Supplemental material

In this supplemental material, we provide further insight into our 5PN-e^{10} adiabatic model, including (i) a comparison with another adiabatic-evolution scheme due to Hughes et al. [S1], (ii) a sketch of the derivation of the evolution equations across resonances, (iii) additional plots of relative flux errors, and (iv) an estimate of the dephasing between adiabatic and first post-adiabatic model.

Notations, conventions and symbols are all the same as those used in the body of the letter, unless stated otherwise.

INITIAL PHASES AND CHOICE OF ANGLE VARIABLES

Our adiabatic evolution differs in one detail from the evolutions of Hughes et al. [S1]. The difference arises in our treatment of initial phases. Although we write the Teukolsky mode amplitudes as functions of the orbital parameters I_A, they are also implicitly functions of the initial phases. Here we show how this dependence affects the evolving waveform in the context of the two-timescale expansion.

The initial-phase dependence of the Teukolsky amplitudes is inherited directly from the initial-phase dependence of the modes of the point-particle stress-energy tensor $T^{\mu\nu}$. We can therefore restrict our attention to $T^{\mu\nu}$, which reads

$$T^{\mu\nu} = \mu \frac{u^\mu u^\nu}{u^t \Sigma}\, \delta[r - r_{\rm p}(t)]\, \delta[z - z_{\rm p}(t)]\, \delta[\varphi - \varphi_{\rm p}(t)], \tag{S1}$$

where $z \equiv \cos\theta$, $x^\mu_{\rm p}(t)$ denotes the particle's trajectory as a function of Boyer-Lindquist time t, $u^\mu \equiv dx^\mu_{\rm p}/d\tau = u^t dx^\mu/dt$, and $\Sigma \equiv r^2 + a^2 z^2$.

Whether we consider snapshots or the evolving orbit, the functions $r_{\rm p}(t)$, $z_{\rm p}(t)$, and $u^\mu(t)$ are periodic functions of the phase variables $\Phi^r(t)$ and $\Phi^\theta(t)$. The azimuthal angle $\varphi_{\rm p}(t)$ can be written as $\varphi_{\rm p}(t) = \Phi^\varphi(t) + \Delta\varphi(t)$, where $\Delta\varphi(t)$ is an oscillatory function of $\Phi^r(t)$ and $\Phi^\theta(t)$. It follows that we can write $T^{\mu\nu}$ as a discrete Fourier series in the phases,

$$T^{\mu\nu} = \sum_{mkn} T^{\mu\nu}_{mkn}(r,\theta) e^{-i\Phi_{mkn}(t) + im\varphi}, \tag{S2}$$

where we write $T^{\mu\nu} = T^{\mu\nu}(\Phi^A(t), r, \theta, \varphi)$ and define $T^{\mu\nu}_{mkn} \equiv \frac{1}{(2\pi)^3} \oint d^3\Phi\, T^{\mu\nu}(\Phi^A, r, \theta, \varphi) e^{i\Phi_{mkn}}$. The values of $T^{\mu\nu}_{mkn}$ obviously depend on how the phases relate to the coordinate position. For concreteness, choose the origin of the Φ^A phase space such that $\Phi^A = 0$ at $r_{\rm p} = r_{\rm min}$, $\theta_{\rm p} = \theta_{\rm min}$ ($z_{\rm p} = z_{\rm max}$), and $\varphi_{\rm p} = 0$ (where these coordinate positions are defined for fixed parameters I_A); this corresponds to the choice in Pound & Wardell [S2], and the coefficients $T^{\mu\nu}_{mkn}$ are then given explicitly by Eq. (411) of that reference. Note that this is simply a choice of coordinates on the phase space rather than a statement about any particular orbit $x^\mu_{\rm p}(t)$ in that phase space. The orbit might not pass through the point $(r_{\rm min}, \theta_{\rm min}, 0)$, and likewise it might never pass through the origin in the phase space.

In the case of a geodesic snapshot, the phases are given by $\Phi^A(t) = \Omega^A t + \Phi^A(0)$. We then have

$$T^{\mu\nu} = \sum_{mkn} T^{\mu\nu}_{mkn}(r,\theta) e^{-i\Phi_{mkn}(0)} e^{-i\omega_{mkn}t + im\varphi}. \tag{S3}$$

In typical applications, one solves field equations for coefficients of $e^{-i\omega_{mkn}t+im\varphi}$, with sources $\hat{T}^{\mu\nu}_{mkn} = T^{\mu\nu}_{mkn} e^{-i\Phi_{mkn}(0)}$. Without loss of generality, we can choose t and φ such that $t = 0 = \varphi_{\rm p}(0)$ at a periapsis passage, implying $\Phi^\varphi(0) = \Phi^r(0) = 0$. With these choices, $\hat{T}^{\mu\nu}_{mkn} = T^{\mu\nu}_{mkn} e^{-ik\Phi^\theta(0)}$. For a fiducial geodesic with $\theta_{\rm p}(0) = \theta_{\rm min}$, we also have $\Phi^\theta(0) = 0$, implying that for this geodesic, $\hat{T}^{\mu\nu}_{mkn} = T^{\mu\nu}_{mkn}$. If one computes the Teukolsky mode amplitude $Z_{\ell mkn}$ for this fiducial geodesic, one can then recover the $Z_{\ell mkn}$ for any other geodesic simply by multiplying by $e^{-ik\Phi^\theta(0)}$.

In Hughes et al. [S1], they computed amplitudes $Z_{\ell mkn}$ for a family of fiducial geodesics. To perform adiabatic evolutions, they then took inspiration from the geodesic form (S3) (or the analogous form for the asymptotic wave) by replacing $Z_{\ell mkn}$ with $\hat{Z}_{\ell mkn} = Z_{\ell mkn} e^{-i\Phi_{mkn}(0)}$ in our Eq. (2) and then defining adiabatically evolving phases $\Phi^A(t)$ as the solution to our Eq. (1) with vanishing initial values; see their (4.1)–(4.2) and (3.18). By treating $\Phi^\theta(0)$ (or an equivalent quantity) as a function of I_A and evolving it along with I_A, they then effectively included some post-adiabatic information in their waveform.

Such a scheme is equivalent to ours at adiabatic order, but it does not fit neatly into the two-timescale scheme. In the two-timescale scheme, rather than solving field equations with geodesic sources and then feeding their outputs into an adiabatic evolution, one works directly with the expansion (S2) throughout the calculation; the underlying assumption is that the physical, evolving system is periodic with respect to each of the physical, evolving phases (rather than being triperiodic with respect to t as is the case for a geodesic source). Given the expansion (S2), one then solves field equations with the source $T^{\mu\nu}_{mkn}$ rather than the source $\hat{T}^{\mu\nu}_{mkn}$. Since $\hat{T}^{\mu\nu}_{mkn} = T^{\mu\nu}_{mkn}$ for a fiducial geodesic, *the solutions to these field equations are precisely the $Z_{\ell mkn}$ one would obtain for the fiducial geodesic*. But one need not invoke geodesics of any kind, fiducial or otherwise, when solving the field equations, and one entirely loses the association with geodesics at post-adiabatic orders. Instead one works with the genuine, evolving parameters and phases throughout, and the evolving inspiral and waveform are determined entirely by the evolution equations for I_A and Φ^A.

However, suppose one has already computed $Z_{\ell mkn}$ for a given family of geodesics. One can always use this family as input for the two-timescale expansion by appropriately choosing the origin of the Φ^A phase space. For example, in the 5PN-e^{10} calculations we use geodesics with $r_{\rm p}(0) = r_{\rm max}$, $\theta_{\rm p}(0) = \pi/2$, and $\frac{d\theta_{\rm p}}{dt}(0) < 0$. We can then define Φ^A such that $\Phi^A = 0$ at $r_{\rm p} = r_{\rm max}$, $\theta_{\rm p} = \pi/2$, and $\varphi_{\rm p} = 0$. The Teukolsky amplitudes $Z_{\ell mkn}$ obtained for the geodesics with $r_{\rm p}(0) = r_{\rm max}$ and $\theta_{\rm p}(0) = \pi/2$ then precisely correspond to the amplitudes in the two-timescale expansion. Note that this choice of origin in phase space does not restrict the initial conditions for Eq. (1), which are freely specified.

PASSAGE THROUGH RESONANCE

Next, we briefly describe the derivation of the evolution equations of orbital parameters to compute the resonant jumps. This requires the oscillatory pieces of the first-order GSF in terms of the phase Φ^A. Below we focus our attention exclusively on the dissipative sector of the evolution equations, as we do in the body of the letter. The conservative sector is discussed in Ref. [S3].

The evolution equations are most easily extracted from $dJ^{\rm osc}_A/dt$, where the alternative orbital parameters $J^{\rm osc}_A = \{\hat{E},\, \hat{L},\, \hat{Q}\}$ are the (specific) orbital energy, azimuthal angular momentum, and Carter constant, defined as $\hat{E} = -u_t$, $\hat{L} = u_\phi$, and $\hat{Q} = u_\alpha u_\beta K^{\alpha\beta}[= \hat{C} + (a\hat{E} - \hat{L})^2]$, where $K^{\alpha\beta}$ is the Killing tensor of Kerr spacetime (see, e.g., Sec. 2 in Ref. [S4]). The rates of change of these quantities are (see Appendix. C of Ref. [S5])

$$\frac{dJ^{\rm osc}_A}{dt} = -\frac{1}{u^t}\left(\frac{\partial J^{\rm osc}_A}{\partial u_\alpha}\right)_x \left(\frac{\partial H^{(1)}}{\partial x^\alpha}\right)_J \equiv F_A\,, \tag{S4}$$

using $u_\alpha = u_\alpha(x,\, J^{\rm osc})$ [S6] and $d/dt = (d/d\tau)/u^t$ with the proper time τ compatible with the background Kerr metric. $H^{(1)}$ is the dissipative (i.e., time-antisymmetric) piece of the particle's perturbed Hamiltonian to describe the self-acceleration, defined by

$$H^{(1)} \equiv -\frac{1}{2} h^{\alpha\beta}_{\rm rad}\, u_\alpha u_\beta\,, \tag{S5}$$

where $h^{\rm rad}_{\alpha\beta} = \frac{1}{2}\left(h^{\rm ret}_{\alpha\beta} - h^{\rm adv}_{\alpha\beta}\right)$ is the (half-retarded-minus-half-advanced) radiative piece of the metric perturbation, and all quantities are evaluated on the orbit.

Similarly to Eq. (S2), we can expand $h^{\rm rad}_{\alpha\beta}$ in the Fourier modes

$$h^{\rm rad}_{\alpha\beta} = \sum_{mkn} h^{{\rm rad},mkn}_{\alpha\beta}(J^{\rm res}_A, r, \theta)\, e^{-i\Phi_{mkn} + im\varphi}, \tag{S6}$$

where we have evaluated $J^{\rm osc}_A$ at $J^{\rm res}_A \equiv J^{\rm osc}_A(t_{\rm res})$ inside the metric perturbation, using the fact that $J^{\rm osc}_A$ only changes by an amount $\sim \epsilon^{1/2}$ across the resonance, and hence it effectively remains constant. On the orbit we have $\varphi = \varphi_{\rm p} = \Phi^\varphi + \Delta\varphi(\Phi^r, \Phi^\theta)$, such that Φ^φ cancels out in Eq. (S6) and in Eq. (S5). $r_{\rm p}$, $\theta_{\rm p}$, and u^α are likewise independent of Φ^φ and biperiodic in (Φ^r, Φ^θ), such that

$$\frac{dJ^{\rm osc}_A}{dt} = \sum_{mkn,k'n'} F^{mkn,k'n'}_A(J^{\rm res}_B) e^{i(n'-n)\Phi^r + i(k'-k)\Phi^\theta}. \tag{S7}$$

Here, mkn are the mode numbers from Eq. (S6), and $k'n'$ are those from the Fourier expansion of the $(r_{\rm p}, \theta_{\rm p}, u^\alpha)$ dependence.

Since F_A is biperiodic in (Φ^r, Φ^θ), we can write it as $F_A = \sum_{\mathsf{k}_r,\mathsf{k}_\theta} F^{\mathsf{k}_r,\mathsf{k}_\theta}_A(J^{\rm res}_B) e^{i\Phi_{\mathsf{k}_r,\mathsf{k}_\theta}}$, with $\Phi_{\mathsf{k}_r,\mathsf{k}_\theta} \equiv (\mathsf{k}_r\,\Phi^r + \mathsf{k}_\theta\,\Phi^\theta)$ and with coefficients $F^{\mathsf{k}_r,\mathsf{k}_\theta}_A \equiv \frac{1}{(2\pi)^2}\oint d^2\Phi\, F_A e^{-i\Phi_{\mathsf{k}_r,\mathsf{k}_\theta}} = \sum_{mkn}\sum_{\substack{k'=k+\mathsf{k}_\theta\\ n'=n+\mathsf{k}_r}} F^{mkn,k'n'}_A(J^{\rm res}_B)$. The modes $(\mathsf{k}_r, \mathsf{k}_\theta) = (0,0)$, whether the orbit is away from resonance or on resonance, do not oscillate. At resonance, all modes $(\mathsf{k}_r, \mathsf{k}_\theta) = (s\beta^r, -s\beta^\theta)$ satisfying $\Omega_r\beta^r - \Omega_\theta\beta^\theta = 0$ also become stationary. We then have

$$F^{{\rm res},s}_A \equiv \sum_{mkn} F^{mkn,k-s\beta^\theta,n+s\beta^r}_A(J^{\rm res}_B). \tag{S8}$$

Writing $I^{\rm osc}_A = I^{\rm osc}_A(J^{\rm osc}_B)$, where the one-to-one relationship is the geodesic one (see, e.g., Appendix B in Ref. [S6]), we obtain $G^{{\rm res},s}_A = \frac{\partial I_A}{\partial J_B} F^{{\rm res},s}_B$.

The coefficients in Eq. (S8) can be expressed in terms of the Teukolsky mode amplitudes $Z_{\ell mkn}$, as in Eq. (8) in the body of our letter, by expressing the expansion (S6) in terms of a radiative Green's function $G^{\rm rad}_{\alpha\beta\alpha'\beta'}(x,x')$, as in Appendix A of Sago et al. [S4]. In broad strokes, we have $h^{\rm rad}_{\alpha\beta}(x) = \int G^{\rm rad}_{\alpha\beta\alpha'\beta'}(x,x')T^{\alpha'\beta'}(x')dV'$ and $G^{\rm rad}_{\alpha\beta\alpha'\beta'} \sim \sum_{\ell mkn}\left[\Pi^+_{\ell mkn}(x)\bar{\Pi}^+_{\ell mkn}(x') + \Pi^-_{\ell mkn}(x)\bar{\Pi}^-_{\ell mkn}(x')\right]$, where $\Pi^\pm_{\ell mkn} = (\Pi^\pm_{\ell mkn})_{\alpha\beta}$ are constructed from homogeneous solutions to the Teukolsky equation, with $\Pi^{+/-}_{\ell mkn}$ regular at infinity / the horizon (and an overbar denoting the complex conjugate of a quantity). The integral of $\bar{\Pi}^\pm_{\ell mkn}(x')T^{\alpha'\beta'}(x') \sim (\bar{\Pi}^\pm_{\ell mkn})_{\alpha'\beta'}u^{\alpha'}u^{\beta'}$ can be expressed in terms of $Z^\pm_{\ell mkn}$, leaving terms of the form $h^{\rm rad}_{\alpha\beta}(x) \sim Z^\pm_{\ell mkn}\Pi^\pm_{\ell mkn}$. When we evaluate $H^{(1)}(x) \sim h^{\alpha\beta}_{\rm (rad)}(x)\,u_\alpha u_\beta \sim (\Pi^\pm_{\ell mkn})^{\alpha\beta}u_\alpha u_\beta$ on the worldline and integrate against $e^{-i\Phi_{k'n'}}$ to construct $F^{mkn,k'n'}_A$, the integral can again be expressed in terms of $Z^\pm_{\ell mkn}$ (for a given k_r and k_θ), leading to Eq. (8). Very similar calculations in the case of $\mathsf{k}_r = 0 = \mathsf{k}_\theta$ are detailed in, e.g., Sago et al. [S7]; see the steps in going from Eq. (4.3) to (4.6) in that reference, and Sec. 8 and 9 of Drasco et al. [S8] (in the scalar toy-model case).

Equation (8) can also be obtained from calculations of 'resonantly enhanced (or diminished) fluxes' in geodesic snapshots [S5, S9, S10]. Unlike our method, those calculations used the snapshot phase $\Phi^A(t) = \Omega^A t + \Phi^A(0)$ and explicitly took a long-time average of Eq. (S7) (such that $\Phi^A(0)$ created an 'offset'-phase dependence in the time-averaged fluxes at resonances [S11]). But prior to taking the long-time average, we can immediately promote the snapshot phase $\Phi^A(t)$ in the existing calculations to the adiabatic phase of our two-timescale scheme, leaving their intermediate results unaffected. Indeed, the essence of Eq. (8) in the letter can be straightforwardly extracted from, for example, Eq. (54) in Isoyama et al. [S5]; see also Eq. (4.6) in Sago et al. [S7] and Eqs. (5.1) and (5.2) in Ruangsri & Hughes [S12].

RELATIVE FLUX ERRORS OF OTHER ORBITAL PARAMETERS

In FIG. 2 of this letter, we showed a sample of the relative flux error of p. Here, for completeness, we plot additional flux errors of 'traditional' orbital parameters $\{E, L, C\}(\equiv \{\mu\hat{E}, \mu\hat{L}, \mu^2\hat{C}\})$ in FIG. S1 for the worst case $q = -0.9$, $\iota \approx 20°$, and in FIG. S2 for the best case $q = +0.9$, $\iota \approx 80°$; recall that the mapping between $\{E, L, C\}$ and $\{p, e, \iota\}$ is one-to-one. These errors are qualitatively similar to those in $dp/d\tilde{t}$.

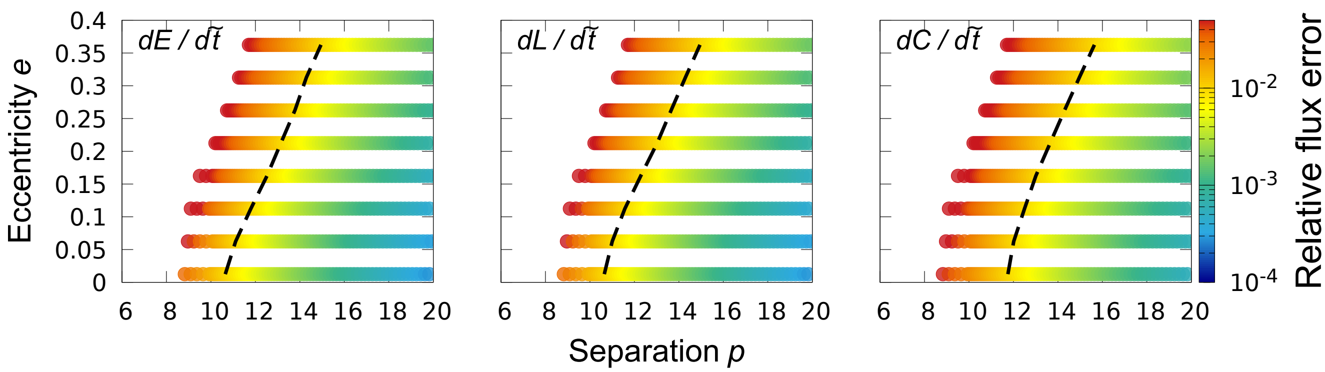

FIG. S1. Error of 5PN-e^{10} fluxes relative to numerical flux data in the worst case of $q = -0.9$, $\iota \approx 20°$. This is the same parameters used in the right panel of FIG. 2. We show the errors of $dE/d\tilde{t}$ (*Left*), $dL/d\tilde{t}$ (*Middle*), and $dC/d\tilde{t}$ (*Right*), respectively. The dashed curves indicate an error of $\approx 1.0 \times 10^{-2}$.

DEPHASING BETWEEN ADIABATIC AND FIRST POST-ADIABATIC MODELS

As an example of the dephasing time between the 5PN-e^{10} and numerical adiabatic models, here we report a specific comparison for an equatorial EMRI (the only type for which there are many numerical adiabatic evolutions). We choose an example for which data is available in Ref. [S13] and which has parameters close to those of our sample generic, inclined EMRI in the letter, with masses and Kerr spin $(\mu, M, q) = (10M_\odot, 10^6 M_\odot, 0.9)$, initial phases $\Phi^A(0) = 0$, and initial orbital parameters $(p(0), e(0)) = (10.0, 0.2)$. In this case we find a phase difference $(\delta\Phi_r, \delta\Phi_\varphi) \approx (0.60, 0.77)$ after the first ≈ 8 weeks ($\tilde{t} = 10.0M$) of evolution.

Next we justify our estimate that this time is comparable to the time over which an adiabatic waveform will maintain phase coherence with a post-adiabatic one. At first post-adiabatic order, away from resonances, the evolution equations can be put in

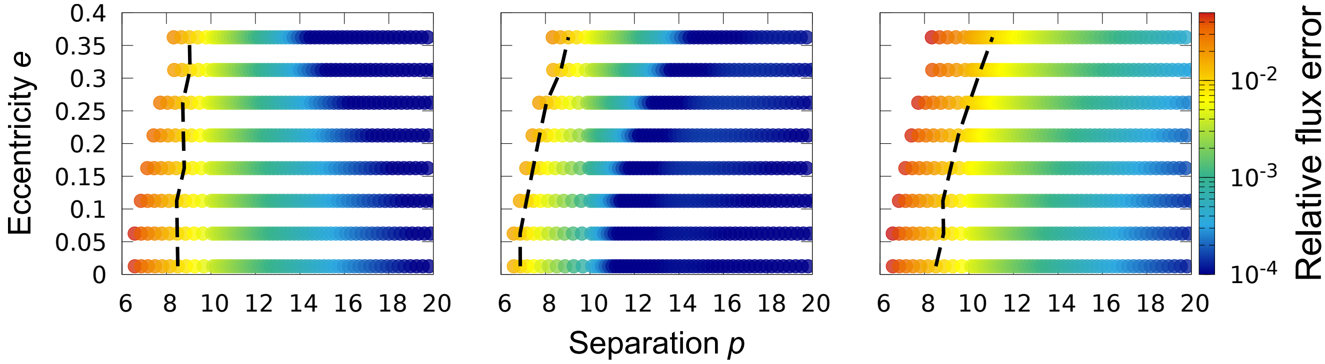

FIG. S2. The same figures as FIG. S1 [the errors of $dE/d\tilde{t}$ (*Left*), $dL/d\tilde{t}$ (*Middle*), and $dC/d\tilde{t}$ (*Right*)], but in the best case of $q = +0.9$, $\iota \approx 80°$. This is the same parameters used in the left panel of FIG. 2.

the form

$$\frac{d\Phi^A}{d\tilde{t}} = \eta^{-1}\Omega^A(I_B) \quad \text{and} \quad \frac{dI_A}{d\tilde{t}} = G_A^{(0)}(I_B) + \eta G_A^{(1)}(I_B). \tag{S9}$$

These equations admit an asymptotic solution of the form

$$\Phi^A = \eta^{-1}\Phi^A_{(0)}(\tilde{t}) + \eta^0\Phi^A_{(1)}(\tilde{t}) + O(\eta), \tag{S10}$$

$$\Omega^A = \Omega^A_{(0)}(\tilde{t}) + \eta\Omega^A_{(1)}(\tilde{t}) + O(\eta^2), \tag{S11}$$

where $\frac{d\Phi^A_{(n)}}{d\tilde{t}} = \Omega^A_{(n)}(\tilde{t})$.

An adiabatic evolution captures $\Phi^A_{(0)}$ but omits $\Phi^A_{(1)}$. This implies that for each mode of the waveform, it omits a post-adiabatic phase correction $\Phi^{(1)}_{mkn} = \eta\int_0^t \omega^{(1)}_{mkn}(\eta t')dt'$, which is bounded by

$$|\Phi^{(1)}_{mkn}| < \eta\omega^{(1)\max}_{mkn} t, \tag{S12}$$

where $\omega^{(1)\max}_{mkn} \equiv \max_{0<t'<t} |\omega^{(1)}_{mkn}(\eta t')|$.

Equation (S12) is an upper bound on the error. We can limit it to an error tolerance N by restricting to a time interval

$$t < N/(\eta\omega^{(1)\max}_{mkn}). \tag{S13}$$

A typical case might be $N \sim 0.1$ rad, a mass ratio $\eta = 10^{-5}$, and $\omega^{(1)\max}_{mkn} \sim 10^{-2}$ rad/s (i.e., frequencies in the LISA band, and the typical post-adiabatic correction to the frequency of the innermost stable circular orbit, caused by the first-order conservative GSF [S14–S18]). This implies a time interval $t \lesssim 16$ weeks.

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