Amplitude in Different Regions: Supplementary Material for Scattering Amplitudes, the Tail Effect, and Conservative Binary Dynamics at $\mathcal{O}(G^4)$ by Z. Bern, J. Parra-Martinez, R. Roiban, M. S. Ruf. C.-H. Shen, M P. Solon, and M. Zeng.

In this appendix we present the scattering amplitudes in the (ppp) and (prr) regions separately. The contribution from the (ppp) region already appeared in Ref. [31] and is given by

$$\begin{aligned}\mathcal{M}_4^{\rm ppp}(\boldsymbol{q}) = G^4 M^7 \nu^2 |\boldsymbol{q}| \pi^2 2^{2\epsilon} \left(\frac{\boldsymbol{q}^2}{\tilde{\mu}^2}\right)^{-3\epsilon} \left[\mathcal{M}_4^{\rm p} + \nu\left(\frac{\mathcal{M}_4^{\rm t}}{\epsilon} + \mathcal{M}_4^{\pi^2} + \mathcal{M}_4^{\rm rem,ppp}\right)\right] \\ + \int_{\boldsymbol{\ell}} \frac{\tilde{I}_{r,1}^4}{Z_1 Z_2 Z_3} + \int_{\boldsymbol{\ell}} \frac{\tilde{I}_{r,1}^2 \tilde{I}_{r,2}}{Z_1 Z_2} + \int_{\boldsymbol{\ell}} \frac{\tilde{I}_{r,1} \tilde{I}_{r,3}}{Z_1} + \int_{\boldsymbol{\ell}} \frac{\tilde{I}_{r,2}^2}{Z_1}\,,\end{aligned} \tag{1}$$

where $\mathcal{M}_4^{\rm p}$, $\mathcal{M}_4^{\pi^2}$, $\mathcal{M}_4^{\rm t}$ and the iteration integrals are as given in the main text in Eq. (3), and $\mathcal{M}_4^{\pi^2} + \mathcal{M}_4^{\rm rem,ppp} = \mathcal{M}_4^{\rm f}$ defined in Eq. (6) of Ref. [31]. The new result from the (prr) region is

$$\mathcal{M}_4^{\rm prr}(\boldsymbol{q}) = G^4 M^7 \nu^3 |\boldsymbol{q}| \pi^2 2^{6\epsilon} p_\infty^{-4\epsilon} \left(\frac{\boldsymbol{q}^2}{\tilde{\mu}^2}\right)^{-3\epsilon} \left(-\frac{\mathcal{M}_4^{\rm t}}{\epsilon} + \mathcal{M}_4^{\rm rem,prr}\right). \tag{2}$$

The remainder functions in both regions are given by

$$\begin{aligned}\mathcal{M}_4^{{\rm rem},x} &= r_8^x + r_9^x \log\left(\tfrac{\sigma+1}{2}\right) + r_{10}^x \tfrac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}} + r_{11}^x \log(\sigma) + r_{12}^x \log^2\left(\tfrac{\sigma+1}{2}\right) + r_{13}^x \tfrac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}} \log\left(\tfrac{\sigma+1}{2}\right) \\ &+ r_{14}^x \tfrac{\operatorname{arccosh}^2(\sigma)}{\sigma^2-1} + r_{15}^x {\rm Li}_2\left(\tfrac{1-\sigma}{2}\right) + r_{16}^x {\rm Li}_2\left(\tfrac{1-\sigma}{1+\sigma}\right) + r_{17}^x \tfrac{1}{\sqrt{\sigma^2-1}}\left[{\rm Li}_2\left(-\sqrt{\tfrac{\sigma-1}{\sigma+1}}\right) - {\rm Li}_2\left(\sqrt{\tfrac{\sigma-1}{\sigma+1}}\right)\right] + r_{18}^x \tfrac{1}{\sqrt{\sigma^2-1}} F(\sigma)\,,\end{aligned} \tag{3}$$

where the relevant coefficients r_i^x in each region, $x = {\rm ppp}, {\rm prr}$, are given in Tables II and III respectively in terms of the polynomials g_i in Table I of the paper. The transcendental function $F(\sigma)$ in Eq. (3) above is defined as

$$F(\sigma) = {\rm Li}_2\left(1-\sigma-\sqrt{\sigma^2-1}\right) - {\rm Li}_2\left(1-\sigma+\sqrt{\sigma^2-1}\right) + 3{\rm Li}_2\left(\sqrt{\tfrac{\sigma-1}{\sigma+1}}\right) - 3{\rm Li}_2\left(-\sqrt{\tfrac{\sigma-1}{\sigma+1}}\right) + 2\log\left(\tfrac{\sigma+1}{2}\right)\operatorname{arccosh}(\sigma)\,, \tag{4}$$

and its coefficients cancel when both regions are combined.

$$\begin{aligned}
r_8^{\rm ppp} &= \frac{1}{144(\sigma^2-1)^2\sigma^7}(-45 + 207\sigma^2 - 1471\sigma^4 + 13349\sigma^6 \\
&\quad - 37566\sigma^7 + 104753\sigma^8 - 12312\sigma^9 - 102759\sigma^{10} - 105498\sigma^{11} \\
&\quad + 134745\sigma^{12} + 83844\sigma^{13} - 101979\sigma^{14} + 13644\sigma^{15} + 10800\sigma^{16}) \\
r_9^{\rm ppp} &= \frac{1}{4(\sigma^2-1)}\left(1759 - 4768\sigma + 3407\sigma^2 - 1316\sigma^3 + 957\sigma^4\right. \\
&\quad \left. - 672\sigma^5 + 341\sigma^6 + 100\sigma^7\right) \\
r_{10}^{\rm ppp} &= \frac{1}{24(\sigma^2-1)^2}\left(1237 + 7959\sigma - 25183\sigma^2 + 12915\sigma^3 + 18102\sigma^4\right. \\
&\quad \left. - 12105\sigma^5 - 9572\sigma^6 + 2973\sigma^7 + 5816\sigma^8 - 2046\sigma^9\right) \\
r_{11}^{\rm ppp} &= \frac{2\sigma\left(-852 - 283\sigma^2 - 140\sigma^4 + 75\sigma^6\right)}{3(\sigma^2-1)} \\
r_{12}^{\rm ppp} &= 4g_1 - 7g_2 - \frac{3}{4}g_4 \\
r_{13}^{\rm ppp} &= 0 \\
r_{14}^{\rm ppp} &= -\frac{\sigma^2\left(-3+2\sigma^2\right)^2}{8(\sigma^2-1)^2}g_3 + 2(\sigma^2-1)g_2 \\
r_{15}^{\rm ppp} &= 24g_1 - 14g_2 + 2g_3 - \frac{3}{2}g_4 \\
r_{16}^{\rm ppp} &= -g_4 \\
r_{17}^{\rm ppp} &= -\frac{\sigma(-3+2\sigma^2)}{\sigma^2-1}(8g_1 - 4g_2) \\
r_{18}^{\rm ppp} &= \frac{\sigma\left(-3+2\sigma^2\right)}{2(\sigma^2-1)}g_3
\end{aligned}$$

TABLE II. Functions specifying the amplitude in the ppp region.

$$\begin{aligned}
r_8^{\rm prr} &= \frac{1}{144\sigma^7\left(\sigma^2-1\right)^2}(-45 + 207\sigma^2 - 1471\sigma^4 + 13349\sigma^6 \\
&\quad - 38704\sigma^7 + 24095\sigma^8 - 52042\sigma^9 + 72647\sigma^{10} + 55208\sigma^{11} \\
&\quad - 78841\sigma^{12} - 17346\sigma^{13} + 31259\sigma^{14} - 5004\sigma^{15} - 3600\sigma^{16}) \\
r_9^{\rm prr} &= \frac{1}{12\left(\sigma^2-1\right)}(-4061 + 34464\sigma - 9133\sigma^2 + 9860\sigma^3 + 2025\sigma^4 \\
&\quad + 5344\sigma^5 - 1023\sigma^6 - 900\sigma^7) \\
r_{10}^{\rm prr} &= \frac{1}{24(\sigma^2-1)^2}(-1237 + 5853\sigma - 16865\sigma^2 + 15653\sigma^3 + 3018\sigma^4 \\
&\quad - 9011\sigma^5 + 5540\sigma^6 - 3165\sigma^7 - 56\sigma^8 + 366\sigma^9) \\
r_{11}^{\rm prr} &= \frac{2\sigma\left(-852 - 283\sigma^2 - 140\sigma^4 + 75\sigma^6\right)}{3(\sigma^2-1)} \\
r_{12}^{\rm prr} &= -4g_1 + 13g_2 + g_3 + \frac{1}{4}g_4 \\
r_{13}^{\rm prr} &= -8\frac{\sigma\left(-3+2\sigma^2\right)}{(\sigma^2-1)}g_1 \\
r_{14}^{\rm prr} &= -\frac{\sigma^2\left(-3+2\sigma^2\right)^2}{8(\sigma^2-1)^2}g_3 - 2(\sigma^2-1)g_2 \\
r_{15}^{\rm prr} &= -40g_1 + 10g_2 - 2g_3 + \frac{1}{2}g_4 \\
r_{16}^{\rm prr} &= -g_4 \\
r_{17}^{\rm prr} &= 0 \\
r_{18}^{\rm prr} &= -\frac{\sigma\left(-3+2\sigma^2\right)}{2(\sigma^2-1)}g_3
\end{aligned}$$

TABLE III. Functions specifying the amplitude in the prr region.