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LETTER TO THE EDITOR

Similarity solutions of the Einstein and Einstein–Maxwell equations

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Abstract. Exact solutions of the equations governing vacuum cylindrical gravitational wave spacetimes and colliding plane electromagnetic and plane gravitational wave spacetimes are presented. Both solutions are found by using the geometric technique of Harrison and Estabrook to find appropriate similarity variables to reduce partial differential equations to ordinary differential equations. One of the solutions is transformed into a solution of the Ernst equations.

The problem of finding exact solutions to the Einstein or Einstein–Maxwell equations for various axisymmetric fields has received much attention in recent years (see Kinnersley 1975, Bell and Szekeres 1974). Some of these physically different problems turn out to possess the same field equations, as shown in Harrison (1968), Fischer (1977) and Catenacci and Alonso (1976). In the following, it is shown how a systematic approach to finding similarity variables for these field equations due to Harrison and Estabrook (1971) leads to exact solutions.

The equations considered are

$$U_{,\rho\rho} + U_{,\rho}/\rho - U_{,\tau\tau} = \mathrm{e}^{-2U}(\Omega_{,\tau}^2 - \Omega_{,\rho}^2), \tag{1a}$$

$$\Omega_{,\rho\rho} + \Omega_{,\rho}/\rho - \Omega_{,\tau\tau} = 2(\Omega_{,\rho}U_{,\rho} - \Omega_{,\tau}U_{,\tau}). \tag{1b}$$

Equations (1) are both the Einstein equations for cylindrical gravitational waves (Kinnersley 1975) (in which case $Q = -U + \log\rho$ and Ω are metric coefficients) and the Einstein–Maxwell equations for colliding plane gravitational and plane electromagnetic waves (Bell and Szekeres 1974) (in which case U is a metric coefficient and Ω is an electromagnetic potential). ρ and τ are the cylindrical radial and time coordinates respectively.

For a discussion on writing differential equations as differential forms, the reader is referred to Harrison and Estabrook (1971), where the concepts of isovector and generalised isovector are discussed. For present purposes all that need be known is that an isovector or generalised isovector can be used to find functional forms for the dependent variables of a set of partial differential equations that will reduce the number of independent variables by one. The geometric techniques are related to the classical techniques discussed, for example, in Bluman and Cole (1974).

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For example, isovector 2 of Harrison and Estabrook leads to the functional form

$$U = U(\rho/\tau) = U(\eta), \tag{2a}$$

$$\Omega = \Omega(\rho/\tau) = \Omega(\eta), \tag{2b}$$

where $\eta = \rho/\tau$. Substitution of equations (2) into equations (1) yields the ordinary differential equations

$$U''(\eta^2-1) + U'(2\eta - 1/\eta) = \mathrm{e}^{-2U}\Omega'^2(1-\eta^2), \tag{3a}$$

$$\Omega''(\eta^2-1) + \Omega'(2\eta - 1/\eta) = 2U'\Omega'(\eta^2-1), \tag{3b}$$

where a prime denotes differentiation with respect to η. To solve equation (3b), rewrite it as

$$\frac{\Omega''}{\Omega'} + \frac{2\eta - 1/\eta}{\eta^2-1} = 2U'$$

and integrate to obtain

$$\Omega' = C_1\,\mathrm{e}^{2U}/(\eta^2-1)^{1/2}, \tag{4}$$

where C_1 is a constant of integration. Substituting equation (4) into equation (3a) yields

$$U''(\eta^2-\eta^4) + U'(\eta - 2\eta^3) = \mathrm{e}^{2U}C_1^2, \tag{5}$$

which may be multiplied by U' and integrated to yield

$$U'^2(\eta^2-\eta^4) = C_1^2\,\mathrm{e}^{2U}, \tag{6}$$

where we choose the constant of integration to be zero to obtain an explicit solution as follows.

Integration of equation (6) yields

$$\mathrm{e}^{-U} = C_1\,\mathrm{sech}^{-1}\,\eta + C_2, \tag{7a}$$

where C_2 is another integration constant. Equation (4) then gives Ω as a quadrature,

$$\Omega = C_1 \int_{C_3}^{\eta} \frac{\mathrm{e}^{2U(\sigma)}\,\mathrm{d}\sigma}{\sigma(\sigma^2-1)^{1/2}}, \tag{7b}$$

where C_3 is a third integration constant. Equations (7) comprise an exact solution of (1) and may be used to represent the appropriate spacetimes. Similarly, the generalised isovector found by Fischer (1977) leads to the functional form

$$U = U(\rho^2-\tau^2) = U(\eta), \tag{8a}$$

$$\Omega = \Omega(\rho^2-\tau^2) = \Omega(\eta). \tag{8b}$$

Substitution of equation (8) into equation (1) yields

$$2\eta U'' + 3U' = -2\eta\,\mathrm{e}^{-2U}\Omega'^2, \tag{9a}$$

$$2\eta\Omega'' + 3\Omega' = 4\eta\Omega' U'. \tag{9b}$$

Equation (9b) integrates to

$$\Omega' = C_1\,\mathrm{e}^{2U}\eta^{-2/3}, \tag{10}$$

where C_1 is an integration constant. Substituting equation (10) into equation (9*a*) yields

$$2\eta^3 U'' + 3\eta^2 U' = -2C_1^2\, e^{-2U},$$

which may be multiplied by U' and integrated to yield

$$\eta^3 U'^2 = -C_1^2\, e^{2U} + C_2,$$

where C_2 is another constant of integration. This may be rewritten as

$$dU/(C_2 - C_1^2\, e^{2U})^{1/2} = \eta^{-3/2}\, d\eta.$$

We see that C_2 must be chosen so that $C_2 > C_1^2\, e^{2U}$. Integration then yields

$$e^{2U} = (C_2/C_1^2)\{1 - 4\coth^2[\sqrt{C_2}(2\eta^{-1/2} + C_3)]\}, \tag{11a}$$

where C_3 is another integration constant. Equation (10) then gives Ω as a quadrature,

$$\Omega = C_1 \int_{C_4}^{\eta} e^{2U(\sigma)} \sigma^{-3/2}\, d\sigma, \tag{11b}$$

where C_4 is a fourth integration constant. Equations (11) yield another Einstein or Einstein–Maxwell solution.

If we set $\tau = iZ$, equations (1) become

$$U_{,\rho\rho} + U_{,\rho}/\rho + U_{,ZZ} = -e^{-2U}(\Omega_{,Z}^2 + \Omega_{,\rho}^2), \tag{12a}$$

$$\Omega_{,\rho\rho} + \Omega_{,\rho}/\rho + \Omega_{,ZZ} = 2(\Omega_{,\rho} U_{,\rho} + \Omega_{,Z} U_{,Z}), \tag{12b}$$

which are the Ernst equations for the external field of an axially symmetric, rotating body where Ω is the 'twist' potential, as shown in Kinnersley (1975). The functional form, equation (8), becomes

$$U = U(\rho^2 + Z^2) = U(\eta), \tag{13a}$$

$$\Omega = \Omega(\rho^2 + Z^2) = \Omega(\eta), \tag{13b}$$

which is of the same functional form as the Curzon (1924) solution. A solution of equations (12) is equations (11) with the similarity variable $\eta = \rho^2 + Z^2$. This solution is in fact asymptotically flat, since as $\eta \to \infty$ we see from equation (11*a*) that

$$e^{2U} \to (C_2/C_1^2)(1 - 4\coth^2\sqrt{C_2 C_3}).$$

The integral for Ω, equation (11*b*), also converges as $\eta \to \infty$, since for large σ the integrand behaves like $\sigma^{-2/3}$. The constants C_1, C_2, C_3 may be chosen so that $e^{2U} \to 1$ as $\eta \to \infty$. Since the potential Ω is only determined to within an additive constant (Kinnersley 1975), we may use this freedom to give $\Omega \to 0$ as $\eta \to \infty$. This asymptotically flat solution represents the external gravitational field of a stationary, axially symmetric, rotating body.

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