nature portfolio

https://doi.org/10.1038/s41586-022-04606-3

Supplementary information

Engineered jumpers overcome biological limits via work multiplication

In the format provided by the
authors and unedited

Supplementary Methods

Energy Utilisation Model

Here we include further details of the energy utilisation model presented in the Methods. This derives the six reduction stages, discusses the scaling behaviours, and notes a minimum body/payload mass. We assume the ground is perfectly rigid and of infinite mass.

Derivation: We step through the derivation of the individual stages of energy flow during a jump. Numerical calculations may require assumptions and approximations as noted.

1) For the first stage, we consider the ability of the jumper to produce up to the maximum possible payload-free specific jump energy, e_{jump}. Any viscoelastic losses in the motor or any impedance mismatches between components that do not allow the motor to operate at full force or the spring to fully discharge will lead to a reduce efficiency of

$$e_{prod} = e_{jump}\, \eta_{prod}\,. \quad (1)$$

2) The second stage describes the effect of adding a payload to the jumper and determines the initial specific energy, e_0, utilised for a single jump. It represents an apportionment of the produced specific energy of the payload-free jumper across the entire mass, m, of the jumper. Defining the payload mass ratio $\frac{m_{payload}}{m}$, we have

$$e_0 = e_{prod}\left(1 - \frac{m_{payload}}{m}\right). \quad (2)$$

3) The third stage considers the specific kinetic energy when the jumper has fully extended, the instant before take-off. It deducts from the initial specific energy, e_0, the specific potential energy surrendered in raising the body of mass m_{body} (including any payload) to stand at height, $z = L$,

$$e_{KE} = \frac{m\, e_o - m_{body}\, gL}{m} e_0 = e_0 - Lg\frac{m_{body}}{m}\,. \quad (3)$$

It is important to note that jump height is defined as the change in height of the centre of mass above its position when the jumper is fully standing[21,22]. We should not consider the height above ground or above the crouch position. Indeed, simply standing up from a crouch is not a jump, and we need to deduct any potential energy increase contained therein. While this inefficiency is negligible for jumpers that jump many body heights, it must be considered if the height gained during standing, L, becomes a significant proportion of the jump. In particular, at large scales, the initial specific energy is scale invariant, capping the maximum jump height and subjecting jumpers with increasing size to an increasing reduction due to this "energy-to-stand."

4) Next, we consider the portion of the specific kinetic energy in (3) due to vertical movements of the body parts. We note that any horizontal and rotational movements, and the energies they

contain, do not contribute to the jump height. Consider the total kinetic energy, E_{KE}, of all component parts, i,

$$E_{KE} = m\, e_{KE} = \sum_i \tfrac{1}{2} m_i v_{xi}^2 + \tfrac{1}{2} m_i v_{yi}^2 + \tfrac{1}{2} m_i v_{zi}^2 + \tfrac{1}{2} I_i \omega_i^2$$

where v_x, v_y, v_z are the translational velocities, and ω is the rotational velocity about the instantaneous axis of rotation. We isolate

$$E_{xy} = m\, e_{xy} = \sum_i \tfrac{1}{2} m_i v_{xi}^2 + \tfrac{1}{2} m_i v_{yi}^2 \quad , \quad E_z = m\, e_z = \sum_i \tfrac{1}{2} m_i v_{zi}^2 \quad , \quad E_\theta = m\, e_\theta = \sum_i \tfrac{1}{2} I_i \omega_i^2$$

and define

$$\beta_{xy} = \frac{E_{xy}}{E_{KE}} = \frac{e_{xy}}{e_{KE}} \quad , \quad \beta_\theta = \frac{E_\theta}{E_{KE}} = \frac{e_\theta}{e_{KE}}$$

where β_{xy} and β_θ are the fractions of the kinetic energy due to horizontal and rotational movements respectively. We note that linkages generally enforce a particular ratio between the component velocities and hence we expect constant percentages independent of energy levels. The stage is summarized as

$$e_{vert} = e_z = e_{KE} \left[1 - \beta_{xy} - \beta_\theta\right]. \quad (4)$$

5) We then examine the portion of the specific kinetic energy in (4) that contributes to the vertical centre of mass motion. By definition, some part of the jumper must be in contact with the ground and static, while other part(s) are moving. Such relative movements contain energy without contributing to the COM motion. Equivalently, after launch, all parts must move with the same average velocity and internal forces have to be applied, e.g. to accelerate the static part.

While mass and momentum are conserved during this transfer process, energy is not. Considering only vertical velocities of all component parts, i, we can write

$$\frac{e_{COM}}{e_{vert}} = \frac{\frac{\frac{1}{2} m v_{COM}^2}{m}}{\frac{\sum_i \frac{1}{2} m_i v_i^2}{m}} = \frac{\frac{1}{2} m v_{COM}^2}{\sum_i \frac{1}{2} m_i v_i^2} = \frac{\frac{1}{2}(m v_{COM})^2}{m \ \sum_i \frac{1}{2} m_i v_i^2} = \frac{\frac{1}{2}(\sum_i m_i v_i)^2}{\sum_i m_i \ \sum_i \frac{1}{2} m_i v_i^2}$$

while conserving $m = \sum_i m_i$ and $m v_{COM} = \sum_i m_i v_i$.

Now, it is possible to describe any set of masses comprising a jumper with a lumped sum model containing only two masses, one static (the "foot") without momentum or energy and one moving (the "body"). To form this description, we constrain

$$v_{foot} = 0$$

while matching the system's mass, linear momentum, and energy

$$m_{body} + m_{foot} = \sum_i m_i \quad = m$$

$$m_{body} v_{body} = \sum_i m_i v_i \quad = m\, v_{COM}$$

$$\frac{1}{2}m_{body}v_{body}^2 = \sum_i \frac{1}{2}m_i v_i^2$$

This inherently apportions all components (e.g., motor or springs) to the foot and body, depending on what proportions are moving with the body or stationary with the foot, such that any non-idealities from the self-mass of a spring are accounted for[34] - we see the effective body and foot masses

$$m_{body} = \frac{\frac{1}{2}(\sum_i m_i v_i)^2}{\sum_i m_i \ \sum_i \frac{1}{2}m_i v_i^2} \quad , \quad m_{foot} = \sum_i m_i - m_{body} \ .$$

With this two-body model we find

$$\frac{e_{COM}}{e_{vert}} = \frac{\frac{1}{2}(m v_{COM})^2}{m \ \sum_i \frac{1}{2}m_i v_i^2} = \frac{\frac{1}{2}\left(m_{body}v_{body}\right)^2}{m \ \frac{1}{2}m_{body}v_{body}^2} = \frac{m_{body}}{m} = \left[1 - \frac{m_{foot}}{m}\right]$$

and can simply write:

$$e_{COM} = e_{vert}\left[1 - \frac{m_{foot}}{m}\right] . \quad (5)$$

6) Finally, we consider the specific energy at the apex of the jump, which is reduced from (5) by the aerodynamic drag losses. We assume that these are negligible during launch (drag forces are orders of magnitude lower than ground reaction force) and only consider the flight phase after take-off. Assume the drag force can be modelled as

$$F_d = \frac{1}{2}C_d \ \rho_{air} \ A \ v^2,$$

where C_D is the drag coefficient, ρ_{air} is the air density, A is the frontal area, and v is the vertical velocity. Then the equation of motion during vertical flight can be written as

$$m\frac{d^2z}{dt^2} + \frac{C_d \ \rho_{air} \ A}{2}\left(\frac{dz}{dt}\right)^2 + mg = 0 \ ,$$

where z is the vertical distance the centre of mass has travelled after take-off[30]. Solving this differential equation leads to

$$h = \frac{m}{C_d \ \rho_{air} \ A}\ln\left[\frac{C_d \ \rho_{air} \ A \ v^2}{2 \ m \ g} + 1\right] .$$

For convenience we define the constant

$$D_s = \frac{C_d \ \rho_{air} \ A}{m \ g} .$$

Knowing $e_{COM} = \frac{1}{2}v^2$, the jump height can then be rewritten as

$$h = \frac{1}{gD_s}\ln[D_s e_{COM} + 1] .$$

We then note that the specific energy at the apex can be written as

$$e_{apex} = gh = \frac{1}{D_s}\ln[D_s e_{COM} + 1] .$$

And the ratio of the specific energy at the apex to that at launch is

$$\frac{e_{apex}}{e_{COM}} = \frac{\ln[D_s e_{COM}+1]}{D_s e_{COM}}.$$

Next, we note that using a Taylor series expansion, we can approximate this ratio as

$$\frac{e_{apex}}{e_{COM}} \approx 1 - \frac{D_s e_{COM}}{2} + \frac{(D_s e_{COM})^2}{3} - \cdots$$

Finally, we can write the specific energy at the apex as a portion of the specific energy at launch by using the first two terms of the series

$$e_{apex} = e_{COM}\left[1 - \frac{D_s e_{COM}}{2}\right]. \quad (6)$$

We note that we incorporate the Taylor series approximation into the overall model equation to lend a certain level of intuitive understanding to the model. While less accurate than the logarithmic function, we believe the loss in accuracy is worth the gain in understanding. Where precision is required, the full logarithmic model can be used.

Putting these stages together leads us to the full model with the maximum available energy input and six reductions incorporated

$$h = \frac{1}{g} e_{apex} = \frac{1}{g}\left[e_{jump}^{mech}\, \eta_{prod}\left(1 - \frac{m_{payload}}{m}\right) - Lg\frac{m_{body}}{m}\right]\left[1 - \beta_{xy} - \beta_\theta\right]\left[1 - \frac{m_{foot}}{m}\right]\left[1 - \frac{D_s e_{COM}}{2}\right]$$

Scaling: Knowing the scaling behaviour of the maximum payload-free specific jump energy, we also review the scaling effects of the six reductions in the energetic jump model. Thus, we can characterise how the overall jump height predictions vary across scale.

We first note that isometric scaling retains geometry and mass distribution, leaving all but energy-to-stand and aerodynamic losses scale-invariant. Energy-to-stand losses increase linearly with scale, leading to a large-scale drop off. The aerodynamic losses dominate at small scale. Together we find a central plateau region, between small- and large-scale drop offs. For biological jumpers without power amplification this region is quite narrow, yet wider for our work multiplied engineered jumpers.

The large-scale drop off is dictated by the specific energy lost while raising from crouch to stand, $gL\frac{m_{body}}{m}$, increasing with increasing scale. This leads to a maximum standing height of

$$L_{stand} = \frac{1}{g}\frac{m - m_{payload}}{m - m_{foot}}\eta_{prod}\, e_{jump}$$

The maximum payload-free specific jump energy, e_{jump}, remains scale-invariant at large scale and implies a maximum size at which jumping is possible, for all but direct-actuated engineered jumpers.

These direct-actuated engineered jumpers can effectively utilise a large gear-reduction to raise an arbitrary mass up to an arbitrary height. However, working against gravity with limited motor specific power, p_m, the vertical body velocity is limited to

$$v_{vert} = \frac{m_m \, p_m}{m_{body} \, g}$$

which bounds the vertical specific kinetic energy and subsequently the jump height independent of scale.

At small-scale, the aerodynamic losses dominate. As the frontal area, A, scales with L^2, and the mass scales with L^3, the aerodynamic constant $D_s = \frac{C_d \rho A}{mg}$ increases for smaller scales. The losses reach 50%

$$\frac{e_{apex}}{e_{COM}} = \frac{\ln[D_s e_{COM} + 1]}{D_s e_{COM}} = 50\%.$$

when

$$D_S e_{COM} = \frac{C_d \, \rho_{air} \, A}{m \, g} \, e_{COM} = 2.513.$$

Setting $m = \rho_{jumper} L^3$ and approximating $A = L^2$ yields the critical scale

$$L_{aero} = 0.398 \, c_d \, \frac{\rho_{air}}{\rho_{jumper}} \, \frac{1}{g} \, e_{COM} \approx 0.398 \, c_d \, \frac{\rho_{air}}{\rho_{jumper}} \, \frac{1}{g} \, e_{jump}^{mech}$$

if we also assume the dominant aerodynamics make other reductions negligible. If the actuator is further power-limited, the two small-scale drop offs may combine.

Ideal body/payload mass: The above model can predict various relationships, in particular understanding the effect of adding payload or body mass. Stages 2 and 3 show that increasing mass lowers the specific energy – more mass needs to be accelerated as well as raise to stand. At the same time, stages 5 and 6 show reduced losses; foot mass losses and aerodynamic losses become less important.

Considering the simplified case, ignoring energy-to-stand and aerodynamics, and assuming a given foot and payload-free jumper mass, we determine the ideal total mass using the partial derivative

$$\frac{\partial}{\partial m}\left[\left(1 - \frac{m_{payload}}{m}\right)\left(1 - \frac{m_{foot}}{m}\right)\right] = \frac{\partial}{\partial m} \frac{m_{jump}(m - m_{foot})}{m^2} = m_{jump} \frac{2m_{foot} - m}{m^3} = 0$$

where the total mass can be separated by jumper/payload ($m = m_{jump} + m_{payload}$) or foot/body ($m = m_{foot} + m_{body}$). We see an ideal mass of

$$m = 2m_{foot} \quad \text{or} \quad m_{body} = m_{foot} \, ,$$

that is, an optimal body mass equal to the foot mass. Should the body mass fall below the foot mass, it is beneficial to add a payload to the system body. (Imagine a heavy foot and massless body; only by adding mass to the body would this system jump.)

For small systems (with aerodynamics dominating), the ideal system mass is shifted higher, negating aerodynamic losses. For large systems, the ideal system mass is shifted lower, reducing the energy-to-stand effect.

Comparing model to jump data: We calculated the take-off velocity with using a 4k, 60 fps video recording to be between 28 and 29 m/s. With these camera properties, there was minimal motion blur, and approximately 46 pixels for the length of the jumper (or 1 pixel per 0.66 centimetre). Each frame represents 0.0167 s, or about 45 cm for our speeds, meaning the digital error is approximately 1 part in 68, or 0.4 m/s at our speeds. We then used the utilisation model to estimate the take-off velocity, inputting masses measured with a scale with 0.001 g precision. We determined the non-vertical losses *(β_{xy}, β_θ)* via a lumped parameter physics simulation of the jumper, with the bow constructed of 20 segments and 19 torsional springs and the rubber springs simply as tension elements. The spring constant of the bows was set to match the force-displacement data of the real bow (equivalent to a flexural modulus of 105 GPa) and a damping ratio of 0.02 (see Methods: Energy Utilisation Model). With these measurements, the model predicts a take-off velocity of 28.2 m/s.

State-Space Model: Adding Jumper Specifics

We revisit the state-space model and provide the derivation, analytic solutions, and some additional observations. Recall we assume a single lumped body mass, m_b, moving vertically with position z, velocity v, and acceleration a. We further assume a leg-extension or length scale, L, such that the body mass accelerates from crouch ($z = 0$) to stand ($z = L$) where the take-off occurs and the jump begins.

The following computes the acceleration time, velocity, and energy for three distinct cases: (i) direct actuation assuming a fixed motor-to-leg gear reduction, (ii) direct actuation with a variable reduction allowing the motor to operate at maximum power, and (iii) spring actuation, where the preloaded spring delivers the energy.

Direct-drive Transmission with Fixed Reduction

Assume for direct-drive transmission that the body mass (m_b), composed of a linkage mass (m_l) and motor mass (m_m), is driven, via a fixed reduction G, by an inertia-free motor with linear viscous losses:

$$m_b\, a = GF_m \left(1 - \frac{Gv}{v_m}\right) - m_b\, g$$

where F_m and v_m are the motor's maximum force and velocity respectively. We also note two restrictions on the reduction G: First the motor must be able to lift the body, so that

$$G > \frac{m_b\, g}{F_m}$$

Second, for biological muscles, the reduction must be less than unity to ensure the muscle can operate within its finite stroke while the leg fully extends

$$G < 1$$

General Solution:

This first-order linear differential equation can be simplified to

$$a = \lambda\,(v_{ss} - v)$$

where the velocity converges exponentially at rate, λ, to a steady-state speed, v_{ss}:

$$v_{ss} = \frac{v_m\,(GF_m - m_b g)}{G^2 F_m} \quad , \quad \lambda = \frac{G^2\,F_m}{m_b\,v_m}$$

We can integrate the resultant velocity

$$v(t) \;=\; v_{ss}\left(1 - \mathrm{e}^{-\lambda t}\right)$$

into the vertical position

$$z(t) = \; v_{ss}\left(t + \frac{\mathrm{e}^{-\lambda t} - 1}{\lambda}\right)$$

where e represents the exponential function.

The acceleration time, t_0, is the time it takes the robot to reach the standing position, $z(t_0) = L$. As function of scale, the expression for the acceleration time is

$$t_0(L) = \frac{1}{v_{ss}}\,L \;+\; \frac{1}{\lambda}\left(1 + W\left(-\mathrm{e}^{-1 - \frac{\lambda}{v_{ss}}L}\right)\right)$$

And the corresponding velocity is

$$v_0(L) \;=\; v_{ss}\left(1 + W\left(-\mathrm{e}^{-1 - \frac{\lambda}{v_{ss}}L}\right)\right)$$

where $W()$ is the Lambert W or product log function. Finally, the specific energy production at t_0 is

$$e_{jump}^{motor} = \frac{1}{2}v_0^2 + gL$$

Explicit equations:
Substituting the motor parameters, we re-write the relations as

$$v_0(L) = \frac{v_m\,(GF_m - m_b g)}{G^2 F_m}\left(1 \;+\; W\left(-\;\mathrm{e}^{-1 - \frac{G^4\,F_m^2}{m_b\,v_m^2\,(GF_m - m_b g)}L}\right)\right)$$

and

$$t_0(L) = \frac{G^2 F_m}{v_m\,(GF_m - m_b g)}\,L \;+\; \frac{m_b\,v_m}{G^2\,F_m}\left(1 \;+\; W\left(-\;\mathrm{e}^{-1 - \frac{G^4\,F_m^2}{m_b\,v_m^2\,(GF_m - m_b g)}L}\right)\right)$$

Optimal Fixed Gear Ratio:
We define the optimal fixed gear ratio as the one that maximizes v_0. Computing this requires a numerical solution of

$$\frac{dv_0}{dG} = 0$$

Biological Systems Scaling Law:
For biological systems, we assume the specific power (p_m^{bio}), specific linkage energy transfer ability (e_l^{bio}), and specific energy (e_m^{bio}) are scale-invariant. Without loss of generality, we declare a unity reduction, $G = 1$, when the motor's full stroke matches the leg extensions.

Thus, the specific jumper power and energy for a payload-free jumper are

$$p_{jump}^{bio} = \frac{F_m\, v_m}{4\, m_b} = p_m^{bio}\, \frac{e_l^{bio}}{e_m^{bio} + e_l^{bio}} = \text{constant}$$

$$e_{jump}^{bio} = \frac{F_m\, L}{m_b} = e_m^{bio}\, \frac{e_l^{bio}}{e_m^{bio} + e_l^{bio}} = \text{constant}$$

We also compute the constants v_{ss} and λ as

$$v_{ss} = \frac{4Lp_{jump}^{bio}\left(Ge_{jump}^{bio} - Lg\right)}{G^2 {e_{jump}^{bio}}^2} \quad , \quad \lambda = \frac{G^2 {e_{jump}^{bio}}^2}{4L^2 p_{jump}^{bio}}$$

Therefore,

$$t_0(L) = \frac{G^2 {e_{jump}^{bio}}^2}{4\, p_{jump}^{bio}\left(Ge_{jump}^{bio} - Lg\right)} + \frac{4L^2\, p_{jump}^{bio}}{G^2 {e_{jump}^{bio}}^2}\left(1 + W\left(-\mathrm{e}^{-1-\frac{G^4 {e_{jump}^{bio}}^4}{16L^2\, {p_{jump}^{bio}}^2\left(Ge_{jump}^{bio} - Lg\right)}}\right)\right)$$

and

$$v_0(L) = \frac{4Lp_{jump}^{bio}\left(Ge_{jump}^{bio} - Lg\right)}{G^2 {e_{jump}^{bio}}^2}\left(1 + W\left(-\mathrm{e}^{-1-\frac{G^4 {e_{jump}^{bio}}^4}{16L^2\, {p_{jump}^{bio}}^2\left(Ge_{jump}^{bio} - Lg\right)}}\right)\right)$$

Engineered Systems Scaling Laws:

For engineering system, we define the specific jumper force (f_j) and power (p_j) as follows:

$$f_j = \frac{F_m}{m_b} \quad , \quad p_j = \frac{f_m v_m}{4}$$

so that, v_{ss} and λ are

$$v_{ss} = \frac{4\, p_j\left(Gf_j - g\right)}{G^2 f_j^2} \quad , \quad \lambda = \frac{\mathrm{G}^2 f_j^2}{4\, p_j}$$

Therefore,

$$t_0(L) = \frac{G^2 f_j^2}{4\, p_j\left(Gf_j - g\right)}\, L + \frac{4\, p_j}{\mathrm{G}^2 f_j^2}\left(1 + W\left(-\mathrm{e}^{-1-\frac{\mathrm{G}^4 f_j^4}{16 p_j^2\left(Gf_j - g\right)}L}\right)\right)$$

and

$$v_0(L) = \frac{4\, p_j\left(Gf_j - g\right)}{G^2 f_j^2}\left(1 + W\left(-\mathrm{e}^{-1-\frac{\mathrm{G}^4 f_j^4}{16 p_j^2\left(Gf_j - g\right)}L}\right)\right)$$

Direct-drive Transmission Operating at Maximum Power

We again assume the body is driven by an inertia-free motor of mass m_m, but allow the reduction to vary such the motor is continually operating a maximum power output. Therefore

$$m_b\, a = \frac{F_m v_m}{4\, v} - m_b g = \frac{m_m\, p_m}{v} - m_b g$$

We note the instantaneous reduction is governed by

$$G = \frac{v_m}{2\, v}$$

so that the motor is always spinning at $\frac{v_m}{2}$. The maximizes both power and body acceleration.

General Solution:

We notice that this system reaches a steady-state velocity when the maximum power balances the gravity forces and change in potential energy, i.e. at

$$v_{ss} = \frac{F_m v_m}{4 m_b g}$$

We also declare the rate

$$\zeta = \frac{g}{v_{ss}} = \frac{4 m_b\, g^2}{F_m v_m}$$

which produces the differential equation

$$a = \zeta\, v_{ss} \left(\frac{v_{ss}}{v} - 1\right)$$

Generally, we can write

$$z(t) = v_{ss}\, t - \frac{v_{ss}}{2\zeta}\left(1 + W\left(-\mathrm{e}^{-1-\zeta t}\right)\right)\left(2 + W\left(\mathrm{e}^{-1-\zeta t}\right)\right)$$
$$v(t) = v_{ss}\left(1 + W\left(-\mathrm{e}^{-1-\zeta t}\right)\right)$$

Unfortunately, an analytic solution for the take-off velocity and time does not exist. Instead, we integrate until the height, L, is reached. Note for biological systems, with a finite stroke, the time is inherently bound when the motor reaches full stroke, L,

$$t < \frac{2\, L}{v_m}$$

Explicit Equations:

Substituting the motor parameters, we obtain the explicit equations:

$$z(t) = \frac{F_m v_m}{4 m_b g}\, t - \frac{F_m^2 v_m^2}{32\, m_b^2 g^3}\left(1 + W\left(-\mathrm{e}^{-1-\frac{4 m_b\, g^2}{F_m v_m} t}\right)\right)\left(2 + W\left(-\mathrm{e}^{-1-\frac{4 m_b\, g^2}{F_m v_m} t}\right)\right)$$
$$v(t) = \frac{F_m v_m}{4 m_b g}\left(1 + W\left(-\mathrm{e}^{-1-\frac{4 m_b\, g^2}{F_m v_m} t}\right)\right)$$

Scaling Laws:
For both biological and engineering systems, the motor specific power is scale invariant. Therefore,

$$a = \frac{m_m\, p_m}{m_b\, v} - g$$

v_{ss} and λ are

$$v_{ss} = \frac{m_m\, p_m}{m_b\, g} \quad , \quad \zeta = \frac{m_b\, g^2}{m_m\, p_m}$$

The biological stroke time constraint is

$$t < \frac{e_{jump}^{bio}}{2p_{jump}^{bio}} = \frac{e_m^{bio}}{2p_m^{bio}}$$

The expanded solution is

$$z(t) = \frac{m_m\, p_m}{m_b\, g}\, t - \frac{m_m^2\, p_m^2}{2\, m_b^2\, g^3}\left(1 + W\left(-\mathrm{e}^{-1-\frac{m_b\, g^2}{m_m\, p_m}t}\right)\right)\left(2 + W\left(\mathrm{e}^{-1-\frac{m_b\, g^2}{m_m\, p_m}t}\right)\right)$$

$$v(t) = \frac{m_m\, p_m}{m_b\, g}\left(1 + W\left(-\mathrm{e}^{-1-\frac{m_b\, g^2}{m_m\, p_m}t}\right)\right)$$

However, when linkages are considered, then the ratio between m_m and m_b may not be scale invariant (i.e., not isometric). For the optimal design with linkages, the linkage mass scales with the total energy production, $m_m p_m t_0$, where t_0 is the acceleration time, such that

$$m_b = m_m + m_l = m_m\left(1 + \frac{p_m t_0}{e_l}\right)$$

Thus,

$$a = \frac{p_m}{\left(1 + \frac{p_m t_0}{e_l}\right)\, v} - g$$

v_{ss} and λ are

$$v_{ss} = \frac{p_m}{\left(1 + \frac{p_m t_0}{e_l}\right) g} \quad , \quad \zeta = \frac{\left(1 + \frac{p_m t_0}{e_l}\right) g^2}{p_m}$$

Note that the biological time constraint remains the same:

$$t < \frac{e_{jump}^{bio}}{2p_{jump}^{bio}} = \frac{e_m^{bio}}{2p_m^{bio}}$$

Acceleration Time Scaling Relationship:
Extended Data Fig. 3c shows that the acceleration time, t_0, is proportional to a 2/3 power of the scale, L, for small scales. We confirm this scaling law using the differential equations without computing explicit solutions. We begin with

$$a = \frac{d}{dt}v = \frac{m_m\, p_m}{m_b\, v} - g = \frac{m_m p_m - m_b g v}{m_b v}$$

Note that we assume that at small scales the ratio between m_m and m_b is constant at small scales.

To eliminate the time dependence, note that

$$a\,v\,dt = \;a\,dz = v\,dv$$

can be rearranged into

$$dz \;=\; \frac{v}{a}\,dv \;=\; \frac{m_b v^2}{m_m p_m - m_b g v}\;\,dv$$

Integrating the equation from the start condition (crouch: $z = 0,\ v = 0$) to the end condition (stand: $z = L, v = v_0$), we get

$$L = \int_0^{v_0} \frac{m_b v^2}{m_m p_m - m_b g v}\;\,dv$$

which, in turn, solves to

$$L \;=\; \left[-\,\frac{1}{2g}\,v^2\; - \frac{m_m p_m}{m_b\;g^2}\,v\; - \;\frac{m_m^2 p_m^2}{m_b^2\;g^3}\ln\left(\frac{m_m p_m - m_b g v}{m_m p_m}\right)\right]_0^{v_0}$$
$$=\; -\;\frac{1}{2g}\,v_0^2\; - \frac{m_m p_m}{m_b\;g^2}\,v_0\; - \;\frac{m_m^2 p_m^2}{m_b^2 g^3}\ln\left(\frac{m_m p_m - m_b g v_0}{m_m p_m}\right)$$

At small scales, the Taylor expansion for L results in

$$L \cong \frac{m_b}{3 m_m p_m} v_0^3 \qquad \text{or} \qquad v_0 = \left(\frac{3\;m_m p_m\;L}{m_b}\right)^{\frac{1}{3}}$$

This leads to the total mechanical energy at take-off

$$e_{jump} = \frac{1}{2} v_0^2 + gL = \left(\frac{9}{8}\,\frac{m_m^2}{m_b^2}\,p_m^2\right)^{\frac{1}{3}}\;L^{\frac{2}{3}} + gL$$

For small L, with $L^{\frac{2}{3}}$>>L, we see,

$$e^{jump} \;\cong\; \left(\frac{9}{8}\;\frac{m_m^2}{m_b^2}\,p_m^2\right)^{\frac{1}{3}}\;L^{\frac{2}{3}}$$

and as $e_{jump} = \frac{m_m p_m}{m_b}\;t_0$,

$$t_0 \;\cong\; \left(\frac{9}{8}\,\frac{m_b}{m_m\;p_m}\right)^{\frac{1}{3}}\;L^{\frac{2}{3}}\;\propto\;L^{\frac{2}{3}}$$

Spring Actuation

Assume a body mass, m_b, sits atop a latched motor pre-stretched heavy spring-linkage. In turn, when fully stretched and released, the spring and linkage assembly propels the body upward. We allow the spring and linkage assembly to show uniformly distributed mass and uniform strain rates. This implies a third of the spring's mass contributes to the inertial forces, while a half of the spring's mass adds gravity forces. Thus

$$\left(m_b + \tfrac{1}{3} m_s\right) a = k\,(L - z) - (m_b + \tfrac{1}{2} m_s) g$$

where the stiffness, k, relates to the effective spring specific energy

$$e_s = \frac{\frac{1}{2}\,k\,L^2}{m_s}$$

General Solution:

This equation has the form of a simple harmonic oscillator. The equilibrium position is

$$z_{eq} = L - \frac{\left(m_b + \frac{1}{2}m_s\right)g}{k}$$

while the natural frequency is given by

$$\omega = \sqrt{\frac{k}{m_b + \frac{1}{3}m_s}}$$

We note the jumper will only reach the take-off height, L, if the equilibrium point lies above the halfway mark:

$$z_{eq} > \frac{L}{2}$$

Assuming an initial condition of $z = 0, v = 0,$ the solution to the differential equation is:

$$z(t) = z_{eq}\,(1 - \cos(\omega t))$$
$$v(t) = z_{eq}\,\omega\sin(\omega t)$$

As a function of scale, the acceleration time $(z(t_0) = L)$ relations are

$$t_0(L) = \frac{1}{\omega}\cos^{-1}\left(\frac{z_{eq} - L}{z_{eq}}\right)$$
$$v_0(L) = \omega\sqrt{2z_{eq}L - L^2}$$

Explicit Equations:

Substituting the equilibrium and natural frequency, we have

$$v_0(L) = \sqrt{\frac{kL^2 - 2gL\left(m_b + \frac{1}{2}m_s\right)}{m_b + \frac{1}{3}m_s}}$$

$$t_0(L) = \sqrt{\frac{m_b + \frac{1}{3}m_s}{k}}\;\cos^{-1}\left(\frac{\left(m_b + \frac{1}{2}m_s\right)g}{\left(m_b + \frac{1}{2}m_s\right)g - kL}\right)$$

The specific energy production is simply:

$$e_{jump} = \frac{\frac{1}{2}\,k\,L^2}{m_s + m_b}$$

Scaling Laws:

Using the spring's specific energy, e_s, we write the stiffness as

$$k = \frac{2\, m_s\, e_s}{L^2}$$

We also define the mass ratio

$$\mu = \frac{m_s}{m_b + m_s}$$

For a payload-free jumper, the body mass is just the motor mass, thus

$$\mu = \frac{m_s}{m_m + m_s}$$

Therefore, we find the equilibrium position and natural frequency

$$z_{eq} = L\left(1 - \frac{2-\mu}{4\mu}\frac{gL}{e_s}\right)$$

$$\omega = \frac{1}{L}\sqrt{\frac{6\mu}{3-2\mu}\; e_s}$$

And the acceleration time relations

$$t_0(L) = \sqrt{\frac{3-2\mu}{6\,\mu\, e_s}}\; L\; \cos^{-1}\left(\frac{(2-\mu)gL}{(2-\mu)gL - 4\,\mu\, e_s}\right)$$

$$v_0(L) = \sqrt{\frac{6\,\mu\, e_s - 3gL\,(2-\mu)}{3-2\mu}}$$

The centre of mass velocity at that time is

$$v_{0,COM}(L) = v_0(L)(1 - \frac{\mu}{2})$$

Finally, the specific energy production is

$$e_{jump} = \mu\, e_s$$

And the specific centre of mass kinetic energy is

$$e_{COM} = \frac{1}{2} v_{0,COM}^2$$

Extended Data Figure 6 Calculations

To estimate the effect of varying the effective spring-mass ratio on our jumper, we first calculated the theoretical centre of mass kinetic energy based on the spring actuation state space model and experimentally measured parameters. To account for the unmodeled inertias and losses, we modified the jump height-centre of mass kinetic energy relation described earlier with a fit factor, η.

$$h = \frac{m}{C_d\, \rho_{air}\, A} \ln\left[\eta \frac{C_d\, \rho_{air}\, A\, e_{COM}}{mg} + 1\right].$$

We assume η holds constant across scales effective spring-mass ratios. For the plot, all masses are held constant except the motor mass.

Simulation for Figure 1a

The energy curve of Fig. 1a was generated using a simplified, symmetric, two-legged rigid body model, as depicted in Extended Data Fig. 7. The body of the jumper is centred about point A. The body has a mass of $m_1 = m_d + m_m$, where m_d is the dead mass and m_m is the motor mass. The legs are modelled as rigid links, driven by torsion springs. Each half of the leg (e.g., linkages AB and BC), has a mass of m_2, rotational inertia of I, and length of l. The torsion spring acts between the top and bottom linkage and exerts a torque of $F = k(\pi - 2\theta)$ about the joint. Extended Data Fig. 7b-d show the free body diagrams for the jumper components.

Due to the symmetry of the problem, the masses can all be lumped to one side of the jumper. The forces on body are:

$$m_1\ddot{x}_1 = A_x - R = 0$$
$$m_1\ddot{y}_1 = A_y - m_1 g$$

The forces and torques on the top linkage are:

$$m_2\ddot{x}_2 = B_x - A_x$$
$$m_2\ddot{y}_2 = B_y - A_y - m_2 g$$
$$I\ddot{\theta} = \frac{l}{2}\sin\theta\,(A_x + B_x) - \frac{l}{2}\cos\theta\,(A_y + B_y) + F$$

The forces and torques on the bottom linkage are:

$$m_2\ddot{x}_3 = C_x - B_x$$
$$m_2\ddot{y}_3 = C_y - B_y - m_2 g$$
$$I\ddot{\theta} = F - \frac{l}{2}sin\theta(B_x + C_x) - \frac{l}{2}\cos\theta\,(B_y + C_y)$$

The cartesian coordinates are related to θ by the following relations:

$$x_1 = 0; y_1 = 2l\sin\theta$$
$$x_2 = -\frac{l}{2}\cos\theta\,; y_2 = \frac{3l}{2}\sin\theta$$
$$x_3 = -\frac{l}{2}\cos\theta\,; y_3 = \frac{l}{2}\sin\theta$$

Solving the systems of equations yields the following angular acceleration ($\ddot{\theta}$) and ground reaction force (C_y):

$$\ddot{\theta} = \frac{8k(\frac{\pi}{2} - \theta) - 4(m_1 + m_2)gl\cos\theta + 2l^2(2m_1 + m_2)\dot{\theta}^2\sin 2\theta}{4\mathrm{I} + l^2(4m_1 + 3m_2) + 2l^2(2m_1 + m_2)\cos 2\theta}$$
$$C_y = (m_1 + 2m_2)g - 2l(m_1 + m_2)\left(\dot{\theta}^2\sin\theta - \ddot{\theta}\cos\theta\right)$$

The total kinetic (T) and potential energy (V) of the system are:

$$T = \frac{1}{4}(4I + l^2(4m_1 + 3m_2) + 2l^2(2m_1 + m_2)\cos 2\theta)\dot{\theta}^2$$
$$V = 2(m_1 + m_2)gl\sin\theta + 2k\left(\frac{\pi}{2} - \theta\right)^2$$

The vertical COM kinetic energy (T_{COM}) is:

$$T_{COM} = \frac{1}{2}\left(4m_1 + \frac{5}{2}m_2\right)l^2\dot{\theta}^2\cos^2\theta$$

For the pre-stretch, we assume that the motor operates at constant power, p and the system starts at θ slightly less than $\pi/2$. The spring is loaded until $\theta = \pi/6$.

$$p = \frac{dT}{dt} + \frac{dV}{dt}$$

Rearranging, results in the following differential equation:

$$\ddot{\theta} = \frac{2p + (8k(\frac{\pi}{2} - \theta) - 4(m_1 + m_2)gl\cos\theta + 2l^2(2m_1 + m_2)\dot{\theta}^2\sin 2\theta)\dot{\theta}}{(4\mathrm{I} + l^2(4m_1 + 3m_2) + 2l^2(2m_1 + m_2)\cos 2\theta)\dot{\theta}}$$

For the push phase, the motor is turned off and the jumper is released from $\theta = \pi/6$. The following differential equation governs the takeoff process:

$$\ddot{\theta} = \frac{8k(\frac{\pi}{2} - \theta) - 4(m_1 + m_2)gl\cos\theta + 2l^2(2m_1 + m_2)\dot{\theta}^2\sin 2\theta}{4\mathrm{I} + l^2(4m_1 + 3m_2) + 2l^2(2m_1 + m_2)\cos 2\theta}$$

Take-off occurs when $C_y = 0$

When the jumper is in flight, its motion is governed by the following differential equation:

$$m\ddot{y} = -mg - \frac{1}{2}C_d A\rho\dot{y}^2$$

m, C_d, A, ρ are the total mass, drag coefficient, frontal area, and air density, respectively.

For clarity, the time axis in the plot is distorted. In the plot, the push and flight times are stretched by a factor of 1000 and 10 respectively.

The parameters chosen for the model are based on the physical jumper.

$$m_1 = 0.018\ \mathrm{kg};\ m_2 = 0.006\ \mathrm{kg};\ I = 1.875 \times 10^{-5}\ \mathrm{kg\ m^2};\ l = 0.15\ \mathrm{m}$$

$$g = 9.81\frac{\mathrm{m}}{\mathrm{s}^2};\ k = 20.97\ \ \mathrm{Nm};\ p = 10\ \mathrm{W};\ C_d = 1;\ A = 3.24 \times 10^{-4}\ \mathrm{m}^2;\ \rho = 1.225\frac{\mathrm{kg}}{\mathrm{m}^3}$$

30m Jumper Design

Here we present further details of the jumper design beyond those found in the Methods.

Spring Material Selection: To maximize the specific energy of the spring, we choose carbon fibre and latex rubber to construct our hybrid compression-tension spring. We select the candidate materials using an Ashby plot[43]. In particular, we search the material database to maximize the "material factor," or the ratio of the elastic stored energy during axial extension to mass:

$$\kappa = \frac{\sigma_y^2}{E\rho},$$

where σ_y is the yield stress, E is the modulus of elasticity, and ρ is the density. The results are plotted in log-log space, with the yield stress over density on the x-axis and elastic modulus over density on the y-axis, as shown in Extended Data Fig. 5a. A line with a slope of two represents points of constant material factor, as

$$y = \frac{1}{\kappa}x^2 \quad \text{or} \quad \frac{E}{\rho} = \frac{1}{\kappa}\left(\frac{\sigma_y}{\rho}\right)^2$$

The largest material factor occurs among two main groups of materials: elastomers at lower values of E/ρ, and fibre-reinforced composites at higher values of E/ρ.

While stretched elastomers, such as natural rubber, would offer the highest spring specific energy, as shown in Fig. 2,3, we note that a passive structural compressive linkage is required to load the stretched elastomer, meaning the functional specific energy decreases. This is why we designed a hybrid compression-tension spring, allowing us to store additional energy in the linkage.

Within the fibre-reinforced composites, a glass fibre has a slightly higher material factor than carbon fibre, due to its lower elastic modulus. However, the material factor assumes axial strain, while our bow spring design places the material in bending. We are constrained by a maximum curvature (the two ends touching), which sets a peak strain, based on the thickness of the beam. At the same time, we are constrained by a force limit (based on the miniature gear motor) and the fact that a bending bow cannot have a narrower cross-section than a square and remain stable. Taken together, at the thickness of bow that the motor can deflect, the strain is not high enough to reach near the yield stress of the glass fibre, but it approaches the yield stress of the carbon fibre. Thus, we choose carbon fibre.

Spring Simulation and Design: To explore the design space of springs, we built a simulation framework that combines linear constitutive behavior (that neglects the impact of large deformation), large rotation kinematics, and cubic Bernoulli-Euler beam elements (i.e., shear is neglected). In the framework, a non-linear set of equations describing quasi-static equilibrium is formulated using stiffness matrices referencing the rotated state of the element; Newton-Raphson iteration is used to solve this non-linear set of equations using conventional incrementation algorithms. Convergence studies were conducted to ensure that the element density was sufficient to produce mesh-independent results (generally, this involved elements approximately 5 mm in length.) Strains were computed from the deformed state after accounting for rigid body rotations using the Bernoulli-Euler interpolation functions.

With this framework, we compare three springs, the first two seen in previous jumping devices: (i) a tension linkage in which the compressive elements do not buckle, (ii) a compression bow with no tension elements, and (iii) our hybrid tension-compression spring-linkage. To compare the three designs, we broke each design into two components: rods (high modulus, supporting compression and bending) and ties (low modulus, supporting tension). For the rods, we used carbon-fibre composite (nominal fibre volume: 65%, tensile strength: 1.7 GPa, tensile modulus: 140 GPa, flexural strength: 1.8 GPa, flexural modulus: 130 GPa, ultimate tensile strain: 1.9%, density: 1.56 g/cm^3). For the ties, we used latex rubber (tensile strength: 3 MPa, modulus (approximated as constant): 300 kPa, ultimate tensile strain: 660%, density: 0.74 g/cm^3). We matched the unloaded length of all springs and compressed all springs to 23% of the unloaded length.

To find the spring constants (i.e., thickness of rods and cross-section of the ties) for the comparison, we swept through the parameter space for each design, matching the peak compression force for all cases (assuming a peak force from a given motor). For the simplest design, the compression bow, we increased the thickness until it reached the strain value of 1.6% (just below the ultimate limit). This resulted in a thickness-to-length aspect ratio of 0.0051. We then increased the width until peak force was met resulting in a width-to-length aspect ratio of 0.026. For the hybrid tension-compression spring, we followed the same procedure for the rods, finding a thickness-to-length aspect ratio of 0.0054 and a width-to-length aspect ratio of 0.017. Interestingly, we found that the addition of ties decreases the peak strain at a given thickness of rod. This is because the ties help distribute the strain more evenly through the rod, resulting in a more constant curvature. As a result, thicker rods can be used for the hybrid case yet have the same peak strain. Finally, for the tension linkage spring, we selected the tie thickness to produce the same peak force, and then varied the rod thickness to be just above the buckling limit. This resulted in a rod thickness-to-length ratio of 0.013 and a width-to-length aspect ratio of 0.017, and a tie thickness- (and width-) to-length ratio of 0.029.

With these parameters, we compressed each of the springs in simulation, and found the relationship between compression and specific force for a nominal length of 270 mm (Fig. 3a). The specific energy of each was calculated as the area under each curve. As noted in the main text, the hybrid spring outperforms both of the other designs.

Importantly, for design of hybrid springs, we note that as mentioned above, there is an ideal amount of tie that should be added. Too little tie results in the simple compression-bow spring case with lower specific energy and higher peak strain in the bow (closer to failure). Too much tie results again in peak strain increasing. For rods of width-to-length ratio of ~0.017 and thickness-to-length of ~0.0054 and ties with a width-to-length ratio of ~0.11, we find that a thickness-to-length ratio of 0.073 is indeed in this sweet spot. For rods of other dimensions, simulations can be performed in a similar manner to find the correct parameters for the ties.

Height Measurement:

Determining Height: Knowing the differential equation governing the motion of the jumper after take-off (Methods, Energetic Model, Derivation, (5)), and using the time-of-flight and initial conditions (initial height of zero and initial velocity from video footage of jumps), both the C_dA and the height can be calculated. The estimate of C_dA is $3.2\text{x}10^{-4}$ m^2.

The time-of-flight of the device was measured for three jumps as 5.13, 5.18, and 5.25 s. We note that the term C_dA has little effect on the relationship between time-of-flight and jump height in the range of values that are reasonable for the jumper. For instance, varying the value of C_dA from 0 to $9\text{x}10^{-4}$ m^2 changes the estimate of height by only 0.2 m. Using the estimated C_dA of $3.2\text{x}10^{-4}$ m^2, we find the jump heights to be 32.2, 32.8, and 33.6 m for the three jumps (32.9$\pm$0.7 m). Although we focus on height in this work, jumping off an inclined surface of 45° would allow a jump that clears a height of 17 m and covers over 50 m in length.