let $P(H)$ denote the positive elements of H . Since the g_i with $i \leq k$ are not in K , it follows that there are choices $\delta_1, \dots, \delta_k \in \pm 1$ such that $\varphi(g_i^{\delta_i}) \in P(H)$. Moreover, since n was chosen to be minimal, there exist choices $\delta_{k+1}, \dots, \delta_n \in \pm 1$ such that no positive product of elements of $g_{k+1}^{\delta_{k+1}}, \dots, g_n^{\delta_n}$ is equal to Id.

On the other hand, by the definition of g_i , there are positive integers n_i such that

$$\text{Id} = g_{i(1)}^{n_1 \delta_{i(1)}} \cdots g_{i(s)}^{n_s \delta_{i(s)}}$$

where each $i(j)$ is between 1 and n . By hypothesis, $i(j) \leq k$ for at least one j . But this implies that the image of the right hand side of this equation under φ is in $P(H)$, which is a contradiction. \square

Theorem 2.1.10 has the corollary that a locally indicable group is LO. It is this corollary that will be most useful to us.

2.2. Circular orders. The approach we take in this section is modelled on [31], although an essentially equivalent approach is found in [11].

We first define a circular ordering on a set. Suppose p is a point in an oriented circle S^1 . Then $S^1 \setminus p$ is homeomorphic to \mathbb{R} , and the orientation on \mathbb{R} defines a natural total order on $S^1 \setminus p$. In general, a circular order on a set S is defined by a choice of total ordering on each subset of the form $S \setminus p$, subject to certain compatibility conditions which we formalize below.

Definition 2.2.1. Let S be a set. A *circular ordering* on a set S with at least 4 elements is a choice of total ordering on $S \setminus p$ for every $p \in S$, such that if \langle_p is the total ordering defined by p , and $p, q \in S$ are two distinct elements, the total orderings \langle_p, \langle_q differ by a *cut* on their common domain of definition. That is, for any x, y distinct from p, q , the order of x and y with respect to \langle_p and \langle_q is the same unless $x \langle_p q \langle_p y$, in which case we have $y \langle_q p \langle_q x$. We also say that the order \langle_q on $S \setminus \{p, q\}$ is obtained from the order \langle_p on $S \setminus p$ by *cutting at q* .

If S has exactly 3 elements $S = \{x, y, z\}$, we must add the condition that $y \langle_x z$ iff $z \langle_y x$. Note that this condition is implied by the condition in the previous paragraph if S has at least 4 elements.

To understand the motivation for the terminology, consider the operation of *cutting* a deck of cards.

Example 2.2.2. The oriented circle S^1 is circularly ordered, where for any p , the ordering \langle_p is just the ordering on $S^1 \setminus p \cong \mathbb{R}$ induced by the orientation on \mathbb{R} .

Definition 2.2.3. A set with three elements x, y, z admits exactly two circular orders, depending on whether $y <_x z$ or $z <_x y$. In the first case, we say the triple (x, y, z) is *positively ordered* and in the second case, we say it is *negatively ordered*.

We also refer to a positively ordered triple of points as *anticlockwise* and a negatively ordered triple as *clockwise*, by analogy with the standard circular order on triples of points in the positively oriented circle.

A circular ordering on a set S induces a circular ordering on any subset $T \subset S$. If T_α is a family of subsets of S which are all circularly ordered, we say the circular orderings on the T_α are *compatible* if they are simultaneously induced by some circular ordering on S .

It is clear that a circular ordering on a set S is determined by the family of circular orderings on all triples of elements in S . Conversely, the following lemma characterizes those families of circular orderings on triples of elements which arise from a circular ordering on all of S :

Lemma 2.2.4. *Suppose S is a set. A circular ordering on all triples of distinct elements on S is compatible iff for every subset $Q \subset S$ with four elements, the circular ordering on triples of distinct elements of Q is compatible. In any case, these circular orderings are uniquely compatible, and determine a circular ordering on S .*

Proof. A circular ordering on triples in S defines, for any $p \in S$, a binary relation $<_p$ on $S \setminus p$ by $x <_p y$ iff the triple (p, x, y) is positively ordered. To see that this binary relation defines a total ordering on $S \setminus p$, we must check transitivity of $<_p$. But this follows from compatibility of the circular ordering on quadruples Q . It is straightforward to check that the total orders $<_p$ defined in this way differ by a cut for distinct p, q . \square

Definition 2.2.5. Let C_1, C_2 be circularly ordered sets. A map $\phi : C_1 \rightarrow C_2$ is *monotone* if for each $c \in C_2$ and each $d \in \phi^{-1}(c)$, the restriction map between totally ordered sets

$$\phi : (C_1 \setminus \phi^{-1}(c), <_d) \rightarrow (C_2 \setminus c, <_c)$$

is monotone.

There is a natural topology on a circularly ordered set for which monotone maps are continuous.

Definition 2.2.6. Let $O, <$ be a totally ordered set. The *order topology* on O is the topology generated by open sets of the form $\{x | x > p\}$ and $\{x | x < p\}$ for all $p \in O$. Let S be a circularly ordered set. The *order topology* on S is the topology generated on each $S \setminus p$ by the (usual) order topology on the totally ordered set $S \setminus p, <_p$.

We now turn to the analogue of left ordered groups for circular orderings.

Definition 2.2.7. A group G is *left circularly ordered* if it admits a circular order as a set which is preserved by the action of G on itself on the left. A group is *left circularly orderable* if it can be left circularly ordered.

We usually abbreviate this by saying that a group is *circularly orderable* if it admits a *circular order*.

Example 2.2.8. A left orderable group $G, <$ is circularly orderable as follows: for each element $g \in G$, the total order $<_g$ on $G \setminus g$ is obtained from the total order $<$ by cutting at g .

Definition 2.2.9. The group of orientation-preserving homeomorphisms of \mathbb{R} is denoted $\text{Homeo}^+(\mathbb{R})$. The group of orientation-preserving homeomorphisms of the circle is denoted $\text{Homeo}^+(S^1)$.

An action of G on \mathbb{R} or the circle by orientation-preserving homeomorphisms is the same thing as a representation in $\text{Homeo}^+(\mathbb{R})$ or $\text{Homeo}^+(S^1)$. We will see that for countable groups G , being LO is the same as admitting a faithful representation in $\text{Homeo}^+(\mathbb{R})$, and CO is the same as admitting a faithful representation in $\text{Homeo}^+(S^1)$. First we give one direction of the implication.

Lemma 2.2.10. *If G is countable and admits a left-invariant circular order, then G admits a faithful representation in $\text{Homeo}^+(S^1)$.*

Proof. Let g_i be a countable enumeration of the elements of G . We define an embedding $e : G \rightarrow S^1$ inductively on i by the following conditions. Firstly, at every stage, the map

$$e : \bigcup_{i \neq n} g_i \rightarrow \bigcup_{i \leq n} e(g_i)$$

should be injective and circular order preserving, where the $e(g_i)$ are circularly ordered by the natural circular ordering on S^1 . Secondly, at every stage, the element $e(g_n)$ should be taken to the midpoint of the unique interval complementary to $\bigcup_{i < n} e(g_i)$ compatible with the first condition. This defines e uniquely.

The first two elements g_1, g_2 map to arbitrary distinct points in S^1 . Then, if e has been defined for all g_i with $i < n$, the element g_n will be mapped to the midpoint of the interval between $e(g_j)$ and $e(g_k)$, uniquely determined by the properties that

- (1) $e(g_j), e(g_n), e(g_k)$ are positive in the circular ordering
- (2) for all $l < n$ which are not equal to j, k the triples $e(g_j), e(g_l), e(g_n)$ and $e(g_n), e(g_l), e(g_k)$ are negative in the circular ordering

It is easy to see that the left action of G on itself extends uniquely to a continuous order preserving homeomorphism of the closure $\overline{e(G)}$ to itself. The complementary intervals I_i to $\overline{e(G)}$ are permuted by the action of G ; we choose an identification $\varphi_i : I_i \rightarrow I$ of each interval with I , and extend the action of G so that if $g(I_i) = I_j$ then the action of g on I_i is equal to

$$g|_{I_i} = \varphi_j^{-1} \varphi_i$$

This defines a faithful representation of G in $\text{Homeo}^+(S^1)$, as claimed. \square

Remark 2.2.11. Note that basically the same argument shows that a left orderable countable group is isomorphic to a subgroup of $\text{Homeo}^+(\mathbb{R})$. Notice further that this construction has an important property: if G is a countable left- or circularly ordered group, then G is circular or acts on \mathbb{R} in such a way that *some point has trivial stabilizer*. In particular, any point in the image of e has trivial stabilizer.

Short exact sequences intertwine circularity and left orderability:

Lemma 2.2.12. *Suppose*

$$0 \rightarrow K \rightarrow G \rightarrow H \rightarrow 0$$

is a short exact sequence, where K is left ordered and H is circularly ordered. Then G can be circularly ordered in such a way that the inclusion of K into G respects the order on $G \setminus g$ for any g not in K , and the map from G to H is monotone.

Proof. Let $\phi : G \rightarrow H$ be the homomorphism in the short exact sequence. Let g_1, g_2, g_3 be three distinct elements of G . We define the circular order as follows:

- (1) If $\phi(g_1), \phi(g_2), \phi(g_3)$ are distinct, circularly order them by the circular order on their image in H
- (2) If $\phi(g_1) = \phi(g_2)$ but these are distinct from $\phi(g_3)$, then $g_2^{-1}g_1 \in K$. If $g_2^{-1}g_1 < \text{Id}$ then g_1, g_2, g_3 is positively ordered, otherwise it is negatively ordered
- (3) If $\phi(g_1) = \phi(g_2) = \phi(g_3)$ then $g_3^{-1}g_1, g_3^{-1}g_2, \text{Id}$ are all in K , and therefore inherit a total ordering. The three corresponding elements of G in the same total order are negatively ordered

One can check that this defines a left-invariant circular order on G . \square

Here our convention has been that the orientation-preserving inclusion of \mathbb{R} into $S^1 \setminus p$ is order-preserving.

We will show that for countable groups, being LO or CO is equivalent to admitting a faithful representation in $\text{Homeo}^+(\mathbb{R})$ or $\text{Homeo}^+(S^1)$ respectively. But first we must describe an operation due to Denjoy [9] of *blowing up* or *Denjoying* an action.

Construction 2.2.13 (Denjoy). Let $\rho : G \rightarrow \text{Homeo}^+(S^1)$ be an action of a countable group on S^1 . For convenience, normalize S^1 to have length 1. Let $p \in S^1$ be some point. Let O denote the countable orbit of p under G , and let $\phi : O \rightarrow \mathbb{R}^+$ assign a positive real number to each $o \in O$ such that $\sum_{o \in O} \phi(o) = 1$. Choose some point q not in O , and define $\tau : [0, 1] \rightarrow S^1$ to be an orientation-preserving parameterization by length, which takes the two endpoints to q . Define $\sigma : [0, 1] \rightarrow [0, 2]$ by

$$\sigma(t) = t + \sum_{o \in O: \tau^{-1}(o) \leq t} \phi(o)$$

Then σ is discontinuous on $\tau^{-1}(O)$, and its graph can be completed to a continuous image of I in $[0, 1] \times [0, 2]$ by adding a vertical segment of length $\phi(o)$ at each point $\tau^{-1}(o)$ where $o \in O$. Identify opposite sides of $[0, 1] \times [0, 2]$ to get a torus, in which the closure of the graph of σ closes up to become a $(1, 1)$ curve which, by abuse of notation, we also refer to as σ . Notice that projection π_h onto the horizontal factor defines a monotone map from σ to S^1 .

Then the action of G on S^1 extends in an obvious way to an action on this torus which leaves the $(1, 1)$ curve invariant, and also preserves the foliations of the torus by horizontal and vertical curves. Up to conjugacy in $\text{Homeo}^+(\sigma)$, the action of G on σ is well-defined, and is called the *blown-up action at p* . The push-forward of this blown-up action under $(\pi_h)_*$ recovers the original action of G on S^1 ; that is, the two actions are related by a degree one *monotone map*, and are said to be *semi-conjugate*. The equivalence relation that this generates is called *monotone equivalence*.

With this construction available to us, we demonstrate the equivalence of CO with admitting a faithful representation in $\text{Homeo}^+(S^1)$.

Theorem 2.2.14. *Let G be a countable group. Then G is left or circularly ordered iff G admits a faithful homomorphism to $\text{Homeo}^+(\mathbb{R})$ or $\text{Homeo}^+(S^1)$ respectively. Moreover, the action on \mathbb{R} or S^1 can be chosen so that some point has a trivial stabilizer.*

Proof. In lemma 2.2.10 we have already showed how a left or circular order gives rise to a faithful action on \mathbb{R} or S^1 . So it remains to prove the converse.

Let $\phi : G \rightarrow \text{Homeo}^+(\mathbb{R})$ be faithful. Let p_i be some sequence of points such that the intersection of the stabilizers of the p_i is the identity. Some such sequence p_i exists, since G is countable, and any nontrivial element acts nontrivially at some point. Then each p_i determines a (degenerate) left-invariant order on G , by setting $g >_i h$ if $g(p_i) > h(p_i)$, and $g =_i h$ if $g(p_i) = h(p_i)$. Then we define $g > h$ if $g >_i h$ for some i , and $g =_j h$ for all $j < i$.

The definition of a circular order is similar: pick some point $p \in S^1$, and suppose that the stabilizer $\text{stab}(p)$ is nontrivial. Then $\text{stab}(p)$ acts faithfully on $S^1 - p = \mathbb{R}$, so by the argument above, $\text{stab}(p)$ is left orderable and acts on \mathbb{R} . In fact, we know $\text{stab}(p)$ acts on \mathbb{R} in such a way that some point has trivial stabilizer. Let $\varphi : \text{stab}(p) \rightarrow \text{Homeo}^+(\mathbb{R})$ be such a representation. We construct a *new* representation $\phi' : G \rightarrow \text{Homeo}^+(S^1)$ from ϕ by *blowing up* p as in construction 2.2.13. The representation ϕ' is monotone equivalent to ϕ ; that is, there is a monotone map $\pi : S^1 \rightarrow S^1$ satisfying

$$\pi_*\phi' = \phi$$

Let $C \subset S^1$ be the set where the monotone map π is not locally constant. We will modify the action of G on $S^1 \setminus C$ as follows. Note that G acts on C by the pullback under π of the action on S^1 by ϕ . We extend this action to $S^1 \setminus C$ to define ϕ'' . Let I be the open interval obtained by blowing up p . We identify I with \mathbb{R} , and then let $\text{stab}(p)$ act on I by the pullback of φ under this identification. Then for each other component I_i in $S^1 \setminus C$ is of the form $g(I)$ for some $g \in G$. Choose such a g_i for each I_i , and pick an arbitrary (orientation preserving) identification $\varphi_i : I \rightarrow I_i$, and define $\phi''(g_i)|_I = \varphi_i$. Now, for any $g \in G$, define $g|_{I_i}$ as follows: suppose $g(I_i) = I_j$. Then $g_j^{-1}gg_i \in \text{stab}(p)$, so define

$$\phi''(g)|_{I_i} = \varphi_j \varphi(g_j^{-1}gg_i) \varphi_i^{-1} : I_i \rightarrow I_j$$

It is easy to see that this defines a faithful representation $\phi'' : G \rightarrow \text{Homeo}^+(S^1)$, monotone equivalent to ϕ , with the property that some point $q \in S^1$ has trivial stabilizer.

Now define a circular order on distinct triples g_1, g_2, g_3 by restricting the circular order on S^1 to the triple $g_1(q), g_2(q), g_3(q)$. \square

Notice that in this theorem, in order to recover a left- or circular order on G from a faithful action, all we used about \mathbb{R} and S^1 was that they were ordered and circularly ordered sets respectively.

With this theorem, and our lemmas on short exact sequences, we can deduce the existence of left- or circular orders on countable groups from the existence of actions on ordered or circularly ordered sets, with left orderable kernel.

Theorem 2.2.15. *Suppose a countable group G admits an action by order preserving maps on a totally ordered or circularly ordered set S in such a way that the kernel K is left orderable. Then G admits a faithful, order preserving action on \mathbb{R} or S^1 , respectively.*

Proof. We discuss the case that S is circularly ordered, since this is slightly more complicated. Since G is countable, it suffices to look at an orbit of the action, which will also be countable. By abuse of this notation, we also denote the orbit by S . As in lemma 2.2.10, the set S with its order topology is naturally order isomorphic to a subset of S^1 . Let \bar{S} denote the closure of S under this identification. Then the

action of G on S extends to an orientation-preserving action on S^1 , by permuting the complementary intervals to \bar{S} . It follows that the image of G in $\text{Homeo}^+(S^1)$ is CO, with kernel K . By lemma 2.2.12, G is CO.

The construction for S totally ordered is similar. \square

2.3. Homological characterization of circular groups. Circular orders on groups G can be characterized homologically. There are at least two different ways of doing this, due to Thurston and Ghys respectively, which reflect two different ways of presenting the theory of group cohomology.

First, we recall the definition of group cohomology. For details, we refer to [16] or [15].

Definition 2.3.1. Let G be a group. The *homogeneous chain complex* of G is a complex $C_*(G)_h$ where $C_i(G)_h$ is the free abelian group generated by equivalence classes of $(i+1)$ -tuples $(g_0 : g_1 : \dots : g_n)$, where two such tuples are equivalent if they are in the same coset of the left diagonal action of G on the co-ordinates. That is,

$$(g_0 : g_1 : \dots : g_n) \sim (gg_0 : gg_1 : \dots : gg_n)$$

The boundary operator in homogeneous co-ordinates is very simple, defined by the formula

$$\partial(g_0 : \dots : g_n) = \sum_{i=0}^n (-1)^i (g_0 : \dots : \hat{g}_i : \dots : g_n)$$

The *inhomogeneous chain complex* of G is a complex $C_*(G)_i$ where $C_i(G)_i$ is the free abelian group generated by i -tuples (f_1, \dots, f_n) . The boundary operator in inhomogeneous co-ordinates is more complicated, defined by the formula

$$\begin{aligned} \partial(f_1, \dots, f_n) &= (f_2, \dots, f_n) + \sum_{i=1}^{n-1} (-1)^i (f_1, \dots, f_i f_{i+1}, \dots, f_n) \\ &\quad + (-1)^n (f_1, \dots, f_{n-1}) \end{aligned}$$

The relation between the two co-ordinates comes from the following bijection of generators

$$(g_0 : g_1 : \dots : g_n) \rightarrow (g_0^{-1}g_1, g_1^{-1}g_2, \dots, g_{n-1}^{-1}g_n)$$

which correctly transforms one definition of ∂ to the other. It follows that the two chain complexes are canonically isomorphic, and therefore by abuse of notation we denote either by $C_*(G)$, and write an element either in homogeneous or inhomogeneous co-ordinates as convenient.

Let R be a commutative ring. The homology of the complex $C_*(G) \otimes R$ is denoted $H_*(G; R)$, and the homology of the adjoint complex $\text{Hom}(C_*(G), R)$ is denoted $H^*(G; R)$. If $R = \mathbb{Z}$, we abbreviate these groups to $H_*(G)$ and $H^*(G)$ respectively. If G is a topological group, and we want to stress that this is the abstract group (co)homology, we denote these groups by $H_*(G^\delta)$ and $H^*(G^\delta)$ respectively (δ denotes the discrete topology).

We give a geometrical interpretation of this complex. The simplicial realization of the complex $C_*(G)$ is a model for the classifying space BG , where G has the discrete topology. If G is torsion free, an equivalent model for EG is the complete simplex on the elements of G . In this case, since G is torsion free, it acts freely and properly discontinuously on this simplex, with quotient BG . If we label vertices of

EG , the labels on each simplex give homogeneous co-ordinates on the quotient. If we label edges of EG by the difference of the labels on the vertices at the end, then the labels are well-defined on the quotient; the labels on the i edges between consecutive vertices, with respect to a total order of the vertices, give inhomogeneous co-ordinates.

In particular, the cohomology $H^*(G)$ is just the cohomology of the $K(G, 1)$, that is, of the unique (up to homotopy) aspherical space with fundamental group isomorphic to G . If G is not torsion free, this equality of groups is nevertheless true.

The cohomology of the group $\text{Homeo}^+(S^1)$ is known by a general theorem of Mather and Thurston (see [29] or [33] for details and more references):

Theorem 2.3.2 (Mather, Thurston). *For any manifold M , there is an isomorphism of cohomology rings*

$$H^*(\text{Homeo}(M)^\delta; \mathbb{Z}) \cong H^*(B\text{Homeo}(M); \mathbb{Z})$$

where $B\text{Homeo}(M)$ denotes the classifying space of the topological group of homeomorphisms of M , and the left hand side denotes the group cohomology of the abstract group of homeomorphisms of M .

In particular, for $M = S^1$ or \mathbb{R}^2 , the group $\text{Homeo}^+(M)$ is homotopy equivalent to a circle. It follows that $B\text{Homeo}^+(M)$ in either case is homotopic to $\mathbb{C}\mathbb{P}^\infty$, and therefore there is an isomorphism of rings

$$H^*(\text{Homeo}^+(\mathbb{R}^2); \mathbb{Z}) \cong H^*(\text{Homeo}^+(S^1); \mathbb{Z}) \cong \mathbb{Z}[e]$$

where $[e]$ is a free generator in degree 2 called the *Euler class*.

An algebraic characterization of the Euler class can be given.

Definition 2.3.3. For any group G with $H^1(G; \mathbb{Z}) = 1$, there is a *universal central extension*

$$0 \rightarrow A \rightarrow \widehat{G} \rightarrow G \rightarrow 0$$

where A is abelian, with the property that for any other central extension

$$0 \rightarrow B \rightarrow G' \rightarrow G \rightarrow 0$$

there is a unique homomorphism from $\widehat{G} \rightarrow G'$, extending uniquely to a morphism of short exact sequences.

A central extension G' can be characterized as the universal central extension of G iff G is perfect (i.e. $H^1(G; \mathbb{Z}) = 1$) and every central extension of G' splits. See Milnor [19] for more details.

For $G = \text{Homeo}^+(S^1)$, the central extension is denoted $\widetilde{\text{Homeo}}^+(S^1)$, and can be identified with the preimage of $\widetilde{\text{Homeo}}^+(S^1)$ in $\text{Homeo}^+(\mathbb{R})$ under the covering map $\mathbb{R} \rightarrow S^1$. The center of $\widetilde{\text{Homeo}}^+(S^1)$ is \mathbb{Z} , and the class of this \mathbb{Z} extension is called the Euler class. By the universal property of this extension, one sees that this class is the generator of $H^2(\text{Homeo}^+(S^1); \mathbb{Z})$. This can be summarized by a short exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \widetilde{\text{Homeo}}^+(S^1) \rightarrow \text{Homeo}^+(S^1) \rightarrow 0$$

The following construction is found in [31]. An equivalent construction is given in [14].

Construction 2.3.4 (Thurston). Let G be a countable CO group. Let $\rho : G \rightarrow \text{Homeo}^+(S^1)$ be constructed as in theorem 2.2.14 so that the point p has trivial stabilizer. For each triple $g_0, g_1, g_2 \in G$ of distinct elements, define the cocycle

$$c(g_0 : g_1 : g_2) = 1 \text{ if } (g_0(p), g_1(p), g_2(p)) \text{ is positively oriented, and } -1 \text{ otherwise}$$

It is clear that c is well-defined on the homogeneous co-ordinates for $C_2(G)$. Then extend c to degenerate triples by setting it equal to 0 if at least two of its coefficients are equal.

The fact that the circular order on triples of points in S^1 is compatible on quadruples is exactly the condition that the coboundary of c is 0 — that is, c is a cocycle, and defines an element $[c] \in H^2(G; \mathbb{Z})$.

The following (related) construction is found in [11]:

Construction 2.3.5 (Ghys). Let G be a countable CO group. Let $\rho : G \rightarrow \text{Homeo}^+(S^1)$ be constructed as in theorem 2.2.14. By abuse of notation, we identify G with its image $\rho(G)$. Let \widehat{G} denote the preimage of G in $\widetilde{\text{Homeo}^+(S^1)} \subset \text{Homeo}^+(\mathbb{R})$. Define a section $s : G \rightarrow \widehat{G}$ uniquely by the property that $s(g)(0) \in [0, 1)$. For each pair of elements $g_0, g_1 \in G$, define the cocycle

$$e(g_0, g_1) = s(g_0 g_1)^{-1} s(g_0) s(g_1)(0)$$

Then one can check that e is a cocycle on $C_2(G)$ in inhomogeneous co-ordinates, and defines an element $[e] \in H^2(G; \mathbb{Z})$. Moreover, e takes values in $\{0, 1\}$.

The following lemma can be easily verified; for a proof, we refer to [31] or [14].

Lemma 2.3.6 (Ghys, Jekel, Thurston). *Let G be a countable circularly ordered group. The cocycles e, c satisfy*

$$2[e] = [c]$$

Moreover, the class $[e]$ is the Euler class of the circular order on G .

Actually, the restriction to countable groups is not really necessary. One can define the cocycles c, e directly from a circular order on an arbitrary group G . This is actually done in [31] and [11]; we refer the reader to those papers for the more abstract construction.

Theorem 2.3.7. *Let G be a circularly ordered group with Euler class $[e] \in H^2(G; \mathbb{Z})$. If $[e] = 0$, then G is left ordered. In any case, the central extension of G corresponding to the class $[e]$ is left orderable.*

Proof. We prove the theorem for G countable; the general case is proved in [11].

From the definition of s in construction 2.3.5 and lemma 2.3.6, we see that e is the obstruction to finding some (possibly different) section $G \rightarrow \widehat{G}$. But \widehat{G} is a subgroup of the group $\text{Homeo}^+(\mathbb{R})$. Now, every finitely generated subgroup of $\text{Homeo}^+(\mathbb{R})$ is left orderable, by theorem 2.2.14. It follows by lemma 2.1.6 that the entire group $\text{Homeo}^+(\mathbb{R})$ is left orderable; in particular, so is \widehat{G} . \square

2.4. Bounded cohomology and the Milnor–Wood inequality. Construction 2.3.4 and construction 2.3.5 do more than give an explicit representative cocycle of the Euler class; they verify that this cocycle has a further additional property, namely that the Euler class is a *bounded cocycle* on G .

Definition 2.4.1. Suppose $R = \mathbb{R}$ or \mathbb{Z} . Define an L_1 norm on $C_i(G)$ in the obvious way by

$$\left\| \sum_j s_j (g_0(j) : g_1(j) : \cdots : g_i(j)) \right\|_1 = \sum_j |s_j|$$

Dually, the L_∞ norm is partially defined on $\text{Hom}(C_i(G); R)$, and the subspace consisting of homomorphisms of finite L_∞ norm is denoted $\text{Hom}_b(C_i(G); R)$. The coboundary takes cochains of finite L_∞ norms to cocycles of finite L_∞ norm, and therefore we can take the homology of the subcomplex. This homology is denoted $H_b^*(G; \mathbb{R})$ and is called the *bounded cohomology* of G . For an element $\alpha \in H_b^*(G; \mathbb{R})$, the norm of α , denoted $\|\alpha\|_\infty$ or just $\|\alpha\|$, is the infimum of $\|c\|_\infty$ over cocycles c with $[c] = \alpha$.

When we do not make coefficients explicit, the *norm* of a bounded cocycle refers to its norm amongst representatives with \mathbb{R} coefficients.

In this language, the famous Milnor–Wood inequality [20], [34] can be expressed as follows:

Theorem 2.4.2 (Milnor–Wood). *Let G be a circularly ordered group. Then the Euler class $[e]$ of G is an element of $H_b^2(G)$ with norm $\|[e]\| \leq \frac{1}{2}$.*

Proof. Let e be the cocycle constructed by Ghys. Then $\frac{e}{2} = e - \frac{1}{2}$ is homologous to e , and has norm $\leq \frac{1}{2}$. \square

We will see in §4 that although $\text{Homeo}^+(S^1)$ and $\text{Homeo}^+(\mathbb{R}^2)$ have the same cohomology as abstract groups, their *bounded cohomology* groups are very different. This difference persists to the smooth category, as we shall see.

3. PLANAR GROUPS WITH BOUNDED ORBITS

The purpose of this section is to show that every group of C^1 orientation-preserving homeomorphisms of \mathbb{R}^2 with a bounded orbit is circularly orderable. The main tools will be the Thurston stability theorem, and certain generalizations of the braid groups. In the course of the proof we also show that the mapping class group of a compact totally disconnected set in the plane is circularly orderable, and the mapping class group rel. boundary of such a set in the disk is left orderable. This generalizes a theorem of Dehornoy [7] on orderability of the usual (finitely generated) braid groups.

3.1. Prime ends. In this section we describe some elements of the theory of prime ends. For details of proofs and references, consult [25] or [17].

Prime ends are a technical tool, developed in conformal analysis, to study the boundary behaviour of conformal maps which take the unit disk in \mathbb{C} to the interior of a region K whose boundary is not locally connected. They were introduced by Carathéodory in [6]. If ∂K is locally connected, the the prime ends of ∂K are just the proper homotopy classes of proper rays in U .

Definition 3.1.1. Let U be a bounded open subset of \mathbb{R}^2 , and let K denote its closure. Fix the notation $\partial K = K \setminus U$. Notice that this might not be the frontier of K in the usual sense. A *null chain* is a sequence of proper arcs $(C_i, \partial C_i) \subset (K, \partial K)$ where $\text{int}(C_i) \subset U$ is an embedding, but whose endpoints are not necessarily distinct, such that the following conditions are satisfied:

- (1) $\overline{C_n} \cap \overline{C_{n+1}} = \emptyset$

- (2) C_n separates C_0 from C_{n+1} in U
- (3) The diameter of the C_n converges to 0 as $n \rightarrow \infty$.

A *prime end* of U is an equivalence class of null chains, where two null chains $\{C_i\}, \{C'_i\}$ are equivalent if for every sufficiently large m , there is an n such that C'_m separates C_0 from C_n , and C_m separates C'_0 from C'_n .

For such a set U , denote the set of prime ends of U by $\mathcal{P}(U)$.

Example 3.1.2. Let D denote the open unit disk in \mathbb{C} . Then $\mathcal{P}(D)$ is in natural bijective correspondence with $\partial\bar{D}$.

Notice that any homeomorphism of \mathbb{R}^2 which fixes U as a set will induce an automorphism of $\mathcal{P}(U)$. On the other hand, a self-homeomorphism of U which does not extend continuously to \bar{U} will not typically induce an automorphism of $\mathcal{P}(U)$, since for instance the image of a properly embedded arc C_i with endpoints on $\bar{U} \setminus U$ will not necessarily limit to well-defined endpoints in U .

Lemma 3.1.3. *Let U be a simply-connected, bounded, open subset of \mathbb{R}^2 . Then the set of prime ends $\mathcal{P}(U)$ admits a natural circular order.*

Proof. Let $\varphi : U \rightarrow D$ be a uniformizing map. Then the set of prime ends of U is taken bijectively to the set of prime ends of D . This is not entirely trivial; it is contained in proposition 2.14 and theorem 2.15 in [25]. In any case, this map identifies $\mathcal{P}(U)$ with $\partial\bar{D}$. Thus $\mathcal{P}(U)$ inherits a natural circular ordering from $\partial\bar{D}$. If φ' is another uniformizing map, then $\varphi' \circ \varphi^{-1}$ is a Möbius transformation of $\partial\bar{D}$, and therefore preserves the circular order. \square

If $\varphi : U \rightarrow V$ is a conformal map between simply connected domains, then let $\varphi_* : \mathcal{P}(U) \rightarrow \mathcal{P}(V)$ denote the corresponding map between prime ends.

Lemma 3.1.4. *Suppose $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ fixes U as a set, and fixes the prime ends of U . Then φ fixes $\bar{U} \setminus U$ pointwise.*

Proof. If $f : D \rightarrow U$ is a uniformizing map, then f has a radial limit at $\zeta \in S^1$ iff the prime end $f_*(\zeta) \in \mathcal{P}(U)$ is *accessible*; that is, if there is a Jordan arc that lies in U except for one endpoint, and that intersects all but finitely many crosscuts of some null-chain of $f_*(\zeta)$. The endpoints of this Jordan arc is called an *accessible point*. It is known (see [25]) that the set of accessible points is *dense* in $\bar{U} \setminus U$. But an automorphism which fixes all prime ends must fix all accessible points, and therefore must fix $\bar{U} \setminus U$ pointwise. \square

3.2. Groups which stabilize a point. In this subsection we state the Thurston stability theorem, and from this deduce information about the group of C^1 homeomorphisms of \mathbb{R}^2 which stabilizes a point.

The Thurston stability theorem is proved in [30]. Since many people will only be familiar with the 1-dimensional version of this theorem, we indicate the idea of the proof.

Theorem 3.2.1 (Thurston stability theorem). *Let p be a point in a smooth manifold M^n . Let G be the group of germs at p of C^1 homeomorphisms of M^n which fix p . Let $L : G \rightarrow GL(n, \mathbb{R})$ denote the natural homomorphism obtained by linearizing G at p . Let $L(G)$ denote the image of G , and $K(G)$ the kernel of L . Then $K(G)$ is locally indicable.*

Proof. The idea of the proof is as follows. Let $H < K(G)$ be a finitely generated subgroup, with generators h_1, \dots, h_m . Let $p_i \rightarrow p$ be some convergent sequence. If we rescale the action near p_i so that every h_j moves points a bounded distance, but some $h_k(i)$ moves points distance 1, then the rescaled actions vary in a pre-compact family. It follows that we can extract a limiting nontrivial action, which by construction will be an action by *translations*. In particular, H is indicable, and $K(G)$ is locally indicable.

Now we make this more precise. Change co-ordinates so that p is at the origin. Then each generator h_i can be expressed in local co-ordinates as a sum

$$h_i(x) = x + y(h_i)(x)$$

where $|y(h_i)(x)| = o(x)$ and satisfies $y(h_i)'|_0 = 0$. For each $\epsilon > 0$, let U_ϵ be an open neighborhood of 0 on which $|y(h_i)'| < \epsilon$ and $|y(h_i)| < |x|\epsilon$. Now, for two indices i, j the composition has the form

$$\begin{aligned} h_i \circ h_j(x) &= x + y(h_j)(x) + y(h_i)(x + y(h_j)(x)) \\ &= x + y(h_j)(x) + y(h_i)(x) + O(\epsilon y(h_j)(x)) \end{aligned}$$

In particular, the composition deviates from $x + y(h_i)(x) + y(h_j)(x)$ by a term which is small compared to $\max(y(h_i)(x), y(h_j)(x))$.

Now, choose some sequence of points $x_i \rightarrow 0$. For each i define the map $v_i : H \rightarrow \mathbb{R}^n$ where $v_i(h) = y(h)(x_i)$. Let $w_i = \sup_{j \leq m} |v_i(h_j)|$, and define $v'_i(h) = v_i/w_i$. It follows that the functions w_i are uniformly bounded on each $h \in H$, and therefore there is some convergent subsequence. Moreover, by the estimate above, on this subsequence, the maps w_i converge to a homomorphism $w : H \rightarrow \mathbb{R}^n$. On the other hand, by construction, there is some index j such that $|v'_i(h_j)| = 1$. In particular, the homomorphism w is nontrivial, and H surjects onto a free abelian group, and we are done. \square

Lemma 3.2.2. *Let G be a group of germs of orientation preserving C^1 homeomorphisms of \mathbb{R}^2 at a fixed point p . Then G is circularly orderable. Moreover, if the image $L(G)$ of G in $GL^+(2, \mathbb{R})$ obtained by linearizing the action at p is left orderable, then G itself is left orderable.*

Proof. By the Thurston stability theorem, if $L : G \rightarrow L(G)$ denotes the homomorphism onto the linear part of $L(G)$, then the kernel $K(G)$ is locally indicable, and therefore by theorem 2.1.10, $K(G)$ is LO. That is, we have a short exact sequence

$$0 \rightarrow K(G) \rightarrow G \rightarrow L(G) \rightarrow 0$$

where $K(G)$ is LO. If the image $L(G)$ is LO, then so is G by lemma 2.1.8.

Moreover, since G is orientation preserving, $L(G) < GL^+(2, \mathbb{R})$ where GL^+ denotes the subgroup of GL with positive determinant. There is a homomorphism from $GL^+(2, \mathbb{R})$ to $SL(2, \mathbb{R})$ with kernel \mathbb{R}^+ . As a short exact sequence,

$$0 \rightarrow \mathbb{R}^+ \rightarrow GL^+(2, \mathbb{R}) \rightarrow SL(2, \mathbb{R}) \rightarrow 0$$

The group $SL(2, \mathbb{R})$ double covers $PSL(2, \mathbb{R})$, and can be thought of as the group of orientation-preserving transformations of the connected double cover of \mathbb{RP}^1 which are the pullback of projective transformations of \mathbb{RP}^1 by $PSL(2, \mathbb{R})$. In particular, $SL(2, \mathbb{R})$ is a subgroup of $\text{Homeo}^+(S^1)$, and is therefore CO. It follows by lemma 2.2.12 that $GL^+(2, \mathbb{R})$ is CO, and therefore so is $L(G)$.

By another application of lemma 2.2.12, G is CO. \square

3.3. Groups which stabilize one or more compact regions. The main point of this subsection is to prove theorem 3.3.6, which says that a group of C^1 orientation-preserving homeomorphisms of the plane which fixes a compact set K with connected complement is circularly orderable, and a group which fixes at least two such sets is left orderable.

Now, suppose that a group G stabilizes the disjoint compact connected sets K_1, K_2, \dots . For each K_i , exactly one complementary component is unbounded. So without loss of generality, we can fill in these bounded complementary regions and assume G stabilizes disjoint compact connected sets K_1, \dots with connected complement for all i .

Remark 3.3.1. A compact set K_i which is an absolute neighborhood retract satisfies the hypotheses of Alexander duality, and has connected complement in \mathbb{R}^2 iff $H_1(K_i; \mathbb{Z}) = 0$. However, the well-known example of the “topologist’s circle” shows that in general, the vanishing of homology is not sufficient to show that the complement is connected.

We show how each region K_i gives rise to a CO group G_i which is naturally a quotient of G .

Construction 3.3.2. If K_i consists of more than one point, the complement of K_i in the sphere $S^2 = \mathbb{R}^2 \cup \infty$ is conformally a disk, and we let G_i denote the image of G in $\text{Aut}(\mathcal{P}(S^2 \setminus K_i))$. Note that G_i is CO, by lemma 3.1.3.

If K_i consists of a single point and G acts C^1 near K_i , let G_i denote the germ of G at K_i . By lemma 3.2.2, G_i is CO, and is actually LO if the linear part $L(G)$ of G at K_i is LO.

Remark 3.3.3. Notice that there is a natural circular order on G_i in the first case, but *not* in the second. However, in the second case, the *Euler class* of the circular order provided by lemma 3.2.2 is just the circular Euler class of the linear part $L(G)$ of G at K_i , acting projectively on the unit tangent bundle, and is therefore natural.

In particular, the set K_i gives a homomorphism from G to a product

$$G \rightarrow \prod_i G_i$$

of CO groups. The product of any number of left orderable groups is left orderable. This follows immediately from lemma 2.1.8 and lemma 2.1.6. But a product of CO groups is not necessarily CO. In this section we will show that in the context above, if G stabilizes two disjoint compact connected sets K_1, K_2 with connected complement, then the groups G_1, G_2 are actually both LO.

Lemma 3.3.4. *Let G act by C^1 orientation-preserving homeomorphisms on the plane. Suppose that G stabilizes two disjoint compact connected sets K_1, K_2 with connected complement. Then the groups G_1, G_2 provided by construction 3.3.2 are left orderable, and their Euler classes are zero.*

Proof. First suppose K_1 contains at least two points.

For a compact, connected set $K_1 \subset \mathbb{R}^2$ with connected complement, the complement of $\mathbb{R}^2 \setminus K_1$ is homeomorphic to an annulus A_1 . Let \widetilde{A}_1 denote the universal

cover of A_1 , and denote by \widehat{G}_1 the central extension of G_1

$$0 \rightarrow \mathbb{Z} \rightarrow \widehat{G}_1 \rightarrow G_1 \rightarrow 0$$

This requires some explanation. The Riemann surface \widetilde{A}_1 is noncompact, and topologically an open disk. Under a uniformizing map, there is a unique point $p \in S^1$ such that every proper ray r in \widetilde{A}_1 which projects to an unbounded proper ray in A_1 is asymptotic to p . If we cut S^1 at p , we get a copy of \mathbb{R} , which by abuse of notation we refer to as the set of prime ends $\mathcal{P}(\widetilde{A}_1)$

If K_1 consists of more than one point, there is a natural map $\mathcal{P}(\widetilde{A}_1) \rightarrow \mathcal{P}(S^2 \setminus K_1)$ which is just a covering map $\mathbb{R} \rightarrow S^1$ under the identification of $\mathcal{P}(S^2 \setminus K_1)$ with $\mathcal{P}(D) = S^1$ by a uniformizing map. Then the group \widehat{G}_1 is just the usual preimage of a subgroup of $\text{Homeo}^+(S^1)$ in $\widetilde{\text{Homeo}}^+(S^1)$.

In particular, the class of this extension is the Euler class of the circular ordering on G_i . In order to show that this Euler class is trivial, it suffices to show that this extension splits. To do this, it suffices to show that the restriction homomorphism $\text{res} : G \rightarrow \text{Homeo}^+(A_1)$ lifts to the covering space $\widehat{\text{res}} : G \rightarrow \text{Homeo}^+(\widetilde{A}_1)$.

Let \widehat{K}_2 be a lift of K_2 to \widetilde{A}_1 . Then for each element $g \in G$, there is a unique lift $\widehat{res} : g \rightarrow \text{Homeo}^+(\widetilde{A}_1)$ which stabilizes \widehat{K}_2 . By uniqueness, this defines the desired section.

The case that K_1 consists of a single point is very similar. The complement $\mathbb{R}^2 \setminus K_1$ is again an annulus, and there is a covering $\widetilde{A}_1 \rightarrow A_1$. There is a central extension

$$0 \rightarrow \mathbb{Z} \rightarrow \widehat{G}_1 \rightarrow G_1 \rightarrow 0$$

where G_1 is the germ of G at K_1 , and \widehat{G}_1 is the germ of the preimage of G in $\text{Homeo}^+(\widetilde{A}_1)$ near infinity. Again, choosing a lift \widehat{K}_2 of K_2 to \widetilde{A}_1 determines a unique section $G \rightarrow \text{Homeo}^+(\widetilde{A}_1)$ whose germ is \widehat{G}_1 . So the extension splits, and the Euler class is trivial in this case too. \square

We now establish a technical lemma about groups which act in a C^1 fashion and have indiscrete fixed point set.

Lemma 3.3.5. *Suppose G acts faithfully by C^1 orientation-preserving homeomorphisms of the plane, and suppose that the fixed point set $\text{fix}(G)$ is not discrete. Then G is left orderable.*

Proof. Let $p \in \text{fix}(G)$ in the frontier of $\text{fix}(G)$ be a limit point of distinct $p_i \in \text{fix}(G)$. Then the p_i contain a subsequence which approach p radially along some vector v . It follows that the linearization of G at p , denoted $L(G)$, fixes the vector v , and therefore the image is contained in some conjugate of the affine group of the line $\text{Aff}^+(\mathbb{R})$. The affine group of the line is an extension of \mathbb{R} by \mathbb{R} :

$$0 \rightarrow \mathbb{R} \rightarrow \text{Aff}^+(\mathbb{R}) \rightarrow \mathbb{R} \rightarrow 0$$

and is therefore left orderable by lemma 2.1.8.

It follows from lemma 3.2.2 that the germ of G at p is LO. In particular, if H is any finitely generated subgroup of G , then we can apply this reasoning to a limit point in the frontier of $\text{fix}(H)$ and see that the germ of H at this limit point is left orderable. In particular, H is LO-surjective. Since H was arbitrary, by theorem 2.1.10 the group G is LO as required. \square

We now have all the tools to prove the main result of this subsection:

Theorem 3.3.6. *Let G act faithfully by C^1 orientation–preserving homeomorphisms of the plane. Suppose that G stabilizes disjoint compact connected sets $\{K_i\}$ such that each K_i has connected complement. Then G is CO. If there is more than one region K_i , then G is LO.*

Proof. By lemma 3.3.4 there is a homomorphism

$$G \rightarrow \prod_i G_i$$

where the image group is CO, and is LO if there is more than one region K_i .

Let K denote the kernel of this homomorphism. For each region K_i with more than one point, lemma 3.1.4 implies that K fixes ∂K_i pointwise. For each region K_i consisting of a single point, every element of K fixes some neighborhood of K_i pointwise. It follows that for each finitely generated subgroup H of K the set of fixed points of H is not isolated. In particular, by lemma 3.3.5, every finitely generated subgroup of K is LO, and therefore K is LO by lemma 2.1.6.

By lemma 2.2.12, G is CO, or LO if there is more than one region K_i . \square

3.4. Totally disconnected planar sets. If G acts faithfully on \mathbb{R}^2 and merely *permutes* a collection of disjoint compact regions $\{K_i\}$ we can study the image of G in the group of homotopy classes of *all* orientation preserving homeomorphisms of \mathbb{R}^2 which permute the $\{K_i\}$. We will call such a group a generalized braid group. The following construction is a convenient simplification.

Construction 3.4.1. Suppose G acts faithfully on \mathbb{R}^2 and permutes a collection of disjoint compact connected sets $\{K_i\}$. Suppose further that the union $\cup_i K_i$ is bounded. Then G preserves the closure of the union of the K_i , whose components we label and add to the collection $\{K_i\}$.

Fill in each component with disconnected complement by adding the bounded open complementary regions, and throw away K_i which are engulfed by this process. This gives a new compact collection of disjoint compact connected sets, which by abuse of notation we relabel $\{K_i\}$ where now each K_i has connected complement. Then we can define an equivalence relation on \mathbb{R}^2 by quotienting each K_i to a point.

Definition 3.4.2. A *decomposition* \mathcal{G} of a space is a partition into compact subsets. A decomposition is *upper semi–continuous* if for every decomposition element $\zeta \in \mathcal{G}$ and every open set U with $\zeta \subset U$, there exists an open set $V \subset U$ with $\zeta \subset V$ such that every $\zeta' \in \mathcal{G}$ with $\zeta' \cap V \neq \emptyset$ has $\zeta \subset U$. The decomposition is *monotone* if its elements are connected.

It is clear that the equivalence relation in construction 3.4.1 is monotone and upper semi–continuous.

The following theorem is proved by R. L. Moore in [22]:

Theorem 3.4.3 (Moore). *Let \mathcal{G} be a monotone upper semi–continuous decomposition of a topological sphere S such that no decomposition element separates S . Then the quotient space obtained by collapsing each decomposition element to a point is a topological sphere.*

Definition 3.4.4. Suppose G acts on a space X and preserves an equivalence relation \sim . Then the action of G descends to an action on X/\sim . We say the two actions are *monotone equivalent*.

Lemma 3.4.5. *Let G act on \mathbb{R}^2 in such a way as to permute a union of disjoint compact regions. Then this action is monotone equivalent to an action which leaves invariant a compact, totally disconnected, G -invariant subset.*

Proof. The fact that \mathbb{R}^2 / \sim is homeomorphic to \mathbb{R}^2 follows from Moore’s theorem 3.4.3. Since construction 3.4.1 is natural, the equivalence relation it defines is G -invariant, and therefore the action of G descends to the quotient. It is clear from the construction that C is totally disconnected. \square

Since G acts on \mathbb{R}^2 and permutes C , we can study a *minimal subset* of C ; that is, a closed invariant nonempty subset of \mathbb{R}^2 which is minimal with respect to these properties. Amongst all invariant subsets, a minimal subset C can be characterized by the property that every G orbit contained in C is dense in C . Such a subset will be either finite or *perfect*. Here, a set is *perfect* if every point is a limit point.

The Moore–Kline theorem gives a useful “normal form” for totally disconnected compact planar sets. This theorem is proved in [23]:

Theorem 3.4.6 (Moore, Kline). *Let C be a totally disconnected compact set in \mathbb{R}^2 . Then there is a homeomorphism of \mathbb{R}^2 to itself which takes C to a closed subset of the arc $[0, 1]$ contained in the x -axis.*

We will use this theorem in the next subsection. Note that this theorem implies that a perfect, totally disconnected subset of \mathbb{R}^2 is a tame Cantor set.

3.5. Generalized braid groups.

Definition 3.5.1. For a surface S possibly with boundary ∂S , the *relative orientation-preserving mapping class group*, denoted $\text{MCG}^+(S, \partial S)$, is the group of homotopy classes of orientation-preserving homeomorphisms of S to itself which are fixed on ∂S . We generally just call this the mapping class group for short.

Let C be a compact, totally disconnected subset of the open unit disk D . The *generalized braid group* B_C is the relative mapping class group $\text{MCG}^+(\overline{D} \setminus C, \partial \overline{D})$ and the *planar generalized braid group* B'_C is the mapping class group $\text{MCG}^+(\mathbb{R}^2 \setminus C)$.

By extending a homeomorphism of D fixed on $\partial \overline{D}$ by the identity, we see that there is a natural homomorphism $B_C \rightarrow B'_C$. This gives rise to a central extension

$$0 \rightarrow \mathbb{Z} \rightarrow B_C \rightarrow B'_C \rightarrow 0$$

where the generator of \mathbb{Z} is the (positive) Dehn twist of D^2 along a boundary parallel loop.

If C is a finite set with n points, then B_n is the usual braid group, and B'_n is the usual quotient of B_n by its center. A reference for the theory of braid groups, with particular reference to questions of orderability or circular orderability, is [8].

We come to the first main result of this paper.

The proof of this theorem follows the same general approach as Thurston’s proof of the orderability of the usual (finitely generated) braid groups [27], but several technical complications arise because the surfaces $D^2 \setminus C$ and $\mathbb{R}^2 \setminus C$ are not of finite type.

Theorem A. *Let C be a compact, totally disconnected subset of the open unit disk D . Then B'_C is circularly orderable, and B_C is left orderable.*

Proof. If C consists of a single point, both groups are trivial. If C consists of two points, $B'_C = \mathbb{Z}/2\mathbb{Z}$ and $B_C = \mathbb{Z}$, so the theorem is satisfied in that case. So without loss of generality we assume that C contains at least 3 points.

First we show that B'_C is circularly orderable. Let $S = S^2 \setminus C$. Then S is a hyperbolic surface, and we can identify the universal cover $\tilde{S} = \mathbb{H}^2$. We work in the unit disk model of \mathbb{H}^2 , so that \tilde{S} can be compactified by adding the circle at infinity S^1_∞ . There is a short exact sequence

$$0 \rightarrow \pi_1(S) \rightarrow \widehat{\text{MCG}}(S) \rightarrow \text{MCG}(S) \rightarrow 0$$

Here $\widehat{\text{MCG}}(S)$ denotes the group of homotopy classes of homeomorphisms of \tilde{S} which commute with the action of $\pi_1(S)$, and cover homeomorphisms of S .

It turns out that there is a naturally defined injective homomorphism σ from the mapping class group $\text{MCG}(\mathbb{R}^2 \setminus C)$ to $\widehat{\text{MCG}}(S)$. We define σ as follows. Let $[\phi] \in \text{MCG}(\mathbb{R}^2 \setminus C)$ denote a typical element, and let ϕ be a representative. Then ϕ extends continuously to a homeomorphism ϕ_S of S which fixes the point ∞ . Choose a lift $\widehat{\infty}$ of ∞ in \tilde{S} . Then there is a unique lift of ϕ_S to \tilde{S} which fixes $\widehat{\infty}$; call this lift $\phi_{\tilde{S}}$. Then define σ by

$$\sigma([\phi]) = [\phi_{\tilde{S}}]$$

Since the construction of $\phi_{\tilde{S}}$ depends continuously on ϕ , it follows that the class $[\phi_{\tilde{S}}]$ is well-defined.

We show that it is injective. If $[\phi_{\tilde{S}}] = [\psi_{\tilde{S}}]$ for some ϕ, ψ then there is a homotopy $\Psi_{\tilde{S}} : \tilde{S} \times I \rightarrow \tilde{S}$ with $\Psi_{\tilde{S}}(\cdot, 0) = \psi_{\tilde{S}}$ and $\Psi_{\tilde{S}}(\cdot, 1) = \phi_{\tilde{S}}$ which at every point is $\pi_1(S)$ equivariant. It follows that $\Psi_{\tilde{S}}$ descends to a map $\Psi_S : S \times I \rightarrow S$ which is a homotopy between ψ_S and ϕ_S . Moreover, under the track of this homotopy, the point ∞ moves in a homotopically inessential loop, since by construction, this loop lifts to \tilde{S} . So we can find another homotopy Ψ'_S , homotopic to Ψ_S , which fixes ∞ . It follows that Ψ'_S restricts to $\Psi : \mathbb{R}^2 \setminus C \times I \rightarrow \mathbb{R}^2 \setminus C$ which is a homotopy from ψ to ϕ . This proves that the map is an injection.

In fact, it is not hard to show that the map σ is actually an isomorphism, but this is unnecessary for our purposes.

We now show that the group $\widehat{\text{MCG}}(S)$ is circular. First of all we construct an action of $\widehat{\text{MCG}}(S)$ on S^1 .

The natural hyperbolic structure on S is complete, and therefore the limit set Λ of $\pi_1(S)$ is the entire circle S^1_∞ . A dense subset of S^1_∞ consists of endpoints of axes of hyperbolic translations $\alpha \in \pi_1(S)$. Notice that since $\pi_1(S)$ is discrete, the stabilizer of a point in $\pi_1(S)$ is cyclic, so distinct axes have distinct endpoints. This induces a circular order on the set \mathcal{E} whose elements are maximal hyperbolic cyclic subgroups $\langle \alpha \rangle$ of $\pi_1(S)$ together with a choice of generator for $H_1(\langle \alpha \rangle; \mathbb{Z})$. The group $\widehat{\text{MCG}}(S)$ induces an automorphism of $\pi_1(S)$ by conjugation, and therefore induces an action on the set \mathcal{E} . We verify that this action preserves the cyclic order.

To see this, let α, β be hyperbolic elements of $\pi_1(S)$ with axes l_α, l_β . Let $[\phi] \in \widehat{\text{MCG}}(S)$, and let ϕ be a representative. The images of the axes $\phi(l_\alpha), \phi(l_\beta)$ are periodic under the elements $[\phi]_*\alpha = [\phi]\alpha[\phi]^{-1}$ and $[\phi]_*\beta = [\phi]\beta[\phi]^{-1}$ in $\pi_1(S)$ respectively, so they converge to well-defined end points in S^1_∞ , which are exactly the endpoints of the axes of $l_{[\phi]_*\alpha}, l_{[\phi]_*\beta}$. Moreover, they are either disjoint or cross transversely in exactly one point, according to whether l_α, l_β are disjoint or

cross transversely in exactly one point respectively. It follows that l_α, l_β cross iff $l_{[\phi]_*(\alpha)}, l_{[\phi]_*(\beta)}$ cross. See figure 1.

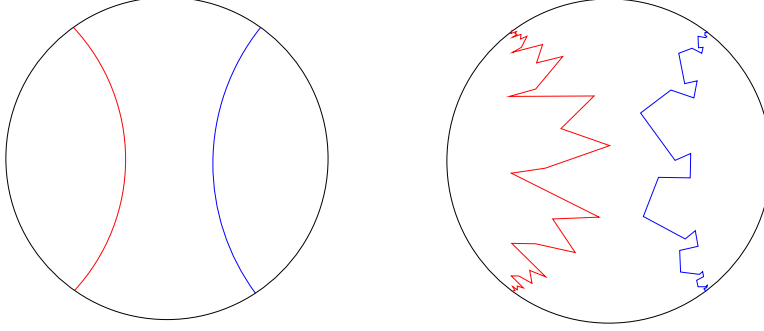


FIGURE 1. The axes l_α, l_β are disjoint iff their images under ϕ are disjoint. But this is true iff the axes $l_{[\phi]_*(\alpha)}, l_{[\phi]_*(\beta)}$ are disjoint. This defines a natural action of $\widehat{\text{MCG}}(S)$ on the circularly ordered set of endpoints of axes.

The circular ordering on the endpoints of l_α, l_β is determined by this crossing information, the orientation of the axes, and if the axes do not intersect, the relationship between the orientation of the axes and the orientation of the subsurface of \tilde{S} that they both cobound. It follows that the circular order of the endpoints of the l_α, l_β is taken to the circular order of the endpoints of the axes $l_{[\phi]_*(\alpha)}, l_{[\phi]_*(\beta)}$ under $[\phi]$, and therefore the action of $\widehat{\text{MCG}}(S)$ preserves the circular order on \mathcal{E} .

It remains to show this action is faithful. One way to do this is to find a maximal collection of pairwise disjoint geodesic loops $\Gamma = \{\gamma_i\}$ in S . We want to be slightly careful about the choice of Γ ; we want it to be maximal, and also *closed* as a subset of S . In particular, the geodesics γ_i should not accumulate. We show how to choose such a collection.

By the Moore–Kline theorem 3.4.6 we can assume, after a homeomorphism of \mathbb{R}^2 , that the set C is contained as a closed, totally disconnected subset of a horizontal arc I . In particular, the complement $I \setminus C$ consists of a countable collection of open intervals I_i . For each connected complementary open interval I_i we let p_i denote the midpoint. We inductively produce a maximal set of round circles S_i which are symmetric in the x -axis, and intersect the x axis in points p_i, p_j subject to the constraint that the circles are nonintersecting, and the i th circle is chosen to be the biggest circle with diameter $\leq 2^{-i}$. Note that a maximal collection of such circles has the property that each complementary domain contains at most finitely many unmatched p_i .

Our initial guess for Γ is the set of geodesics isotopic to S_i . Note that some S_i might be boundary parallel; we throw these out and just look at the ones with geodesic representatives. This is not quite maximal, but each complementary region is of finite type, and there are only finitely many complementary regions whose diameter (in the Euclidean metric in \mathbb{R}^2) is $> \epsilon$ for any ϵ . So we just take a maximal (finite) collection of geodesic loops in each such complementary region, and add these loops to Γ . Notice that this union of geodesic loops is *closed*, doesn't accumulate anywhere (in S) and is maximal.

Now, by hypothesis, $[\phi]$ preserves every hyperbolic cyclic subgroup $\langle \alpha \rangle$ of $\pi_1(S)$; in particular, the projection ϕ_S of a representative ϕ to S preserves every oriented free homotopy class of loop which is not boundary parallel. So the collection Γ is taken to a collection of curves which are curvewise isotopic to Γ . By straightening these geodesics inductively, we can homotop ϕ_S to ϕ'_S which fixes Γ pointwise. Since Γ was maximal, every complementary surface to Γ is either a thrice punctured sphere, a twice punctured disk, a once punctured annulus, or a pair of pants. An automorphism of one of these subsurfaces which preserves every hyperbolic cyclic subgroup is isotopic to the identity, so ϕ_S is actually isotopic to the identity, and therefore $[\phi]$ is in $\pi_1(S)$. Now, an element of $\pi_1(S)$ which, under conjugation, takes a hyperbolic cyclic subgroup to itself in an orientation preserving way must actually commute with that subgroup. But this contradicts the fact that $\pi_1(S)$ is nonelementary.

The conclusion is that $\widehat{\text{MCG}}(S)$ acts faithfully in an order preserving way on a circularly ordered set, and therefore $\widehat{\text{MCG}}(S)$ is circularly ordered, by theorem 2.2.14. Since $\sigma : \widehat{B}_C \rightarrow \widehat{\text{MCG}}(S)$ is injective, \widehat{B}_C is circularly ordered too.

Proving that B_C is left orderable is similar; let E be an open collar neighborhood of the disk D , and let $p \in E \setminus D$ be some point. Define S to be obtained from $E \setminus (C \cup p)$ by adding a point to compactify E , and then removing $(C \cup p)$. Exactly as in the previous case, there is an injective homomorphism from B_C to $\widehat{\text{MCG}}(S)$, and an invariant circular order on $\widehat{\text{MCG}}(S)$. Moreover, the image of B_C stabilizes the parabolic subgroup corresponding to a small loop around p , since representatives of B_C act trivially there. In particular, the image of B_C stabilizes a point in S_∞^1 , and therefore it is left orderable, as required. \square

3.6. Groups with bounded orbits. We are now in a position to prove the following:

Theorem B. *Let G be a group of orientation preserving C^1 homeomorphisms of \mathbb{R}^2 with a bounded orbit. Then G is circularly orderable.*

Proof. Let p be a point with a bounded orbit, and consider the closure of the orbit of p . This is a compact union of disjoint compact components which is permuted by G .

We apply construction 3.4.1 to fill in the sets with disconnected complement to get a new collection of connected sets $\{K_i\}$ which are compact with connected complements, whose union is compact, and whose components are permuted by G . We then apply lemma 3.4.5 to get a monotone equivalent action of G on \mathbb{R}^2 which leaves invariant a totally disconnected compact set C .

By theorem A, the image of G in the mapping class group of $\mathbb{R}^2 \setminus C$ is circularly orderable. Denote the kernel by K . Then K fixes C pointwise, and therefore with respect to the original action, K preserves the components $\{K_i\}$ componentwise.

By theorem 3.3.6, K is circularly orderable, and is left orderable if there are at least two components of K_i . If there is only one component K_i , the quotient C is a single point, and the mapping class group of $\mathbb{R}^2 \setminus C$ is trivial, in which case $G = K$. So either G is circularly orderable, or else it maps to a circularly orderable group with left orderable kernel. By lemma 2.2.12, G is circularly orderable, as claimed. \square

4. THE EULER CLASS AND SMOOTHNESS

The purpose of this section is to show that the Euler class of the group of C^∞ orientation-preserving diffeomorphisms of the plane is an unbounded class, in stark contrast with the case of $\text{Homeo}^+(S^1)$ and the Milnor–Wood inequality. In particular, if S is a closed surface of genus at least 2, there is a C^∞ action of $\pi_1(S)$ on the plane with arbitrary Euler class. By contrast, the genus one case is more rigid: we show that an orientation-preserving C^1 action of $\mathbb{Z} \oplus \mathbb{Z}$ on the plane has vanishing Euler class.

Using these facts, we prove the existence of a finitely generated torsion free group of homeomorphisms of the plane which is not circularly orderable.

4.1. The Euler class and winding number. We give two descriptions of the Euler class of a group G acting on \mathbb{R}^2 by a representation $\rho : G \rightarrow \text{Homeo}^+(\mathbb{R}^2)$. The first description is more algebraic.

Construction 4.1.1. Let G_∞ denote the *germ of G at ∞* . Let A be a punctured disk neighborhood of ∞ , and let \tilde{A} be the universal cover of A . Then there is a central extension

$$0 \rightarrow \mathbb{Z} \rightarrow \hat{G}_\infty \rightarrow G_\infty \rightarrow 0$$

where \hat{G}_∞ denotes the subgroup of periodic germs of homeomorphisms of \tilde{A} . The class of this extension pulls back by the natural restriction homomorphism $G \rightarrow G_\infty$ to give the Euler class $\rho^*[e] \in H^2(G; \mathbb{Z})$.

The next description is geometric, in terms of a winding number. Notice that for this definition to make sense, we must assume the action is at least C^1 .

Construction 4.1.2. Now we give a more geometric construction of the Euler class, in terms of how it pairs with elements of $H_2(G; \mathbb{Z})$. Assume the action is at least C^1 . It is a fact that for any group G , an integral class $\sigma \in H_2(G; \mathbb{Z})$ can be realized as the image of a map from a closed orientable surface S of genus ≥ 1 into a $K(G, 1)$

$$f : S \rightarrow K(G, 1)$$

Given a representation ρ , we get an induced action

$$\rho : \pi_1(S) \rightarrow \text{Homeo}^+(\mathbb{R}^2)$$

Let g be the genus of S , and suppose that we obtain S by gluing the sides of a $4g$ -sided polygon P_g in pairs in the usual way. Let $d_\rho : \tilde{S} \rightarrow \mathbb{R}^2$ be an equivariant C^1 map, called a *developing map* and look at the restriction

$$d_\rho : \partial P_g \rightarrow \mathbb{R}^2$$

Let $\delta : S^1 \rightarrow \mathbb{R}^2$ be obtained from d_ρ by smoothing the corners at each vertex of ∂P_g in such a way that the tangent vector rotates *positively* at each vertex. Then the Euler class of ρ , evaluated on $[S]$, can be calculated by the formula

$$\rho^*([e])([S]) = \text{index}(\delta) + 1 - 2g$$

where $\text{index}(\delta)$ denotes the usual winding number of the tangent vector $T\delta$ around S^1 . See figure 2 for an example of a hypothetical action of $\mathbb{Z} \oplus \mathbb{Z}$ with Euler number 1.

The justification for this formula is as follows. Associated to a representation $\rho : G \rightarrow \text{Homeo}^+(\mathbb{R}^2)$ there is an associated *foliated \mathbb{R}^2 bundle* E_ρ over BG . Here

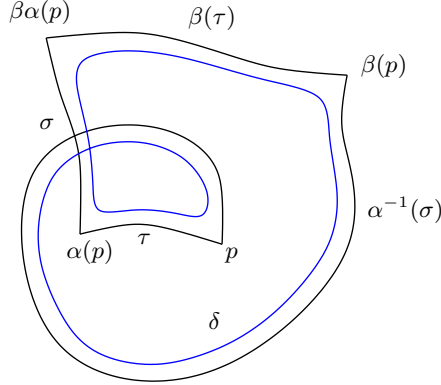


FIGURE 2. This hypothetical action of $\mathbb{Z} \oplus \mathbb{Z}$ satisfies $\text{index}(\delta) = 2$, so $\rho^*([e])([\Sigma]) = 1$.

BG is the classifying space for G as a discrete group; that is, it is a $K(G, 1)$. It is defined as the quotient of the product $\mathbb{R}^2 \times EG$ by the action of G , where EG is the universal cover of BG , and

$$\gamma(p, q) = (\rho(\gamma)(p), \gamma(q))$$

By basic obstruction theory, the Euler class $\rho^*([e])$ is the obstruction to trivializing E_ρ as an \mathbb{R}^2 bundle over the 2-skeleton of BG . See for example [13] for details.

If Σ is a closed, orientable surface of genus g , and $G = \pi_1(\Sigma)$, then $BG = \Sigma$. Since \mathbb{R}^2 is contractible, we can always find a section $\sigma : \Sigma \rightarrow E_\rho$ of E_ρ over Σ .

Suppose ρ is C^1 . Then E_ρ is naturally a C^1 manifold, and we can choose σ to be smooth. The pullback of the unit tangent bundle in the fiber direction by σ defines an orientable circle bundle E'_ρ over Σ . A trivialization of E'_ρ defines a trivialization of E_ρ by exponentiation, and conversely, a (smooth) trivialization of E_ρ defines a trivialization of E'_ρ by restriction. Since $\text{Diffeo}^+(\mathbb{R}^2)$ and $\text{Homeo}^+(\mathbb{R}^2)$ are homotopic as topological groups, E_ρ can be trivialized iff it can be smoothly trivialized.

It follows that the Euler class of ρ is the Euler class of the circle bundle E'_ρ . The section σ defines a developing map d_ρ . If necessary, perturb d_ρ to be an immersion on each edge of P_g . This immersion gives a trivialization of E'_ρ along each edge with a singularity of order $2 - 2g$ at the vertex. This section extends nonsingularly over a fundamental domain P_g iff the index of the smoothed curve $d_\rho(\delta)$ is 1. It follows that the Euler class, evaluated on Σ , is $\text{index}(\delta) - 1 + 2 - 2g$. The formula follows.

4.2. Unboundedness of the Euler class. In this subsection we construct C^∞ actions on the plane of $\pi_1(S)$, where S is the closed surface of genus 2, with arbitrary Euler class.

We begin with the following example, due to Bestvina:

Example 4.2.1 (Bestvina). Let C_i be the round circle in \mathbb{R}^2 centered at the origin, with radius 2^i , for $i \in \mathbb{Z}$. That is, the C_i are a bi-infinite nested sequence of circles about 0. Let α be the composition of a positive Dehn twist in each circle C_i . This composition is infinite, but locally finite away from the global fixed point 0, so it makes sense. Let β be the dilation centered at 0 sending $z \rightarrow 2z$. Then the

commutator $[\alpha, \beta] = \text{Id}$, and it is easy to see that

$$\langle \alpha, \beta \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$$

On the other hand, the image of the boundary of the polygon ∂P_1 under d_ρ from construction 4.1.2 is isotopic to the configuration in figure 2. In particular, the Euler number of this $\mathbb{Z} \oplus \mathbb{Z}$ action is 1.

If G_i is a subgroup of $\mathbb{Z} \oplus \mathbb{Z}$ of index i , then G_i is also isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, and the induced action has Euler number i . In particular, the Euler class $[e] \in H^2(\text{Homeo}^+(\mathbb{R}^2); \mathbb{Z})$ is *unbounded*.

The action of $\mathbb{Z} \oplus \mathbb{Z}$ is not C^1 at the origin. Bestvina posed the question of whether the Euler class was unbounded for C^1 actions. We show that the answer to this question is positive, and in fact show that the Euler class is unbounded for C^∞ actions.

Theorem C. *For each integer i , there is a C^∞ action*

$$\rho_i : \pi_1(S) \rightarrow \text{Diffeo}^+(\mathbb{R}^2)$$

where S denotes the closed surface of genus 2, satisfying

$$\rho_i^*([e])([S]) = i$$

In particular, the Euler class $[e] \in H^2(\text{Diffeo}^+(\mathbb{R}^2); \mathbb{Z})$ is unbounded.

Proof. As in example 4.2.1, let C_j be the round circle in \mathbb{R}^2 with radius 2^j centered at the origin. Let α be the composition of a positive Dehn twist in each circle C_j for $j \geq 0$, and let β be the dilation centered at 0 sending $z \rightarrow 2z$. Then α, β are both C^∞ , and the commutator $[\alpha^i, \beta]$ is the i th power of a positive Dehn twist in the circle C_0 . Now let D_j be the round circle in \mathbb{R}^2 with radius 1 centered at the point with x co-ordinate $2j$. Note that $C_0 = D_0$. Let γ be the composition of a positive Dehn twist in each circle D_j for $j \geq 0$, and let δ be the translation parallel to the x -axis through length 2. Then γ, δ are both C^∞ , and the commutator $[\gamma^i, \delta]$ is the i th power of a positive Dehn twist in the circle D_0 . It follows that there is an identity

$$[\alpha^i, \beta] = [\gamma^i, \delta]$$

and the group generated by $\alpha^i, \beta, \gamma^i, \delta$ is a quotient of $\pi_1(S)$ for S the closed surface of genus 2.

We calculate the Euler class of this action. We can find a polygon P_2 and a developing map $d_\rho : \partial P_2 \rightarrow \mathbb{R}^2$ which is smoothed to $\delta : \partial P_2 \rightarrow \mathbb{R}^2$ with $\text{index}(\delta) = 3+i$. It follows by construction 4.1.2 that

$$\rho_i^*([e])([S]) = 3 + i + 1 - 2g = i$$

□

Theorem C raises the following natural question:

Question 4.2.2. *Is the Euler class a bounded class on the group of real analytic orientation-preserving diffeomorphisms of \mathbb{R}^2 ?*

Some weak evidence for a positive answer to this question is the following:

Theorem 4.2.3. *The group $\text{Poly}^+(\mathbb{R}^2)$ of polynomial orientation-preserving homeomorphisms is circularly orderable.*

Proof. Let r_0 be the positive real axis, and let R denote the set of germs at infinity of translates of r_0 by orientation-preserving polynomial homeomorphisms. Then R is a circularly ordered set. To see this, observe that if $\alpha : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a polynomial homeomorphism, then $\alpha(r_0)$ is real algebraic, and therefore its germ at infinity is either equal to the germ at infinity of r_0 , or is disjoint from it. If r_1, r_2, r_3 are three disjoint, properly embedded rays in \mathbb{R}^2 , then they are naturally circularly ordered as follows. Let D be a large closed disk whose boundary intersects r_1, r_2, r_3 transversely. For each r_i , the intersection $r_i \cap \partial D$ is a finite set. This set inherits a natural ordering from the orientation on r . For each i , let p_i be the greatest element of this set. Then the rays r_i are circularly ordered by the circular ordering of the p_i in ∂D . One can verify that this ordering is independent of the choice of (sufficiently large) D , and therefore defines a circular ordering on the germs at infinity of the r_i .

Now, a polynomial homeomorphism of \mathbb{R}^2 which preserves the germ at infinity of a straight line must preserve the entire line as a set. In particular, the natural homomorphism $\text{Poly}^+(\mathbb{R}^2) \rightarrow \text{Aut}(R)$ is injective. By theorem 2.2.15, $\text{Poly}^+(\mathbb{R}^2)$ is circularly orderable. \square

4.3. Torsion free groups which are not left orderable. In this section we construct groups of orientation-preserving homeomorphisms of the plane which are not left-orderable. Of course, there are some very simple examples of such groups: any finite cyclic group acts on the plane by rotations, but the only finite left-orderable group is the trivial group. Benson Farb asked whether there are any examples of torsion free groups which act on the plane but are not left or circularly orderable. In this section, we construct an explicit example of such a group which is not left orderable. In the next section, we show how to promote this example to show the existence of a finitely presented torsion free subgroup of $\text{Homeo}^+(\mathbb{R}^2)$ which is not circularly orderable.

Theorem 4.3.1. *There is a finitely presented torsion free C^∞ subgroup of $\text{Diffeo}^+(\mathbb{R}^2)$ which is not left orderable.*

Proof. Let G be given by the following presentation:

$$G = \langle a, b, c, t \mid a^2 = t, b^3 = t, c^7 = t, abc = t^3 \rangle$$

Then G is the fundamental group of the Seifert fibered space

$$M = (Oo0 \mid 3; (2, 1), (3, 1), (7, 1))$$

in the notation of Montesinos [21]. In particular, this is a 3-manifold with $\widetilde{SL}(2, \mathbb{R})$ geometry, so \widetilde{M} is homeomorphic to \mathbb{R}^3 , and G is torsion free.

On the other hand, $\pi_1(M)$ is not left orderable, and in fact admits no nontrivial homomorphism to $\text{Homeo}^+(\mathbb{R})$. A more general fact is proved in [26], but it is easy enough to give a direct proof in our case. Suppose G is left orderable, so that we can disjointly decompose $G = P \cup N \cup \text{Id}$ with $P \cdot P \subset P$ and $P^{-1} = N$, by lemma 2.1.2. Without loss of generality, we can assume $t \in P$. Then since t is a nontrivial positive power of each of the generators a, b, c we must have $a, b, c \in P$ and also $a, b, c < t$. But then $abc < t^3$ contradicting the fourth relation.

On the other hand, since t is central, we can form the quotient Δ of G by $\langle t \rangle$ with the presentation

$$\Delta = \langle a, b, c \mid a^2 = b^3 = c^7 = \text{Id} \rangle$$

so that there is a short exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow G \rightarrow \Delta \rightarrow 0$$

Now, the group Δ acts faithfully and discretely on \mathbb{R}^2 by C^∞ homeomorphisms. To see this, observe that Δ is the fundamental group of the $(2, 3, 7)$ triangle orbifold, and therefore admits a (unique up to conjugacy and orientation) discrete faithful representation into $\mathrm{PSL}(2, \mathbb{R}) = \mathrm{Isom}^+(\mathbb{H}^2)$. So after choosing a diffeomorphism of $\mathbb{H}^2 \rightarrow \mathbb{R}^2$, we obtain a representation $\rho : \Delta \rightarrow \mathrm{Diffeo}^+(\mathbb{R}^2)$. We will construct a monotone equivalent (smooth) faithful action of G . Since $\rho(\Delta)$ is discrete, and the set of points with nontrivial stabilizer in Δ are discrete, we let O be the orbit of a point with trivial stabilizer in Δ , and choose a diffeomorphism of $\mathbb{R}^2 \setminus O$ with $\mathbb{R}^2 \setminus X$ where X is the union of infinitely many nowhere accumulating disjoint closed disks. Let $\rho' : G \rightarrow \mathrm{Diffeo}^+(\mathbb{R}^2 \setminus X)$ be the pushforward of the representation ρ under this diffeomorphism. We extend this action over X in the following way. Since the components of X are permuted by Δ with trivial stabilizer, we can identify $X = \Delta \times D$ where D denotes the closed unit disk. Now let the kernel $\langle t \rangle$ of ρ act on D by some smooth diffeomorphism which is fixed on the boundary and has infinite order, and take the product action on $\Delta \times D$. This defines an extension of ρ' to all of \mathbb{R}^2 which is smooth and faithful. \square

Remark 4.3.2. In [26], it is actually proved that *every* action of $\pi_1(M)$ of an exceptional Seifert fibered space on \mathbb{R} is equivalent to the action of $\pi_1(M)$ on the space of leaves of some taut foliation of M transverse to the circle fibration. By the Milnor–Wood inequality, such a foliation can only exist if the Euler class of the bundle is smaller in absolute value than the Euler characteristic of the base orbifold, which is a sphere with three exceptional fibers. In particular, for any such base orbifold, there are infinitely many Seifert fibered spaces, but only finitely many whose fundamental groups are left orderable. On the other hand, by more or less repeating the construction in theorem 4.3.1, one can see that the fundamental group of every Seifert fibered 3–manifold with infinite fundamental group admits a faithful C^∞ representation in $\mathrm{Diffeo}^+(\mathbb{R}^2)$. Every such group is torsion free. On the other hand, all of these groups are circularly orderable, and contain finite index subgroups which are left orderable.

4.4. Torsion free groups which are not circularly orderable. In this subsection we show that we can combine theorem C with the example from theorem 4.3.1 to give an example of a torsion free subgroup of $\mathrm{Homeo}^+(\mathbb{R}^2)$ which is not circularly orderable, thereby giving a complete positive answer to Farb’s question. At one point, we make use of a powerful theorem of Tsuboi ([32]) about the homology of the group of C^1 homeomorphisms of \mathbb{R}^n , but aside from this, our construction is basically elementary.

The use of Tsuboi’s theorem is not logically essential; one could avoid it by an explicit construction. But such a construction would add considerably to the length of the example, without being any more enlightening.

Our proof follows the model of an argument due to Ghys which shows that there exist planar groups which are not circularly orderable, using example 4.2.1, and the fact that $\mathrm{Homeo}^+(\mathbb{R}^2)$ contains torsion elements.

First we prove a couple of lemmas.

Lemma 4.4.1. *Let G be the group from theorem 4.3.1. If $\rho : G \rightarrow \text{Homeo}^+(S^1)$ is a homomorphism for which $\rho(t)$ has a fixed point, then ρ is semi-conjugate to an action which factors through $\Delta = G/\langle t \rangle$. In particular, ρ is either trivial, or is semi-conjugate to the “standard” hyperbolic action coming from a faithful representation of Δ in $\text{PSL}(2, \mathbb{R})$.*

Proof. Let $T \subset S^1$ be the fixed point set of $\rho(t)$. By blowing up the orbit of some $p \in T$, we can assume that T has nonempty interior. Since t is central, the subset T is permuted by $\rho(G)$, so we can blow down $S^1 \setminus T$ to get a monotone equivalent action on which t acts trivially, so up to monotone equivalence, the action factors through $\Delta = G/\langle t \rangle$, as claimed.

Now, it is well-known that every homomorphism of Δ to $\text{Homeo}^+(S^1)$ is either trivial, or monotone equivalent to a hyperbolic action. See [5] for a proof, which makes use of a theorem of Matsumoto [18]. \square

Lemma 4.4.2. *Let $\text{Diffeo}_c^+(\mathbb{R}^2)$ denote the group of diffeomorphisms of \mathbb{R}^2 with compact support. There is a subgroup of $\text{Diffeo}_c^+(\mathbb{R}^2)$ isomorphic to*

$$H = \langle q, r, s, t \mid [r, s] = t, qtq^{-1} = t^2 \rangle$$

Proof. Such subgroups are very easy to find. For example, let $\phi : \mathbb{R}^2 \rightarrow D^2$ be a diffeomorphism such that $\|\phi'\|$ decays very fast at infinity, and let t be the pushforward by ϕ of a translation. Then t is certainly both a commutator, and conjugate to its square. For suitably generic choice of r, s, q the representation of H is faithful. \square

With these lemmas proved, we can now give a positive answer to Farb’s question.

Theorem 4.4.3. *There is a finitely generated torsion free subgroup of $\text{Homeo}^+(\mathbb{R}^2)$ which is not circularly orderable.*

Proof. Let G be the example from theorem 4.3.1. By abuse of notation, we will actually think of G as a subgroup of $\text{Diffeo}^+(\mathbb{R}^2)$. We use the same notation for the generators of G . By construction, the support of t is a nonaccumulating union of closed disks $\Delta \times D$. Moreover, the action of t on $1 \times D$ can be chosen to be arbitrary, and the action on each other translate $g \times D$ is conjugate to the action on $1 \times D$. By lemma 4.4.2, we can insist that the restriction of t to $1 \times D$ has the following properties:

- There are diffeomorphisms r, s whose commutator satisfies $[r, s] = t$.
- There is a diffeomorphism q such that $qtq^{-1} = t^2$.

Moreover, we can choose q, r, s to have support equal to $\Delta \times D$, and further we can insist that the action on each translate $g \times D$ is conjugate to the action on $1 \times D$. It is easy to choose q, r, s, t sufficiently generically so that the group

$$G' = \langle G, q, r, s, t \mid [r, s] = t, qtq^{-1} = t^2 \rangle$$

acts faithfully on \mathbb{R}^2 .

Now, if $\rho' : G' \rightarrow \text{Homeo}^+(S^1)$ is a homomorphism, then $\rho'(t)$ is conjugate in $\text{Homeo}^+(S^1)$ to $\rho'(t^2)$. In particular, the rotation number of $\rho'(t)$ satisfies

$$\text{rot}(\rho'(t)) = \text{rot}(\rho'(t^2)) = 2\text{rot}(\rho'(t))$$

and therefore this rotation number is equal to 0, and $\rho'(t)$ has a fixed point. For the definition and properties of rotation number, see [5].

It follows by lemma 4.4.1 that up to semi-conjugacy, ρ' is equivalent to the standard action of the fundamental group of the $(2, 3, 7)$ hyperbolic triangle orbifold on its circle at infinity.

The group G contains a subgroup H of index 168 corresponding to the congruence cover of the $(2, 3, 7)$ hyperbolic triangle orbifold. This subgroup H is a \mathbb{Z} extension of the fundamental group of a closed surface Σ of genus 3. For any m , we can find a subgroup H_m of H of index m which is a \mathbb{Z} extension of the fundamental group of a closed surface Σ_m of genus $2m + 1$. The Euler class of the extension corresponding to H is some constant a , and the Euler class of the extension corresponding to H_m is ma . So there are elements

$$\alpha_{1,m}, \beta_{1,m}, \dots, \alpha_{2m+1,m}, \beta_{2m+1,m} \in G$$

such that

$$\prod_i [\alpha_{i,m}, \beta_{i,m}] = t^{ma}$$

Now, $[q^n r q^{-n}, q^n s q^{-n}] = t^{2^n}$, so for each m , we can express t^{ma} as a product of approximately $\log ma$ commutators of this form. Together with the $\alpha_{i,m}, \beta_{i,m}$ we can produce surface subgroups S_m of G' of genus approximately $2m + 2 \log ma$. We know that the action of $\alpha_{i,m}, \beta_{i,m}$ on S^1 is "standard", up to semi-conjugacy. So by the Milnor–Wood inequality theorem 2.4.2, we can estimate the circular Euler class

$$(\rho')^*([e])[S_m] \leq -4m + 4 \log ma$$

which is as negative as we like for big m . Moreover, since r, s have support contained in the union of the disks $\Delta \times D$, the formula from construction 4.1.2 implies that the planar Euler class of S_m is $-4m$.

Now, by a theorem of Tsuboi, the homology of the group of C^1 orientation-preserving diffeomorphisms of \mathbb{R}^2 is equal to \mathbb{Z} in dimension 2; see [33] and [32]. Let F_2 denote the fundamental group of a surface of genus 2. Let ρ_{-4m} be the representation constructed in theorem C with $i = -4m$. Then $\rho_{-4m}(F_2)$ and S_m are homologous as subgroups of the group of C^1 orientation-preserving homeomorphisms of \mathbb{R}^2 . By vanishing of the oriented cobordism group in dimension 3, we can take this homology to be an oriented 3-manifold. That is, there is a 3-manifold M with two boundary components, $\partial(M) = \partial_1(M) \cup \partial_2(M)$, where $\partial_1(M)$ has genus 2 and $\partial_2(M)$ has genus approximately $2m + 2 \log ma$, and a C^1 representation $\sigma_M : \pi_1(M) \rightarrow \text{Homeo}^+(\mathbb{R}^2)$ where $\sigma_M|_{\partial_1(M)}$ is conjugate to $\rho_{-4m}|_{F_2}$, and $\sigma_M|_{\partial_2(M)}$ is conjugate to the action of S_m . In fact, C^1 is superfluous for our application: it suffices that each element of $\sigma_M(\pi_1(M))$ is C^1 on an open dense subset. Such a homology can be constructed by more elementary methods, since the boundary representation is C^∞ .

We can find an oriented irreducible 3-manifold N with torsion-free fundamental group, and a degree 1 map $\varphi : N \rightarrow M$ which is a homeomorphism on the boundary. Then σ_M induces a C^1 representation $\sigma_N : \pi_1(N) \rightarrow \text{Homeo}^+(\mathbb{R}^2)$ with the same boundary behaviour. It follows that σ_N extends to a C^1 action of the amalgamated free product

$$G'' = G' *_{S_m} \pi_1(N)$$

which is faithful on the subgroup G' . Let $\sigma : G'' \rightarrow \text{Homeo}^+(\mathbb{R}^2)$ denote the homomorphism defining this action. Note that G'' is torsion-free, but we do not yet know that σ is faithful, and therefore we don't know that $\sigma(G'')$ is torsion free.

Let K be the kernel of σ . Since σ is injective on G' , for generic σ , the kernel is the free product of subgroups of $\pi_1(N)$. But $\pi_1(N)$ is left orderable. Actually, [1] observed that every compact irreducible 3-manifold with boundary has locally indicable fundamental group, and therefore by theorem 2.1.10, has LO fundamental group. A free product of LO groups is LO, by lemma 2.1.6. It follows that K is LO. We show how to modify σ to make it faithful.

Pick a point p , and blow up its orbit. That is, replace each translate $g(p)$ of p by a closed disk D_g in such a way that the diameters of these disks in any compact region go to zero. Since σ is C^1 at p , we can extend the action of G'' on $\mathbb{R}^2 \setminus G''(p)$ to the boundary circle ∂D_g by the projective linear action on the unit tangent bundle $UT_{g(p)}\mathbb{R}^2$. Let G''_p be the stabilizer of p in G'' . Then $\sigma(G''_p)$ is circularly orderable, and the stabilizer of ∂D_g in $\sigma(G''_p)$ is left orderable, by lemma 3.2.2. Let K_p be the stabilizer of ∂D_g in G''_p , so that $\sigma(K_p)$ is the stabilizer of ∂D_g in $\sigma(G''_p)$. Since K is left orderable, and $\sigma(K_p)$ is left orderable, it follows that K_p is left orderable by lemma 2.1.8, and G''_p is circularly orderable, by lemma 2.2.12. We can insert a faithful action of G''_p on D_g by thinking of D_g as the cone on S^1 , and coning the (positive) projectivized linear action of G''_p on S^1 by a faithful action of K_p on I , which exists by theorem 2.2.14. We can then translate this action to the other disks D_h . The resulting action defines a *faithful* representation σ' . Notice that even though σ' is only C^0 , we could not perform the blow-up without knowing that σ was C^1 at p . So σ' exhibits G'' as a torsion-free subgroup of $\text{Homeo}^+(\mathbb{R}^2)$.

If there were a representation $\rho'' : G'' \rightarrow \text{Homeo}^+(S^1)$ extending ρ' , then the circular Euler class of $\rho_{-4m}(F_2)$ would be equal to the circular Euler class of S_m , which as we calculated before, is $\leq -4m + 4 \log ma$, violating the Milnor–Wood inequality (theorem 2.4.2). This contradiction shows that ρ'' cannot exist, and G'' is not circularly orderable, as claimed. \square

Remark 4.4.4. Notice that in order to find a C^∞ example, one just needs to find a torsion-free C^∞ homology from S_m to $\rho_{-4m}(F_2)$. Such a homology can actually be constructed by hand, but the construction is somewhat complicated and unenlightening, so we do not include it here.

4.5. Homological rigidity for $\mathbb{Z} \oplus \mathbb{Z}$ actions. In the remainder of the paper, we will show, in contrast to theorem C, and example 4.2.1 that for C^1 actions of $\mathbb{Z} \oplus \mathbb{Z}$, the Euler class *always* vanishes.

We do this by studying the possible dynamics of two commuting C^1 homeomorphisms α, β , examining the cases based on the dynamics of β on the fix point set of α . First we treat the case that α has a fixed point whose orbit under β is not proper.

Lemma 4.5.1. *Let $\rho : \mathbb{Z} \oplus \mathbb{Z} \rightarrow \text{Homeo}^+(\mathbb{R}^2)$ be C^1 , and suppose that p has nontrivial stabilizer. If p is fixed by a finite index subgroup, or has an indiscrete orbit, then $\rho^*([e])$ is trivial.*

Proof. Suppose the stabilizer of p is a finite index subgroup. Since any finite index subgroup of $\mathbb{Z} \oplus \mathbb{Z}$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, and since the Euler class is multiplicative under covers, it suffices to assume that p is a global fixed point. In this case the Euler class of the planar action is equal to the Euler class of the (projective) action on the unit tangent bundle at p obtained by linearizing. In particular, the Euler class is pulled back from an action on S^1 , and is therefore a bounded class. But $H_b^2(\mathbb{Z} \oplus \mathbb{Z}; \mathbb{Z}) = 0$, so the Euler class vanishes.

If p is fixed by some generator α , fix some other generator β and let $p_i = \beta^i(p)$. Suppose p_i is indiscrete and contains some subsequence which accumulates at q , which is fixed by α . Some subsequence approaches q radially along some tangent vector v . Since the action is C^1 , it follows that α fixes q and $d\alpha$ fixes v . Let p_i, p_j be very close to q , and let τ be an arc joining p_i to p_j so that $\tau'(0)$ is parallel to v , and $d\beta^{j-i}\tau'(0) = \tau'(1)$. Then $d\alpha\tau'(0)$ is very close to $\tau'(0)$, and since α and β commute, $d\alpha\tau'(1)$ is very close to $\tau'(1)$. Moreover, near q , α is very close in the C^1 topology to a linear transformation. So, since $d\alpha$ almost fixes $\tau'(0)$ and $\tau'(1)$, either these vectors are almost parallel, in which case we can take τ very close to a radial straight arc, or else $d\alpha$ is very close to the identity. In either case, for our choice of τ , the arc $\alpha(\tau)$ is very close in the C^1 topology to τ . We let $\tau \cup \alpha(\tau)$ be the image of ∂P_1 under a “degenerate” developing map d_ρ as in construction 4.1.2. Then the index is 1, and the Euler class, evaluated on the fundamental class of $\mathbb{Z} \oplus \mathbb{Z}$, is $1 + 1 - 2 = 0$, as claimed. \square

4.6. The Alexander series of ρ . In this section we prove that if α fixes some point p , then if there exists τ from p to $\beta(p)$ which intersects translates $\beta^i(\tau)$ only at endpoints, then the Euler class is zero. An individual treatment of this case is not logically necessary for the proof of theorem D, but it does make subsequent sections easier to understand.

We fix the following notation. Let $p_i = \beta^i(p)$, and let τ be a smooth arc from p_0 to p_1 such that $d\beta(\tau'(0)) = \tau'(1)$. Let $P = \cup_i p_i$ and $l = \cup_i \beta^i(\tau)$. We also introduce notation $\delta = \alpha(\tau)$.

First we introduce an algebraic tool to study the dynamics of α and β , which we call the *Alexander series*, by analogy with the usual Alexander polynomial of a knot. The condition we will impose on τ will ensure that the series is actually a Laurent polynomial.

We assume the Euler class of our representation ρ is $n \neq 0$. Look at the closed curve $\tau \cup \delta$. This is the image of a fundamental domain for the torus under a “degenerate” developing map, so we can calculate the Euler class of ρ from the formula in construction 4.1.2.

The curves τ, δ both represent elements in relative homology

$$[\tau], [\delta] \in H_1(\mathbb{R}^2, P; \mathbb{Z})$$

Since there is no natural intersection form on this group, we must work harder to find the correct algebraic definition of the Euler class.

There is a natural Laurent series associated to the action ρ , defined as follows:

Definition 4.6.1. The *Alexander series* of ρ , denoted

$$A_\rho \in \mathbb{Z}[[t, t^{-1}]]$$

is defined as follows. For generic τ , we can assume that δ and τ are transverse at 0. The constant term is the *algebraic* intersection number, for a generic choice of representative (subject to the constraints above) of δ and τ , plus the algebraic sign of the intersection at 0. We now define the other coefficients. For $i \neq 0$, let the coefficient of t^i be the algebraic intersection number of the interior of δ with the interior of $\beta^i(\tau)$.

Notation 4.6.2. If μ, ν are two arcs or loops in general position, we denote their algebraic intersection number by $\mu \cdot \nu$. So, with this notation, we have

$$a_i = \delta \cdot \beta^i(\tau)$$

and

$$\mu \cdot \nu = -\nu \cdot \mu$$

Remark 4.6.3. As defined, A_ρ might depend on τ . Nevertheless we suppress this in the notation.

Notice that the coefficients of the power series only depend on the germs of τ, δ at p_0, p_1 , together with their smooth isotopy class in $\mathbb{R}^2 \setminus N(p_0 \cup p_1)$. So if we replace τ, δ with τ', δ' which are smoothly isotopic by an isotopy supported outside a neighborhood of P , the algebraic intersection number of δ' with $\beta^i(\tau')$ is equal to the i th coefficient of A_ρ .

For the moment, we have defined $A_\rho \in \mathbb{Z}[[t, t^{-1}]]$. We show, under the right conditions, that A_ρ is actually a *Laurent polynomial*.

But first we must see how to calculate the Euler class of ρ from the configuration of τ and δ .

Lemma 4.6.4. *Let $\rho : \mathbb{Z} \oplus \mathbb{Z} \rightarrow \text{Homeo}^+(\mathbb{R}^2)$ be C^1 . Let α, β be the generators, and let p be fixed by α . Let τ be an embedded arc from p to $\beta(p)$, and let δ be $\alpha(\tau)$. Let τ^\pm be immersed proper rays in \mathbb{R}^2 such that $\tau^- \cup \tau \cup \tau^+$ is an immersed proper line, and τ^\pm intersect τ only at their endpoints. Then the Euler class $\rho^*([e])$, evaluated on the fundamental cycle of $\mathbb{Z} \oplus \mathbb{Z}$, satisfies the formula*

$$\rho^*([e])([\mathbb{Z} \oplus \mathbb{Z}]) = \delta \cdot \tau^+ - \delta \cdot \tau^-$$

Proof. First we prove the lemma in the case that τ^+, τ^- are properly embedded, and intersect τ only at their endpoints.

For concreteness, let $\tau^- \cup \tau \cup \tau^+$ be the real axis, oriented so that τ^- limits to $-\infty$.

We can deform δ to δ' through a family of smooth arcs δ' which are all embedded, which do not intersect p or $\beta(p)$ except at their endpoints, and whose germ near $p, \beta(p)$ is fixed. This does not change either the index of $\delta \cup \tau$, or the intersection numbers $\delta \cdot \tau^\pm$. After such a deformation we can assume δ' is of a particularly simple form: it consists of three pieces: a concentric spiral around p through $\delta \cdot \tau^-$ revolutions, a concentric spiral around $\beta(p)$ through $\delta \cdot \tau^+$ revolutions, and an arc between the two which has no vertical tangent. To see this, observe that we can deform δ to τ by an isotopy rel. endpoints through smooth arcs, at the cost of twisting the tangent vectors at p and at $\beta(p)$ through $\delta \cdot \tau^-$ and $\delta \cdot \tau^+$ revolutions, respectively. Then we can “undo” this twisting by inserting spirals, to produce δ' .

When we smooth the curve $\tau \cup \delta$ to calculate the index by the formula of construction 4.1.2, we introduce another twist at $\beta(p)$, in order to satisfy the condition of construction 4.1.2 that we must turn *positively* at vertices. It follows that the index of the smoothing of $\tau \cup \delta$ is equal to $\delta \cdot \tau^+ - \delta \cdot \tau^- + 1$ and therefore, by the formula of construction 4.1.2, the Euler class is $\delta \cdot \tau^+ - \delta \cdot \tau^-$, as claimed.

For an example, see figure 3. In this case $\delta \cdot \tau^+ = 2, \delta \cdot \tau^- = -1$ and the Euler class is 3.

Note that this calculation is still valid if τ^-, τ^+ are merely properly *immersed* rays which do not intersect τ except at their endpoints. For, we can always isotope

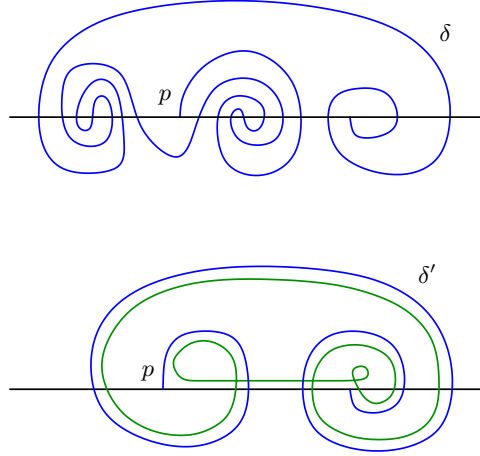


FIGURE 3. The Euler class can be calculated from the intersection numbers $\delta \cdot \tau^\pm$. The green loop is the smoothed boundary of a fundamental domain for the hypothetical $\mathbb{Z} \oplus \mathbb{Z}$ action. The index of the green loop is $\delta \cdot \tau^+ - \delta \cdot \tau^- + 1 = 2 + 1 + 1 = 4$, and the Euler class is 3.

δ to δ' which is contained in a neighborhood of τ , without affecting the intersection number with τ^+ or τ^- , and therefore the calculation is insensitive to intersections of τ^\pm with each other or with themselves away from τ . This completes the proof in general. \square

With this lemma, we can establish the following fundamental properties of A_ρ .

Lemma 4.6.5. *Suppose that τ has the property that for any i , the algebraic intersection number of τ and $\beta^i(\tau)$ is zero. Then the Alexander series $A_\rho(t) = \sum_i a_i t^i$ has the following properties:*

- (1) A_ρ is actually a Laurent polynomial. That is, the coefficient of t^i is zero for all but finitely many i .
- (2) The sum of the coefficients is zero:

$$A_\rho(1) = 0$$

- (3) The Euler class $\rho^*([e])([\mathbb{Z} \oplus \mathbb{Z}])$, which we abbreviate by e_ρ , satisfies

$$e_\rho = \sum_{i>0} a_i - \sum_{i<0} a_i$$

In this case, we call A_ρ the Alexander polynomial of ρ .

Proof. The first fact follows because for large i , the points p_i, p_{i+1} escape any compact set, by lemma 4.5.1. Since $\beta^i(\tau)$ is an arc from p_i to p_{i+1} , the algebraic intersection number of this arc with $\tau \cup \delta$ is zero. But by hypothesis, $\beta^i(\tau)$ and τ have algebraic intersection number zero, so the algebraic intersection with δ alone is also zero.

The second fact follows for essentially the same reason as the first fact: the union $\tau \cup \delta$ is an (immersed) closed loop, and the sum of coefficients is equal to the algebraic intersection number of $\tau \cup \delta$ with the union $l = \cup_i \beta^i(s)$. Since the p_i

are unbounded, the algebraic intersection of the loop and l is zero, and therefore the formula follows.

The third fact follows from lemma 4.6.4, by using $\bigcup_{i<0} \beta^i(\tau)$ as τ^- , and $\bigcup_{i>0} \beta^i(\tau)$ as τ^+ . Now, the union $\bigcup_i \beta^i(\tau)$ is *not* necessarily properly immersed in general, but note that we can isotope each $\beta^i(\tau)$ rel. P (not necessarily equivariantly!) without intersecting τ to new τ'_i until the union $\bigcup_i \tau'_i$ is properly immersed. This does not change the algebraic intersection number with δ . Then

$$\begin{aligned} e_\rho &= \sum_{i>0} \delta \cdot \tau'_i - \sum_{i<0} \delta \cdot \tau'_i \\ &= \sum_{i>0} \delta \cdot \beta^i(\tau) - \sum_{i<0} \delta \cdot \beta^i(\tau) \\ &= \sum_{i>0} a_i - \sum_{i<0} a_i \end{aligned}$$

□

With this algebraic tool, we can now prove the following lemma:

Lemma 4.6.6. *Let $\rho : \mathbb{Z} \oplus \mathbb{Z} \rightarrow \text{Homeo}^+(\mathbb{R}^2)$ be a C^1 action which is not free. Suppose $\mathbb{Z} \oplus \mathbb{Z}$ is generated by α and β , where α fixes some point p , and there is a smooth embedded arc τ from p to $\beta(p)$ with $d\beta(\tau'(0)) = \tau'(1)$ such that the algebraic intersection number of τ with $\beta^i(\tau)$ is zero for any i . Then the Euler class $\rho^*([e])$ is zero.*

Proof. Let a_i be the i th coefficient of A_ρ . Let G_n be the subgroup of $\mathbb{Z} \oplus \mathbb{Z}$ of index $2n + 1$ generated by α and β^{2n+1} . This gives a new representation ρ_n of $\mathbb{Z} \oplus \mathbb{Z}$, whose Euler class is equal to $2n + 1$ times the Euler class of ρ , by multiplicativity. By the properties of A_ρ , we can calculate

$$A_{\rho_n}(t) = \sum_j \sum_{i=-2n}^{2n} (2n + 1 - |i|) a_{i+j(2n+1)} t^j$$

and so by lemma 4.6.5, we have

$$e_{\rho_n} = (2n+1)e_\rho = \sum_{j>0} \sum_{i=-2n}^{2n} (2n+1-|i|) a_{i+j(2n+1)} - \sum_{j<0} \sum_{i=-2n}^{2n} (2n+1-|i|) a_{i+j(2n+1)}$$

On the other hand, by lemma 4.6.5, the coefficients a_i are zero for $|i|$ sufficiently large, so as n increases, the right hand side is eventually constant. On the other hand, it is equal to $(2n + 1)e_\rho$, so e_ρ is equal to zero, as claimed □

4.7. The case that $\tau \cdot \beta^i(\tau) \neq 0$. In general, we cannot find an arc τ with the properties of lemma 4.6.6. An example is constructed in [12] of an orientation-preserving homeomorphism h of \mathbb{R}^2 with a proper orbit $h^i(p)$ such that every arc τ from p to $h(p)$ must intersect its translates. We show how to modify our arguments to deal with the case that $\tau \cdot \beta^i(\tau) \neq 0$ for some (possibly infinitely many) i .

Let τ be any smooth arc joining p to $\beta(p)$, satisfying $d\beta(\tau'(0)) = \tau'(1)$.

Definition 4.7.1. The *self-intersection series*, denoted $B_\rho \in \mathbb{Z}[[t, t^{-1}]]$ is defined by setting the constant term equal to zero, and for $i \neq 0$, the t^i term equal to the algebraic intersection number of τ with $\beta^i(\tau)$.

There is no good way to define the homological self–intersection of τ with itself, so the constant term must be zero. A meaningful quantity for the self–intersection of τ with itself is the *writhe*, defined as follows:

Definition 4.7.2. Let τ be a smooth immersed arc in \mathbb{R}^2 . Let \mathcal{C} denote the space of homotopy classes of C^1 immersed curves in \mathbb{R}^2 with the same germ at the endpoints $\tau(0), \tau(1)$. Note that these curves may intersect the endpoints in their interiors. The class of τ in \mathcal{C} is denoted $[\tau]$ and is called the *writhe* of τ .

Lemma 4.7.3. *The space \mathcal{C} is an affine space for \mathbb{Z} , where the action of 1 is given by connect summing with a small figure 8 on the negative side.*

Proof. This is a very special case of Smale’s immersion theorem. \square

Here by *connect summing* we mean the usual connect sum of planar diagrams. Suppose κ_1, κ_2 are two disjoint curves, and I is an embedded arc joining a point on κ_1 to a point on κ_2 . Thicken I to a 1–handle $I \times I$, with boundary $I \times \{0, 1\} \cup \{0, 1\} \times I$, where $I \times \{0, 1\}$ is contained in the κ_i . Then the connect sum $\kappa_1 \# \kappa_2$ is just the logical XOR of $\kappa_1 \cup \kappa_2$ with $\partial(I \times I)$:

$$\kappa_1 \# \kappa_2 = (\kappa_1 \cup \kappa_2) \cup \partial(I \times I) \setminus (\kappa_1 \cup \kappa_2) \cap \partial(I \times I)$$

Note that there is no natural basepoint for \mathcal{C} , but if τ and δ are two curves with the same endpoints, and if we are given an orientation–preserving isomorphism

$$\phi : T_{\tau(0)}\mathbb{R}^2 \rightarrow T_{\tau(1)}\mathbb{R}^2$$

satisfying

$$\phi(\tau'(0)) = \tau'(1), \phi(\delta'(0)) = \delta'(1)$$

then we can deform δ through curves δ_t satisfying $\phi(\delta_t'(0)) = \delta_t'(1)$ until $\delta_1'(0) = \tau'(0)$, and define the difference of the writhe of τ and δ as

$$[\tau] - [\delta] \in \mathcal{C}$$

Lemma 4.7.4. *With τ, δ as above, the difference in writhe is equal to the Euler class:*

$$[\tau] - [\delta] = e_\rho$$

Proof. This follows immediately from the definition of index and the formula in construction 4.1.2. \square

For τ with the properties of the previous section, B_ρ is identically zero. The Alexander series A_ρ is defined as before. We now prove an analogue of lemma 4.6.5 for the case that B_ρ is not identically zero.

Lemma 4.7.5. *For general τ , the series A_ρ and B_ρ satisfy the following properties:*

- (1) *The coefficients of B_ρ are antisymmetric; that is, if $B_\rho = \sum_i b_i t^i$, then*

$$b_i = -b_{-i}$$

- (2) *The sum $A_\rho - B_\rho$ is a Laurent polynomial.*
(3) *The sum of the coefficients of $A_\rho - B_\rho$ is zero:*

$$(A_\rho - B_\rho)(1) = 0$$

- (4) *The Euler class $\rho^*([e])([\mathbb{Z} \oplus \mathbb{Z}])$, which we abbreviate by e_ρ , satisfies*

$$e_\rho = \sum_{i>0} (a_i - b_i) - \sum_{i<0} (a_i - b_i)$$

Proof. The first fact follows because the intersection number of τ with $\beta^i(\tau)$ is equal to the intersection number of $\beta^{-i}(\tau)$ with τ , and intersection number of 1-cycles is antisymmetric.

The second and third facts follow as in lemma 4.6.5, by the fact that $\tau \cup \delta$ (where τ has the opposite orientation) is an (immersed) closed loop, and the orbit $\beta^i(p)$ is proper.

To calculate the Euler number, we move τ and δ by a homotopy τ_t, δ_t through C^1 immersions whose germs near p and $\beta(p)$ are constant. We move τ, δ equivariantly, so that at each stage, $\delta_t = \alpha(\tau_t)$. We also connect sum τ and δ equivariantly with sufficiently many figure 8's on the positive or negative sides until they are embedded curves from p to $\beta(p)$.

Note that at finitely many times t_0 , we might move τ_{t_0} through some p_i , and change b_i and b_{i-1} . However, since $\delta_{t_0} = \alpha(\tau_{t_0})$, it follows that δ_{t_0} passes through p_i . Therefore $\tau_t \cdot \beta^i(\tau)$ and $\delta_t \cdot \beta^i(\tau)$ change by the same amount at such a time t_0 , and therefore the differences $\delta_t \cdot \beta^i(\tau) - \tau_t \cdot \beta^i(\tau)$ are always constant, and equal to $a_i - b_i$.

We also connect sum τ and δ equivariantly with sufficiently many small figure 8's on positive or negative sides. This does not affect any a_i or b_i , and by lemma 4.7.4, it does not affect the relative write $[\tau_t] - [\delta_t]$ or the calculation of the Euler class.

By finitely many moves of this kind, we can equivariantly homotope τ and δ through immersions whose germ at $p, \beta(p)$ is constant, until they are both embedded (not necessarily disjointly).

It follows that the Euler class can be calculated from the formula of lemma 4.6.4, after finding suitable properly immersed rays τ^\pm such that $\tau^- \cup \tau \cup \tau^+$ is a properly immersed line, and τ^\pm do not intersect τ except at their endpoints. The unions $\bigcup_{i < 0} \beta^i(\tau)$ and $\bigcup_{i > 0} \beta^i(\tau)$ will not do, since $\beta^i(\tau)$ might intersect τ essentially. But we can homotope each $\beta^i(\tau)$ through smooth curves, rel. endpoints, to τ'_i which are disjoint from τ , at the cost of crossing τ'_i b_i times (algebraically) over $p, \beta(p)$. Then for each i ,

$$\delta \cdot \tau'_i - \tau \cdot \tau'_i = \delta \cdot \beta^i(\tau) - \tau \cdot \beta^i(\tau)$$

Moreover, we can homotope τ'_i through immersions rel. endpoints, and without crossing p or $\beta(p)$, to some new τ''_i , so that we have $\bigcup_{i > 0} \tau''_i = \tau^+$ and $\bigcup_{i < 0} \tau''_i = \tau^-$ where τ^\pm are properly immersed, and do not intersect τ except at their endpoints. Then

$$\delta \cdot \tau''_i = \delta \cdot \beta^i(\tau) - \tau \cdot \beta^i(\tau) = a_i - b_i$$

so the formula for the Euler class follows.

Note that since $A_\rho - B_\rho$ is a Laurent polynomial, this is actually a finite sum. \square

Having established the properties of A_ρ and B_ρ , we can now prove the analogue of lemma 4.6.6 for arbitrary τ .

Lemma 4.7.6. *Let $\rho : \mathbb{Z} \oplus \mathbb{Z} \rightarrow \text{Homeo}^+(\mathbb{R}^2)$ be a C^1 action which is not free. Then the Euler class $\rho^*([e])$ is zero.*

Proof. Without loss of generality, we can assume α fixes some p , and the orbit of p under β is proper. Let τ be as above, and define the series A_ρ, B_ρ . Let G_n be the subgroup of $\mathbb{Z} \oplus \mathbb{Z}$ of index $2n + 1$ generated by α and β^{2n+1} as before. As in the

proof of lemma 4.6.6, we have

$$\begin{aligned} e_{\rho_n} &= (2n+1)e_\rho = \sum_{j>0} \sum_{i=-2n}^{2n} (2n+1-|i|)(a_{i+j(2n+1)} - b_{i+j(2n+1)}) \\ &\quad - \sum_{j<0} \sum_{i=-2n}^{2n} (2n+1-|i|)(a_{i+j(2n+1)} - b_{i+j(2n+1)}) \end{aligned}$$

On the other hand, by lemma 4.7.5, the coefficients $a_i + b_i$ are zero for $|i|$ sufficiently large, so as n increases, the right hand side is eventually constant. On the other hand, this sum is equal to $(2n+1)e_\rho$, so $e_{\rho_n} = e_\rho = 0$, as claimed. \square

4.8. Vanishing of the Euler class. From the previous section, we know that a C^1 $\mathbb{Z} \oplus \mathbb{Z}$ action with nontrivial Euler class must be *free*; that is, no nontrivial element has a fixed point.

We make use of several theorems, one of which was originally proved by Brouwer (see [2]), but later given a simpler and more illuminating proof by many people including [28], [10]. The first theorem says that for β a *fixed-point free* orientation preserving homeomorphism of \mathbb{R}^2 , and p arbitrary, we can find an arc τ from p to $\beta(p)$ which does not intersect its translates, except at the endpoints. More formally,

Theorem 4.8.1 (Brouwer). *Let $\alpha : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be orientation-preserving and fixed point free. Then every point has a proper orbit, and for every point $p \in \mathbb{R}^2$ there is an embedded line $l \subset \mathbb{R}^2$ containing p , on which the action of α is conjugate to a translation.*

Remark 4.8.2. The caveat of theorem 4.8.1 is that the invariant line l it produces through every point p is *not* generally properly embedded, even though the orbit of every *point* is properly embedded.

We also make use of a property of fixed-point free orientation-preserving homeomorphism, established by Brown, in [3].

Brown makes the following definition:

Definition 4.8.3. An orientation-preserving homeomorphism $h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is *free* if for every bounded, connected set X with $X \cap h(X) = \emptyset$, we have $X \cap h^i(X) = \emptyset$ for all i .

Theorem 4.8.4 (Brown). *Let $h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be an orientation preserving homeomorphism. Then h is free.*

Finally, we prove an algebraic lemma:

Lemma 4.8.5. *Let $h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a C^1 orientation preserving homeomorphism. Then for any two arcs τ_1, τ_2 from p to $h(p)$ such that the union of translates of τ_i is a C^1 embedded line for $i = 1, 2$, the writhe of τ_1 and τ_2 are equal.*

Proof. Note that for an arc τ with the property in question, the homeomorphism h satisfies $dh(\tau'(0)) = \tau'(1)$ so the relative writhe of τ_1 and τ_2 make sense.

Let τ_t be a 1-parameter deformation of $\tau(0)$ through smooth embedded curves with $dh(\tau_t'(0)) = \tau_t'(1)$.

Initially we have $i \neq 0$, and $\tau_t \cdot h^i(\tau_t) = 0$. When we pass τ_t over some vertex $h^i(p)$ with $i \neq 0, 1$ we get $\tau_t \cdot h^i(\tau_t)$ increases by e , and $\tau_t \cdot h^{i+1}(\tau_t)$ decreases by $-e$, where $e = \pm 1$ depending on the orientation of τ_t at $h^i(p)$. Moreover, if we

introduce a twist at p (thereby changing the writhe by 1), then $\tau_t \cdot h^{-1}(\tau_t)$ changes by 1, and $\tau_t \cdot h(\tau_t)$ changes by -1 . It follows that

$$w_t = \sum_{i>0} \tau_t \cdot h^i(\tau_t) - \sum_{i<0} \tau_t \cdot h^i(\tau_t)$$

is a finite sum. Moreover, by lemma 4.7.4, we have

$$w_t = 2([\tau_t] - [\tau_1])$$

where the notation $[\tau_t] - [\tau_1]$ means the difference in writhe. It follows that w_t is equal to 0 iff the writhe of τ_t is equal to the writhe of τ_1 . On the other hand, if we take an isotopy from τ_1 to τ_2 , then $w_2 = 0$, since τ_2 does not intersect $h^i(\tau_2)$ except at its endpoints. So $[\tau_2] - [\tau_1] = 0$, and the lemma is proved. \square

Remark 4.8.6. In fact, the argument of lemma 4.6.6 can be used to give another proof of lemma 4.8.5. Conversely, lemma 4.8.5 can be used to give a different proof of lemma 4.6.6. But the argument of lemma 4.8.5 does not easily generalize to prove lemma 4.7.6, whereas the argument of lemma 4.6.6 does.

We are now ready to prove the main theorem on homological rigidity of C^1 $\mathbb{Z} \oplus \mathbb{Z}$ actions:

Theorem D. *Let $\rho : \mathbb{Z} \oplus \mathbb{Z} \rightarrow \text{Homeo}^+(\mathbb{R}^2)$ be a C^1 action. Then the Euler class $\rho^*([e]) \in H^2(\mathbb{Z} \oplus \mathbb{Z}; \mathbb{Z})$ is zero.*

Proof. By lemma 4.7.6, we have proved this theorem except in the case that $\rho(\mathbb{Z} \oplus \mathbb{Z})$ is free.

Let p be arbitrary, and let τ be a smooth arc from p to $\beta(p)$ so that $\bigcup_i \beta^i(\tau) = l$ is an embedded line. Such a τ can be found by theorem 4.8.1. Let σ be an arc from p to $\alpha(p)$. For each point $p_t \in \sigma$, we let τ_t be a smooth arc from p_t to $\beta(p_t)$ so that $\bigcup_i \beta^i(\tau_t) = l_t$ is an embedded line. We show that for any t_0 , there is an ϵ so that for $|t - t_0| < \epsilon$, the family τ_t can be chosen to vary smoothly.

To see this, observe that we can certainly choose a smoothly varying family τ_t near τ_{t_0} so that τ_t intersects $\beta(\tau_t)$ only at $\beta(p_t)$. But by theorem 4.8.4, this implies that the union of the $\beta^i(\tau_t)$ is an embedded line, as required.

It follows that we can subdivide $[0, 1]$ into finitely many intervals

$$[0, q_1], [q_1, q_2], \dots, [q_n, 1]$$

and choose smooth families τ_t^i for $q_i \leq t \leq q_{i+1}$ with the property that $\tau_0^0 = \tau$ and $\tau_1^n = \alpha(\tau)$. But $\tau_{q_{i+1}}^i$ and $\tau_{q_{i+1}}^{i+1}$ have equal writhe, by lemma 4.8.5, so we can insert a 1-parameter family of isotopies from $\tau_{q_{i+1}}^i$ to $\tau_{q_{i+1}}^{i+1}$ whose germ at $p_t, \beta(p_t)$ is fixed. After inserting these isotopies, we have constructed a one parameter family of curves τ_t from τ to $\alpha(\tau)$ with $d\beta(\tau_t'(0)) = \tau_t'(1)$ for all t . The vector field τ_t' pulls back to give a trivialization over a fundamental domain P_1 for the torus. It follows from the formula of construction 4.1.2 that the Euler class is zero, and the theorem is proved. \square

Remark 4.8.7. It might be possible to try to prove theorem D by a cancellation argument, by analogy with lemma 4.6.6 and lemma 4.7.6. The problem is that, though the orbit of any point p under any cyclic subgroup of a free action $\rho(\mathbb{Z} \oplus \mathbb{Z})$ is proper, it is *not* true that the orbit under all of $\rho(\mathbb{Z} \oplus \mathbb{Z})$ is proper.

Remark 4.8.8. Notice that the use of Brown's theorem in the proof of theorem D is essential, in order to be able to apply compactness of the interval σ and compare the relative writhe of τ and $\beta(\tau)$. Brown's theorem itself is a kind of compactness theorem, since it lets one establish properties of the entire orbit of a connected set τ from the properties of only two translates τ and $\beta(\tau)$.

REFERENCES

- [1] S. Boyer, D. Rolfsen and B. Wiest, *Left orderability of 3-manifold groups*, preprint.
- [2] L. E. J. Brouwer, *Remark on the plane translation theorem*, *Nederl. Akad. Wetensch. Proc.* **21** (1919), 935–936
- [3] M. Brown, *Homeomorphisms of two-dimensional manifolds*, *Houston J. Math.* **11** (1985), no. 4, 455–469
- [4] R. Burns and V. Hale, *A note on group rings of certain torsion-free groups*, *Canad. Math. Bull.* **15** (1972), 441–445
- [5] D. Calegari, *Dynamical forcing of circular groups*, preprint, math.GT/0312066
- [6] C. Carathéodory, *Über die Begrenzung einfach zusammenhängender Gebiete*, *Math. Ann.* **73** (1913), 323–370
- [7] P. Dehornoy, *Braids and self-distributivity*, *Progress in Mathematics*, **192** Birkhäuser Verlag, Basel (2000)
- [8] P. Dehornoy, I. Dynnikov, D. Rolfsen and B. Wiest, *Why are braids orderable?* *Panoramas et Synthèses* **14** *Soc. Math. de France, Paris* (2002)
- [9] A. Denjoy, *Sur les courbes définies par les équations différentielles à la surface du tore*, *J. Math. Pures Appl.* **11** (1932), 333–375
- [10] J. Franks, *A new proof of the Brouwer plane translation theorem*, *Ergodic Theory Dynam. Systems* **12** (1992), no. 2, 217–226
- [11] E. Ghys, *Groupes d'homéomorphismes du cercle et cohomologie bornée*, *The Lefschetz centennial conference, Part III (Mexico City, 1984)*, 81–106, *Contemp. Math.*, **58**, III, *Amer. Math. Soc., Providence, RI*, (1987)
- [12] M. Handel, *A fixed point theorem for planar homeomorphisms*, *Topology* **38** (1999), no. 2, 235–264
- [13] D. Husemoller, *Fibre bundles*, Third Edition; *Grad. Texts in Math.* **20** Springer-Verlag, Berlin (1994)
- [14] S. Jekel, *A simplicial formula and bound for the Euler class*, *Israel Journal of Mathematics* **66** (1989) nos. 1–3, 247–259
- [15] J-L. Loday, *Cyclic homology*, *Die Grundlehren der Mathematischen Wissenschaften*, **301** Springer-Verlag, Berlin (1998)
- [16] S. Mac Lane, *Homology*, *Die Grundlehren der Mathematischen Wissenschaften*, **114** Springer-Verlag, Berlin (1963)
- [17] J. Mather, *Topological proofs of some purely topological consequences of Caratheodory's Theory of prime ends*, in *Selected Studies*, (Th M Rassias, G M Rassias, editors) North-Holland (1982) 225–255
- [18] S. Matsumoto, *Some remarks on foliated S^1 bundles*, *Invent. Math.* **90** (1987), no. 2, 343–358
- [19] J. Milnor, *Introduction to Algebraic K theory*, *Annals of Mathematics Studies*, **72**, *Princeton University Press*, (1971)
- [20] J. Milnor, *On the existence of a connection with curvature zero*, *Comm. Helv. Math.* **32** (1958), 215–223
- [21] J. Montesinos, *Classical Tessellations and Three-Manifolds*, *Universitext*, Springer-Verlag, Berlin (1985)
- [22] R. L. Moore, *Foundations of point set theory*, *American Mathematical Society Colloquium Publications*, Vol. XIII *Amer. Math. Soc., Providence*, (1962)
- [23] R. L. Moore and J. Kline, *On the most general plane closed point set through which it is possible to pass a simple continuous arc*, *Ann. Math.* **20** (1919), 218–223
- [24] R. Mura and A. Rhemtulla, *Orderable groups*, *Lecture Notes in Pure and Applied Mathematics*, Vol. **27**. *Marcel Dekker, Inc., New York-Basel*, (1977)
- [25] Ch. Pommerenke, *Boundary behaviour of conformal maps*, *Die Grundlehren der Mathematischen Wissenschaften*, **299** Springer-Verlag, Berlin (1992)
- [26] R. Roberts and M. Stein, *Exceptional Seifert group actions on \mathbb{R}* , *J. Knot Theory Ramifications* **8** (1999), no. 2, 241–247
- [27] H. Short, B. Wiest, *Orderings of mapping class groups after Thurston*, *Enseign. Math. (2)* **46** (2000), no. 3–4, 279–312
- [28] E. Slaminka, *A Brouwer translation theorem for free homeomorphisms*, *Trans. Amer. Math. Soc.* **306** (1988), no. 1, 277–291
- [29] W. Thurston, *Foliations and groups of diffeomorphisms*, *Bull. Amer. Math. Soc.* **80** (1974), 304–307
- [30] W. Thurston, *A generalization of the Reeb stability theorem*, *Topology* **13** (1974), 347–352
- [31] W. Thurston, *3-manifolds, foliations and circles II*, preprint
- [32] T. Tsuboi, *On the foliated products of class C^1* , *Ann. Math.* **130** (1989), 227–271
- [33] T. Tsuboi, *Homology of diffeomorphism groups, and foliated structures*, *Translated in Sugaku Expositions* **3** (1990), no. 2, 145–181
- [34] J. Wood, *Bundles with totally disconnected structure group*, *Comm. Math. Helv.* **46** (1971), 257–273

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