

# MOTOHASHI'S FORMULA FOR THE FOURTH MOMENT OF INDIVIDUAL DIRICHLET $L$ -FUNCTIONS AND APPLICATIONS

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**ABSTRACT.** A new reciprocity formula for Dirichlet  $L$ -functions associated to an arbitrary primitive Dirichlet character of prime modulus  $q$  is established. We find an identity relating the fourth moment of individual Dirichlet  $L$ -functions in the  $t$ -aspect to the cubic moment of central  $L$ -values of Hecke–Maaß newforms of level at most  $q^2$  and primitive central character  $\psi^2$  averaged over all primitive nonquadratic characters  $\psi$  modulo  $q$ . Our formulæ would be viewed as reverse versions of recent work of Petrow–Young. Direct corollaries include a version of Iwaniec’s short interval fourth moment bound and the twelfth moment bound for individual Dirichlet  $L$ -functions, which generalise work of Jutila and Jutila–Motohashi, respectively. This work traverses an intersection of analytic number theory and automorphic forms.

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*Date:* October 19, 2021.

*2010 Mathematics Subject Classification.* Primary: 11M06, 11F72; Secondary: 11F03.

*Key words and phrases.* Motohashi’s formula; Dirichlet  $L$ -functions; fourth moment; twelfth moment.

The author is supported in part by the Masason Foundation and the Spirit of Ramanujan STEM Talent Initiative.

## 1. INTRODUCTION

Estimating moments of families of  $L$ -functions is a central problem in analytic number theory not only due to their substantial applications, but also since they give an insight into the behaviour of  $L$ -functions in the critical strip. Of particular interests are the  $2k$ -th moments of the Riemann zeta function and Dirichlet  $L$ -functions:

$$\mathcal{Z}_k(g) = \int_{-\infty}^{\infty} |\zeta(1/2 + it)|^{2k} g(t) dt, \quad \mathcal{Z}_k(g; \chi) = \int_{-\infty}^{\infty} |L(1/2 + it, \chi)|^{2k} g(t) dt,$$

where  $k$  is a positive integer and the function  $g$  is of rapid decay. The initial cases  $k = 1, 2$  have been successfully investigated for  $\mathcal{Z}_k(g)$  and the other cases have remained untouched so far. In this article, we manifest Motohashi's formula for the fourth moment  $\mathcal{Z}_2(g; \chi)$  of individual Dirichlet  $L$ -functions, which was unsolved since the 1990's.

**1.1. Overview and Motivation.** In 1990's, Motohashi [53, Theorem 4.2] procured a mysterious identity relating the smoothed fourth moment of the Riemann zeta function  $\zeta(s)$  to the spectral cubic moment of automorphic  $L$ -functions associated to the group  $\mathrm{SL}_2(\mathbb{Z})$ . We assume a fixed test function  $g$  to be of Schwartz class and denote by  $\mathcal{B}(q, \chi)$  the set of Hecke–Maaß forms of level  $q$  and central character  $\chi$ ; we write  $\mathcal{B}(\Gamma_0(q))$  as usual when  $\chi$  is principal. His formula then asserts that the following spectral decomposition holds up to an explicit description of holomorphic, Eisenstein and residual contributions that we elide here:

$$\int_{-\infty}^{\infty} |\zeta(1/2 + it)|^4 g(t) dt \leftrightarrow \sum_{f \in \mathcal{B}(\Gamma_0(1))} L(1/2, f)^3 \check{g}(t_f), \quad (1.1)$$

where  $\check{g}$  is an elaborate integral transform of  $g$  involving the Gauß hypergeometric function. The right-hand side of (1.1) ought to be understood as a complete integral over the full spectrum of level 1 modular forms, including holomorphic, discrete and continuous spectra. Motohashi has given several approaches in the spirit of analytic number theory and representation theory. Note that all these approaches are in the framework of relative trace formulæ. On the other hand, Michel–Venkatesh ([47, §4.3.3], [48, §4.5.3]) suggested an elegant geometric and spectral strategy to confirm (1.1). Following their method, Nelson [56] studied the cubic moment of automorphic  $L$ -functions on  $\mathrm{PGL}_2$  via the use of the regularised diagonal periods of products of Eisenstein series.

In terms of automorphic representation theoretic language, the spectral reciprocity formula (1.1) in question is a connection between the fourth moment of  $L$ -functions on  $\mathrm{GL}_1$  (geometric side) and the cubic moment of  $L$ -functions on  $\mathrm{GL}_2$  (spectral side). These two sides are derived from an application of a relative trace formula (Kuznetsov formula). The progress towards Motohashi's formula are indicated in the following table<sup>1</sup>.

$L$ -functions	Individual	Character average
Riemann zeta function	Bruggeman–Motohashi [18, 51, 53]	n/a
Dirichlet $L$ -functions	Theorems 1.1	Blomer et al. [9]
Dedekind zeta functions	Bruggeman–Motohashi [16, 17, 52]	unknown

These works share similar structure to Motohashi's original identity (1.1) (cf. [54]). In particular, the heuristics due to Blomer et al. [9] establishes that the character average of the smoothed fourth moment of Dirichlet  $L$ -functions weighted by  $\chi(a)\bar{\chi}(b)$  will be expressed by means of a cubic moment of  $L$ -functions attached to Maaß forms of level  $ab$ . Their method relies on brute force calculations and is simpler than in [53] in the sense that their reasoning is rather symmetric and uses additive reciprocity. There are several articles in the antecedent literature on this kind of fourth moment problem; see [29, 39, 67]. Although these achievements address the problem of obtaining an asymptotic formula, one must go through a harder route in order to establish Motohashi's formula.

There exist two versions of spectral reciprocities:  $\mathrm{GL}_4 \times \mathrm{GL}_2 \leftrightarrow \mathrm{GL}_4 \times \mathrm{GL}_2$  reciprocity and  $\mathrm{GL}_2 \times \mathrm{GL}_2 \leftrightarrow \mathrm{GL}_3 \times \mathrm{GL}_2$  reciprocity. The former one involves work of Andersen–Kıral [1], Blomer–Li–Miller [12], Blomer–Khan [10, 11], Humphries–Khan [32], Kuznetsov [45], Nunes [57], and Zacharias [69]. The latter involves work of Blomer et al. [9], Nelson [56], Petrow [60], Petrow–Young [61, 62], Wu [66], and Young [67]. It behoves us to mention an ongoing project of Humphries–Khan to deduce  $\mathrm{GL}_3 \times \mathrm{GL}_2 \leftrightarrow \mathrm{GL}_4 \times \mathrm{GL}_1$  spectral reciprocity. This renders a natural extension of Motohashi's formula (1.1).

<sup>1</sup>The character average for the fourth moment of Dedekind zeta functions should be meant only for quadratic number fields. In this case, the problem could be solved with classical technology, although this has not been worked out anywhere.

**1.2. Statement of Main Result.** Let  $\chi$  be an arbitrary primitive Dirichlet character modulo an integer  $q$  and let  $\mathcal{R}_4^+$  be a subdomain of  $\mathbb{C}^4$ , where all four parameters have real parts greater than one. In this article, we update the progress towards spectral reciprocity formulæ and generalise (1.1) to the fourth moment of *individual* Dirichlet  $L$ -functions in the  $t$ -aspect associated to  $\chi$ . Thus the set-up we build upon is as follows. For  $q \in \mathbb{N}$ , we consider

$$\mathcal{Z}_2(g; \chi) = \int_{-\infty}^{\infty} |L(1/2 + it, \chi)|^4 g(t) dt. \quad (1.2)$$

This is seen as a character analogue of (1.1). The twist by a character substantially complicates our analysis and we will encounter various intricate character sums. If  $g$  is a sufficiently nice test function, then we define

$$\mathcal{Z}_2(s_1, s_2, s_3, s_4; g; \chi) = \int_{-\infty}^{\infty} L(s_1 + it, \chi) L(s_2 + it, \chi) L(s_3 - it, \bar{\chi}) L(s_4 - it, \bar{\chi}) g(t) dt, \quad (1.3)$$

where the parameters (or shifts) satisfy  $(s_1, s_2, s_3, s_4) \in \mathcal{R}_4^+$ . In order to establish a formula for (1.2), we initially work with (1.3) by exploiting Dirichlet series expansions for the integrand in the region of absolute convergence, and then we take the limit  $(s_1, s_2, s_3, s_4) \rightarrow (1/2, 1/2, 1/2, 1/2)$  after the meromorphic continuation as in [53].

We now proceed to the rigorous statement of the new reciprocity formula, which requires a bit of notation. For technical simplicity, we assume  $q$  is prime. The classification of automorphic forms is convenient in the sequel:

- Cuspidal holomorphic newforms  $f$  of weight  $k \equiv \kappa(\psi) = \kappa \pmod{2}$ , level  $q$ , central character  $\psi$  and Hecke eigenvalues  $\lambda_f(n) \in \mathbb{C}$ ; we denote the set of such forms by  $\mathcal{B}_k^*(q, \psi)$ ;
- Cuspidal Maaß newforms  $f$  of spectral parameter  $t_f \in \mathbb{R} \cup [-i\vartheta, i\vartheta]$ , weight  $\kappa \in \{0, 1\}$ , level  $q$ , central character  $\psi$  and Hecke eigenvalues  $\lambda_f(n) \in \mathbb{C}$ , where at the current state of knowledge  $\vartheta = 7/64$  can be taken (see [43]; although  $\vartheta = 0$  is expected); we denote the set of such forms by  $\mathcal{B}_\kappa^*(q, \psi)$ ;
- Unitary Eisenstein series  $E(z, s, f)$ , where  $s = 1/2 + it$  with  $t \in \mathbb{R} \setminus \{0\}$  and  $\mathcal{B}(\psi_1, \psi_2) \ni f$  with  $\psi = \psi_1 \psi_2$  is a certain finite set depending upon  $\psi_1, \psi_2$  corresponding to an orthonormal basis in the space of the induced representation constructed out of  $(\psi_1, \psi_2)$ . Their  $n$ -th Hecke eigenvalue is written as  $\lambda_f(n, t) = \sum_{ab=n} \psi_1(a) a^{it} \psi_2(b) b^{-it}$  for  $(n, q) = 1$ .

Let  $\tau(\chi)$  be the Gauss sum and  $J(\chi, \psi)$  be the Jacobi sum so that we define  $\mathcal{H}_\pm(\chi, \psi) = \psi(\mp 1) \tau(\psi) \tau(\bar{\chi} \psi) J(\psi, \psi)$ . For a fixed test function as in Convention 1.10 below, we define the twisted spectral mean values<sup>2</sup>

$$\begin{aligned} \mathcal{J}_\pm^{\text{Maaß}} &:= \frac{q^{s_2 - s_1 - 2}}{\tau(\bar{\chi})} \sum_{d|q} \frac{\mu(d)}{d} \sum_{\psi \pmod{q}}^\# \mathcal{H}_\pm(\chi, \psi) \sum_{f \in \mathcal{B}_k^*(dq, \psi^2)} \epsilon_f^{(1 \mp 1)/2} \\ &\quad \times \frac{L\left(\frac{1-s_1+s_2+s_3-s_4}{2}, f \otimes \bar{\psi}\right) L\left(\frac{1-s_1+s_2-s_3+s_4}{2}, f \otimes \bar{\psi}\right) L\left(\frac{s_1+s_2+s_3+s_4-1}{2}, f \otimes \bar{\psi}\right)}{L(1, \text{Ad}^2 f)} \Phi_s^\pm(it_f), \\ \mathcal{J}_\pm^{\text{Eis}} &:= \frac{2q^{s_2 - s_1 - 1}}{\tau(\bar{\chi}) \varphi(q)} \sum_{d|q} \frac{\mu(d)}{d} \sum_{\psi \pmod{q}}^\# \mathcal{H}_\pm(\chi, \psi) \sum_{\psi_1 \psi_2 = \psi^2} \sum_{f \in \mathcal{B}(\psi_1, \psi_2)} \int_{-\infty}^{\infty} \frac{\mathcal{S}_f(t; s_1, s_2, s_3, s_4)}{|L(1 + 2it, \bar{\psi}_1 \psi_2)|^2} \Phi_s^\pm(it) \frac{dt}{2\pi}, \\ \mathcal{J}_+^{\text{hol}} &:= \frac{q^{-2}}{\tau(\bar{\chi})} \left(\frac{2\pi}{q}\right)^{s_1 - s_2} \cos\left(\frac{\pi(s_3 - s_4)}{2}\right) \sum_{d|q} \frac{\mu(d)}{d} \sum_{\psi \pmod{q}}^\# \mathcal{H}_+(\chi, \psi) \sum_{\substack{k > \kappa \\ k \equiv \kappa \pmod{2}}} \sum_{f \in \mathcal{B}_k^*(dq, \psi^2)} i^k \\ &\quad \times \frac{L\left(\frac{1-s_1+s_2+s_3-s_4}{2}, f \otimes \bar{\psi}\right) L\left(\frac{1-s_1+s_2-s_3+s_4}{2}, f \otimes \bar{\psi}\right) L\left(\frac{s_1+s_2+s_3+s_4-1}{2}, f \otimes \bar{\psi}\right)}{L(1, \text{Ad}^2 f)} \Xi_s\left(\frac{k-1}{2}\right), \end{aligned}$$

where  $\mathcal{J}_\pm^{\text{Maaß}}$  and  $\mathcal{J}_\pm^{\text{hol}}$  involve the cubic moment of twisted modular  $L$ -functions and the continuous term  $\mathcal{J}_\pm^{\text{Eis}}$  can be regarded as the sixth moment of Dirichlet  $L$ -functions. Here  $\#$  on the sum signifies that the sum runs over all

<sup>2</sup>These are the main spectral contributions to the cubic moment side. One should compute degenerate terms coming from the principal and quadratic Dirichlet characters in the proof of Theorem 1.1, but they are in principle of the same size as the main spectral contributions in terms of estimations. Hence one can disregard them when applying Motohashi's formula to subconvexity and certain asymptotic evaluations.

primitive nonquadratic characters modulo  $q$ , and we define

$$\begin{aligned} \mathcal{S}_f(t; s_1, s_2, s_3, s_4) &= L\left(\frac{1-s_1+s_2+s_3-s_4}{2} + it, \overline{\psi}\psi_2\right) L\left(\frac{1-s_1+s_2+s_3-s_4}{2} - it, \overline{\psi}\psi_1\right) \\ &\times L\left(\frac{1-s_1+s_2-s_3+s_4}{2} + it, \overline{\psi}\psi_2\right) L\left(\frac{1-s_1+s_2-s_3+s_4}{2} - it, \overline{\psi}\psi_1\right) \\ &\times L\left(\frac{s_1+s_2+s_3+s_4-1}{2} + it, \psi\overline{\psi}_2\right) L\left(\frac{s_1+s_2+s_3+s_4-1}{2} - it, \psi\overline{\psi}_1\right). \end{aligned} \quad (1.4)$$

The factor  $\epsilon_f$  is the parity of a Maaß form  $f$ . In addition,  $\Phi_{\mathfrak{s}}^{\pm}$  and  $\Xi_{\mathfrak{s}}$  are introduced in (3.24a), (3.24b) and (3.24c), respectively. The Dirichlet series expansions of the adjoint square  $L$ -function  $L(s, \text{Ad}^2 f)$  and the twisted modular  $L$ -function  $L(s, f \otimes \psi)$  in  $\Re(s) > 1$  (which serve as a definition of these functions) are given in (2.16) and (2.17), respectively. We are now ready to describe the reciprocity formula to which we have alluded.

**Theorem 1.1.** *Let  $\chi$  be a primitive character modulo a prime  $q$  and let  $\mathfrak{s} = (s_1, s_2, s_3, s_4)$  with  $\overline{\mathfrak{s}} = (\overline{s_3}, \overline{s_4}, \overline{s_2}, \overline{s_1})$ . Let  $(s_1, s_2, s_3, s_4) \in \mathbb{C}^4$  be such that  $|s_1|, |s_2|, |s_3|, |s_4| < B$  where  $B$  is sufficiently large. If a test function  $g$  satisfies the basic assumption in Convention 1.10, then we have that*

$$\mathcal{Z}_2(\mathfrak{s}; g; \chi) = \mathcal{N}(\mathfrak{s}; g; \chi) + \sum_{\pm} \left( \mathcal{J}_{\pm}(\mathfrak{s}; g; \chi) + \mathcal{E}_{\pm}(\mathfrak{s}; g; \chi) + \overline{\mathcal{J}_{\pm}(\overline{\mathfrak{s}}; g; \chi)} + \overline{\mathcal{E}_{\pm}(\overline{\mathfrak{s}}; g; \chi)} \right), \quad (1.5)$$

where  $\mathcal{N}$  is an explicitly calculable main term and we decompose

$$\begin{aligned} \mathcal{J}_{\pm}(\mathfrak{s}; g; \chi) &= \{ \mathcal{J}_{\pm}^{\text{MaaB}} + \mathcal{J}_{\pm}^{\text{Eis}} + \delta_{\pm=+} \mathcal{J}_{+}^{\text{hol}} \}(\mathfrak{s}; g; \chi), \\ \mathcal{E}_{\pm}(\mathfrak{s}; g; \chi) &= \{ \mathcal{E}_{\pm}^{\text{MaaB}} + \mathcal{E}_{\pm}^{\text{Eis}} + \delta_{\pm=+} \mathcal{E}_{+}^{\text{hol}} \}(\mathfrak{s}; g; \chi). \end{aligned}$$

Here  $\mathcal{E}_{\pm}(\mathfrak{s}; g; \chi)$  is the so-called degenerate term defined in §3.5.2, which has similar shape to  $\mathcal{J}_{\pm}(\mathfrak{s}; g; \chi)$ .

It is feasible to extend Theorem 1.1 to the case of any positive integer  $q$ . If  $q$  is not prime, we need to utilise a general version of the transformation formula due to Blomer–Milićević [14, (2.2)]:

$$\sum_{(c,q)=1} \chi(c) S(m, n; c) h(c) = \frac{\chi_1(m)}{\tau(\chi_1)} \sum_{d|q} \mu(d) \sum_{d q_1 | c} S_{\chi_1}(m, n q_1^2; c) h(c/q_1), \quad (1.6)$$

which is valid for an arbitrary Dirichlet character modulo  $q$  induced from a primitive character  $\chi_1$  modulo  $q_1 | q$  and for every  $(m, q_1) = 1$ . The formula (1.6) translates the Kloosterman sum associated to the  $(\infty, 0)$  cusp-pair into the twisted Kloosterman sum associated to the  $(\infty, \infty)$  cusp-pair. The difficulties come from the Kloosterman sum on the right-hand side of (1.6) involving  $q_1^2$  instead of  $q^2$ .

**Remark 1.2.** In the author's recent work [42], the second moment of the product of the Riemann zeta and Dirichlet  $L$ -functions was contemplated:

$$\int_{-\infty}^{\infty} |\zeta(1/2 + it) L(1/2 + it, \chi)|^2 g(t) dt. \quad (1.7)$$

If we replace the Riemann zeta function with a Dirichlet  $L$ -function  $L(1/2 + it, \chi)$ , this is in accordance with (1.2). Although the definition (1.2) is similar to (1.7), the cubic moment side in Theorem 1.1 has quite different shape. This is due to the occurrence of some additional character sums when we consider the fourth moment of Dirichlet  $L$ -functions. Nonetheless the resulting form of an integral transform of the test function  $g$  never becomes altered.

**1.3. Quantitative Applications.** We are able to state various quantitative applications of Theorem 1.1. A hybrid fourth moment bound of interval  $H$  for individual Dirichlet  $L$ -functions is initially established.

**Corollary 1.3.** *Let  $T^{1/2} \leq H \leq T(\log T)^{-1}$ . For any primitive Dirichlet character modulo  $\chi$  a prime  $q$ , we have*

$$\int_T^{T+H} |L(1/2 + it, \chi)|^4 dt \ll_{\epsilon} H^{1+\epsilon} q^{\epsilon} + \left( \frac{qT}{\sqrt{H}} \right)^{1+\epsilon}. \quad (1.8)$$

Note that the proof of Corollary 1.3 looks circular in some sense: we use work of Petrow–Young [61] on cubic moments, who show that bounds for these cubic moments follow from bounds for the fourth moment of Dirichlet  $L$ -functions. Nonetheless, this is not circular: Petrow–Young arrive at a long fourth moment that can be bounded essentially optimally via approximate functional equations and the spectral large sieve, whereas our short fourth moment can never be bounded optimally via the large sieve, namely it never implies subconvexity. We need to first apply Motohashi's formula and then use the large sieve afterwards.

The choice  $H = (qT)^{2/3}$  in Corollary 1.3 yields a variant of Iwaniec's [34] short interval fourth moment bound.

**Corollary 1.4.** *Let  $q \ll T^{1/2-\epsilon}$ . For any primitive Dirichlet character  $\chi$  modulo a prime  $q$ , we have*

$$\int_T^{T+(qT)^{2/3}} |L(1/2 + it, \chi)|^4 dt \ll_\epsilon (qT)^{2/3+\epsilon}.$$

Jutila [40, Theorem 3] obtained the fourth moment bound

$$\sum_{\chi \pmod{D}} \int_T^{T+T^{2/3}} |L(1/2 + it, \chi)|^4 dt \ll_\epsilon D^{1+\epsilon} T^{2/3+\epsilon}. \quad (1.9)$$

His result implies a bound for individual characters which is of the form  $q^{1+\epsilon} T^{2/3+\epsilon}$ . In the range  $q \ll T^{1/2-\epsilon}$ , our result improves upon this bound for individual characters to  $(qT)^{2/3+\epsilon}$ . There is a fundamental obstruction that the parameter  $H$  in Corollary 1.3 cannot exceed  $T$  and the optimisation of  $H$  in the bound (1.8) in turn gives some restriction on  $q$ . Indeed, there is a weaker version where one has no restriction on the conductor, which can be deduced from the choice  $H = T^{2/3}$  in Corollary 1.3. It therefore follows that

$$\int_T^{T+T^{2/3}} |L(1/2 + it, \chi)|^4 dt \ll_\epsilon q^{1+\epsilon} T^{2/3+\epsilon}.$$

This is similar to (1.9), but eventually yields weaker subconvexity bounds for Dirichlet  $L$ -functions.

Corollary 1.4 is equivalent to the claim that Dirichlet  $L$ -functions cannot sustain large values, namely

$$\#\{t \in [T, T + (qT)^{2/3}] : |L(1/2 + it, \chi)| \geq V\} \ll_\epsilon (qT)^{2/3+\epsilon} V^{-4}.$$

In particular, we have that  $V \ll (qT)^{1/6+\epsilon}$  which leads to Weyl-strength hybrid subconvexity bounds for Dirichlet  $L$ -functions when  $q \ll (1+|t|)^{1/2-\epsilon}$  (the exponent  $1/6$  often reoccurs in modern incarnations of these problems):

$$L(1/2 + it, \chi) \ll_\epsilon (q(1+|t|))^{1/6+\epsilon}.$$

This is covered by the results of Petrow–Young [61, 62], who has shown Weyl-strength subconvexity bounds for twisted modular  $L$ -functions, which are hybrid in both the  $t$  and the  $q$ -aspect. We also establish the following:

**Corollary 1.5.** *Let  $(qT)^{1/2+\epsilon} \ll T_0 \ll (qT)^{2/3}$  with  $q \leq T^{1/2-\epsilon}$  and  $T \leq t_1 < \dots < t_R \leq 2T$  with  $t_{r+1} - t_r \geq T_0$ . For any primitive Dirichlet character  $\chi$  modulo a prime  $q$ , we then have that*

$$\sum_{r=1}^R \int_{t_r}^{t_r+T_0} |L(1/2 + it, \chi)|^4 dt \ll_\epsilon \left( RT_0 + qT \sqrt{\frac{R}{T_0}} \right) (qT)^\epsilon.$$

This kind of fourth moment bound is occasionally seen in the literature; we mention Iwaniec [34] and Jutila–Motohashi [39] to name a few. Since one wants to establish Corollary 1.5 as an application of Theorem 1.1, one can exploit the method outlined in the Appendix in Jutila–Motohashi [39]. They were unable to show Motohashi's formula for the fourth moment of Dirichlet  $L$ -functions averaged over primitive Dirichlet characters, but for the second moment of Estermann zeta functions. The proof of Corollary 1.5 is nearly identical to that of Corollary 1.3, but we must detect some cancellations in the spectral sum, which was initially executed by Ivić–Motohashi [33]. We mention that Jutila–Motohashi [39] established with the same notation as in Corollary 1.5 that

$$\sum_{\chi \pmod{D}} \sum_{r=1}^R \int_{t_r}^{t_r+T_0} |L(1/2 + it, \chi)|^4 dt \ll_\epsilon \left( DRT_0 + D^{2(1+\theta)/3} (RT)^{2/3} \right) T^\epsilon$$

with  $0 \leq \theta \leq 7/64$  an admissible exponent towards the Ramanujan–Petersson conjecture. Corollary 1.5 resolves an open problem on the twelfth moment of individual Dirichlet  $L$ -functions when  $q$  is small relative to  $T$ .

**Corollary 1.6.** *Let  $q \ll T^{1/2-\epsilon}$ . For any primitive Dirichlet character  $\chi$  modulo a prime  $q$ , we have*

$$\int_0^T |L(1/2 + it, \chi)|^{12} dt \ll_{\epsilon} (qT)^{2+\epsilon}.$$

This extends the work of Heath-Brown [28] who obtained the twelfth moment bound

$$\int_0^T |\zeta(1/2 + it)|^{12} dt \ll_{\epsilon} T^{2+\epsilon}. \quad (1.10)$$

One must point out that Heath-Brown deduced (1.10) in an entirely different manner via the combination of van der Corput's method with the Halász–Montgomery inequality. Our method is fundamentally limited in this regard due to the fact that an average over characters is not included; we cannot hope to prove subconvexity bounds for  $L(1/2 + it, \chi)$  in the  $q$ -aspect with  $t$  fixed, since we are averaging over too small a family. The result that would be of most interest includes an average over Dirichlet characters modulo  $q$ , namely the conjectural upper bound

$$\sum_{\chi \pmod{q}} \int_0^T |L(1/2 + it, \chi)|^{12} dt \ll_{\epsilon} (qT)^{2+\epsilon}.$$

This is a new proof of Weyl-strength subconvexity. If  $q = 1$ , this is Heath-Brown's bound for the twelfth moment of the Riemann zeta function. If  $T \ll 1$  and  $q$  is a smooth squarefree modulus, this is due to Nunes [58], although his method does not work for arbitrary  $q$ . Meurman [46] has shown the weaker bound  $O_{\epsilon}(q^{3+\epsilon} T^{2+\epsilon})$  that implies Weyl-strength subconvexity in the  $t$ -aspect but convexity in the  $q$ -aspect. Jutila–Motohashi [39] established that

$$\sum_{q \leq Q} \sum_{\chi \pmod{q}} \int_0^T |L(1/2 + it, \chi)|^{12} dt \ll_{\epsilon} Q^{3+\epsilon} T^{2+\epsilon}.$$

Milićević–White [49] have studied this problem when  $T \ll 1$  and  $q$  is growing in the depth aspect.

**Remark 1.7.** It would be interesting to study whether we can obtain sub-Weyl subconvexity in a hybrid regime. It should be feasible to prove that there exists a small  $\epsilon$  such that sub-Weyl subconvexity is derived when  $q \ll T^{\epsilon}$ . The question is whether  $\epsilon$  can be made reasonable. We leave such pursuits and their applications to future work.

Finally, Topaçoğullari [65] has manifested an asymptotic formula for the fourth moment of individual Dirichlet  $L$ -functions. Via the specialisation of the test function  $g$  in Theorem 1.1 and a sequence of standard manipulations such as spectral large sieve inequalities, one may arrive at an asymptotic formula of the same quality.

**Corollary 1.8** (Topaçoğullari [65, Theorem 1.1]). *Let  $\epsilon > 0$  be a small positive quantity which is not necessarily the same at each occurrence. Let  $\chi$  be any primitive Dirichlet character modulo  $q$ . Then we have for  $T \geq 1$  that*

$$\int_0^T |L(1/2 + it, \chi)|^4 dt = \int_0^T P_{\chi}(\log t) dt + O_{\epsilon}(q^{2-3\theta} T^{1/2+\theta+\epsilon} + qT^{2/3+\epsilon}), \quad (1.11)$$

where  $P_{\chi}$  is a polynomial of degree 4 whose coefficients depend only on  $q$ .

We omit the proof of Corollary 1.8 since our method is quite analogous to that of Topaçoğullari [65]. Improving upon the error term in (1.11) in the  $q$ -aspect requires some additional manoeuvres.

**1.4. Sketch of Modus Operandi.** We produce a heuristic overview on the genesis of the automorphic reciprocity shown in Theorem 1.1. This is a high-level sketch geared towards experts. For the sake of argument, we may rely on approximate tools, although our proof will be based on more precise inspection of  $L$ -functions in the region of absolute convergence. We should also ignore all polar terms, correction factors and the  $t$ -average.

The experienced reader understands our method from the shape of (1.5). The overall strategy is inspired by Motohashi's seminal work [53, §4.3–4.7] by which we are able to establish fairly explicit spectral identities. We want to study what happens if we replace the Riemann zeta function  $\zeta(1/2+it)$  in his argument with a Dirichlet  $L$ -function  $L(1/2+it, \chi)$ , and we observe that the presence of the character  $\chi$  substantially complicates our analysis. In a single phrase, we first open the four zeta values as Dirichlet series and apply Atkinson's dissection, then we

work with the shifted convolution problem with the Voronoï summation and the Kloosterman summation formula (Kuznetsov formula) attached to selected Atkin–Lehner cusps. An initial shape of the off-diagonal term looks like

$$\sum_{n,m \geq q^{1/2}} \bar{\chi}(n)\chi(n+m)\tau(n)\tau(n+m) = \sum_{a,b \pmod{q}} \bar{\chi}(a)\chi(a+b) \sum_{\substack{n,m \geq q^{1/2} \\ n \equiv a, m \equiv b \pmod{q}}} \tau(n)\tau(n+m). \quad (1.12)$$

This kind of sum has naturally arisen in the antecedent works such as [62, 65, 67] and  $\tau(m)$  ought to be replaced with the divisor function  $\sigma_\lambda(m)$  in our proof due to the convergence issue. One sifts out the congruence conditions on the right-hand side of (1.12) in terms of additive characters. We apply the orthogonality of the Ramanujan sum

$$\delta_{n \equiv a \pmod{q}} = \frac{1}{q} \sum_{c|q} r_c(n-a).$$

In order to spectrally expand (1.12), it is necessary to separate the two variables in  $\tau(n+m)$ . We then make use of the approximate functional equation for the divisor function

$$\sigma_\lambda(m) = \sum_{(\ell,q)=1} \frac{S(m,0;\ell)}{\ell^{1-\lambda}} \varpi_\lambda\left(\frac{\ell}{\sqrt{m}}\right) + m^\lambda \sum_{(\ell,q)=1} \frac{S(m,0;\ell)}{\ell^{1+\lambda}} \varpi_{-\lambda}\left(\frac{\ell}{\sqrt{m}}\right), \quad (1.13)$$

where for  $a > |\Re(\lambda)|$ ,

$$\varpi_\lambda(x) = \frac{1}{2\pi i} \int_{(a)} x^{-w} \zeta^q(1-\lambda+w) \frac{G(w)}{w} dw.$$

The formula (1.13) is seen as a simple alternative to the  $\delta$ -symbol method of Duke–Friedlander–Iwaniec [21] and plays a rôle in eliminating the pole of the Riemann zeta function appearing in the original Ramanujan expansion. In our actual proof, we create a zero that like a *deus ex machina* kills the pole from the Riemann zeta function. One then utilises the  $\text{GL}_2$  Voronoï summation formula to the  $n$ -sum. In other words, the functional equation of the Estermann zeta function is invoked. Letting  $q\bar{q} \equiv 1 \pmod{\ell}$ ,  $\ell\bar{\ell} \equiv 1 \pmod{q}$  and  $\bar{\tau}(\psi)$  be the normalised Gauß sum, we are eventually led to sums of the product of Kloosterman sums looking roughly like<sup>3</sup>

$$\sum_{(\ell,q)=1} \frac{S(m\bar{q}, \pm n\bar{q}; \ell) S(a\bar{\ell}, \pm n\bar{\ell}; q)}{\ell} \approx \sum_{\psi \pmod{q}} \bar{\psi}(\pm anm) \bar{\tau}(\psi)^2 \sum_{(\ell,q)=1} \psi(\ell)^2 \frac{S(m\bar{q}, \pm n\bar{q}; \ell)}{\ell}. \quad (1.14)$$

Here the second Kloosterman sum  $S(a\bar{\ell}, \pm n\bar{\ell}; q)$  was encoded via the orthogonality of Dirichlet characters modulo  $q$  and the character  $\bar{\psi}(m)$  was also added on the right-hand side for technical brevity. Be aware of the great similarity between (1.14) and Motohashi's conjecture written down in [52]. One decomposes the  $\psi$ -sum into the sum over primitive nonquadratic characters and others, followed by an application of the transformation formula due to Blomer–Miličević [14]. We are in a position to use the Kloosterman summation formula (Theorem 2.11) of level at most  $q^2$  and  $(\infty, \infty)$  cusp-pair. We obtain three twisted modular  $L$ -functions with an explicit calculation of the resulting  $m, n$ -sums, thence deriving

$$\int_{-\infty}^{\infty} |L(1/2 + it, \chi)|^4 g(t) dt \leftrightarrow \sum_{\pm} \sum_{d|q} \frac{\mu(d)}{d} \sum_{\psi \pmod{q}}^{\#} \mathcal{H}_{\pm}(\chi, \psi) \sum_{f \in \mathcal{B}_{\kappa}^*(dq, \psi^2)} \epsilon_f^{(1\mp 1)/2} \frac{L(1/2, f \otimes \bar{\psi})^3}{L(1, \text{Ad}^2 f)} \check{g}(t_f) \quad (1.15)$$

for the character sum  $\mathcal{H}_{\pm}(\chi, \psi)$  defined in §1.2 and the transform  $\check{g}$  which involves the hypergeometric function and depends on  $\pm$ . The contribution of the quadratic character can similarly be described. A merit to work with Motohashi's classical method is that one can achieve an explicit formulation of  $\check{g}$ , which would be alluring from an aesthetic point of view. If  $\psi$  is a primitive character modulo  $q$  and  $f \in \mathcal{B}_{\kappa}^*(dq, \psi^2)$ , then  $f \otimes \bar{\psi}$  has trivial central character and conductor dividing  $q^2$  via Theorem B.1. There is structural beauty in our reciprocity (1.15), since we were forced to decompose the  $\psi$ -sum in terms of whether  $\psi$  is primitive nonquadratic or not. The condition that  $f \otimes \bar{\psi}$  has the trivial central character is decisive as we may rely on the result of Guo [26], which guarantees that  $L(1/2, f \otimes \bar{\psi}) \geq 0$ . One can then evaluate the cubic moment in (1.15) using a standard positivity argument.

<sup>3</sup>In this sketch, we use the symbol  $\approx$  to mean that the left-hand side may roughly be written as an expression resembling the right-hand side with an acceptable error term.

There is also a noteworthy plan to contemplate the fourth moment of individual Dirichlet  $L$ -functions. Fix a primitive Dirichlet character  $\psi$  modulo  $q$ . We take  $b = 1$  in [9, Theorem 1], multiply  $\mathcal{T}_{a,b,q}(s, u, v)$  by  $\overline{\psi}(a)$  and then sum over  $a \pmod{q}$  with application of the orthogonality relation. As an aside, this process necessitates a little bit of modification since  $a$  is supposed to be squarefree. Nevertheless, this assumption was only to keep the formulæ a little simpler, and hence their method works more generally when  $a$  is not squarefree, but the formulæ are much more complicated. We suspect that a substantial simplification on the right-hand side of [9, (1.14)] will happen. We remark that the argument in [9] relies essentially upon the isobaric sum  $4 = 3 + 1$  and dualises the 3 afterwards and its chief novelty is to apply the twisted multiplicativity of Kloosterman sums, which enables us to circumvent the manipulation of Kloosterman sums associated to various Atkin–Lehner cusps.

**Remark 1.9.** This research is relevant to the sixth moment of the Riemann zeta function [41] whose calculations were initially contained in this article, but the author decided to remove this part since they were heuristics.

**1.5. Organisation of the Article.** We devote §2 to compile a preparatory toolbox, in particular the evaluation of various multiplicative functions, followed by the  $GL_2$  Voronoï summation formula and Kloosterman summation formula. We also render an exhaustive exposition of Kloosterman sums. In §3, we prove Theorem 1.1 via a shifted convolution problem. In §4, we establish Corollaries 1.3, 1.4, 1.5 and 1.6. The methods involve a combination of analytic number theory and automorphic forms. Additionally, we establish the Voronoï–Oppenheim summation formula in Appendix A and evaluate the size of the conductor for twists of Maaß newforms in Appendix B.

**1.6. Basic Notation and Conventions.** Throughout this article, the letter  $\epsilon$  represents an arbitrarily small positive quantity, not necessarily the same at each occurrence. An implicit constant may depend on  $\epsilon$ , but this will be often suppressed from the notation. The Vinogradov symbol  $A \ll B$  or the big O notation  $A = O(B)$  signifies that  $|A| \leq C|B|$  for some constant  $C$ . We write  $e(\zeta) := \exp(2\pi i\zeta)$  and  $e_\alpha(\zeta) := e(\zeta/\alpha)$  with  $\zeta \in \mathbb{C}$  and  $\alpha \in \mathbb{R}$ . We assume that a test function  $g$  satisfies the following basic assumption:

**Convention 1.10.** *The function  $g$  is real valued on  $\mathbb{R}$  and there exists a large positive constant  $A$  such that  $g(t)$  is regular and  $\ll (1+|t|)^{-A}$  on a sufficiently wide horizontal strip  $|\Im(t)| \leq A$ . All implicit constants in Vinogradov symbols and big O notation may possibly depend on  $A$  (where applicable).*

## 2. ARITHMETIC AND AUTOMORPHIC TOOLS

We provide background materials which we shall need later to establish Theorem 1.1. For starters, we prepare elementary lemmata that are suitable for evaluating the character sums arising in this work. Our focus is on the simplification of sums involving multiplicative characters, which are akin to [62, §6.1]. Secondly, we define the double Estermann zeta function and mention some properties including a functional equation. Finally, we briefly present automorphic tools as well as an introductory explanation of Kloosterman sums. An exhaustive account of the theoretical background can also be found in [8, 20, 22] and references therein.

**2.1. Manipulations of Character Sums.** The orthogonality relation asserts that

$$\sum_{a \pmod{q}} \chi(a) = \begin{cases} \varphi(q) & \text{if } \chi = \chi_0, \\ 0 & \text{otherwise,} \end{cases} \quad \sum_{\chi \pmod{q}} \chi(a) = \begin{cases} \varphi(q) & \text{if } a \equiv 1 \pmod{q}, \\ 0 & \text{otherwise.} \end{cases} \quad (2.1)$$

For any Dirichlet character  $\chi$  modulo  $q$ , let

$$\tau(\chi, h) = \sum_{b \pmod{q}} \chi(b) e\left(\frac{bh}{q}\right) \quad (2.2)$$

denote the Gauß sum associated to characters on residue classes modulo  $q$ . One writes  $\tau(\chi) = \tau(\chi, 1)$  as usual. Multiplying (2.2) by  $\overline{\chi}(a)$  and summing over  $\chi$ , we derive via orthogonality (2.1) that

$$e\left(\frac{ah}{q}\right) = \frac{1}{\varphi(q)} \sum_{\chi \pmod{q}} \overline{\chi}(a) \tau(\chi, h) \quad \text{if } (a, q) = 1. \quad (2.3)$$

This is the Fourier expansion of additive characters in terms of the multiplicative ones. One can evaluate Gauß sums for general Dirichlet characters.

**Lemma 2.1.** *Let  $\chi$  be a nontrivial Dirichlet character modulo  $q$  induced by the primitive character  $\chi^*$  modulo  $q^*$ . For an integer  $n \geq 1$ , we have that*

$$\tau(\chi, n) = \tau(\chi^*) \sum_{d|(n, q/q^*)} d \bar{\chi}^* \left( \frac{n}{d} \right) \chi^* \left( \frac{q}{dq^*} \right) \mu \left( \frac{q}{dq^*} \right).$$

*Proof.* See [37, Lemma 3.2] which is corrected in the list of errata on Kowalski's website.  $\square$

We indicate by  $r_c(n)$  and  $S(m, n; c)$  the Ramanujan sum and Kloosterman sum respectively as follows:

$$r_c(n) := \sum_{a \pmod{c}}^* e \left( \frac{an}{c} \right) = \sum_{d|(n, c)} d \mu \left( \frac{c}{d} \right), \quad S(m, n; c) := \sum_{d \pmod{c}}^* e \left( \frac{md + n\bar{d}}{c} \right),$$

where the asterisk means that the summation is restricted to a reduced system of residues. We have the Weil bound

$$|S(m, n; c)| \leq (m, n, c)^{1/2} c^{1/2} \tau(c). \quad (2.4)$$

This gives the best possible bound for individual Kloosterman sums, whilst we apt to make use of the Kloosterman summation formula (Theorem 2.11) to obtain additional savings from the sum over the modulus  $c$ . Indeed, various arithmetic problems can be transformed into bounding sums of Kloosterman sums. The twisted multiplicativity for Kloosterman sums is occasionally exploited. One also uses the Jacobi sum

$$J(\chi, \psi) = \sum_{a \pmod{q}} \chi(a) \psi(1 - a).$$

In particular, when  $\chi$  and  $\psi$  are of the same modulus and  $\chi\psi$  is primitive, the relation between the Gauss sum and the Jacobi sum is illustrated as

$$J(\chi, \psi) = \frac{\tau(\chi)\tau(\psi)}{\tau(\chi\psi)}.$$

In general, we have the following lemma:

**Lemma 2.2.** *Assume that  $q$  is prime. Suppose that  $\psi_1, \psi_2$  are primitive characters modulo  $q$  satisfying  $\psi_1 \neq \overline{\psi_2}$ , and let  $a, b, c, d \in \mathbb{Z}$  with  $(a, c, q) = 1$ . Then*

$$\sum_{t \pmod{q}} \psi_1(at + b) \psi_2(ct + d) = \overline{\psi_1}(c) \overline{\psi_2}(a) \psi_1 \psi_2(ad - bc) \frac{\tau(\psi_1) \tau(\overline{\psi_1 \psi_2})}{\tau(\overline{\psi_2})}.$$

Moreover, when  $\psi_1 = \overline{\psi_2} = \psi$ , we have that

$$\sum_{t \pmod{q}} \psi(at + b) \overline{\psi}(ct + d) = \psi(a) \overline{\psi}(c) r_q(ad - bc).$$

*Proof.* As in [62], we observe that the sum vanishes unless  $(a, q) = (c, q) = 1$ . We assume  $(a, q) = (c, q) = 1$  in what follows. To prove the first assertion, we initially expand the character  $\psi_1$  into exponentials:

$$\psi_1(at + b) = \frac{1}{\tau(\overline{\psi_1})} \sum_{x \pmod{q}} \overline{\psi_1}(x) e_q((at + b)x).$$

We want to calculate the  $t$ -sum to reduce the problem to the manipulation of the  $x$ -sum. Then

$$\sum_{t \pmod{q}} \psi_2(ct + d) e_q(atx) = \sum_{t \pmod{q}} \psi_2(t) e_q(a\bar{c}(t - d)x) = \overline{\psi_2}(ax) \psi_2(c) e_q(-a\bar{c}dx) \tau(\psi_2).$$

On the other hand, one has

$$\sum_{x \pmod{q}} \overline{\psi_1 \psi_2}(x) e_q(x(b - a\bar{c}d)) = \psi_1 \psi_2(ad - bc) \overline{\psi_1 \psi_2}(-c) \tau(\overline{\psi_1 \psi_2}).$$

Putting the above formulae together, we arrive at

$$\sum_{t \pmod{q}} \psi_1(at + b) \psi_2(ct + d) = \overline{\psi_1}(c) \overline{\psi_2}(a) \psi_1 \psi_2(bc - ad) \frac{\tau(\psi_2) \tau(\overline{\psi_1 \psi_2})}{\tau(\overline{\psi_1})}.$$

Applying the identities  $\tau(\overline{\psi_1}) = \psi_1(-1)\tau(\psi_1)^{-1}q$  and  $\tau(\psi_2) = \psi_2(-1)\tau(\overline{\psi_2})^{-1}q$ , we reach the desired expression. The second claim is the same as in [62, Lemma 6.3].  $\square$

As mentioned in the introduction, we encounter sums of the product of Kloosterman sums. One should resolve  $\overline{\ell}$  inside the argument of the Kloosterman sum and the following lemma is helpful when we expand  $S(a\overline{\ell}, \pm n\overline{\ell}; q)$  into multiplicative characters.

**Lemma 2.3.** *Suppose that  $q \geq 1$  and  $a, b \in \mathbb{Z}$ . Write  $a = a_0a'$  and  $b = b_0b'$  where  $a_0b_0 \mid q^\infty$  and  $(a'b', q) = 1$ . Then*

$$S(a, b; q) = \frac{1}{\varphi(q)} \sum_{\psi \pmod{q}} \tau(\psi, a_0)\tau(\psi, b_0)\overline{\psi}(a'b'),$$

where the sum runs over all Dirichlet characters modulo  $q$  and  $\varphi(q)$  is Euler's totient function.

*Proof.* We exploit (2.3) so that the Kloosterman sum  $S(a, b; q)$  equals

$$\begin{aligned} \sum_{d \pmod{q}}^* e\left(\frac{a_0d + b_0a'b'\overline{d}}{q}\right) &= \frac{1}{\varphi(q)} \sum_{d \pmod{q}}^* e\left(\frac{a_0d}{q}\right) \sum_{\psi \pmod{q}} \overline{\psi}(a'b'\overline{d})\tau(\psi, b_0) \\ &= \frac{1}{\varphi(q)} \sum_{\psi \pmod{q}} \tau(\psi, a_0)\tau(\psi, b_0)\overline{\psi}(a'b'). \end{aligned}$$

This finishes the proof of the lemma.  $\square$

Direct corollaries of Lemma 2.3 include

**Corollary 2.4.** *Suppose that  $q \geq 1$  and  $a, b \in \mathbb{Z}$ . For  $(ab, q) = 1$ , we then have*

$$S(a, b; q) = \frac{1}{\varphi(q)} \sum_{\psi \pmod{q}} \tau(\psi)^2\overline{\psi}(ab).$$

This idea of separation of variables traces back to important work of Blomer–Milićević in [14]. The number of the Gauß sums  $\tau(\psi)$  is the crux in Lemma 2.3, since the  $n$ -th power determines the hyper-Kloosterman sum of  $n$  variables. The variation of Lemma 2.3 was employed by Petrow–Young [62, Lemma 8.8], where they handled the hyper-Kloosterman sum  $\text{Kl}_3(x, y, z; q)$ .

**2.2. Double Estermann Zeta Function.** We introduce the divisor function

$$\sigma_w(m) = \sum_{d|m} d^w.$$

As far as we know, the Hecke relation for  $\sigma_w(m)$  has never appeared in the antecedent literature.

**Lemma 2.5.** *With the notation as above, the divisor function enjoys the following multiplicativity relation:*

$$\sigma_w(mn) = \sum_{c|(m,n)} \mu(c)c^w \sigma_w(m/c)\sigma_w(n/c).$$

*Proof.* If  $m = p_1^{k_1}p_2^{k_2}\cdots p_r^{k_r}$ ,  $n = p_1^{\ell_1}p_2^{\ell_2}\cdots p_r^{\ell_r}$  and such a formula is shown for powers of primes, then

$$\begin{aligned} \sigma_w(mn) &= \sigma_w(p_1^{k_1+\ell_1})\sigma_w(p_2^{k_2+\ell_2})\cdots\sigma_w(p_r^{k_r+\ell_r}) \\ &= \prod_{i=1}^r \sum_{j=0}^{\min(k_i, \ell_i)} \mu(p_i^j)p_i^{jw} \sigma_w(p_i^{k_i-j})\sigma_w(p_i^{\ell_i-j}) \\ &= \sum_{c|(m,n)} \mu(c)c^w \sigma_w(m/c)\sigma_w(n/c). \end{aligned} \tag{2.5}$$

By multiplicativity, it suffices to prove the result for  $m = p^k$ ,  $n = p^\ell$ . The formula is obvious if  $\min(k, \ell) = 0$ , so we assume both the variables are at least 1. Setting  $X = p^w$ , we find that the right-hand side of (2.5) reads

$$\text{RHS} = \sigma_w(p^k)\sigma_w(p^\ell) - p^w \sigma_w(p^{k-1})\sigma_w(p^{\ell-1})$$

$$\begin{aligned}
&= (1 + X + \cdots + X^k)(1 + X + \cdots + X^\ell) - X(1 + X + \cdots + X^{k-1})(1 + X + \cdots + X^{\ell-1}) \\
&= \frac{X^{k+\ell+1} - 1}{X - 1} = (1 + p^w + p^{2w} + \cdots + p^{(k+\ell)w}) = \sigma_w(p^{k+\ell}).
\end{aligned}$$

This establishes Lemma 2.5.  $\square$

For any positive integers  $h, \ell$  with  $(h, \ell) = 1$  and  $\Re(s) > 1$ , we define the double Estermann zeta function as

$$D_2(s, \lambda; h/\ell) = \sum_{n=1}^{\infty} \sigma_\lambda(n) e(nh/\ell) n^{-s}. \quad (2.6)$$

Most of the analytic properties of  $D_2(s, \lambda; h/\ell)$  follow from the identity

$$D_2(s, \lambda; h/\ell) = \ell^{\lambda-2s} \sum_{a, b \pmod{\ell}} e(abh/\ell) \zeta(s, a/\ell) \zeta(s - \lambda, b/\ell),$$

where for  $\alpha \in \mathbb{R}$  and  $\Re(s) > 1$ ,

$$\zeta(s, \alpha) := \sum_{n+\alpha>0} (n + \alpha)^{-s}$$

is the Hurwitz zeta function. It has meromorphic continuation to the whole complex plane  $\mathbb{C}$  with a simple pole at  $s = 1$  of residue 1 and satisfies the functional equation

$$\zeta(s, \alpha) = \sum_{\pm} G^\mp(1-s) \zeta^{\pm(\alpha)}(1-s), \quad (2.7)$$

where  $G^\pm(s) = (2\pi)^{-s} \Gamma(s) \exp(\pm i\pi s/2)$  and  $\zeta^{(\alpha)}(s)$  is the meromorphic continuation of  $\sum_{n \geq 1} e(\alpha n) n^{-s}$ . For  $\alpha \in \mathbb{Q}$ , the formula (2.7) is seen as a restatement of the Poisson summation in residue classes. Hence, as a function of the single variable  $s$ , the double Estermann zeta function (2.6) also has meromorphic continuation to all  $s \in \mathbb{C}$  with two simple poles at  $s = 1$  and  $s = 1 + \lambda$  with respective residues  $\ell^{-\lambda-1} \zeta(1 - \lambda)$  and  $\ell^{-\lambda-1} \zeta(1 + \lambda)$ , provided  $\lambda \neq 0$ . In the case of  $\lambda = 0$ , there is a double pole at  $s = 1$  and the Laurent expansion is given by

$$D_2(s, 0; h/\ell) = \frac{1/\ell}{(s-1)^2} + \frac{2(\gamma - \log \ell)/\ell}{(s-1)} + \cdots,$$

where  $\gamma$  is the Euler–Mascheroni constant. The functional equation for  $D_2(s, \lambda; h/\ell)$  reads as follows:

**Theorem 2.6** ([53, Lemma 3.7]). *The Estermann zeta function satisfies the functional equation*

$$\begin{aligned}
D_2(s, \lambda; h/\ell) &= 2(2\pi)^{2s-\lambda-2} \ell^{1+\lambda-2s} \Gamma(1-s) \Gamma(1+\lambda-s) \\
&\times \left[ D_2(1-s, -\lambda; \bar{h}/\ell) \cos(\pi\lambda/2) - D_2(1-s, -\lambda; -\bar{h}/\ell) \cos(\pi(s-\lambda/2)) \right], \quad (2.8)
\end{aligned}$$

where  $\bar{h}$  is the multiplicative inverse of  $h$  modulo  $\ell$ , i.e.  $h\bar{h} \equiv 1 \pmod{\ell}$ .

The formula (2.8) was first proven by Hecke and Estermann in connection with an integral representation of the Hurwitz zeta function. We already find fragments of the Kloosterman sum in (2.8), which incline us to apply the Kloosterman summation formula. The functional equation of the Estermann zeta function  $D_2$  is essentially equivalent to  $\text{GL}_2$  Voronoï summation. Indeed, the Estermann zeta function involves a divisor function instead of Hecke eigenvalues and this corresponds to the standard Eisenstein series instead of a cusp form.

**2.3. Kloosterman Sums at Singular Cusps.** Our presentation of cusps and scaling matrices is inspired by [36]. We restrict ourselves to cusps with respect to the Hecke congruence subgroup

$$\Gamma = \Gamma_0(q) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{q} \right\}.$$

Let  $q_0 \mid q$ . In the following lines, the letter  $\psi$  denotes a Dirichlet character modulo  $q_0$  with  $\kappa = (1 - \psi(-1))/2$  such that  $\psi(-1) = (-1)^\kappa$ . Then we extend  $\psi$  via the identification

$$\psi \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \psi(d) = \bar{\psi}(a).$$

The group  $\Gamma$  acts transitively on  $\mathbb{P}^1(\mathbb{Q})$  by fractional linear transformations. An element  $\mathfrak{a} \in \mathbb{P}^1(\mathbb{Q})$  is called a cusp (denoted by a Gothic letter). Two cusps  $\mathfrak{a}$  and  $\mathfrak{b}$  are termed equivalent under  $\Gamma$  if there exists  $\gamma \in \Gamma$  satisfying  $\mathfrak{a} = \gamma\mathfrak{b}$ . We now write  $q = rs$  with  $(r, s) = 1$  and  $q_0 \mid s$ . Then we call a cusp of the form  $\mathfrak{a} = 1/r$  an Atkin–Lehner cusp. The Atkin–Lehner cusps are equivalent to  $\infty$  under the Atkin–Lehner operator as in [44, Definition 2.4]. Motohashi [55] singled out the scaling matrices corresponding to the Atkin–Lehner operators and this is crucial since the Fourier coefficients around the cusp  $1/r$  are proportional to the Fourier coefficients around the cusp  $\infty$  for an eigenform of the Atkin–Lehner operators.

Let  $\Gamma_{\mathfrak{a}} = \{\gamma \in \Gamma : \gamma\mathfrak{a} = \mathfrak{a}\}$  be the stabiliser of the cusp  $\mathfrak{a}$  in  $\Gamma$ . A matrix  $\sigma_{\mathfrak{a}} \in \mathrm{SL}_2(\mathbb{R})$ , satisfying

$$\sigma_{\mathfrak{a}}\infty = \mathfrak{a} \quad \text{and} \quad \sigma_{\mathfrak{a}}^{-1}\Gamma_{\mathfrak{a}}\sigma_{\mathfrak{a}} = \Gamma_{\infty} = \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}$$

is called a scaling matrix for the cusp  $\mathfrak{a}$ . Because the scaling matrix  $\sigma_{\mathfrak{a}}$  is not uniquely determined, the choice of  $\sigma_{\mathfrak{a}}$  will be important in our subsequent discussions.

**Definition 2.7.** Let  $\mathfrak{a}, \mathfrak{b}$  be cusps and  $\sigma_{\mathfrak{a}}, \sigma_{\mathfrak{b}}$  be scaling matrices. The set

$$\mathcal{C}(\mathfrak{a}, \mathfrak{b}) = \left\{ \gamma > 0 : \begin{pmatrix} * & * \\ \gamma & * \end{pmatrix} \in \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}} \right\}$$

is called the set of allowed moduli.

For a cusp  $\mathfrak{a}$  and a scaling matrix  $\sigma_{\mathfrak{a}}$ , let  $u_{\mathfrak{a}}$  be such that  $\sigma_{\mathfrak{a}}^{-1}u_{\mathfrak{a}}\sigma_{\mathfrak{a}} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ . If  $\psi$  satisfies  $\psi(u_{\mathfrak{a}}) = 1$ , then we say that  $\mathfrak{a}$  is *singular* with respect to  $\psi$ . It behoves us to mention the following proposition:

**Proposition 2.8** ([44, Proposition 2.6]). *Assume that  $q = rs$  with  $(r, s) = 1$  with  $q_0 \mid s$ , where  $q_0$  is the modulus of  $\psi$ . The two cusps  $\infty$  and  $1/r$  are then singular with respect to  $\psi$ . We choose a scaling matrix  $\sigma_{1/r}$  associated to the Atkin–Lehner cusp  $1/r$  to be an Atkin–Lehner operator, namely*

$$\sigma_{1/r} = \tau_r \nu_s \quad \text{with} \quad \tau_r = \begin{pmatrix} 1 & (s\bar{s} - 1)/r \\ r & s\bar{s} \end{pmatrix}, \quad \nu_s = \begin{pmatrix} \sqrt{s} & \\ & 1/\sqrt{s} \end{pmatrix}$$

with  $s\bar{s} \equiv 1 \pmod{r}$ . Then the set of allowed moduli is given by

$$\mathcal{C}(\infty, 1/r) = \{\gamma = c\sqrt{s} : c \equiv 0 \pmod{r}, (c, s) = 1\}.$$

An important example is when  $r = 1$  and  $s = q$ . In this case, one has  $\mathcal{C}(\infty, 0) = \{c\sqrt{q} : c \geq 1, (c, q) = 1\}$ . We are now in a position to define Kloosterman sums with respect to a pair of cusps and general central character.

**Definition 2.9.** If  $\mathfrak{a}, \mathfrak{b}$  are singular cusps for  $\psi$  modulo  $q_0$ , then the Kloosterman sum associated to  $\mathfrak{a}, \mathfrak{b}$  and  $\psi$  with modulus  $c$  is defined as

$$S_{\mathfrak{a}\mathfrak{b}}(m, n; c; \psi) = \sum_{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{\infty} \setminus \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}/\Gamma_{\infty}} \psi(\mathrm{sgn}(c)) \overline{\psi(\sigma_{\mathfrak{a}}\gamma\sigma_{\mathfrak{b}}^{-1})} e\left(\frac{am + dn}{c}\right). \quad (2.9)$$

The occurrence of  $-I \in \Gamma_{\infty}$  signifies that the lower-left entry  $c$  is only defined up to  $\pm$  sign, so that the factor  $\psi(\mathrm{sgn}(c))$  accounts for this. Definition 2.9 is sensitively dependent upon the choice of  $\sigma_{\mathfrak{a}}$  and  $\sigma_{\mathfrak{b}}$ . Moreover, if  $|c| \notin \mathcal{C}(\mathfrak{a}, \mathfrak{b})$ , then the sum appearing in (2.9) is empty, thus  $S_{\mathfrak{a}\mathfrak{b}}(m, n; c; \psi) = 0$ . If we stick to the case of  $\mathfrak{a} = \infty$  and  $\mathfrak{b} = 0$ , we have (see [44, (2.20)])

$$S_{\infty 0}(m, n; c\sqrt{q}; \psi) = \bar{\psi}(c) S(m, n\bar{q}; c) \quad (2.10)$$

with  $(c, q) = 1$  and  $q\bar{q} \equiv 1 \pmod{c}$ . This differs from an identity shown in [35, p.58] by an additive character, which is due to a different choice of the scaling matrix. In the formula (2.10), the presence of the character  $\bar{\psi}(c)$  is a nice feature of the  $(\infty, 0)$  cusp-pair as opposed to the  $(\infty, \infty)$  cusp-pair:

$$S_{\infty\infty}(m, n; c; \psi) = S_{\psi}(m, n; c) = \sum_{d \pmod{c}}^* \psi(d) e\left(\frac{md + n\bar{d}}{c}\right). \quad (2.11)$$

**2.4. Normalisation and Hecke Eigenvalues.** In order to formulate our Kloosterman summation formula, we borrow the normalisation from [8, 22]. We refer to [63] for an exposition in the case of general multiplier systems. Let  $\Gamma \backslash \mathbb{H}$  be the modular surface, where  $\Gamma = \Gamma_0(q)$  is the Hecke congruence subgroup and  $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$  is the upper half-plane. We mention various self-adjoint and pairwise commuting operators acting on the space  $L^2(\Gamma \backslash \mathbb{H})$ : the hyperbolic Laplacian

$$\Delta = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + i\kappa y \frac{\partial}{\partial x},$$

the Hecke operators (the non-archimedean counterparts of  $\Delta$ )

$$(T_n f)(z) = \frac{1}{\sqrt{n}} \sum_{ad=n} \psi(a) \sum_{b \pmod{d}} f\left(\frac{az+b}{d}\right)$$

for  $n \geq 1$ , and the reflection operator  $(T_{-1}f)(z) = f(-\bar{z})$  which flips positive and negative Fourier coefficients. As shall be explained later, we have the spectral decomposition of  $L^2(\Gamma \backslash \mathbb{H})$  in terms of pure point spectrum, residual spectrum and continuous spectrum, namely

$$L^2(\Gamma \backslash \mathbb{H}) = L^2_{\text{cusp}}(\Gamma \backslash \mathbb{H}) \oplus L^2_{\text{res}}(\Gamma \backslash \mathbb{H}) \oplus L^2_{\text{cont}}(\Gamma \backslash \mathbb{H}).$$

For an integer  $k > 0$  with  $k \equiv \kappa \pmod{2}$ , we choose a basis  $\mathcal{B}_k(q, \psi)$  of holomorphic cusp forms. It is taken orthonormal with respect to the weight  $k$  Petersson inner product

$$\langle h_1, h_2 \rangle = \int_{\Gamma \backslash \mathbb{H}} h_1(z) \overline{h_2(z)} y^k \frac{dx dy}{y^2},$$

where  $z = x + iy$  as usual. We let  $\mathcal{B}_\kappa(q, \psi)$  denote a basis of the space of Maaß cusp forms. In particular, they are eigenfunctions on  $\mathbb{H}$ , are automorphic of weight  $\kappa \in \{0, 1\}$ , are square-integrable on a fundamental domain and vanish at all the cusps. Furthermore, they are eigenfunctions of the  $L^2$ -extension of the hyperbolic Laplacian  $\Delta$ . For  $f \in \mathcal{B}_\kappa(q, \psi)$ , we write  $\Delta f = s(1-s)f$  with  $s = 1/2 + it_f$  and  $t_f \in \mathbb{R} \cup i[-1/2, 1/2]$ . One may choose the basis  $\mathcal{B}_\kappa(q, \psi)$  orthonormal with respect to the weight zero Petersson inner product introduced above. We define

$$\vartheta := \sup_{f \in \mathcal{B}_\kappa(q, \psi)} |\Im t_f|.$$

Then the Selberg eigenvalue conjecture asserts  $\vartheta = 0$ , whilst Selberg only established the upper-bound  $\vartheta \leq 1/4$ . The current world record is  $\vartheta \leq 7/64$  due to Kim–Sarnak [43] (see Blomer–Brumley [6] for the treatment in a more general scenario). The decomposition of the space of square-integrable, weight  $\kappa$  automorphic forms on  $\mathbb{H}$  with respect to the eigenspaces of the hyperbolic Laplacian involves the Eisenstein spectrum  $\mathcal{E}(q, \psi)$ , which turns out to be the orthogonal complement to the space of Maaß cusp forms. It is explicitly described in terms of the Eisenstein series  $E_\alpha(z, 1/2 + it)$  where  $\alpha$  runs over singular cusps and  $t \in \mathbb{R}$ . In this article, instead of using the classical Eisenstein series indexed by singular cusps as a basis of the continuous spectrum, we use another basis of Eisenstein series indexed by a set of parameters of the form

$$\{(\psi_1, \psi_2, f) : \psi_1 \psi_2 = \psi, f \in \mathcal{B}(\psi_1, \psi_2)\},$$

where  $(\psi_1, \psi_2)$  ranges over the pairs of characters of modulus  $q$  such that  $\psi_1 \psi_2 = \psi$  and  $\mathcal{B}(\psi_1, \psi_2)$  is some finite set dependent on  $(\psi_1, \psi_2)$ . We do not need to be more explicit here and we refer the reader to [23] for a precise definition of these parameters. The main advantage of such a basis is that the Eisenstein series are eigenforms of the Hecke operators  $T_n$  with  $(n, q) = 1$ : we have

$$T_n E(z, 1/2 + it, f) = \lambda_f(n, t) E(z, 1/2 + it, f)$$

with

$$\lambda_f(n, t) = \sum_{ab=n} \psi_1(a) a^{it} \psi_2(b) b^{-it}.$$

For convenience, we introduce the *ad hoc* notation

$$i(g, z) = cz + d, \quad j(g, z) = (cz + d)|cz + d|^{-1}, \quad g = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R}).$$

We write the Fourier expansion of  $f \in \mathcal{B}_\kappa(q, \psi)$  around a singular cusp  $\mathfrak{a}$  with a scaling matrix  $\sigma_{\mathfrak{a}}$  as

$$i(\sigma_{\mathfrak{a}}, z)^{-k} f(\sigma_{\mathfrak{a}} z) = \left( \frac{(4\pi)^k}{\Gamma(k)} \right)^{1/2} \sum_{n=1}^{\infty} \rho_{f, \mathfrak{a}}(n) n^{(k-1)/2} e(nz).$$

Let  $f \in \mathcal{B}_\kappa(q, \psi)$  be an orthonormal basis of the space of Maaß cusp forms of weight  $\kappa$  with respect to  $\Gamma_0(q)$  and nebentypus  $\psi$ . As is customary, we assume that each  $f$  is either even or odd depending on whether  $T_{-1}f = f$  or  $T_{-1}f = -f$ . If we denote the corresponding spectral parameter by  $t_f$ , we have

$$j(\sigma_{\mathfrak{a}}, z)^{-\kappa} f(\sigma_{\mathfrak{a}} z) = \sqrt{\cosh(\pi t_f)} \sum_{n \neq 0} \rho_{f, \mathfrak{a}}(n) |n|^{-1/2} W_{\frac{n}{|n|}, \frac{\kappa}{2}, it_f}(4\pi|n|y) e(nx)$$

with  $W_{\alpha, \beta}$  the standard Whittaker function. For an Eisenstein series  $E(z, 1/2 + it, f)$ , we write

$$\begin{aligned} j(\sigma_{\mathfrak{a}}, z)^{-\kappa} E(\sigma_{\mathfrak{a}} z, 1/2 + it, f) &= c_{1, f, \mathfrak{a}}(t) y^{1/2+it} + c_{2, f, \mathfrak{a}}(t) y^{1/2-it} \\ &\quad + \sqrt{\cosh(\pi t)} \sum_{n \neq 0} \rho_{f, \mathfrak{a}}(n, t) |n|^{-1/2} W_{\frac{n}{|n|}, \frac{\kappa}{2}, it}(4\pi|n|y) e(nx). \end{aligned}$$

Now let  $\mathcal{B}_\kappa^*(q, \psi)$  be an orthonormal basis consisting of Hecke–Maaß newforms of level  $q$  and central character  $\psi \pmod{q_0}$  normalised so that  $\lambda_f(1) = 1$  and define similarly  $\mathcal{B}_\kappa^*(q, \psi)$ . By Atkin–Lehner–Li theory [2, 3], we have the following direct sum decomposition as in [14]:

$$\mathcal{B}_\kappa(q, \psi) = \bigsqcup_{q_1 q_2 = q} \bigsqcup_{f \in \mathcal{B}_\kappa^*(q_1, \psi)} \mathcal{S}_{q_2}^\square(f), \quad (2.12)$$

where an element of the orthonormal basis  $\mathcal{S}_{q_2}^\square(f) = \{f_{(d)}(z) : d \mid q_2\}$  is written as a linear combination of  $f(cz)$  with  $c \mid d$ . A description of this linear combination is given by Iwaniec–Luo–Sarnak [38] for the squarefree level and principal nebentypus. Blomer–Milićević [13] subsequently orthonormalised the collection of Maaß forms  $\{f(dz) : d \mid q_2\}$  for a newform  $f$  and an integer  $q_2$  which is not necessarily squarefree. Their technique focuses on the principal nebentypus, but the generalisation to nontrivial central characters is possible; see [30]. There is an Atkin–Lehner–Li theory for the continuous spectrum and a decomposition into spaces of oldforms analogous to (2.12) due to Young [68]. We make use of (2.12) to apply Hecke relations and the proportionality of the Fourier coefficients to the Hecke eigenvalues without restrictions to the variables.

In what follows, we exploit the shorthand  $\rho_{f, \infty}(n) = \rho_f(n)$  and  $\rho_{f, \infty}(n, t) = \rho_f(n, t)$ . We consider the Fourier coefficients  $\rho_{f, \mathfrak{a}}(n)$  in more detail and we stick to the case  $\mathfrak{a} \sim \infty$ . Let  $f \in \mathcal{B}_\kappa(q, \psi)$  be any Hecke eigenform and let  $\lambda_f(n)$  denote the corresponding eigenvalue for  $T_n$ , namely  $T_n f = \lambda_f(n) f$ . Then we will often use the Hecke multiplicativity relations for  $(mn, q) = 1$ :

$$\lambda_f(mn) = \sum_{d \mid (m, n)} \mu(d) \psi(d) \lambda_f(m/d) \lambda_f(n/d), \quad \lambda_f(m) \lambda_f(n) = \sum_{d \mid (m, n)} \psi(d) \lambda_f(mn/d^2). \quad (2.13)$$

We will see the structural beauty of (2.13) in §3.5 with the formula  $\lambda_f(n) = \psi(n) \overline{\lambda_f(n)}$  for  $(n, q) = 1$ . Notice that the relation (2.13) is valid for all  $m, n \geq 1$  if  $f$  is a newform. We invoke the bounds for the Hecke eigenvalues: if  $f$  belongs to  $\mathcal{B}_\kappa(q, \psi)$  or is an Eisenstein series, there follows that

$$|\lambda_f(n)| \leq \tau(n) \ll_\epsilon n^\epsilon$$

for any  $\epsilon > 0$ . For  $f \in \mathcal{B}_\kappa(q, \psi)$ , the general upper bound is available in the form

$$|\lambda_f(n)| \leq \tau(n) n^\vartheta \ll_\epsilon n^{\vartheta+\epsilon}. \quad (2.14)$$

There is a connection between the Fourier coefficients and the Hecke eigenvalues

$$\rho_f(n) = \rho_f(1) \lambda_f(n), \quad |\rho_f(1)|^2 = \frac{\varphi(q)}{2q^2} \frac{1}{L(1, \text{Ad}^2 f)} \quad (2.15)$$

for  $(n, q) = 1$  (we refer the reader to [22, (6.14)] for the former formula and to [5, (2.10)] for the latter formula). Here the adjoint square  $L$ -function is defined as

$$L(s, \text{Ad}^2 f) = \zeta(2s) \sum_{n=1}^{\infty} \frac{\lambda_f(n^2)}{n^s}. \quad (2.16)$$

The formula (2.15) is valid for all  $n \geq 1$  if  $f$  is a newform.

**Remark 2.10.** If  $\pi$  is an automorphic representation of  $\mathrm{GL}_n$  with a contragredient representation  $\tilde{\pi}$ , then it holds that  $L(s, \tilde{\pi}) = \overline{L(\bar{s}, \pi)}$ . In particular the Dirichlet coefficients of  $L(s, \pi)$  are real if and only if  $\pi \cong \tilde{\pi}$ . If  $n = 2$ , one has  $\tilde{\pi} \cong \pi \otimes \omega^{-1}$ , where  $\omega$  is the central character of  $\pi$ . Hence they are not necessarily real in our context.

The above formulæ for Hecke operators acting on holomorphic and Maaß cusp forms apply to the Eisenstein series via a bit of rectification. The Fourier coefficients  $\rho_f(n, t)$  are proportional to the Hecke eigenvalues  $\lambda_f(n, t)$  (see [22, (7.13)]): we have for  $n \neq 0$  that

$$\rho_f(n, t) = i^k \left( \frac{\pi}{vq} \right)^{1/2+it} \psi_1 \left( \frac{n}{|n|} \right) \overline{\tau(\psi_1)} \Gamma \left( \frac{1}{2} + \frac{n}{|n|} \frac{\kappa}{2} + it \right)^{-1} L(1 + 2it, \overline{\psi_1} \psi_2)^{-1} \frac{\lambda_f(n, t)}{\sqrt{\cosh(\pi t)}}.$$

One also derives

$$\rho_f(-n, t) = \frac{\Gamma(s + \kappa/2)}{\Gamma(s - \kappa/2)} \rho_f(n, t).$$

In §3, we consider primitive Dirichlet characters  $\psi^2$  for primitive nonquadratic characters  $\psi$  modulo a prime. In this case, one observes that  $\kappa(\psi^2) = 0$ , deriving specifically that

$$|\rho_f(n, t)|^2 = \frac{|\lambda_f(n, t)|^2}{q |L(1 + 2it, \overline{\psi_1} \psi_2)|^2},$$

where the formula  $\Gamma(s)\Gamma(1-s) = \pi/\sin(\pi s)$  was used.

We also need to define the twisted automorphic  $L$ -function

$$L(s, f \otimes \psi) := \sum_{n=1}^{\infty} \frac{\psi(n) \lambda_f(n)}{n^s}, \quad \Re(s) > 1, \quad (2.17)$$

which has a meromorphic continuation to the whole complex plane  $\mathbb{C}$ .

**2.5. A Version of the Kuznetsov Formula.** A spectral summation formula that we deploy here is an asymmetric trace formula relating sums of Kloosterman sums to Fourier coefficients of automorphic forms in the sense that the spectral side and the geometric side have a quite different shape. This technology can establish that there is considerable cancellation in sums of Kloosterman sums. Moreover, it provides a separation of variables of  $m$  and  $n$  in the sum  $\sum_c S(m, n; c)/c$  that is conducive to obtaining additional savings in summations over  $m$  and  $n$ .

For  $x > 0$ , we introduce the three integral kernels

$$\begin{aligned} \mathcal{H}^+(x, t) &:= \frac{2\pi i t^\kappa}{\sinh(\pi t)} (J_{2it}(x) - (-1)^\kappa J_{-2it}(x)), \\ \mathcal{H}^-(x, t) &:= \frac{2\pi i^{1-\kappa}}{\sinh(\pi t)} (I_{2it}(x) - I_{-2it}(x)) = 8i^{-\kappa} \cosh(\pi t) K_{2it}(x), \\ \mathcal{H}^{\mathrm{hol}}(x, k) &:= 4i^k J_{k-1}(x), \quad k \in 2\mathbb{N}, \end{aligned}$$

where we have borrowed the normalisation from [65]. We decided to suppress  $\psi$  from the notation and we adopt this kind of abuse of notation unless otherwise specified. For  $F \in C_c^\infty(\mathbb{R}_+)$ , we introduce

$$\mathcal{L}^\diamond F = \int_0^\infty \mathcal{H}^\diamond(\eta, \cdot) F(\eta) \frac{d\eta}{\eta}$$

for  $\diamond \in \{+, -, \mathrm{hol}\}$ . With the whole notation set up, we formulate the Kloosterman summation formula, which in our case reads as follows:

**Theorem 2.11** ([65, Theorem 3.2]). *Assume  $F \in C^3(0, \infty)$  satisfies  $x^j F^{(j)}(x) \ll \min(x, x^{-3/2})$  for  $0 \leq j \leq 3$ . Let  $\mathfrak{a}, \mathfrak{b}$  be singular cusps and  $\psi \pmod{q_0}$  be a Dirichlet character and let  $m, n \geq 1$ . Then we have that*

$$\mathcal{O}_{\mathfrak{ab}}^q(m, \pm n; F; \psi) = \mathcal{A}_{\mathfrak{ab}}^{\mathrm{MaaB}}(m, \pm n; \mathcal{L}^\pm F; \psi) + \mathcal{A}_{\mathfrak{ab}}^{\mathrm{Eis}}(m, \pm n; \mathcal{L}^\pm F; \psi) + \delta_{\pm=+} \mathcal{A}_{\mathfrak{ab}}^{\mathrm{hol}}(m, n; \mathcal{L}^{\mathrm{hol}} F; \psi).$$

Here we set

$$\mathcal{O}_{\mathfrak{ab}}^q(m, \pm n; F; \psi) := \sum_{c \in \mathcal{C}(\mathfrak{a}, \mathfrak{b})} \frac{S_{\mathfrak{ab}}(m, \pm n; c; \psi)}{c} F\left(\frac{4\pi\sqrt{mn}}{c}\right), \quad (2.18)$$

$$\begin{aligned}
\mathcal{A}_{\text{ab}}^{\text{MaaB}}(m, \pm n; F; \psi) &:= \sum_{f \in \mathcal{B}_\kappa(q, \psi)} \overline{\rho_{f\text{a}}(m)} \rho_{f\text{b}}(\pm n) F(t_f), \\
\mathcal{A}_{\text{ab}}^{\text{Eis}}(m, \pm n; F; \psi) &:= \sum_{\psi_1 \psi_2 = \psi} \sum_{f \in \mathcal{B}(\psi_1, \psi_2)} \int_{-\infty}^{\infty} \overline{\rho_{f\text{a}}(m, t)} \rho_{f\text{b}}(n, t) F(t) \frac{dt}{4\pi}, \\
\mathcal{A}_{\text{ab}}^{\text{hol}}(m, n; F; \psi) &:= \sum_{\substack{k > \kappa \\ k \equiv \kappa \pmod{2}}} \sum_{f \in \mathcal{B}_\kappa(q, \psi)} \overline{\rho_{f\text{a}}(m)} \rho_{f\text{b}}(n) F(k),
\end{aligned}$$

where the  $c$ -sum on the right-hand side of (2.18) ranges over all positive real numbers for which the Kloosterman sum  $S_{\text{ab}}(m, \pm n; c; \psi)$  is non-empty.

The above formulation mimics the Kuznetsov formula that Motohashi [53] used to show a spectral reciprocity for the smoothed fourth moment of the Riemann zeta function. The name of the Kloosterman summation formula stems from the fact that it expresses sums of Kloosterman sums weighted by a function  $F$  in terms of sums of automorphic forms weighted by transformed functions  $\mathcal{L}^\diamond F$  with  $\diamond \in \{+, -, \text{hol}\}$ . We emphasise that there is no delta term in the Kloosterman summation formula.

### 3. PROOF OF THEOREM 1.1

In this section, we prove Theorem 1.1 by applying the triad of Ramanujan–Voronoi–Kuznetsov to attack the shifted divisor sum (1.12) for a primitive Dirichlet character  $\chi$  modulo a prime  $q$ .

**3.1. Dissection Argument of Atkinson.** Recall (1.3) and Convention 1.10. Let  $\mathcal{R}_4^+$  (resp.  $\mathcal{R}_4^-$ ) be the subdomain of  $\mathbb{C}^4$  where all four parameters have real parts larger than (resp. less than) one; namely

$$\begin{aligned}
\mathcal{R}_4^+ &:= \{(s_1, s_2, s_3, s_4) \in \mathbb{C}^4 : \Re(s_i) > 1, 1 \leq i \leq 4\}, \\
\mathcal{R}_4^- &:= \{(s_1, s_2, s_3, s_4) \in \mathbb{C}^4 : \Re(s_i) < 1, 1 \leq i \leq 4\}.
\end{aligned}$$

For notational convenience, we set  $\mathcal{Z}_2 := \mathcal{Z}_2(\mathbf{s}; g; \chi) = \mathcal{Z}_2(s_1, s_2, s_3, s_4; g; \chi)$  for a vector  $\mathbf{s} = (s_1, s_2, s_3, s_4)$ . In the domain  $\mathcal{R}_4^+$ , we open the Dirichlet series to write  $\mathcal{Z}_2$  as

$$\int_{-\infty}^{\infty} \sum_{n_1, n_2, n_3, n_4=1}^{\infty} \frac{\chi(n_1 n_2) \overline{\chi}(n_3 n_4)}{n_1^{s_1} n_2^{s_2} n_3^{s_3} n_4^{s_4}} \left( \frac{n_1 n_2}{n_3 n_4} \right)^{-it} g(t) dt = \sum_{m, n=1}^{\infty} \frac{\chi(m) \overline{\chi}(n) \sigma_{s_1-s_2}(m) \sigma_{s_3-s_4}(n)}{m^{s_1} n^{s_3}} \hat{g}\left(\frac{1}{2\pi} \log \frac{m}{n}\right), \quad (3.1)$$

where  $\hat{g}$  is the Fourier transform<sup>4</sup>

$$\hat{g}(\xi) = \int_{-\infty}^{\infty} g(t) e(-\xi t) dt.$$

A shift of the contour in (1.3), either upwards or downwards, yields that  $\mathcal{Z}_2$  is meromorphic over the domain

$$\mathcal{B}_4 = \{(s_1, s_2, s_3, s_4) \in \mathbb{C}^4 : |s_1|, |s_2|, |s_3|, |s_4| < B\},$$

where  $B = cA$  and  $c$  is a small positive constant so that  $B$  is sufficiently large. Although it is possible to make  $B$  tend to infinity, we deal with the regime  $\mathcal{B}_4$  for technical convenience. Since we are assuming  $\chi$  to be primitive, the fourth moment  $\mathcal{Z}_2$  is regular in the vicinity of the point  $\mathbf{s} = (1/2, 1/2, 1/2, 1/2)$ . As an initial manipulation, we utilise Atkinson’s dissection argument, splitting the double sum (3.1) into three parts. We then have that

$$\mathcal{Z}_2 = \mathcal{D}_4 + \mathcal{O}\mathcal{D}_4^\dagger + \mathcal{O}\mathcal{D}_4^\ddagger \quad (3.2)$$

with  $\mathcal{D}_4$  being the diagonal contribution associated to  $m = n$ , whereas  $\mathcal{O}\mathcal{D}_4^\dagger$  (resp.  $\mathcal{O}\mathcal{D}_4^\ddagger$ ) being the off-diagonal contribution which involves terms with  $m > n$  (resp.  $m < n$ ). Note that the case of  $m < n$  is symmetrical to the case of  $m > n$  and we derive

$$\mathcal{O}\mathcal{D}_4^\ddagger(s_1, s_2, s_3, s_4; g; \chi) = \overline{\mathcal{O}\mathcal{D}_4^\dagger(\overline{s_3}, \overline{s_4}, \overline{s_1}, \overline{s_2}; g; \chi)}. \quad (3.3)$$

Splitting the summation in this way usually leads to sums of Kloosterman sums and ultimately to automorphic forms. For an explanation of why this is not as surprising as it initially seems, see the discussion in [53, §4.2].

<sup>4</sup>A slightly different definition of the Fourier transform was used in Motohashi’s monograph [53].

**Lemma 3.1.** *With the notation as above, we have that*

$$\mathcal{D}_4 = \hat{g}(0) \frac{\zeta^q(s_1 + s_3) \zeta^q(s_1 + s_4) \zeta^q(s_2 + s_3) \zeta^q(s_2 + s_4)}{\zeta^q(s_1 + s_2 + s_3 + s_4)},$$

where, for any  $L$ -function, the notation  $L^q$  signifies the removal of the Euler factor at primes dividing  $q$ .

*Proof.* We only need to calculate

$$\mathcal{D}_4 = \hat{g}(0) \sum_{(m,q)=1} \frac{\sigma_{s_1-s_2}(m) \sigma_{s_3-s_4}(m)}{m^{s_1+s_3}},$$

where the Dirichlet coefficient is a multiplicative function of  $m$ . Since the sum is over positive integers coprime to  $q$ , we can apply Ramanujan's identity (see [27, §17.8, Theorem 305])

$$\sum_{m=1}^{\infty} \frac{\sigma_{\alpha}(m) \sigma_{\beta}(m)}{m^s} = \frac{\zeta(s) \zeta(s-\alpha) \zeta(s-\beta) \zeta(s-\alpha-\beta)}{\zeta(2s-\alpha-\beta)},$$

omitting the Euler factors dividing  $q$ . This identity is valid as long as  $\Re(s_i) > 1/2$ .  $\square$

We stress that  $\hat{g}(0) = \int_{-\infty}^{\infty} g(t) dt$  is the mass of the weight function  $\hat{g}_T(0) = T\hat{g}(0)$ , where  $g_T(x) = g(x/T)$  with  $T > 0$ . Therefore, if  $g$  has support in the interval  $[1, 2]$  for example, then  $\text{supp}(g_T) \subseteq [T, 2T]$ . Because  $\mathcal{D}_4$  has a pole of order four at  $s = (1/2, 1/2, 1/2, 1/2)$ , we should have cancellation with a similar term coming from the off-diagonal terms. This is a typical feature in the study of moment problems.

The continuation of the off-diagonal terms requires the full machinery of the spectral theory of automorphic forms associated to congruence subgroups. We compute the second term in (3.2) because of the symmetry (3.3) and the contribution of the third term will be incorporated at the end.

**3.2. Evaluation of Off-Diagonal Terms.** For notational convenience, we call

$$G(y, s) = (1+y)^{-s} \hat{g}\left(\frac{1}{2\pi} \log(1+y)\right) = \int_{(1+\delta)}^{\infty} \hat{g}(\tau, s) y^{-\tau} \frac{d\tau}{2\pi i} \quad (3.4)$$

for a suitable  $\delta > 0$ , where  $\hat{g}$  is the Mellin transform

$$\hat{g}(\tau, s) = \int_0^{\infty} y^{\tau-1} (1+y)^{-s} \hat{g}\left(\frac{1}{2\pi} \log(1+y)\right) dy = \Gamma(\tau) \int_{-\infty}^{\infty} \frac{\Gamma(s-\tau+it)}{\Gamma(s+it)} g(t) dt, \quad (3.5)$$

provided  $\Re(s) > \Re(\tau) > 0$ . We have the following lemma:

**Lemma 3.2.** *As a function of two complex variables,  $\hat{g}(\tau, s)/\Gamma(\tau)$  is entire in  $\tau$  and  $s$ . Besides,  $\hat{g}(\tau, s)$  is of rapid decay in  $\tau$  as far as  $s$  and  $\Re(\tau)$  are bounded, say*

$$\hat{g}(\tau, s) \ll (1+|\tau|)^{-A}.$$

*Proof.* Shifting the contour  $\Im(t) = 0$  in (3.5) downwards appropriately, we prove the first assertion. The second claim is a consequence of an upward shift. See [53, Lemma 4.1] for an analogous statement.  $\square$

We embark on the calculation of  $\mathcal{O}\mathcal{D}_4^{\dagger}$  to solve the shifted convolution problem. Pulling the Dirichlet characters out of the summations, one sees that

$$\mathcal{O}\mathcal{D}_4^{\dagger} = \sum_{a,b \pmod{q}} \bar{\chi}(a) \chi(a+b) \sum_{\substack{n \equiv a \pmod{q} \\ m \equiv b \pmod{q}}} \frac{\sigma_{s_3-s_4}(n) \sigma_{s_1-s_2}(n+m)}{n^{s_1+s_3}} G(m/n, s_1). \quad (3.6)$$

We invoke the Ramanujan expansion (also known as the Ramanujan formula) for the divisor function which is a precise formulation of (1.13). For any integers  $q, n$  with  $(n, q) = 1$  and  $\Re(\xi) < 0$ , we have that

$$\sigma_{\xi}(n) = \zeta^q(1-\xi) \sum_{(\ell, q)=1} \ell^{\xi-1} r_{\ell}(n). \quad (3.7)$$

This follows readily from  $r_{\ell}(n) = \sum_{d|(\ell, n)} d\mu(\ell/d)$  and a reversal of summations. Therefore, we appeal to the fact that the divisor function  $\sigma_{\xi}(n)$  emerges in the Fourier expansion of the Eisenstein series for the modular

group  $\mathrm{SL}_2(\mathbb{Z})$ . On account of the appearance of  $\chi(a+b)$ , we have  $(n+m, q) = 1$ . By appealing to (3.7), the right-hand side of (3.6) equals

$$\zeta^q(1-s_1+s_2) \sum_{a,b \pmod{q}} \bar{\chi}(a)\chi(a+b) \sum_{(\ell,q)=1} \ell^{s_1-s_2-1} \sum_{\substack{n \equiv a \pmod{q} \\ m \equiv b \pmod{q}}} \frac{r_\ell(n+m)\sigma_{s_3-s_4}(n)}{n^{s_1+s_3}} G(m/n, s_1).$$

In order to reduce the innermost sums over  $m$  and  $n$  to the ordinary sums of the divisor function, we detect the congruences modulo  $q$  in an additive manner, obtaining

$$\begin{aligned} \mathcal{O}\mathcal{D}_4^\dagger &= \zeta^q(1-s_1+s_2)q^{-2} \sum_{a,b \pmod{q}} \bar{\chi}(a)\chi(a+b) \sum_{(\ell,q)=1} \ell^{s_1-s_2-1} \\ &\times \sum_{m,n=1}^{\infty} \sum_h^* \sum_{(\ell,q)=1}^* \sum_{r|q}^* \sum_{c \pmod{r}}^* \sum_{d \pmod{q}} e_r(c(n-a))e_q(d(m-b)) \frac{e_\ell(h(n+m))\sigma_{s_3-s_4}(n)}{n^{s_1+s_3}} G(m/n, s_1). \end{aligned}$$

The reason for the occurrence of the  $r$ -sum is to make subsequent formulæ simpler. If we pursue the change of variables  $(a, b) \mapsto (-a, -b)$ , then the form of the sum does not change according as whether  $\chi$  is even or odd, so we altogether use (3.4) to show

$$\begin{aligned} \mathcal{O}\mathcal{D}_4^\dagger &= \zeta^q(1-s_1+s_2)q^{-2} \sum_{a,b \pmod{q}} \sum_{r|q} \sum_{c \pmod{r}}^* \sum_{d \pmod{q}} \bar{\chi}(a)\chi(a+b)e_r(ac)e_q(bd) \sum_{(\ell,q)=1} \ell^{s_1-s_2-1} \\ &\times \sum_{h \pmod{\ell}}^* \int_{(1+\delta)}^* \hat{g}(\tau, s_1) \zeta^{(h/\ell+d/q)}(\tau) D_2(s_1+s_3-\tau, s_3-s_4; h/\ell+c/r) \frac{d\tau}{2\pi i}, \quad (3.8) \end{aligned}$$

where the right-hand side of (3.8) is absolutely convergent in a suitable region such as

$$\mathcal{R}_{4,\delta} = \{(s_1, s_2, s_3, s_4) \in \mathcal{R}_4^+ : \Re(s_1+s_3) > 2+2\delta, \Re(s_1)+1 < \Re(s_2), |s_3-s_4| < \delta\}$$

for an arbitrary but fixed  $\delta$ . We then want to shift the contour to the right. In anticipation of the future application of the functional equation of the Estermann zeta function (Theorem 2.6), we define

$$\mathcal{E}_4 = \{(s_1, s_2, s_3, s_4) \in \mathcal{B}_4 : \Re(s_1+s_3) < B/3, \Re(s_1+s_4) < B/3, \Re(s_1+s_2+s_3+s_4) > 3B\}. \quad (3.9)$$

We ascertain that  $\mathcal{R}_{4,\delta} \cap \mathcal{E}_4 \neq \emptyset$  if  $\delta$  is small. One confines all the variables  $(s_1, s_2, s_3, s_4)$  to be in the codomain  $\mathcal{E}_4$ , and then our shift of the path in (3.8) to  $\Re(\tau) = B$  results in

$$\begin{aligned} &\sum_{h \pmod{\ell}}^* \left\{ \int_{(1+\delta)}^* - \int_{(B)}^* \right\} \hat{g}(\tau, s_1) \zeta^{(h/\ell+d/q)}(\tau) D_2(s_1+s_3-\tau, s_3-s_4; h/\ell+c/r) \frac{d\tau}{2\pi i} \\ &= (\ell r)^{s_3-s_4-1} \zeta(1-s_3+s_4) \hat{g}(s_1+s_3-1, s_1) \sum_{n=1}^{\infty} r_\ell(n) e_q(nd) n^{1-s_1-s_3} \\ &\quad + (\ell r)^{s_4-s_3-1} \zeta(1+s_3-s_4) \hat{g}(s_1+s_4-1, s_1) \sum_{n=1}^{\infty} r_\ell(n) e_q(nd) n^{1-s_1-s_4}. \quad (3.10) \end{aligned}$$

Since the above polar terms do not contain the parameter  $c$  anywhere, we may express the sum over  $c$  in (3.8) as  $r_r(a)$ . This yields a further simplification of the polar terms. At this stage, we notice that

$$\sum_{a,b \pmod{q}} \bar{\chi}(a)\chi(a+b)e_q(ac+bd) = \tau(\chi, d) \overline{\tau(\chi, d-c)} = \bar{\chi}(d)\chi(d-c)q$$

when  $r = q$ . We will not exploit this evaluation, because the expanded  $a, b$ -sum is more amenable. Consequently, after some rearrangements, our formula is transformed into

$$\begin{aligned} \mathcal{O}\mathcal{D}_4^\dagger &= (\text{polar terms}) + q^{-2} \sum_{a,b,d \pmod{q}} \sum_{r|q} \zeta_r(1-s_1+s_2) \sum_{c \pmod{r}}^* \bar{\chi}(a)\chi(a+b)e_r(ac)e_q(bd) \\ &\times \sum_{(\ell,r)=1} \ell^{s_1-s_2-1} \sum_{h \pmod{\ell}}^* \int_{(B)}^* \hat{g}(\tau, s_1) \zeta^{(h/\ell+d/q)}(\tau) D_2(s_1+s_3-\tau, s_3-s_4; h/\ell+c/r) \frac{d\tau}{2\pi i}, \end{aligned}$$

which is now in shape to make use of the Poisson summation twice, namely the Voronoï summation once. This is because we have that  $\Re(s_1 + s_3 - \tau) < 0$  and  $\Re(s_1 + s_4 - \tau) < 0$ ; so we replace the Estermann zeta function  $D_2(s_1 + s_3 - \tau, s_3 - s_4; h/\ell + c/r)$  with the absolutely convergent series implied by (2.8).

**3.3. Application of Voronoï Summation.** This subsection aims at applying the functional equation to the  $m$ -sum and simplifying the resulting expression to a certain sum of Kloosterman sums of the form  $S_{\text{ab}}(m, \pm n; c; \psi)$ . To this end, the following elementary formula is necessary: if we set  $v = hr + c\ell$ , then

$$\bar{v} = \overline{hr + c\ell} \equiv \bar{h}r\bar{r}^2 + \bar{c}\bar{\ell}^2 \pmod{\ell r}$$

with  $h\bar{h} \equiv 1$ ,  $r\bar{r} \equiv 1 \pmod{\ell}$  and  $c\bar{c} \equiv 1$ ,  $\bar{\ell}\bar{\ell} \equiv 1 \pmod{r}$ . This congruence expression holds under the condition  $(\ell, r) = 1$ . The application of Theorem 2.6 hence leads to

$$\begin{aligned} \mathcal{O}\mathcal{D}_4^\dagger &= (\text{polar terms}) + 2q^{-2} \sum_{a,b,d \pmod{q}} \sum_{r|q} \frac{\zeta_r(1-s_1+s_2)}{r} \sum_{c \pmod{r}}^* \bar{\chi}(a)\chi(a+b)e_r(ac)e_q(bd) \\ &\times \sum_{(\ell,r)=1} \ell^{s_1-s_2-2} \sum_{h \pmod{\ell}}^* \int_{(B)} \left(\frac{2\pi}{\ell r}\right)^{2s_1+s_3+s_4-2\tau-2} \zeta^{(h/\ell+d/q)}(\tau)\Gamma(1+\tau-s_1-s_3) \\ &\times \Gamma(1+\tau-s_1-s_4) \left\{ D_2(1+\tau-s_1-s_3, s_4-s_3; \bar{h}\bar{r}^2/\ell + \bar{c}\bar{\ell}^2/r) \cos\left(\frac{\pi(s_3-s_4)}{2}\right) \right. \\ &\left. - D_2(1+\tau-s_1-s_3, s_4-s_3; -\bar{h}\bar{r}^2/\ell - \bar{c}\bar{\ell}^2/r) \cos\left(\pi\left(\frac{s_3+s_4}{2} + s_1 - \tau\right)\right) \right\} \hat{g}(\tau, s_1) \frac{d\tau}{2\pi i}. \end{aligned}$$

We expand the integrand as the Dirichlet series again so that the sums over  $h$  and  $c$  boil down to two Kloosterman sums, whence we are left with

$$\begin{aligned} \mathcal{O}\mathcal{D}_4^\dagger &= (\text{polar terms}) + 2(2\pi)^{s_1-s_2-1} q^{-2} \sum_{a,b,d \pmod{q}} \bar{\chi}(a)\chi(a+b)e_q(bd) \\ &\times \sum_{r|q} r^{s_2-s_1} \zeta_r(1-s_1+s_2) \sum_{m,n=1}^\infty m^{(1-s_1-s_2-s_3-s_4)/2} n^{(s_1-s_2+s_3-s_4-1)/2} \\ &\times e_q(md)\sigma_{s_4-s_3}(n) \sum_{\pm} \sum_{(\ell,r)=1} \frac{1}{\ell} S(m, \pm n\bar{r}^2; \ell) S(a, \pm n\bar{\ell}^2; r) \Psi_{\mathfrak{s}}^\pm\left(\frac{4\pi\sqrt{mn}}{\ell r}\right), \end{aligned} \quad (3.11)$$

where

$$\Psi_{\mathfrak{s}}^+(x) := \cos\left(\frac{\pi(s_3-s_4)}{2}\right) \int_{(B)} \left(\frac{x}{2}\right)^{s_1+s_2+s_3+s_4-1-2\tau} \Gamma(1+\tau-s_1-s_3)\Gamma(1+\tau-s_1-s_4) \hat{g}(\tau, s_1) \frac{d\tau}{2\pi i}, \quad (3.12)$$

$$\Psi_{\mathfrak{s}}^-(x) := - \int_{(B)} \left(\frac{x}{2}\right)^{s_1+s_2+s_3+s_4-1-2\tau} \cos\left(\pi\left(\frac{s_3+s_4}{2} + s_1 - \tau\right)\right) \Gamma(1+\tau-s_1-s_3)\Gamma(1+\tau-s_1-s_4) \hat{g}(\tau, s_1) \frac{d\tau}{2\pi i}. \quad (3.13)$$

The formula (3.11) affirmatively answers Motohashi's conjecture spelled out in his article [52] on the reciprocity law for the second moment of Dedekind zeta functions. One ascertains that the integrand in (3.12) has exponential decay in  $\tau$ , whereas the definition (3.13) requires the properties shown in Lemma 3.2. Convention 1.10 therefore applies to control  $\Psi_{\mathfrak{s}}^-(x)$  which involves the opposite-sign case of Kloosterman sums. In other words, the sum of  $\Psi_{\mathfrak{s}}^-(x)$  would dominate the bulk of the evaluation of  $\mathcal{O}\mathcal{D}_4^\dagger$ . We will ensure this phenomenon in Proposition 3.5.

We then manage to simplify the product of Kloosterman sums appearing in (3.11). It is desirable to collapse the second Kloosterman sum to perform the separation of variables. To this end, we write  $n = n_0 n'$  with  $n_0 \mid r^\infty$  and  $(n', r) = 1$ . We are in a position to use Lemma 2.3, getting

$$S(a, \pm n\bar{\ell}^2; r) = \frac{1}{\varphi(r)} \sum_{\psi \pmod{r}} \psi(\ell)^2 \bar{\psi}(\pm a n') \tau(\psi) \tau(\psi, n_0), \quad (3.14)$$

where we exploited the condition  $(a, r) = 1$  (due to the presence of the character  $\bar{\chi}(a)$ ) and  $\tau(\psi)$  is the Gauß sum associated to  $\psi$ . Dropping the primes for notational simplicity, we observe that  $\mathcal{O}\mathcal{D}_4^\dagger$  equals the polar terms plus

$$\begin{aligned}
& 2(2\pi)^{s_1-s_2-1}q^{-2} \sum_{a,b,d \pmod{q}} \bar{\chi}(a)\chi(a+b)e_q(bd) \sum_{r|q} \frac{r^{1-s_1+s_2}}{\varphi(r)} \zeta_r(1-s_1+s_2) \sum_{\pm} \sum_{\psi \pmod{r}} \\
& \times \sum_{m,n=1}^{\infty} \sum_{n_0|r^{\infty}} \frac{\bar{\psi}(\pm na)\tau(\psi)\tau(\psi, n_0)e_q(md)\sigma_{s_4-s_3}(n_0n)}{m^{(s_1+s_2+s_3+s_4-1)/2}(n_0n)^{(1-s_1+s_2-s_3+s_4)/2}} \sum_{(\ell,r)=1} \psi(\ell)^2 \frac{S(m, \pm n_0 n \bar{r}^2; \ell)}{\ell r} \Psi_{\pm}^{\pm} \left( \frac{4\pi\sqrt{mn_0n}}{\ell r} \right), \quad (3.15)
\end{aligned}$$

In order to utilise the Kloosterman summation formula in its suitable form, there are two roads to proceed. The first route is to represent  $S(m, \pm n_0 n \bar{r}^2; \ell)$  in terms of the Kloosterman sum associated to the  $(\infty, 0)$  cusp-pair and apply Theorem 2.11 to the  $\ell$ -sum. The second is to extract the principal and quadratic character from the  $\psi$ -sum in (3.14) and then replace  $S(m, \pm n_0 n \bar{r}^2; \ell)$  with the twisted Kloosterman sum, namely we make use of the formula of Blomer–Milićević [14]:

$$\sum_{(\ell,r)=1} \psi(\ell)^2 \frac{S(m, \pm n_0 n \bar{r}^2; \ell)}{\ell r} \Psi_{\pm}^{\pm} \left( \frac{4\pi\sqrt{mn_0n}}{\ell r} \right) = \frac{\psi(m)^2}{\tau(\psi^2)} \sum_{d|r} \mu(d) \sum_{dr|c} \frac{S_{\psi^2}(m, \pm n_0 n; c)}{c} \Psi_{\pm}^{\pm} \left( \frac{4\pi\sqrt{mn_0n}}{c} \right), \quad (3.16)$$

where  $S_{\psi}(m, n; c)$  is defined in (2.11) and  $\psi$  is a primitive nonquadratic character modulo  $r$  so that  $\psi^2$  is primitive. This is where the assumption of  $q$  being prime is used, since the square of a nonquadratic character is not always primitive in general. However, one can remove this assumption via the use of an extended form of (3.16). Notice that (3.16) is only available when  $(m, r) = 1$ . As shall be seen later, the character  $\bar{\psi}(m)$  arises when we calculate the exponential sums in (3.15). We have to take the second route for technical simplicity. The off-diagonal term  $\mathcal{O}\mathcal{D}_4^{\dagger}$  is thus equal to the polar terms plus

$$\begin{aligned}
& 2(2\pi)^{s_1-s_2-1}q^{-2} \sum_{a,b,c \pmod{q}} \bar{\chi}(a)\chi(a+b)e_q(bc) \sum_{r|q} \frac{r^{1-s_1+s_2}}{\varphi(r)} \zeta_r(1-s_1+s_2) \\
& \times \sum_{\pm} \left\{ \sum_{\psi \pmod{r}}^{\#} \frac{\bar{\psi}(\pm a)\tau(\psi)}{\tau(\psi^2)} \sum_{m,n=1}^{\infty} \sum_{n_0|r^{\infty}} \frac{\psi(m)^2 \bar{\psi}(n)\tau(\psi, n_0)e_q(mc)\sigma_{s_4-s_3}(n_0n)}{m^{(s_1+s_2+s_3+s_4-1)/2}(n_0n)^{(1-s_1+s_2-s_3+s_4)/2}} \right. \\
& \left. \times \sum_{d|r} \mu(d) \sum_{dr|c} \frac{S_{\psi^2}(m, \pm n_0 n; c)}{c} \Psi_{\pm}^{\pm} \left( \frac{4\pi\sqrt{mn_0n}}{c} \right) + \mathcal{R} \right\},
\end{aligned}$$

where  $\#$  on the  $\psi$ -sum means that the sum runs over all primitive nonquadratic characters modulo  $r$  and  $\mathcal{R}$  serves as a remainder term

$$\mathcal{R} = \sum_{\psi \pmod{r}}^b \sum_{m,n=1}^{\infty} \sum_{n_0|r^{\infty}} \frac{\psi(\pm na)\tau(\psi)\tau(\psi, n_0)e_q(mc)\sigma_{s_4-s_3}(n_0n)}{m^{(s_1+s_2+s_3+s_4-1)/2}(n_0n)^{(1-s_1+s_2-s_3+s_4)/2}} \sum_{(\ell,r)=1} \frac{S(m, \pm n_0 n \bar{r}^2; \ell)}{\ell r} \Psi_{\pm}^{\pm} \left( \frac{4\pi\sqrt{mn_0n}}{\ell r} \right).$$

Here  $b$  on the  $\psi$ -sum means that the sum runs over the principal and the quadratic character modulo  $r$ . Since we have  $\tau(\psi, n_0) = \bar{\psi}(n_0)\tau(\psi)$  in the case when  $\psi$  is a primitive nonquadratic character, we can take  $n_0 = 1$  owing to the condition  $n_0 | r^{\infty}$ . To reduce the character sums over  $a, b, c$ , we start with simplifying the  $c$ -sum, obtaining

$$\sum_{a,b,c \pmod{q}} \bar{\chi}(a)\bar{\psi}(a)\chi(a+b)e_q((b+m)c) = \psi(-1)q \sum_{a \pmod{q}} \bar{\chi}(a)\bar{\psi}(a)\chi(a+m).$$

When  $\chi\psi$  is primitive, one can set  $\psi_1 = \bar{\chi}\bar{\psi}$  and  $\psi_2 = \chi$  in Lemma 2.2. To be more general, we deduce

$$\sum_{a \pmod{q}} \bar{\chi}(a)\bar{\psi}(a)\chi(a+m) = \frac{1}{\tau(\bar{\chi})} \sum_{a,b \pmod{q}} \bar{\chi}(ab)\bar{\psi}(a)e\left(\frac{(a+m)b}{q}\right) = \frac{\tau(\bar{\chi}\bar{\psi})\tau(\psi, m)}{\tau(\bar{\chi})}.$$

Since the sum over primitive nonquadratic characters is empty when  $r = 1$ , we arrive at the following proposition:

**Proposition 3.3.** *For any primitive Dirichlet character  $\chi$  modulo a prime  $q$ , the function  $\mathcal{O}\mathcal{D}_4^{\dagger}$  can be meromorphically continued to the domain  $\mathcal{E}_4$ , and there we have*

$$\mathcal{O}\mathcal{D}_4^{\dagger} = \mathcal{P} + \sum_{\pm} \{\mathcal{J}_{\pm} + \mathcal{E}_{\pm}\}.$$

Here for the multiplicative function

$$A_q(s_1, s_2, s_3, s_4) := \frac{1}{q} \sum_{c|q} \mu(c) c^{2-s_1-s_2-s_3-s_4} \sum_{d|q/c} \mu\left(\frac{q}{cd}\right) d^{2-s_1-s_3} \sigma_{s_1-s_2+s_3-s_4-1}(d),$$

we set

$$\begin{aligned} \mathcal{P} &= A_q(s_1, s_2, s_3, s_4) \mathring{g}(s_1 + s_3 - 1, s_1) \frac{\zeta^q(1-s_1+s_2) \zeta^q(1-s_3+s_4) \zeta(s_2+s_4) \zeta(s_1+s_3-1)}{\zeta^q(s_2+s_4-s_1-s_3+2)} \\ &\quad + A_q(s_1, s_2, s_4, s_3) \mathring{g}(s_1 + s_4 - 1, s_1) \frac{\zeta^q(1-s_1+s_2) \zeta^q(1+s_3-s_4) \zeta(s_2+s_3) \zeta(s_1+s_4-1)}{\zeta^q(s_2+s_3-s_1-s_4+2)}, \\ \mathcal{J}_\pm &= \frac{2\zeta^q(1-s_1+s_2)}{\tau(\bar{\chi})\varphi(q)q} \left(\frac{2\pi}{q}\right)^{s_1-s_2-1} \sum_{\psi \pmod{q}}^\# \psi(\mp 1) \tau(\psi) \tau(\bar{\chi}\psi) J(\psi, \psi) \sum_{d|q} \mu(d) \\ &\quad \times \sum_{m,n=1}^\infty \psi(m) \bar{\psi}(n) m^{(1-s_1-s_2-s_3-s_4)/2} n^{(s_1-s_2+s_3-s_4-1)/2} \sigma_{s_4-s_3}(n) \mathcal{O}_{\infty\infty}^{dq}(m, \pm n; \Psi_S^\pm, \psi^2), \quad (3.17) \end{aligned}$$

$$\begin{aligned} \mathcal{E}_\pm &= \frac{2(2\pi)^{s_1-s_2-1}}{\tau(\bar{\chi})q} \sum_{r|q} \frac{r^{1-s_1+s_2}}{\varphi(r)} \zeta_r(1-s_1+s_2) \sum_{\psi \pmod{r}}^b \psi(\mp 1) \tau(\psi) \tau(\bar{\chi}\psi) \sum_{m,n=1}^\infty \sum_{n_0|r^\infty} \psi(n) \tau(\psi, m) \tau(\psi, n_0) \\ &\quad \times m^{(1-s_1-s_2-s_3-s_4)/2} (n_0 n)^{(s_1-s_2+s_3-s_4-1)/2} \sigma_{s_4-s_3}(n_0 n) \mathcal{O}_{\infty 0}^{r^2}(m, \pm n_0 n; \Psi_S^\pm, \psi_0), \quad (3.18) \end{aligned}$$

where  $\mathcal{O}_{ab}^q(m, \pm n; \Psi_S^\pm; \psi)$  is defined in Theorem 2.11.

**Remark 3.4.** The square of a primitive nonquadratic Dirichlet character of a composite modulus having at least two prime factors is not primitive. For instance, we choose a primitive quadratic character  $\chi_1$  (resp. nonquadratic character  $\chi_2$ ) modulo 3 (resp. modulo 5) and combine them to obtain a character  $\chi_1\chi_2$  modulo 15 as follows.

$n$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\psi_1(n)$	1	-1	0	1	-1	0	1	-1	0	1	-1	0	1	-1	0
$\psi_2(n)$	1	$i$	$-i$	-1	0	1	$i$	$-i$	-1	0	1	$i$	$-i$	-1	0
$\psi_1\psi_2(n)$	1	$-i$	0	-1	0	0	$i$	$i$	0	0	-1	0	$-i$	1	0

The multiplication of Dirichlet characters of different moduli gives a Dirichlet character modulo a least common multiple. We define  $\psi = \psi_1\psi_2$ , which is nonquadratic since it involves  $i$  and  $-i$  as values and is primitive since it cannot be induced from a Dirichlet character modulo 1, modulo 3 or modulo 5. However,  $\psi^2 = \psi_1^2\psi_2^2$  is induced from a quadratic character modulo 5. This is why we assume that  $q$  is prime in this article.

*Proof.* For the first assertion, we can easily check that both  $L$ -series in (3.17) and (3.18) converge absolutely and uniformly in our regime  $\mathcal{E}_4$ , giving the meromorphic continuation of  $\mathcal{O}\mathcal{D}_4^\dagger$  to  $\mathbb{C}^4$ . A remarkable point is that we only need the Weil bound (2.4) for Kloosterman sums. To establish the second assertion, it suffices to compute the polar terms. We decompose  $\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2$ , where  $\mathcal{P}_1$  (resp.  $\mathcal{P}_2$ ) stems from first term (resp. second term) on the right-hand side of (3.10). First of all, we remark that the polar contribution  $\mathcal{P}$  is expressed as

$$\zeta^q(1-s_1+s_2)q^{-2} \sum_{a,b \pmod{q}} \sum_{r|q} \sum_{c \pmod{r}}^* \sum_{d \pmod{q}} \bar{\chi}(a)\chi(a+b)e_r(ac)e_q(bd) \sum_{(\ell,q)=1} \ell^{s_1-s_2-1} \times (\text{RHS of (3.10)}).$$

In the following lines, we work with  $\mathcal{P}_1$ . The  $\ell$ -sum is reduced to

$$\sum_{(\ell,q)=1} \ell^{s_1-s_2+s_3-s_4-2} r_\ell(n) = \zeta^q(s_2+s_4-s_1-s_3+2)^{-1} \sigma_{s_1-s_2+s_3-s_4-1}(n).$$

We are in a position to manipulate the exponential sums. The  $d$ -sum gives rise to

$$\sum_{d \pmod{q}} e_q((b+n)d) = \begin{cases} q & \text{if } b \equiv -n \pmod{q}, \\ 0 & \text{otherwise,} \end{cases}$$

This renders

$$\sum_{a \pmod{q}} \bar{\chi}(a) \chi(a-n) r_r(a) = \frac{\mu(r)}{\tau(\bar{\chi})} \sum_{a,b \pmod{q}} \bar{\chi}(ab) e_q((a+n)b) = \mu(r) r_q(n).$$

In this way, the sums over  $a, b, c, d$  can be eliminated. In general, the  $r$ -sum turns into

$$\sum_{r|q} \mu(r) r^{s_3-s_4-1} = \prod_{p|q} \left(1 - \frac{1}{p^{1-s_3+s_4}}\right),$$

which yields  $\zeta^q(1-s_3+s_4)$  when combined with  $\zeta(1-s_3+s_4)$ . We hence obtain

$$\mathcal{P}_1 = \frac{\mathring{g}(s_1+s_3-1, s_1)}{q} \frac{\zeta^q(1-s_1+s_2) \zeta^q(1-s_3+s_4)}{\zeta^q(s_2+s_4-s_1-s_3+2)} \sum_{n=1}^{\infty} \frac{\sigma_{s_1-s_2+s_3-s_4-1}(n) r_q(n)}{n^{s_1+s_3-1}}.$$

Having mentioned this expression, we need to calculate the the  $n$ -sum via Lemma 2.5, getting

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\sigma_w(n)}{n^s} \sum_{d|(n,q)} \mu\left(\frac{q}{d}\right) d &= \sum_{d|q} \mu\left(\frac{q}{d}\right) d^{1-s} \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{c|(n,d)} \mu(c) \sigma_w(n/c) \sigma_w(d/c) c^w \\ &= \sum_{d|q} \mu\left(\frac{q}{d}\right) d^{1-s} \sum_{c|d} \mu(c) c^{w-s} \sigma_w(c) \sum_{n=1}^{\infty} \frac{\sigma_w(n)}{n^s} \\ &= \zeta(s) \zeta(s-w) \sum_{c|q} \mu(c) c^{1+w-2s} \sum_{d|q/c} \mu\left(\frac{q}{cd}\right) d^{1-s} \sigma_w(d), \end{aligned}$$

where  $s = s_1 + s_3 - 1$  and  $w = s_1 - s_2 + s_3 - s_4 - 1$ . Notice that the right-hand side can be written as a Dirichlet convolution of multiplicative functions. Recall that the Dirichlet convolution is defined as

$$(f * g)(m) = \sum_{d|m} f(d) g(m/d),$$

which is an associative and commutative binary operation on arithmetic functions. If  $f$  and  $g$  are multiplicative functions, so is  $f * g$ . Setting  $f(n) := \sigma_w(n) n^{1-s}$  and  $g(n) := \mu(n) n^{1+w-2s}$ , we have that

$$\sum_{n=1}^{\infty} \frac{\sigma_w(n) r_q(n)}{n^s} = (f * g * \mu)(q) \zeta(s) \zeta(s-w).$$

This finishes the proof of Proposition 3.3.  $\square$

**3.4. Application of Kuznetsov Formula.** We are in a position to apply the automorphic machinery in §2 to our sums of Kloosterman sums along with a careful analysis of integrals involving Bessel functions. Let us focus on the treatment of  $\mathcal{O}_{\infty \mathfrak{b}}^{dq}(m, \pm n; \Psi_{\mathfrak{s}}^{\pm}; \psi)$  with  $\mathfrak{b} = \infty, 0$  in an across-the-board manner and pursue the substitutions at the end to calculate  $\mathcal{J}_{\pm}$  and  $\mathcal{E}_{\pm}$ . The three-time differentiability and the decay condition at  $x = 0$  in Theorem 2.11 are fulfilled; moreover the decay condition at positive infinity follows by shifting the contours  $\Re(s) = B$  in (3.12) and (3.13) to the right and then applying Lemma 3.2. We forthwith derive the spectral decomposition

$$\mathcal{O}_{\infty \mathfrak{b}}^{dr}(m, \pm n; \Psi_{\mathfrak{s}}^{\pm}; \psi) = \mathcal{A}_{\infty \mathfrak{b}}^{\text{MaaB}}(m, \pm n; \mathcal{L}^{\pm} \Psi_{\mathfrak{s}}^{\pm}; \psi) + \mathcal{A}_{\infty \mathfrak{b}}^{\text{Eis}}(m, \pm n; \mathcal{L}^{\pm} \Psi_{\mathfrak{s}}^{\pm}; \psi) + \delta_{\pm=+} \mathcal{A}_{\infty \mathfrak{b}}^{\text{hol}}(m, n; \mathcal{L}^{\text{hol}} \Psi_{\mathfrak{s}}^+; \psi). \quad (3.19)$$

Complying with Motohashi's book [53], we initially observe how the case of the plus sign contributes. In order to reduce the integral transform  $\mathcal{L}^+$ , we consider, in the light of the definition (3.12), the double integral

$$\int_0^{\infty} J_{2it}(\eta) \int_{(B)} \Gamma(1+\tau-s_1-s_3) \Gamma(1+\tau-s_1-s_4) \left(\frac{\eta}{2}\right)^{s_1+s_2+s_3+s_4-2-2\tau} \mathring{g}(\tau, s_1) d\tau d\eta, \quad (3.20)$$

where  $\mathring{g}$  is the Mellin transform (3.5). This double integral is regular in the domain

$$\left\{ (s_1, s_2, s_3, s_4) \in \mathbb{C}^4 : \begin{array}{l} \Re(s_1+s_3) < 1+B, \Re(s_1+s_4) < 1+B, \\ 1+2B < \Re(s_1+s_2+s_3+s_4) < 3/2+2A \end{array} \right\}, \quad (3.21)$$

which contains  $\mathcal{E}_4$  defined in (3.9) and  $A > B$  is the same as in Convention 1.10. To see this fact, we divide (3.20) into two parts according as  $0 \leq \eta < 1$  and  $\eta \geq 1$  and note that

$$J_{2it}(\eta) \ll \begin{cases} 1 & \text{as } \eta \rightarrow 0, \\ \eta^{-1/2} & \text{as } \eta \rightarrow \infty, \end{cases}$$

where we implicitly assumed that  $t$  is real. The first part is clearly regular in the domain (3.21). To estimate the second part, we move the contour in the  $\tau$ -integral to  $\Re(\tau) = A$ . We want to consider the subdomain of (3.21) where we have  $1 + 2B < \Re(s_1 + s_2 + s_3 + s_4) < 3/2 + 2B$ . There the integral (3.20) is absolutely and uniformly convergent in view of the power series expansion of the  $J$ -Bessel function

$$J_\nu(y) = \sqrt{\frac{2}{\pi y}} \left\{ \cos\left(y - \frac{\pi}{2}\left(\nu + \frac{1}{2}\right)\right) \sum_{m=0}^M (-1)^m \binom{\nu - 1/2}{2m} \frac{\Gamma(\nu + 1/2 + 2m)}{\Gamma(\nu + 1/2)(2y)^{2m}} \right. \\ \left. - \sin\left(y - \frac{\pi}{2}\left(\nu + \frac{1}{2}\right)\right) \sum_{m=0}^M (-1)^m \binom{\nu - 1/2}{2m+1} \frac{\Gamma(\nu + 3/2 + 2m)}{\Gamma(\nu + 1/2)(2y)^{2m+1}} \right\} + O(y^{-\Re(\nu) - 3/2 - 2M}),$$

provided  $\Re(\nu) > -2M - 3/2$ . Hence, the double integral (3.20) equals

$$\int_{(B)} \frac{\Gamma\left(\frac{s_1 + s_2 + s_3 + s_4 - 1}{2} - \tau + it\right)}{\Gamma\left(\frac{3 - s_1 - s_2 - s_3 - s_4}{2} + \tau + it\right)} \Gamma(1 + \tau - s_1 - s_3) \Gamma(1 + \tau - s_1 - s_4) \mathring{g}(\tau, s_1) d\tau, \quad (3.22)$$

where we have utilised the formula

$$\int_0^\infty J_\nu(\eta) \left(\frac{\eta}{2}\right)^{-\mu} d\eta = \Gamma\left(\frac{1 + \nu - \mu}{2}\right) \Gamma\left(\frac{1 + \nu + \mu}{2}\right)^{-1}$$

valid for  $1/2 < \Re(\mu) < 1 + \Re(\nu)$ . Since the integral (3.22) is regular in (3.21), the analytic continuation implies that the double integral (3.20) equals (3.22) throughout the domain (3.21) or  $\mathcal{E}_4$ . At this stage, we notice that the following identities hold:

$$\frac{\Gamma(a - \tau + it)}{\Gamma(1 - a + \tau + it)} - \frac{\Gamma(a - \tau - it)}{\Gamma(1 - a + \tau - it)} = \frac{2}{\pi i} \sinh(\pi t) \cos(\pi(a - \tau)) \Gamma(a - \tau + it) \Gamma(a - \tau - it), \\ \frac{\Gamma(a - \tau + it)}{\Gamma(1 - a + \tau + it)} + \frac{\Gamma(a - \tau - it)}{\Gamma(1 - a + \tau - it)} = \frac{2}{\pi} \cosh(\pi t) \sin(\pi(a - \tau)) \Gamma(a - \tau + it) \Gamma(a - \tau - it).$$

which follows from the functional equation  $\Gamma(s)\Gamma(1-s) = \pi/\sin(\pi s)$ . Substituting  $a = (s_1 + s_2 + s_3 + s_4 - 1)/2$ , we have after some rearrangement that<sup>5</sup>

$$\mathcal{L}^+ \Psi_s^+(t) = (it \coth(\pi t))^\kappa \cos\left(\frac{\pi(s_3 - s_4)}{2}\right) \int_{(B)} \sin\left(\frac{\pi(s_1 + s_2 + s_3 + s_4 - \kappa - 2\tau)}{2}\right) \\ \times \Gamma\left(\frac{s_1 + s_2 + s_3 + s_4 - 1}{2} + it - \tau\right) \Gamma\left(\frac{s_1 + s_2 + s_3 + s_4 - 1}{2} - it - \tau\right) \\ \times \Gamma(1 + \tau - s_1 - s_3) \Gamma(1 + \tau - s_1 - s_4) \mathring{g}(\tau, s_1) \frac{d\tau}{\pi i}.$$

In a similar fashion, the holomorphic contribution turns into

$$\mathcal{L}^{\text{hol}} \Psi_s^+(k) = i^{k-1} \cos\left(\frac{\pi(s_3 - s_4)}{2}\right) \int_{(B)} \frac{\Gamma\left(\frac{k + s_1 + s_2 + s_3 + s_4 - 2}{2} - \tau\right)}{\Gamma\left(\frac{k + 2 - s_1 - s_2 - s_3 - s_4}{2} + \tau\right)} \Gamma(1 + \tau - s_1 - s_3) \Gamma(1 + \tau - s_1 - s_4) \mathring{g}(\tau, s_1) d\tau.$$

To proceed, we concentrate on the context of the minus sign. In this case, the bound shown in Lemma 3.2 plays a crucial rôle. By definition, the integral transform in question is written as

$$\mathcal{L}^- \Psi_s^-(t) = 8i^{-\kappa} \cosh(\pi t) \int_0^\infty \Psi_s^-(\eta) K_{2it}(\eta) \frac{d\eta}{\eta}.$$

<sup>5</sup>Motohashi [53, p.161] made a minor mistake in the argument of a gamma function.

Inserting (3.13), we obtain an absolutely convergent integral, for we have the growth condition

$$K_{2it}(\eta) \ll \begin{cases} |\log \eta| & \text{as } \eta \rightarrow 0, \\ \exp(-\eta) & \text{as } \eta \rightarrow \infty. \end{cases}$$

Substituting Heaviside's integral formula

$$\int_0^\infty K_{2v}(\eta) \left(\frac{\eta}{2}\right)^{2s-1} d\eta = \frac{\Gamma(s+v)\Gamma(s-v)}{2} \quad \text{for } \Re(s) > |\Re(v)|, \quad (3.23)$$

we are left with the expression

$$\begin{aligned} \mathcal{L}^- \Psi_s^-(t) &= -i^{-\kappa} \cosh(\pi t) \int_{(B)} \cos\left(\pi\left(s_1 + \frac{s_3 + s_4}{2} - \tau\right)\right) \Gamma\left(\frac{s_1 + s_2 + s_3 + s_4 - 1}{2} + it - \tau\right) \\ &\quad \times \Gamma\left(\frac{s_1 + s_2 + s_3 + s_4 - 1}{2} - it - \tau\right) \Gamma(1 + \tau - s_1 - s_3) \Gamma(1 + \tau - s_1 - s_4) \mathring{g}(\tau, s_1) \frac{d\tau}{\pi i}. \end{aligned}$$

In anticipation of future simplifications of  $\mathcal{L}^\pm \Psi_s^\pm(t)$ , we introduce the following three functions:

$$\begin{aligned} \Phi_s^+(\xi) &= -2i(2\pi)^{s_1-s_2-2} (i\xi \cot(\pi\xi))^\kappa \cos\left(\frac{\pi(s_3 - s_4)}{2}\right) \\ &\quad \times \int_{-i\infty}^{i\infty} \sin\left(\frac{\pi(s_1 + s_2 + s_3 + s_4 - \kappa - 2\tau)}{2}\right) \Gamma\left(\frac{s_1 + s_2 + s_3 + s_4 - 1}{2} + \xi - \tau\right) \\ &\quad \times \Gamma\left(\frac{s_1 + s_2 + s_3 + s_4 - 1}{2} - \xi - \tau\right) \Gamma(1 + \tau - s_1 - s_3) \Gamma(1 + \tau - s_1 - s_4) \mathring{g}(\tau, s_1) d\tau, \end{aligned} \quad (3.24a)$$

$$\begin{aligned} \Phi_s^-(\xi) &= 2i^{1-\kappa} (2\pi)^{s_1-s_2-2} \cos(\pi\xi) \int_{-i\infty}^{i\infty} \cos\left(\pi\left(s_1 + \frac{s_3 + s_4}{2} - \tau\right)\right) \Gamma\left(\frac{s_1 + s_2 + s_3 + s_4 - 1}{2} + \xi - \tau\right) \\ &\quad \times \Gamma\left(\frac{s_1 + s_2 + s_3 + s_4 - 1}{2} - \xi - \tau\right) \Gamma(1 + \tau - s_1 - s_3) \Gamma(1 + \tau - s_1 - s_4) \mathring{g}(\tau, s_1) d\tau, \end{aligned} \quad (3.24b)$$

$$\Xi_s(\xi) = \int_{-i\infty}^{i\infty} \frac{\Gamma\left(\xi + \frac{s_1 + s_2 + s_3 + s_4 - 1}{2} - \tau\right)}{\Gamma\left(\xi + \frac{3 - s_1 - s_2 - s_3 - s_4}{2} + \tau\right)} \Gamma(1 + \tau - s_1 - s_3) \Gamma(1 + \tau - s_1 - s_4) \mathring{g}(\tau, s_1) \frac{d\tau}{2\pi i}. \quad (3.24c)$$

Here the path in the formula for  $\Phi_s^+(\xi)$  is curved to ensure that the poles of the first two gamma factors in the integrand lie to the right of the path and those of other factors are on the left of the path; moreover the variables  $\xi, s_1, s_2, s_3, s_4$  are assumed to be such that the path can be drawn. The path in the definition of  $\Phi_s^-(\xi)$  is chosen in just the same way. On the other hand, the contour in  $\Xi$  separates the poles of  $\Gamma(\xi + (s_1 + s_2 + s_3 + s_4 - 1)/2 - \tau)$  and those of  $\Gamma(1 + \tau - s_1 - s_3) \Gamma(1 + \tau - s_1 - s_4) \mathring{g}(\tau, s_1)$  to the left and the right of the path, respectively.

**Proposition 3.5.** *With the notation above, we have that*

$$\begin{aligned} \Phi_s^+(\xi) &= -(-i\xi)^\kappa \frac{(2\pi)^{s_1-s_2}}{2 \sin(\pi\xi)} \cos\left(\frac{\pi(s_3 - s_4)}{2}\right) \left\{ \Xi_s(\xi) - (-1)^\kappa \Xi_s(-\xi) \right\}, \\ \Phi_s^-(\xi) &= i^{-\kappa} \frac{(2\pi)^{s_1-s_2}}{2 \sin(\pi\xi)} \left\{ \sin\left(\pi\left(\frac{s_2 - s_1}{2} + \xi\right)\right) \Xi_s(\xi) - \sin\left(\pi\left(\frac{s_2 - s_1}{2} - \xi\right)\right) \Xi_s(-\xi) \right\}, \end{aligned}$$

provided the left-hand sides are well-defined. We also have for real  $t$  and  $s = (s_1, s_2, s_3, s_4) \in \mathcal{E}_4$  that

$$\begin{aligned} \mathcal{L}^+ \Psi_s^+(t) &= (2\pi)^{1-s_1+s_2} \Phi_s^+(it), \\ \mathcal{L}^- \Psi_s^-(t) &= (2\pi)^{1-s_1+s_2} \Phi_s^-(it). \end{aligned}$$

For integral  $k \equiv \kappa \pmod{2}$  and  $(s_1, s_2, s_3, s_4) \in \mathcal{E}_4$ , we have that

$$\mathcal{L}^{\text{hol}} \Psi_s^+(k) = 2\pi i^k \cos\left(\frac{\pi(s_3 - s_4)}{2}\right) \Xi_s\left(\frac{k-1}{2}\right).$$

*Proof.* The claim follows in the same way as in [53, Lemma 4.4].  $\square$

**Remark 3.6.** From Proposition 3.3, we see that all Dirichlet characters we are interested in are of the form  $\psi^2$ . It therefore suffices to handle the case of  $\kappa(\psi^2) = 0$  in what follows. This should not trigger any problem, although we suppressed the dependence on  $\psi$  from the functions  $\Phi_s^\pm(\xi)$  and  $\Xi_s(\xi)$ .

**3.5. Cubic Moment of Central  $L$ -Values.** We devote this subsection to the derivation of the cubic moment side. The absolute convergence we would like to check is apparent as far as the double summation over the variables  $m, n$  is concerned, since we have the bound (2.14) on the Hecke eigenvalues and  $\mathbf{s} = (s_1, s_2, s_3, s_4) \in \mathcal{E}_4$ . Hence, the chief issue is reduced to bounding  $\mathcal{L}^\pm \Psi_s^\pm$  and then Proposition 3.5 reduces our task to the analysis of the function  $\Xi$ . If real  $t$  and positive integral  $k$  tend to infinity, we have uniformly for any compact subset of  $\mathcal{E}_4$  that

$$\Xi_s(it) \ll |t|^{-A}, \quad \Xi_s\left(\frac{k-1}{2}\right) \ll k^{-A}.$$

**3.5.1. The Computation of  $\mathcal{J}_\pm$ .** We are able to insert (3.19) into (3.17) and change the order of sums and integrals freely as long as we work inside  $\mathcal{E}_4$ . The replacement of  $\psi$  (in the discussion of §3.4) with  $\psi^2$  is necessary. We now evaluate Rankin–Selberg  $L$ -functions involving the divisor function defined as

$$\mathcal{J}(s, u; f; \psi) := \sum_{m=1}^{\infty} \frac{\bar{\psi}(m) \sigma_u(m) \lambda_f(m)}{m^s}, \quad \mathcal{J}(s, u; E; \psi) := \sum_{m=1}^{\infty} \frac{\bar{\psi}(m) \sigma_u(m) \lambda_f(m, t)}{m^s}.$$

One then establishes the following lemma:

**Lemma 3.7.** For  $f \in \mathcal{B}_\kappa(dq, \psi^2)$  and an Eisenstein series  $E(z, 1/2 + it, f)$  with  $f \in \mathcal{B}(\psi_1, \psi_2)$ , we have that

$$\begin{aligned} \mathcal{J}(s, u; f; \psi) &= \frac{L(s, f \otimes \bar{\psi}) L(s - u, f \otimes \bar{\psi})}{\zeta^q(2s - u)}, \\ \mathcal{J}(s, u; E; \psi) &= \frac{L(s + it, \bar{\psi}\psi_2) L(s - it, \bar{\psi}\psi_1) L(s - u + it, \bar{\psi}\psi_2) L(s - u - it, \bar{\psi}\psi_1)}{\zeta^q(2s - u)}. \end{aligned}$$

**Remark 3.8.** Obviously, the level  $dq$  in Lemma 3.7 can be replaced with any positive integer which is divided by the modulus  $q$  of the central character.

*Proof.* For the first assertion, we exploit the multiplicativity (2.13) of the Hecke eigenvalues, getting

$$\mathcal{J}(s, u; f; \psi) = \sum_{m, n=1}^{\infty} \frac{\bar{\psi}(mn) \lambda_f(mn)}{m^s n^{s-u}} = \sum_{m, n=1}^{\infty} \frac{\bar{\psi}(mn)}{m^s n^{s-u}} \sum_{d|(m, n)} \mu(d) \psi(d)^2 \lambda_f(m/d) \lambda_f(n/d).$$

By exchanging the order of summation and pulling the  $d$ -sum out, the right-hand side equals

$$\sum_{(d, q)=1} \frac{\mu(d)}{d^{2s-u}} \sum_{m=1}^{\infty} \frac{\bar{\psi}(m) \lambda_f(m)}{m^s} \sum_{n=1}^{\infty} \frac{\bar{\psi}(n) \lambda_f(n)}{n^{s-u}} = \frac{L(s, f \otimes \bar{\psi}) L(s - u, f \otimes \bar{\psi})}{\zeta^q(2s - u)}.$$

In order to establish the second assertion, we make use of Lemma 2.5 to derive

$$\begin{aligned} \mathcal{J}(s, u; E; \psi) &= \sum_{m=1}^{\infty} \frac{\bar{\psi}(m) \sigma_u(m)}{m^s} \sum_{a|m} \psi_1(a) \psi_2(m/a) (a^2/m)^{it} \\ &= \sum_{(c, q)=1} \frac{\mu(c)}{c^{2s-u}} \sum_{a=1}^{\infty} \frac{\bar{\psi}(a) \psi_1(a) \sigma_w(a)}{a^{s-it}} \sum_{m=1}^{\infty} \frac{\bar{\psi}(m) \psi_2(m) \sigma_u(m)}{m^{s+it}} \\ &= \frac{L(s + it, \bar{\psi}\psi_2) L(s - it, \bar{\psi}\psi_1) L(s - u + it, \bar{\psi}\psi_2) L(s - u - it, \bar{\psi}\psi_1)}{\zeta^q(2s - u)}. \end{aligned}$$

This concludes our proof of Lemma 3.7. □

We proceed to compute  $A_{\infty\infty}^{\text{Maab}}(m, \pm n; \mathcal{L}^\pm \Psi_s^\pm; \psi^2)$  by virtue of Lemma 3.7, obtaining

$$\sum_{n=1}^{\infty} \frac{\bar{\psi}(n) \sigma_{s_4-s_3}(n) \rho_f(\pm n)}{n^{(1-s_1+s_2-s_3+s_4)/2}} = \epsilon_f^{(1\mp 1)/2} \rho_f(1) \frac{L\left(\frac{1-s_1+s_2+s_3-s_4}{2}, f \otimes \bar{\psi}\right) L\left(\frac{1-s_1+s_2-s_3+s_4}{2}, f \otimes \bar{\psi}\right)}{\zeta^q(1-s_1+s_2)},$$

where  $f \in \mathcal{B}_\kappa(dq, \psi^2)$ . There is beauty in the quotient above, because we have  $\zeta^q(1-s_1+s_2)^{-1}$  which completely cancels out with the term  $\zeta^q(1-s_1+s_2)$  appearing in the expression for  $\mathcal{J}_\pm$  in Proposition 3.3. We can calculate the Dirichlet series for the  $m$ -sum in a similar manner, getting

$$\sum_{m=1}^{\infty} \frac{\psi(m)\overline{\rho_f(m)}}{m^{(s_1+s_2+s_3+s_4-1)/2}} = \overline{\rho_f(1)}L\left(\frac{s_1+s_2+s_3+s_4-1}{2}, f \otimes \overline{\psi}\right),$$

where  $f \in \mathcal{B}_\kappa(dq, \psi^2)$ . Note that we obtain the central  $L$ -value  $L(1/2, f \otimes \overline{\psi})^3$  if we take the limit  $(s_1, s_2, s_3, s_4) \mapsto (1/2, 1/2, 1/2, 1/2)$ . A similar procedure can be applied to the Eisenstein contribution to deduce the six Dirichlet  $L$ -functions in (1.4). The eventual form of the cubic moment looks like

$$\mathcal{J}_\pm = \mathcal{J}_\pm^{\text{MaaB}} + \mathcal{J}_\pm^{\text{Eis}} + \delta_{\pm=+} \mathcal{J}_+^{\text{hol}}$$

with

$$\begin{aligned} \mathcal{J}_\pm^{\text{MaaB}} &:= \frac{q^{s_2-s_1-2}}{\tau(\overline{\chi})} \sum_{d|q} \frac{\mu(d)}{d} \sum_{\psi \pmod{q}}^{\#} \mathcal{H}_\pm(\chi, \psi) \sum_{f \in \mathcal{B}_\kappa^*(dq, \psi^2)} \epsilon_f^{(1\mp 1)/2} \\ &\quad \times \frac{L\left(\frac{1-s_1+s_2+s_3-s_4}{2}, f \otimes \overline{\psi}\right) L\left(\frac{1-s_1+s_2-s_3+s_4}{2}, f \otimes \overline{\psi}\right) L\left(\frac{s_1+s_2+s_3+s_4-1}{2}, f \otimes \overline{\psi}\right)}{L(1, \text{Ad}^2 f)} \Phi_s^\pm(it_f), \\ \mathcal{J}_\pm^{\text{Eis}} &:= \frac{2q^{s_2-s_1-1}}{\tau(\overline{\chi})\varphi(q)} \sum_{d|q} \frac{\mu(d)}{d} \sum_{\psi \pmod{q}}^{\#} \mathcal{H}_\pm(\chi, \psi) \sum_{\psi_1 \psi_2 = \psi^2} \sum_{f \in \mathcal{B}(\psi_1, \psi_2)} \int_{-\infty}^{\infty} \frac{\mathcal{S}_f(t; s_1, s_2, s_3, s_4)}{|L(1+2it, \overline{\psi_1 \psi_2})|^2} \Phi_s^\pm(it) \frac{dt}{2\pi}, \\ \mathcal{J}_+^{\text{hol}} &:= \frac{q^{-2}}{\tau(\overline{\chi})} \left(\frac{2\pi}{q}\right)^{s_1-s_2} \cos\left(\frac{\pi(s_3-s_4)}{2}\right) \sum_{d|q} \frac{\mu(d)}{d} \sum_{\psi \pmod{q}}^{\#} \mathcal{H}_+(\chi, \psi) \sum_{\substack{k > \kappa \\ k \pmod{2}}} \sum_{f \in \mathcal{B}_k^*(dq, \psi^2)} i^k \\ &\quad \times \frac{L\left(\frac{1-s_1+s_2+s_3-s_4}{2}, f \otimes \overline{\psi}\right) L\left(\frac{1-s_1+s_2-s_3+s_4}{2}, f \otimes \overline{\psi}\right) L\left(\frac{s_1+s_2+s_3+s_4-1}{2}, f \otimes \overline{\psi}\right)}{L(1, \text{Ad}^2 f)} \Xi_s\left(\frac{k-1}{2}\right). \end{aligned}$$

Here we set  $\mathcal{H}_\pm(\chi, \psi) = \psi(\mp 1)\tau(\psi)\tau(\overline{\chi\psi})J(\psi, \psi)$  and the formula  $\varphi(dq) = d\varphi(q)$  was used.

**3.5.2. The Computation of  $\mathcal{E}_\pm$ .** We first deal with the contribution  $\mathcal{A}_{\infty 0}^{\text{MaaB}}(m, \pm n_0 n; \mathcal{L}^\pm \Psi_s^\pm; \psi_0)$ . One can rewrite the direct sum decomposition (2.12) in order to imitate the formulation of Blomer–Milićević [13], namely

$$\mathcal{B}_0(\Gamma_0(r^2)) = \bigsqcup_{r_1 r_2 = r^2} \bigsqcup_{f \in \mathcal{B}_0^*(\Gamma_0(r_1))} \bigsqcup_{d|r_2} \iota_d f \cdot \mathbb{C},$$

where the first two sums are orthogonal, but the last one is not orthogonal in general and needs to be orthogonalised by Gram–Schmidt. Here we also define  $(\iota_d f)(z) := f(dz)$ . By [13, Lemma 9], the set of functions

$$\left\{ f^{(g)} := \sum_{d|g} \xi_g(d) (\iota_d f) : g \mid r_2 \right\}$$

is an orthonormal basis of the space  $\bigsqcup_{d|r_2} \iota_d f \cdot \mathbb{C}$  where  $f \in \mathcal{B}_0^*(\Gamma_0(r_1))$ . Then the Fourier coefficients are

$$\rho_{f^{(g)} \mathfrak{a}}(n) = \sum_{d|g} \xi_g(d) d^{1/2} \rho_{f \mathfrak{a}}(n/d)$$

with the convention that  $\rho_{f \mathfrak{a}}(x) = 0$  for  $x \notin \mathbb{Z}$ . One calculates the  $n$ -sum in (3.18) as

$$\sum_{n=1}^{\infty} \frac{\psi(n)\sigma_{s_4-s_3}(n)\rho_{f \mathfrak{a}}(\pm n/d)}{n^{(1-s_1+s_2-s_3+s_4)/2}} = \delta_{d=1} \omega_f \epsilon_f^{(1\mp 1)/2} \rho_f(1) \frac{L\left(\frac{1-s_1+s_2+s_3-s_4}{2}, f \otimes \psi\right) L\left(\frac{1-s_1+s_2-s_3+s_4}{2}, f \otimes \psi\right)}{\zeta_r(1-s_1+s_2)},$$

where  $\omega_f$  stands for the root number for a newform  $f \in \mathcal{B}_0^*(\Gamma_0(r_1))$ . We denote by  $\psi$  a Dirichlet character modulo  $r$  induced by the primitive character  $\psi^*$  modulo  $r^*$ . We now regard the character modulo 1 to be

primitive. Since the central character is trivial and the Hecke eigenvalues are real (see Remark 2.10), we use Lemma 2.1 to derive

$$\sum_{m=1}^{\infty} \frac{\tau(\psi, m) \overline{\rho_f(m/d)}}{m^{(s_1+s_2+s_3+s_4-1)/2}} = \frac{\overline{\rho_f(1)} \tau(\psi^*) \psi^*(d)}{d^{(s_1+s_2+s_3+s_4-1)/2}} \Sigma_{r/r^*}(s_1, s_2, s_3, s_4; \psi) L\left(\frac{s_1+s_2+s_3+s_4-1}{2}, f \otimes \psi^*\right),$$

where  $\Sigma_q(s_1, s_2, s_3, s_4; \psi)$  is a multiplicative function defined as

$$\Sigma_q(s_1, s_2, s_3, s_4; \psi) = \sum_{c|q} \mu\left(\frac{q}{c}\right) \psi^*\left(\frac{q}{c}\right) c^{(3-s_1-s_2-s_3-s_4)/2} \sum_{e|c} \mu(e) \psi_0(e) \psi^*(e) e^{(1-s_1-s_2-s_3-s_4)/2} \lambda_f(c/e)$$

and  $\psi_0$  is the principal character modulo  $r_1$ . On the other hand, we obtain via routine calculations that

$$\sum_{n|r^\infty} \frac{\tau(\psi, n) \sigma_w(n) \lambda_f(n)}{n^s} = \tau(\psi^*) \sum_{c|r/r^*} c \psi^*\left(\frac{r}{cr^*}\right) \mu\left(\frac{r}{cr^*}\right) \sum_{n|r^\infty} \frac{\psi^*(n) \sigma_w(nc) \lambda_f(nc)}{(nc)^s}, \quad (3.25)$$

where  $s = (1 - s_1 + s_2 - s_3 + s_4)/2$  and  $w = s_4 - s_3$ . We thus observe that the left-hand side of (3.25) equals  $\tau(\psi^*)$  when  $\psi$  is the quadratic character modulo  $q$  or the character modulo 1. The problem occurs if  $\psi$  is the principal character modulo a prime  $q$ . In this case, a brute force computation gives

$$\sum_{c|q} \mu\left(\frac{q}{c}\right) c^{1-s} \sum_{n|q^\infty} \frac{\sigma_w(nc) \lambda_f(nc)}{n^s} = \varphi(q) \left(1 - \frac{\lambda_f(q)}{q^s}\right)^{-1} \left(1 - \frac{\lambda_f(q)}{q^{s-w}}\right)^{-1} - q.$$

We then conclude that

$$\sum_{n|r^\infty} \frac{\tau(\psi, n) \sigma_w(n) \lambda_f(n)}{n^{(1-s_1+s_2-s_3+s_4)/2}} = \tau(\psi^*) \Pi_{r/r^*}(s_1, s_2, s_3, s_4; \psi),$$

where  $\Pi_q(s_1, s_2, s_3, s_4; \psi)$  is a multiplicative function defined as

$$\Pi_q(s_1, s_2, s_3, s_4; \psi) = q \sum_{c|q} \mu\left(\frac{q}{c}\right) \prod_{p|c} \left(1 - \frac{\psi^*(p)}{p}\right) \left(1 - \frac{\psi^*(p) \lambda_f(p)}{p^{(1-s_1+s_2+s_3-s_4)/2}}\right)^{-1} \left(1 - \frac{\psi^*(p) \lambda_f(p)}{p^{(1-s_1+s_2-s_3+s_4)/2}}\right)^{-1}.$$

The same analysis works *mutatis mutandis* for the Eisenstein and holomorphic spectra, giving similar expressions (see [68] for an extension of Weisinger's newform theory and newform Eisenstein series). Hence we are left with

$$\mathcal{E}_\pm = \mathcal{E}_\pm^{\text{MaaB}} + \mathcal{E}_\pm^{\text{Eis}} + \delta_{\pm=+} \mathcal{E}_+^{\text{hol}},$$

where

$$\begin{aligned} \mathcal{E}_\pm^{\text{MaaB}} &:= \frac{1}{\tau(\bar{\chi})q} \sum_{r|q} \frac{r^{1-s_1+s_2}}{\varphi(r)} \sum_{\psi \pmod{r}} \mathcal{G}_\pm(\chi, \psi) \sum_{r_1 r_2 = r^2} \sum_{f \in \mathcal{B}_0^*(\Gamma_0(r_1))} \omega_f \epsilon_f^{(1\mp 1)/2} \Omega_{r_1, r_2}(s_1, s_2, s_3, s_4; f; \psi) \\ &\quad \times \frac{L\left(\frac{1-s_1+s_2+s_3-s_4}{2}, f \otimes \psi\right) L\left(\frac{1-s_1+s_2-s_3+s_4}{2}, f \otimes \psi\right) L\left(\frac{s_1+s_2+s_3+s_4-1}{2}, f \otimes \psi^*\right)}{L(1, \text{Ad}^2 f)} \Phi_s^\pm(it_f), \end{aligned}$$

$$\begin{aligned} \mathcal{E}_\pm^{\text{Eis}} &:= \frac{2}{\tau(\bar{\chi})q} \sum_{r|q} \frac{r^{1-s_1+s_2}}{\varphi(r)} \sum_{\psi \pmod{r}} \mathcal{G}_\pm(\chi, \psi) \sum_{r_1 r_2 = r^2} \frac{r_1^2}{\varphi(r_1)} \sum_{\psi_1 \psi_2 = \psi_0}^* \sum_{f \in \mathcal{B}(\psi_1, \psi_2)} \\ &\quad \times \Omega_{r_1, r_2}(s_1, s_2, s_3, s_4; f; \psi) \int_{-\infty}^{\infty} \overline{\rho_f(1, t)} \rho_{f_0}(1, t) \mathcal{S}_f(t; s_1, s_2, s_3, s_4) \Phi_s^\pm(it) \frac{dt}{2\pi}, \end{aligned}$$

$$\begin{aligned} \mathcal{E}_+^{\text{hol}} &:= \frac{(2\pi)^{s_1-s_2}}{\tau(\bar{\chi})q} \cos\left(\frac{\pi(s_3-s_4)}{2}\right) \sum_{r|q} \frac{r^{1-s_1+s_2}}{\varphi(r)} \sum_{\psi \pmod{r}} \mathcal{G}_+(\chi, \psi) \\ &\quad \times \sum_{r_1 r_2 = r^2} \sum_{\substack{k > \kappa \\ k \pmod{2}}} \sum_{f \in \mathcal{B}_k^*(\Gamma_0(r_1))} i^k \omega_f \Omega_{r_1, r_2}(s_1, s_2, s_3, s_4; f; \psi) \\ &\quad \times \frac{L\left(\frac{1-s_1+s_2+s_3-s_4}{2}, f \otimes \psi\right) L\left(\frac{1-s_1+s_2-s_3+s_4}{2}, f \otimes \psi\right) L\left(\frac{s_1+s_2+s_3+s_4-1}{2}, f \otimes \psi^*\right)}{L(1, \text{Ad}^2 f)} \Xi_s\left(\frac{k-1}{2}\right), \end{aligned}$$

where we define  $\mathcal{G}_\pm(\chi, \psi) = \psi(\mp 1)\tau(\psi)\tau(\psi^*)^2\tau(\overline{\chi}\psi)$  and

$$\Omega_{r_1, r_2}(s_1, s_2, s_3, s_4; f; \psi) = \frac{\varphi(r_1)}{r_1^2} \sum_{g|r_2} \xi_g(1) \sum_{d|g} \xi_g(d)\psi^*(d)d^{(2-s_1-s_2-s_3-s_4)/2} \\ \times \Sigma_{r/r^*}(s_1, s_2, s_3, s_4; \psi)\Pi_{r/r^*}(s_1, s_2, s_3, s_4; \psi).$$

These terms are altogether complicated, but they are of the same size as the main contributions  $\mathcal{J}_\pm$  in estimations. Thus one can ignore them in the process of applying Motohashi's formula to subconvexity problems, for instance.

**3.6. Endgame: Analytic Continuation.** The aim of this subsection is to prove that the spectral decomposition obtained in the preceding subsection can be continued to the whole complex plane  $\mathbb{C}^4$  and to conclude the proof of Theorem 1.1. The main step is identical to that of Motohashi's book [53] and we stress the following lemmata:

**Lemma 3.9** (Motohashi [53, Lemma 4.7]). *The function  $\Xi_s(\xi)$  is meromorphic in the domain*

$$\mathcal{B}_4^* = \{\xi : \Re(\xi) > -cA\} \times \mathcal{B}_4$$

for a fixed small constant  $c > 0$  and regular in  $\mathcal{B}_4^* \setminus \mathcal{N}$ , where

$$\mathcal{N} = \left\{ (\xi, s_1, s_2, s_3, s_4) : \begin{array}{l} \text{at least one of } \xi + \frac{s_1 + s_2 + s_3 + s_4 - 1}{2}, \xi + \frac{1 - s_1 + s_2 + s_3 - s_4}{2}, \\ \xi + \frac{1 - s_1 + s_2 - s_3 + s_4}{2} \text{ equals a non-positive integer} \end{array} \right\}.$$

Moreover, if  $|\xi|$  tends to infinity in any fixed vertical or horizontal strips while satisfying  $\Re(\xi) > -cA$ , then we have uniformly in  $\mathcal{B}_4$  that

$$\Xi_s(\xi) \ll |\xi|^{-cA}. \quad (3.26)$$

**Lemma 3.10** (Motohashi [53, Lemma 4.8]). *If  $(\xi, s_1, s_2, s_3, s_4)$  is such that the path in (3.24c) can be drawn in a vertical strip contained in the half plane  $\Re(\tau) > 0$ , then we have that*

$$\Xi_s(\xi) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\gamma)} \int_0^\infty y^{\xi+(s_1+s_2+s_3+s_4-3)/2} G(y, s_1) {}_2F_1(\alpha, \beta; \gamma; -y) dy,$$

where  $G(y, s)$  is defined in (3.4) and  ${}_2F_1(\alpha, \beta; \gamma; y)$  is the hypergeometric function with

$$\alpha = \xi + \frac{1 - s_1 + s_2 + s_3 - s_4}{2}, \quad \beta = \xi + \frac{1 - s_1 + s_2 - s_3 + s_4}{2}, \quad \gamma = 1 + 2\xi.$$

We suppose that  $(s_1, s_2, s_3, s_4) \in \mathcal{B}_4$  and examine the consequence of Lemma 3.9 for the contribution  $\mathcal{J}_+^{\text{Maa}\beta}$  from Maa $\beta$  forms. If  $t_f \geq 3B$ , then  $\Xi_s(\pm it_f)$  is regular and  $O(t_f^{-cA})$  uniformly in  $\mathcal{B}_4$  with an absolute constant  $c$ . Hence via Proposition 3.5, the factor  $\mathcal{L}^+\Psi_s^+(t_f)$  is regular and of exponential decay with respect to  $t_f$  uniformly in  $\mathcal{B}_4$ . This implies that  $\mathcal{J}_+^{\text{Maa}\beta}$  exists as a meromorphic function inside  $\mathcal{B}_4$ . The same observation holds for  $\mathcal{J}_\pm^{\text{hol}}$ . As to the function  $\mathcal{J}_-^{\text{Maa}\beta}$ , we essentially need (3.26). From Proposition 3.5, we have  $\mathcal{L}^-\Psi_s^-(t_f) = O(t_f^{-cA})$  uniformly in  $\mathcal{B}_4$  provided  $t_f \geq 3B$ . Hence  $\mathcal{J}_-^{\text{Maa}\beta}$  is meromorphic inside  $\mathcal{B}_4$ . This discussion works also for  $\mathcal{E}_\pm^{\text{Maa}\beta}$  and  $\mathcal{E}_\pm^{\text{hol}}$ .

Thus it remains to contemplate  $\mathcal{J}_\pm^{\text{Eis}}$  and  $\mathcal{E}_\pm^{\text{Eis}}$ . To this end, we assume first that  $\mathbf{s} = (s_1, s_2, s_3, s_4) \in \mathcal{E}_4$  and set

$$\mathcal{J}^{\text{Eis}}(\mathbf{s}; g; \chi) = \mathcal{J}_+^{\text{Eis}}(\mathbf{s}; g; \chi) + \mathcal{J}_-^{\text{Eis}}(\mathbf{s}; g; \chi).$$

Using Proposition 3.5, one derives

$$\Phi_s^+(it) = -\frac{(2\pi)^{s_1-s_2}}{2i \sinh(\pi t)} \cos\left(\frac{\pi(s_3-s_4)}{2}\right) \left\{ \Xi_s(it) - \Xi_s(-it) \right\}, \\ \Phi_s^-(it) = \frac{(2\pi)^{s_1-s_2}}{2i \sinh(\pi t)} \left\{ \sin\left(\pi\left(\frac{s_2-s_1}{2} + it\right)\right) \Xi_s(it) - \sin\left(\pi\left(\frac{s_2-s_1}{2} - it\right)\right) \Xi_s(-it) \right\}.$$

This yields a certain expression of  $\mathcal{J}^{\text{Eis}}(\mathbf{s}; g; \chi)$  in terms of  $\Xi_s(it)$ . We need not shift the contour in the same way as on pages 170–171 of Motohashi's book [53]. This is because  $\mathcal{S}_f(t; s_1, s_2, s_3, s_4)$  does not have poles ( $\psi^2 = \psi_1\psi_2$  means that  $\overline{\psi}\psi_1$  and  $\overline{\psi}\psi_2$  cannot be the principal character). As for the term  $\mathcal{E}_\pm^{\text{Eis}}$  coming from the contribution of the principal character and quadratic character, we ought to move the contour following Motohashi's method. Gathering these observations together, we have proven the following lemma:

**Lemma 3.11.** *The function  $\mathcal{O}\mathcal{D}_4^\dagger$  continues meromorphically to the domain  $\mathcal{B}_4$ , namely the decomposition (3.2) holds throughout  $\mathcal{B}_4$ .*

This concludes the proof of Theorem 1.1.

#### 4. IMPLICATIONS OF THEOREM 1.1

In this section, we aim at establishing a  $q$ -aspect variant of Iwaniec's short interval fourth moment bound and the twelfth moment bound for Dirichlet  $L$ -functions without an average over Dirichlet characters.

**4.1. Proof of Corollaries 1.3 and 1.4.** It suffices to specialise the test function in Theorem 1.1 as

$$g(t) = \frac{1}{\sqrt{\pi}H} \exp\left(-\left(\frac{t-T}{H}\right)^2\right)$$

with the parameter  $H$  at one's disposal. We assume that

$$T^{1/2} \leq H \leq T(\log T)^{-1}.$$

Motohashi [53, (5.1.40)–(5.1.42)] evaluated the corresponding integral transform  $\Xi$  and his expression can be used in our context. We then derive an asymptotic formula almost identical to [53, Theorem 5.1]. Upon estimating the spectral sum by absolute values, an oscillatory component in the summand vanishes and we obtain the bound

$$\int_T^{T+H} |L(1/2 + it, \chi)|^4 dt \ll_\epsilon H^{1+\epsilon} q^\epsilon + \frac{H^{3/2}}{T} q^{-2+\epsilon} \sum_{\psi \pmod{q}} \sum_{\substack{f \in \mathcal{B}_\kappa^*(q^2, \psi^2) \\ t_f \ll T/H}} L(1/2, f \otimes \bar{\psi})^3, \quad (4.1)$$

The truncation of the spectral sum is justified since the Taylor expansion of  $\exp(-(Ht_f/T)^2/4)$  in [53, (5.1.44)] implies that the rest of the spectral sum is much smaller than the main term in (4.1). Note that the central  $L$ -values in (4.1) are nonnegative and it therefore follows from the work of Petrow–Young [61, Theorems 1.2 & 1.3] that

$$H^{1+\epsilon} q^\epsilon + \frac{H^{3/2}}{T} q^{-1+\epsilon} \left(\frac{qT}{H}\right)^{2+\epsilon} \ll H^{1+\epsilon} q^\epsilon + \left(\frac{qT}{\sqrt{H}}\right)^{1+\epsilon}.$$

Optimising  $H = (qT)^{2/3}$  concludes the proof of Corollaries 1.3 and 1.4.  $\square$

**4.2. Proof of Corollaries 1.5 and 1.6.** This section is devoted to establishing of the twelfth moment bound. We follow the argument of [33] verbatim, obtaining the expression<sup>6</sup>

$$\begin{aligned} \sum_{r=1}^R \int_{t_r}^{t_r+T_0} |L(1/2 + it, \chi)|^4 dt &\ll_\epsilon T_0 q^{-2+\epsilon} \sum_{\psi \pmod{q}} \sum_{\substack{f \in \mathcal{B}_\kappa^*(q^2, \psi^2) \\ t_f \leq TT_0^{-1} \sqrt{\log T}}} t_f^{-1/2} \frac{L(1/2, f \otimes \bar{\psi})^3}{L(1, \text{Ad}^2 f)} \\ &\times \sum_{r=1}^R t_r^{-1/2} \exp\left(-\left(\frac{T_0 t_f}{2t_r}\right)^2\right) \sin\left(t_f \log \frac{t_f}{4e t_r}\right) + RT_0 (qT)^\epsilon. \end{aligned}$$

For technical reasons, it is convenient to remove the exponential factor in the last sum over  $r$  by partial summation. In doing this, we obtain an error term of  $O_\epsilon(RT^\epsilon T_0^{-1})$ . Then we must majorise

$$\sum := \sum_{\substack{K=2^{-m} T T_0^{-1} \sqrt{\log T} \\ m=1,2,\dots}} S_K,$$

where

$$S_K := \sum_{\psi \pmod{q}} \sum_{\substack{f \in \mathcal{B}_\kappa^*(q^2, \psi^2) \\ K \leq t_f \leq 2K}} t_f^{-1/2} \frac{L(1/2, f \otimes \bar{\psi})}{L(1, \text{Ad}^2 f)} \left| \sum_{r=1}^R t_r^{-1/2 - it_f} \right|.$$

<sup>6</sup>The variables  $t_r$  and spectral parameters  $t_f$  should not be confused.

The Cauchy–Schwarz inequality leads to

$$S_K \leq \left( \sum_{\psi \pmod{q}} \sum_{\substack{f \in \mathcal{B}_K^*(q^2, \psi^2) \\ K \leq t_f \leq 2K}} t_f^{-1} \frac{L(1/2, f \otimes \bar{\psi})^4}{L(1, \text{Ad}^2 f)} \right)^{1/2} \left( \sum_{\psi \pmod{q}} \sum_{\substack{f \in \mathcal{B}_K^*(q^2, \psi^2) \\ K \leq t_f \leq 2K}} \frac{L(1/2, f \otimes \bar{\psi})^2}{L(1, \text{Ad}^2 f)} \left| \sum_{r=1}^R t_r^{-1/2-it_f} \right|^2 \right)^{1/2}.$$

Applying the fourth moment bound of Petrow–Young [61, Theorem 7.6], the fourth moment is bounded as

$$\sum_{\psi \pmod{q}} \sum_{\substack{f \in \mathcal{B}_K^*(q^2, \psi^2) \\ K \leq t_f \leq 2K}} t_f^{-1} \frac{L(1/2, f \otimes \bar{\psi})^4}{L(1, \text{Ad}^2 f)} \ll_{\epsilon} q^{3+\epsilon} K^{1+\epsilon}.$$

The spectral large sieve inequality involving the square of twisted modular  $L$ -functions yields

$$\sum_{\psi \pmod{q}} \sum_{\substack{f \in \mathcal{B}_K^*(q^2, \psi^2) \\ K \leq t_f \leq 2K}} \frac{L(1/2, f \otimes \bar{\psi})^2}{L(1, \text{Ad}^2 f)} \left| \sum_{r=1}^R t_r^{-1/2-it_f} \right|^2 \ll_{\epsilon} q^{3+\epsilon} RKT^{\epsilon} T_0^{-1},$$

since  $K \ll TT_0^{-1} \sqrt{\log T}$ . It follows that

$$S_K \ll_{\epsilon} q^{3+\epsilon} R^{1/2} K^{1+\epsilon} T^{\epsilon} T_0^{-1/2}$$

and the summation over  $K$  eventually gives

$$\sum_{r=1}^R \int_{t_r}^{t_r+T_0} |L(1/2+it, \chi)|^4 dt \ll_{\epsilon} \left( RT_0 + qT \sqrt{\frac{R}{T_0}} \right) (qT)^{\epsilon}.$$

This establishes Corollary 1.5. Corollary 1.6 immediately follows from Jutila’s trick described in [40, §6].

## APPENDIX A. $\text{GL}_2$ SUMMATION FORMULÆ

In Appendix A, we summarise extended versions of the classical Voronoï summation formula.

**A.1. Voronoï–Oppenheim Summation.** We include our independent proof of the Voronoï summation formula for the generalised divisor function  $\sigma_{\lambda}(n)$ , although the statement *per se* is tantamount to the functional equation for the Estermann zeta function (Theorem 2.6) as remarked in §2.6.

**Theorem A.1.** *Let  $f \in \mathcal{C}_c^{\infty}(\mathbb{R}_+)$  be a smooth and compactly supported test function. Let  $h, \ell \geq 1$  be coprime integers. Then*

$$\begin{aligned} \sum_{n=1}^{\infty} f(n) \sigma_{\lambda}(n) e(nh/\ell) &= \ell^{\lambda-1} \zeta(1-\lambda) \int_0^{\infty} f(\xi) d\xi + \ell^{-\lambda-1} \zeta(1+\lambda) \int_0^{\infty} \xi^{\lambda} f(\xi) d\xi \\ &+ \frac{4}{\ell} \cos(\pi\lambda/2) \sum_{n=1}^{\infty} \frac{\sigma_{\lambda}(n)}{n^{\lambda/2}} e(n\bar{h}/\ell) \int_0^{\infty} \xi^{\lambda/2} K_{\lambda} \left( \frac{4\pi\sqrt{n\xi}}{\ell} \right) f(\xi) d\xi \\ &- \frac{\pi}{\ell \cos(\pi\lambda/2)} \sum_{n=1}^{\infty} \frac{\sigma_{\lambda}(n)}{n^{\lambda/2}} e(-n\bar{h}/\ell) \int_0^{\infty} \xi^{\lambda/2} \{Y_{\lambda} + Y_{-\lambda}\} \left( \frac{4\pi\sqrt{n\xi}}{\ell} \right) f(\xi) d\xi. \end{aligned} \quad (\text{A.1})$$

Theorem A.1 is relevant to Theorem 2.6 when we choose  $f(n) = n^{-s}$ . This also covers the classical Voronoï summation formula by specialising  $\lambda = 0$ .

**Remark A.2.** Historically, Oppenheim [59] extended Voronoï’s original identity to the divisor function  $\sigma_{\lambda}(n)$  whose associated Dirichlet series is  $\zeta(s)\zeta(s-\lambda)$ . Moreover,  $\sigma_{\lambda}(n)$  are the Fourier coefficients attached to an Eisenstein series on  $\text{GL}_2$  and  $\zeta(s)\zeta(s-\lambda)$  is an  $L$ -function attached to an Eisenstein series. Thus the formula (A.1) is often named the Voronoï–Oppenheim summation formula. Beineke and Bump [4] gave a new proof of it via the construction of a certain Eisenstein series and its Fourier expansion. We refer the reader to [50] for a survey on Voronoï-type formulæ.

*Proof of Theorem A.1.* Our proof hinges upon (2.8). One observes that  $D_2(s, \lambda; h/\ell)$  is of polynomial growth in  $|s|$  for  $\Re(s)$  bounded and  $|\Im(s)| \geq 1$ , say. After expressing the right-hand side of (A.1) in terms of an integral, we then shift the contour with crossing two poles. We therefore deduce

$$\begin{aligned} \sum_{n=1}^{\infty} f(n)\sigma_{\lambda}(n)e(nh/\ell) &= \int_{(\nu_1)} D_2(s, \lambda; h/\ell) f^*(s) \frac{ds}{2\pi i} \\ &= \ell^{\lambda-1} \zeta(1-\lambda) f^*(1) + \ell^{-\lambda-1} \zeta(1+\lambda) f^*(1+\lambda) + \int_{(-\nu_2)} D_2(s, \lambda; h/\ell) f^*(s) \frac{ds}{2\pi i}, \end{aligned} \quad (\text{A.2})$$

where  $\nu_1 > 1 + \max\{0, \Re(\lambda)\}$ ,  $\nu_2 > \max\{0, -\Re(\lambda)\}$ , and  $f^*$  is the Mellin transform of  $f$ . The above formula is available of course for the case of  $\lambda = 0$ . Exploiting the functional equation of the Estermann zeta function, the integral in the right-hand side of (A.2) becomes

$$\begin{aligned} &\frac{\cos(\pi\lambda/2)}{2\pi^2 i} \left(\frac{\ell}{2\pi}\right)^{1+\lambda} \sum_{n=1}^{\infty} \frac{\sigma_{-\lambda}(n)}{n} e(n\bar{h}/\ell) \int_{(-\nu_2)} \Gamma(1-s)\Gamma(1+\lambda-s) \left(\frac{2\pi\sqrt{n}}{\ell}\right)^{2s} f^*(s) ds \\ &- \frac{1}{2\pi^2 i} \left(\frac{\ell}{2\pi}\right)^{1+\lambda} \sum_{n=1}^{\infty} \frac{\sigma_{-\lambda}(n)}{n} e(-n\bar{h}/\ell) \int_{(-\nu_2)} \cos(\pi(s-\lambda/2))\Gamma(1-s)\Gamma(1+\lambda-s) \left(\frac{2\pi\sqrt{n}}{\ell}\right)^{2s} f^*(s) ds. \end{aligned} \quad (\text{A.3})$$

As an aside, the Stirling asymptotics ensures absolute convergence of these integrals. The first term in (A.3) can be computed as

$$\int_{(-\nu_2)} \Gamma(1-s)\Gamma(1+\lambda-s) \left(\frac{2\pi\sqrt{n}}{\ell}\right)^{2s} f^*(s) ds = 4\pi i \left(\frac{2\pi\sqrt{n}}{\ell}\right)^{2+\lambda} \int_0^{\infty} \xi^{\lambda/2} K_{\lambda} \left(\frac{4\pi\sqrt{n\xi}}{\ell}\right) f(\xi) d\xi,$$

where we have borrowed the inversion formula of (3.23) from [25, (17.43.18)]. On the other hand, if we assume  $\cos(\pi\lambda/2) \neq 0$ , an easy computation yields

$$\begin{aligned} &\int_{(-\nu_2)} \cos(\pi(s-\lambda/2))\Gamma(1-s)\Gamma(1+\lambda-s) \left(\frac{2\pi\sqrt{n}}{\ell}\right)^{2s} f^*(s) ds \\ &= \frac{(2\pi\sqrt{n}/\ell)^{\lambda}}{2\cos(\pi\lambda/2)} \int_{(-\nu_3)} \{ \cos(\pi(s+\lambda/2)) + \cos(\pi(s-\lambda/2)) \} \Gamma(1+\lambda/2-s)\Gamma(1-\lambda/2-s) \left(\frac{2\pi\sqrt{n}}{\ell}\right)^{2s} f^*(\lambda/2+s) ds \end{aligned}$$

with  $\nu_3 > |\Re(\lambda)|/2$ . As an ostensive assumption we shall impose a restriction on  $\lambda$  such that  $\sin(\pi\lambda) \neq 0$ . Then we find that the integral containing  $\cos(\pi(s+\lambda/2))$  turns into

$$\frac{\pi}{\sin(\pi\lambda)} \int_{(-\nu_3)} \left( \frac{\Gamma(1-s+\lambda/2)\cos(\pi\lambda)}{\Gamma(s+\lambda/2)} - \frac{\Gamma(1-s-\lambda/2)}{\Gamma(s-\lambda/2)} \right) \left(\frac{2\pi\sqrt{n}}{\ell}\right)^{2s} f^*(s+\lambda/2) ds.$$

Shifting the contour with the evaluation of residues, the above integral equals

$$\begin{aligned} &\frac{2\pi^2 i}{\sin(\pi\lambda)} \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(j+1)} (2\pi\sqrt{n}/\ell)^{2j+2} \left\{ \frac{\cos(\pi\lambda)(2\pi\sqrt{n}/\ell)^{\lambda}}{\Gamma(j+1+\lambda)} f^*(j+1+\lambda) - \frac{(2\pi\sqrt{n}/\ell)^{-\lambda}}{\Gamma(j+1-\lambda)} f^*(j+1) \right\} \\ &= \frac{2\pi i (2\pi\sqrt{n}/\ell)^2}{\sin(\pi\lambda)} \int_0^{\infty} \xi^{\lambda/2} \left\{ J_{\lambda} \left(\frac{4\pi\sqrt{n\xi}}{\ell}\right) \cos(\pi\lambda) - J_{-\lambda} \left(\frac{4\pi\sqrt{n\xi}}{\ell}\right) \right\} f(\xi) d\xi \\ &= 2\pi^2 i (2\pi\sqrt{n}/\ell)^2 \int_0^{\infty} \xi^{\lambda/2} Y_{\lambda} \left(\frac{4\pi\sqrt{n\xi}}{\ell}\right) f(\xi) d\xi. \end{aligned}$$

The same process applies to the integral involving  $\cos(\pi(s-\lambda/2))$ , hence we complete the proof of Theorem A.1 under  $\sin(\pi\lambda) \neq 0$ . Finally, all the integrals appearing in (A.1) lie in the regimes for which we have exponential decay with respect to  $n$  for  $\lambda$  bounded, due to the presence of Bessel functions. Thus the resulting expression extends to an entire function of  $\lambda$  via analytic continuation.  $\square$

**A.2. Twisted Estermann Zeta Function.** Fix two primitive Dirichlet characters  $\chi_1 \pmod{q_1}$  and  $\chi_2 \pmod{q_2}$ . We define the twisted Estermann zeta function by

$$D_2(s, \lambda; h/\ell; \chi_1, \chi_2) := \sum_{n=1}^{\infty} \frac{\sigma_\lambda(n; \chi_1, \chi_2)}{n^s} e(nh/\ell) \quad \text{with} \quad \sigma_\lambda(n; \chi_1, \chi_2) := \sum_{d|n} \chi_1(d) \chi_2\left(\frac{n}{d}\right) d^\lambda.$$

If we choose  $\chi_1 = \chi_2 = \chi$ , we observe  $\sigma_\lambda(n; \chi_1, \chi_2) = \chi(n) \sigma_\lambda(n)$ , which has arisen naturally in our approach in §3. We can then establish the twisted functional equation, which is exactly the incarnation of [53, Lemma 3.7] where the case  $q = 1$  was treated.

**Theorem A.3.** *The twisted Estermann zeta function satisfies the functional equation*

$$D_2(s, \lambda; h/\ell; \chi_1, \chi_2) = (2\pi)^{2s-\lambda-2} \ell_1^{\lambda-s} \ell_2^{-s} \Gamma(1-s) \Gamma(1+\lambda-s) \sum_{\pm} \kappa_{\chi_1, \chi_2}^{\pm}(1-s, \lambda) \hat{D}_2(1-s, \lambda; \pm h/\ell; \chi_1, \chi_2), \quad (\text{A.4})$$

where  $\ell_1 = [\ell, q_1]$  and  $\ell_2 = [\ell, q_2]$  are the least common multiples,

$$\kappa_{\chi_1, \chi_2}^+(s, \lambda) := \chi_1 \chi_2(-1) e^{\pi i(s+\lambda/2)} + e^{-\pi i(s+\lambda/2)}, \quad \kappa_{\chi_1, \chi_2}^-(s, \lambda) := \chi_1(-1) e^{\pi i\lambda/2} + \chi_2(-1) e^{-\pi i\lambda/2},$$

and

$$\begin{aligned} \hat{D}_2(s, \lambda; h/\ell; \chi_1, \chi_2) &:= \sum_{n=1}^{\infty} \frac{\hat{\sigma}_\lambda(n; h/\ell; \chi_1, \chi_2)}{n^s}, \\ \hat{\sigma}_\lambda(n; h/\ell; \chi_1, \chi_2) &:= \sum_{n_1 n_2 = n} \sum_{\substack{b_1 \pmod{\ell_1} \\ b_2 \pmod{\ell_2}}} \chi_1(b_1) \chi_2(b_2) n_1^{-\lambda} e\left(\frac{b_1 b_2 h}{\ell} + \frac{b_1 n_1 \ell_2 + b_2 n_2 \ell_1}{\ell_1 \ell_2}\right). \end{aligned}$$

*Proof.* The proof is akin to that of Theorem 2.6, but a little bit complicated due to the fact that the parameters  $\ell, q_1, q_2$  may have common factors. The left-hand side of (A.4) is equal to

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\sigma_\lambda(n; \chi_1, \chi_2)}{n^s} e(nh/\ell) &= \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \frac{\chi_1(n_1) \chi_2(n_2)}{n_1^{s-\lambda} n_2^s} e(n_1 n_2 h/\ell) \\ &= \sum_{\substack{b_1 \pmod{\ell_1} \\ b_2 \pmod{\ell_2}}} \chi_1(b_1) \chi_2(b_2) e(b_1 b_2 h/\ell) \sum_{\substack{n_1 \equiv b_1 \pmod{\ell_1} \\ n_2 \equiv b_2 \pmod{\ell_2}}} n_1^{\lambda-s} n_2^{-s} \\ &= \ell_1^{\lambda-s} \ell_2^{-s} \sum_{\substack{b_1 \pmod{\ell_1} \\ b_2 \pmod{\ell_2}}} \chi_1(b_1) \chi_2(b_2) e(b_1 b_2 h/\ell) \zeta(s-\lambda, b_1/\ell_1) \zeta(s, b_2/\ell_2) \end{aligned}$$

Using the functional equation for the Hurwitz zeta function (2.7), we now deduce

$$\begin{aligned} \zeta(s-\lambda, b_1/\ell_1) \zeta(s, b_2/\ell_2) &= -(2\pi)^{2s-\lambda-2} \Gamma(1-s) \Gamma(1+\lambda-s) \\ &\quad \times \left[ \exp(\pi i(s-\lambda/2)) \zeta^{(b_1/\ell_1)}(1+\lambda-s) \zeta^{(b_2/\ell_2)}(1-s) \right. \\ &\quad + \exp(-\pi i(s-\lambda/2)) \zeta^{(-b_1/\ell_1)}(1+\lambda-s) \zeta^{(-b_2/\ell_2)}(1-s) \\ &\quad - \exp(\pi i\lambda/2) \zeta^{(-b_1/\ell_1)}(1+\lambda-s) \zeta^{(b_2/\ell_2)}(1-s) \\ &\quad \left. - \exp(-\pi i\lambda/2) \zeta^{(b_1/\ell_1)}(1+\lambda-s) \zeta^{(-b_2/\ell_2)}(1-s) \right]. \end{aligned}$$

One may change  $-b_1 \rightarrow b_1$  or  $-b_2 \rightarrow b_2$  as necessary. Consequently, we have that

$$\hat{D}_2(1-s, \lambda; h/\ell; \chi_1, \chi_2) = \sum_{\substack{b_1 \pmod{\ell_1} \\ b_2 \pmod{\ell_2}}} \chi_1(b_1) \chi_2(b_2) e(b_1 b_2 h/\ell) \zeta^{(b_1/\ell_1)}(1+\lambda-s) \zeta^{(b_2/\ell_2)}(1-s). \quad (\text{A.5})$$

This concludes the proof.  $\square$

**Remark A.4.** We noticed after proving Theorem A.3 that there is the work of Topaçoğullari [65, Theorem 2.3], where the Voronoï–Oppenheim summation formula for the divisor function  $\tau_{\chi_1, \chi_2}(n) = \sigma_0(n; \chi_1, \chi_2)$  is derived. This result also follows as a corollary of Theorem A.3.

The left-hand side of (A.4) will be simplified further when  $\chi_1 = \chi_2 = \chi$  and  $q_1 = q_2 = q$  so that  $\ell_1 = \ell_2 = \ell q$ , where  $q$  is prime. In this example, the twisted Estermann zeta function boils down to

$$D_2(s, \lambda; h/\ell q; \chi) := \sum_{n=1}^{\infty} \frac{\chi(n)\sigma_{\lambda}(n)}{n^s} e(nh/\ell q) = (\ell q)^{\lambda-2s} \sum_{a,b \pmod{\ell q}} \chi(ab) e(abh/\ell q) \zeta(s, a/\ell q) \zeta(s-\lambda, b/\ell q)$$

with  $(h, \ell q) = 1$ . With the formula (A.5) in mind, the right-hand side of (A.4) equals

$$\begin{aligned} D_2(s, \lambda; h/\ell q; \chi) &= 2\chi(-1)(2\pi)^{2s-\lambda-2} (\ell q)^{\lambda-2s} \Gamma(1-s)\Gamma(1+\lambda-s) \\ &\quad \times \left( \cos(\pi\lambda/2) \sum_{a,b \pmod{\ell q}} \chi(ab) e(-abh/\ell q) \zeta^{(a/\ell q)}(1-s) \zeta^{(b/\ell q)}(1+\lambda-s) \right. \\ &\quad \left. - \chi(-1) \cos(\pi(s-\lambda/2)) \sum_{a,b \pmod{\ell q}} \chi(ab) e(abh/\ell q) \zeta^{(a/\ell q)}(1-s) \zeta^{(b/\ell q)}(1+\lambda-s) \right). \end{aligned}$$

At this stage, one may utilise the identity

$$\zeta^{(a/\ell q)}(1-s) \zeta^{(b/\ell q)}(1+\lambda-s) = (\ell q)^{2s-\lambda-2} \sum_{c,d \pmod{\ell q}} e\left(\frac{ac+bd}{\ell q}\right) \zeta(1-s, c/\ell q) \zeta(1+\lambda-s, d/\ell q).$$

This is indeed obvious from routine calculations: we decompose  $\zeta^{(a/\ell q)}(1-s)$  and  $\zeta^{(b/\ell q)}(1+\lambda-s)$  into sums over residue classes modulo  $\ell q$ . We have four sums over  $a, b, c, d \pmod{\ell q}$  and proceed to compute the  $a, b$ -sums to simplify the expression. One derives

$$\begin{aligned} \sum_{a,b \pmod{\ell q}} \chi(ab) e\left(\frac{ac+bd-abh}{\ell q}\right) &= \sum_b \chi(b) e\left(\frac{bd}{\ell q}\right) \sum_a \chi(a) e\left(\frac{a(c-bh)}{\ell q}\right) \\ &= \sum_b \chi(b) e\left(\frac{bd}{\ell q}\right) \sum_{a_1=1}^q \chi(a_1) \sum_{a_2=0}^{\ell-1} e\left(\frac{(a_1+a_2q)(c-bh)}{\ell q}\right) \quad (\text{A.6}) \\ &= \ell\tau(\chi) \sum_b \chi(b) \bar{\chi}\left(\frac{c-bh}{\ell}\right) e\left(\frac{bd}{\ell q}\right) \delta_{c \equiv bh \pmod{\ell}}. \end{aligned}$$

The congruence condition  $c \equiv bh \pmod{\ell}$  can be written as  $b \equiv c\bar{h} + v\ell \pmod{\ell q}$  with  $v \pmod{q}$ . Hence, the right-hand side of (A.6) turns into

$$\ell e\left(\frac{cd\bar{h}}{\ell q}\right) \tau(\chi) \sum_v \sum_{\substack{c \pmod{\ell q} \\ c \equiv bh \pmod{\ell}}} \chi(c\bar{h} + v\ell) \bar{\chi}(-hv) e\left(\frac{dv}{q}\right) = \ell e\left(\frac{cd\bar{h}}{\ell q}\right) \chi(-\bar{h}\ell) \tau(\chi) \sum_v^* \chi(1+cv) e\left(\frac{dh\bar{v}}{q}\right),$$

where we replaced  $v \mapsto \bar{h}\bar{v}$ . Since  $\chi$  is assumed to be a primitive character, we have that

$$\chi(1+cv) = \frac{1}{\tau(\bar{\chi})} \sum_z \bar{\chi}(z) e\left(\frac{(1+cv)z}{q}\right).$$

The  $v$ -sum renders the Kloosterman sum  $S(cz, d\bar{h}\bar{\ell}; q)$  and we altogether obtain

$$\sum_{a,b \pmod{\ell q}} \chi(ab) e\left(\frac{ac+bd-abh}{\ell q}\right) = \ell\chi(-1) \bar{\chi}(h)^2 e\left(\frac{cd\bar{h}}{\ell q}\right) \frac{\tau(\chi)}{\tau(\bar{\chi})} \sum_z \bar{\chi}(z) e\left(\frac{h\bar{\ell}z}{q}\right) S(cz, d; q).$$

where we have made the change of variables  $z \mapsto h\bar{\ell}z$ . We are in a position to expand the Kloosterman sum into multiplicative characters, getting

$$S(cz, d; q) = \frac{1}{\varphi(q)} \sum_{\psi \pmod{q}} \tau(\psi, c) \tau(\psi, d) \bar{\psi}(z).$$

This enables us to calculate the  $z$ -sum as follows:

$$\sum_{z \pmod{q}} \overline{\chi\psi}(z) e\left(\frac{h\ell z}{q}\right) = \chi\psi(h\ell)\tau(\overline{\chi\psi}).$$

The final form of the functional equation is very complicated and thus we are not able to write it down here. This is the main reason why we pulled out the Dirichlet characters in (3.6). Note that there is another method to show the functional equation of the Estermann zeta function; see [7].

## APPENDIX B. TWISTS OF MAASS NEWFORMS

We evaluate the size of the conductor of  $f \otimes \overline{\psi}$  via automorphic representation theory.

**Theorem B.1.** *Let  $\psi$  be a primitive character modulo an odd prime  $q = p$  with  $d \mid p$  and let  $f \in \mathcal{B}_\kappa^*(dp, \psi^2)$  be a Hecke–Maaß newform. Then the twist  $f \otimes \overline{\psi}$  has conductor  $p^2$  when  $\psi$  is nonquadratic. If  $\psi$  is quadratic,  $f \otimes \overline{\psi}$  has conductor dividing  $p^2$  and all three possibilities occur: if  $d = 1$ , then the conductor is  $p^2$  and if  $d = p$ , then*

$$\text{cond}(f \otimes \overline{\psi}) = \begin{cases} 1 & \text{if } f \text{ corresponds to a principal series representation,} \\ p & \text{if } f \text{ corresponds to a special representation,} \\ p^2 & \text{if } f \text{ corresponds to a twist-minimal representation.} \end{cases}$$

An automorphic representation  $f$  is said to be twist-minimal if it has minimal conductor among all their twists  $f \otimes \psi$  by Dirichlet characters, that is  $\text{cond}(f) \leq \text{cond}(f \otimes \psi)$ . Theorem B.1 asserts in particular that  $f \otimes \overline{\psi}$  has conductor  $p^2$  when  $f$  corresponds to a twist-minimal representation at the prime  $p$ . This follows from the work of Booker–Lee–Strömbergsson [15, Lemma 1.4]. The method to prove Theorem B.1 features the use of automorphic representation theory; however, one can also use the local Langlands correspondence and go to the Galois side. The only way the conductor of the tensor product could be smaller than  $p^2$  is if the associated Galois representation has a subrepresentation as an inertia representation isomorphic to  $\psi$ , in which case it has a quotient representation also isomorphic to  $\psi$  since its determinant is  $\psi^2$ . Hence  $f$  is not twist-minimal if  $\text{cond}(f \otimes \overline{\psi})$  does not equal  $p^2$  and there are no twist-minimal principal series representations in our case.

**B.1. Classification of Representations.** Let  $\pi_\nu$  be a generic irreducible admissible unitarisable representation of  $\text{GL}_2(\mathbb{Q}_\nu)$  with central character  $\omega_\nu$ . We remember that such representations can be classified as principal series representations, special representations, or supercuspidal representations. Standard references for the properties of these representations are [19, Chapter 4] and [24, Chapter 6], whilst the articles [31, 64] discuss the conductor exponents of these representations.

**B.1.1. Principal Series Representations of  $\text{GL}_2(\mathbb{Q}_\nu)$ .** A principal series representation  $\pi_\nu$  is unitarily induced from a representation of the Borel subgroup of  $\text{GL}_2(\mathbb{Q}_\nu)$ , and these representations are indexed by two characters

$$\omega_{p,1} = \beta_{p,1} \cdot |\cdot|_\nu^{s_1}, \quad \omega_{p,2} = \beta_{p,2} \cdot |\cdot|_\nu^{s_2}$$

of  $\mathbb{Q}_\nu^\times$ , where  $\beta_{p,1}$  and  $\beta_{p,2}$  are characters of  $\mathcal{O}_\nu^\times$  and  $s_1, s_2 \in \mathbb{C}$ . We write

$$\pi_\nu \cong \omega_{p,1} \boxplus \omega_{p,2}.$$

This representation is irreducible and unitarisable if and only if either  $s_1, s_2 \in i\mathbb{R}$  or  $s_1 + s_2 \in i\mathbb{R}$  with  $s_1 - s_2 \in (-1, 1)$  and  $\beta_{p,1} = \beta_{p,2}$ . The central character of  $\pi_\nu$  is

$$\omega_{\pi_\nu} = \omega_{p,1}\omega_{p,2} = \beta_{p,1}\beta_{p,2} \cdot |\cdot|_\nu^{s_1+s_2}.$$

The conductor exponent  $c(\pi_\nu)$  equals  $c(\omega_{p,1}) + c(\omega_{p,2})$ . If  $\omega'_\nu$  is a character of  $\mathbb{Q}_\nu^\times$ , the twist of  $\pi_\nu$  by  $\omega'_\nu$  is the principal series representation

$$\pi_\nu \otimes \omega'_\nu \cong \omega_{p,1}\omega'_\nu \boxplus \omega_{p,2}\omega'_\nu.$$

If  $\pi_\nu$  has trivial central character, then  $\omega_{p,2} = \omega_{p,1}^{-1}$ , so that  $\beta_{p,2} = \beta_{p,1}^{-1}$  and  $s_2 = -s_1$ , and  $c(\pi_\nu) = 2c(\omega_{p,1})$ .

**B.1.2. Special Representations of  $\mathrm{GL}_2(\mathbb{Q}_v)$ .** A special representation is a twist of the Steinberg representation: it is the unique irreducible subrepresentation

$$\pi_v \cong \omega_v \mathrm{St}_v$$

of codimension one of the reducible principal series representation  $\omega_v | \cdot |_v^{1/2} \boxplus \omega_v | \cdot |_v^{-1/2}$ , where  $\omega_v = \beta_v | \cdot |_v^s$  is a central character of  $\mathbb{Q}_v^\times$  with  $\beta_v$  a character of  $\mathcal{O}_v^\times$  and  $s \in \mathbb{C}$ . The central character of  $\pi_v$  is

$$\omega_{\pi_v} = \omega_v^2 = \beta_v^2 | \cdot |_v^{2s}.$$

The conductor exponent is

$$c(\pi_v) = \begin{cases} 1 & \text{if } c(\omega_v) = 0, \\ 2c(\omega_v) & \text{otherwise.} \end{cases}$$

If  $\omega'_v$  is a character of  $\mathbb{Q}_v^\times$ , the twist of  $\pi_v$  by  $\omega'_v$  is the special representation

$$\pi_v \otimes \omega'_v = \omega_v \omega'_v \mathrm{St}_v.$$

**B.1.3. Supercuspidal Representations of  $\mathrm{GL}_2(\mathbb{Q}_v)$ .** A supercuspidal representation is the compact induction to  $\mathrm{GL}_2(\mathbb{Q}_v)$  of a finite-dimensional representation  $\rho_{\pi_v}$  of a maximal open subgroup  $H$  of  $\mathrm{GL}_2(\mathbb{Q}_v)$  such that  $H$  is compact modulo the centre  $Z(\mathbb{Q}_v)$  of  $\mathrm{GL}_2(\mathbb{Q}_v)$ . Every maximal open subgroup of  $\mathrm{GL}_2(\mathbb{Q}_v)$  that is compact modulo the centre is conjugate to either  $Z(\mathbb{Q}_v) \mathrm{GL}_2(\mathcal{O}_v)$  or  $NK_0(\mathfrak{p}_v)$ , the normaliser of  $K_0(\mathfrak{p}_v)$  in  $\mathrm{GL}_2(\mathbb{Q}_v)$ , where  $\mathfrak{p}_v$  is the maximal ideal of the ring of integers  $\mathcal{O}_v$  of the local field  $\mathbb{Q}_v$ . A supercuspidal representation  $\pi_v$  is said to be of type I if  $H$  is conjugate to  $Z(\mathbb{Q}_v) \mathrm{GL}_2(\mathcal{O}_v)$  and of type II if  $H$  is conjugate to  $NK_0(\mathfrak{p}_v)$ . Supercuspidal representations always have conductor exponent  $c(\pi_v)$  at least 2. The twist  $\pi_v \otimes \omega'_v$  of  $\pi_v$  by a character  $\omega'_v$  of  $\mathbb{Q}_v^\times$  is also a supercuspidal representation.

**B.2. Proof of Theorem B.1.** A representation  $\pi$  of  $\mathrm{GL}_2(F)$  (where  $F$  denotes a nonarchimedean local field) of conductor 1, or equivalently of conductor exponent  $c(\pi) = 0$ , must be a spherical principal series representation  $\pi = \omega_1 \boxplus \omega_2$  with both characters  $\omega_1, \omega_2$  of  $F^\times$  unramified, or of conductor exponent  $c(\omega_1) = c(\omega_2) = 0$ . The central character is  $\omega_\pi = \omega_1 \omega_2$ , which has conductor exponent  $c(\omega_1 \omega_2) = 0$ ; if this central character is trivial, then  $\omega_2 = \omega_1^{-1}$ . It follows that  $\omega_1(x) = |x|_v^{it_v}$  and  $\omega_2(x) = |x|_v^{-it_v}$ , where  $t_v$  is such that the Hecke eigenvalue is  $\lambda_\pi(p) = p^{it_v} + p^{-it_v}$ . We must think of this as being the local component of  $f$  at a prime not dividing the level.

For conductor  $p$ , or equivalently conductor exponent  $c(\pi) = 1$ , there are two possibilities. The first is that  $\pi$  is a special representation  $\omega \mathrm{St}$ , where  $\omega$  is an unramified character (so that  $c(\omega) = 0$ ). The central character is  $\omega_\pi = \omega^2$ , which has conductor exponent  $c(\omega^2) = 0$ . We must think of this as the local component of  $f$  at a prime that exactly divides the level and such that the prime does not divide the conductor of the nebentypus of  $f$ . The other possibility is that  $\pi = \omega_1 \boxplus \omega_2$  is a nonspherical principal series representations, where either  $c(\omega_1) = 1$  and  $c(\omega_2) = 0$  or vice versa. The central character is  $\omega_\pi = \omega_1 \omega_2$ , which has conductor exponent  $c(\omega_1 \omega_2) = 1$ . We must think of this as the local component of  $f$  at a prime that exactly divides the level and such that the prime also divides the conductor of the nebentypus.

For conductor  $p^r$  with  $r \geq 2$ , or equivalently conductor exponent  $c(\pi) = r$ , there are three possibilities. The first is that  $\pi = \omega_1 \boxplus \omega_2$  is a nonspherical principal series representation for which  $r = c(\pi) = c(\omega_1) + c(\omega_2) = r$ . The central character is  $\omega_\pi = \omega_1 \omega_2$ , which has conductor exponent at most  $r$  (and it could be anything in between 0 and  $r$  depending on  $\omega_1$  and  $\omega_2$ ). The second possibility is that  $\pi = \omega \mathrm{St}$  is a special representation, where  $\omega$  is a ramified character, so that  $c(\omega) \geq 1$  and  $r = c(\pi) = 2c(\omega)$  (so that  $r$  is necessarily even). The central character is  $\omega_\pi = \omega^2$ , which has conductor exponent at most  $r$ . The third possibility is that  $\pi$  is supercuspidal.

The twist by  $\chi$  of a principal series representation  $\pi = \omega_1 \boxplus \omega_2$  with central character  $\omega_\pi = \omega_1 \omega_2$  is  $\pi \otimes \chi = \omega_1 \chi \boxplus \omega_2 \chi$  with central character  $\omega_{\pi \otimes \chi} = \omega_1 \omega_2 \chi^2$ . The conductor exponent of  $\pi \otimes \chi$  becomes  $c(\pi \otimes \chi) = c(\omega_1 \chi) + c(\omega_2 \chi)$  and the conductor exponent of  $\omega_{\pi \otimes \chi}$  becomes  $c(\omega_1 \omega_2 \chi^2)$ . The twist of a special representation  $\pi = \omega \mathrm{St}$  with central character  $\omega_\pi = \omega^2$  is  $\pi \otimes \chi = \omega \chi \mathrm{St}$  with central character  $\omega_{\pi \otimes \chi} = \omega^2 \chi^2$ . The conductor exponent of  $\pi \otimes \chi$  becomes  $c(\pi \otimes \chi) = \max\{1, 2c(\omega \chi)\}$  and the conductor exponent of  $\omega_{\pi \otimes \chi}$  becomes  $c(\omega^2 \chi^2)$ .

*Proof of Theorem B.1.* The proof uses representation theory. Given a newform  $f$  of level  $dq$  and central character  $\psi^2$ , there exists a cuspidal automorphic representation  $\pi = \pi_\infty \otimes \bigotimes_v \pi_v$  of  $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ , unique up to isomorphism, with conductor exponent  $c(\pi_v)$  at each prime  $p$  satisfying  $p^{c(\pi_v)} \parallel dq$ , whose central character  $\omega_\pi$  is the idèlic lift

of the primitive character  $\chi$  that induces  $\psi^2$  and whose global newvector  $\xi^\circ$  is the adèlic lift of  $f$ . If  $p \nmid dq$ , then  $\pi_v$  is spherical and the local conductor exponents ought to satisfy  $c(\pi_v) = 0$ . All the action happens when  $p \mid dq$  for which  $\pi_v$  is ramified. The central character  $\omega_{\pi_v}$  is the local component  $\omega_v^2$  of the idèlic lift of  $\chi$  and the local conductor exponent  $c(\omega_v)$  is such that  $p^{c(\omega_v)} \parallel q$ . If  $\omega_v$  is nonquadratic,  $c(\omega_{\pi_v}) = c(\omega_v^2) = c(\omega_v)$ ; however, if  $\omega_v$  is quadratic, then  $c(\omega_{\pi_v}) = 0$ , where  $c(\omega_v) = 1$ . The accurate quantity of the conductor exponent  $c(\pi_v \otimes \chi_v)$  is calculable for given  $\pi_v$ , where  $\chi_v$  is the character corresponding to  $\bar{\psi}$ . There follows that  $\text{cond}(f \otimes \bar{\psi}) \mid q^2$  with equality if  $f$  is twist-minimal or  $d = 1$  via Lemma 1.4 of [15]. Since  $\pi_v$  is a generic irreducible admissible representation of  $\text{GL}_2(\mathbb{Q}_v)$ , one must go through a case-by-case analysis. With the above classification in mind, the local conductor exponents can be calculated, although we cannot deal with all the cases at one fell swoop.

- (1) The case where  $\psi$  is quadratic:
  - (a) If  $d = 1$ , then  $\pi_v \cong \omega_v \text{St}_v$  for some unramified character  $\omega_v$  of  $\mathbb{Q}_v^\times$  by the above classification. The local component of the twist  $f \otimes \bar{\psi}$  is  $\pi_v \otimes \chi_v \cong \omega_v \chi_v \text{St}_v$ . The conductor exponent of  $\pi_v \otimes \chi_v$  thus equals 2 which means that  $f \otimes \bar{\psi}$  has conductor  $p^2$ .
  - (b) If  $d = q$  is a prime  $p$ , then  $\pi_v$  has conductor exponent  $c(\pi_v) = 2$ . Since  $\omega_{\pi_v}$  is trivial,  $\pi_v$  can either be principal series, special, or supercuspidal. Humphries [31] discusses this problem in some detail and classifies everything precisely. The crucial point is that if  $\pi_v = \omega_{p,\text{quad}} \boxplus \omega_{p,\text{quad}}$ , where  $\omega_{p,\text{quad}}$  denotes the quadratic character of conductor exponent 1, then  $\pi_v \otimes \chi_v$  is a spherical principal series representation of conductor exponent 0, so that  $f \otimes \bar{\psi}$  has conductor 1. If  $\pi_v = \omega_v \text{St}_v$  is special, its twist is a special representation of conductor exponent 1, so that  $f \otimes \bar{\psi}$  has conductor  $p$ . In all other cases,  $\pi_v$  is twist-minimal and  $\pi_v \otimes \chi_v$  has conductor exponent 2, so that  $f \otimes \bar{\psi}$  has conductor  $p^2$ .
- (2) The case where  $\psi$  is nonquadratic:
  - (a) We first handle the case  $d = 1$ . Since  $\psi^2$  is primitive modulo  $p$ , the corresponding character  $\omega_{\pi_v}$  has conductor exponent  $c(\omega_{\pi_v}) = 1$ . Via the above classification,  $\pi_v = \omega_{p,1} \boxplus \omega_{p,2}$ , where  $c(\omega_{p,1}) = 1$ ,  $c(\omega_{p,2}) = 0$  and  $\omega_{\pi_v} = \omega_{p,1} \omega_{p,2}$ . The twist is  $\pi_v \otimes \chi_v = \omega_{p,1} \chi_v \boxplus \omega_{p,2} \chi_v$ . Since  $\omega_{p,1} \omega_{p,2} = \chi_v^{-2}$  as  $f$  has nebentypus  $\psi^2$ , we must have that  $\omega_{p,1} \chi_v = \omega_{p,2}^{-1} \chi_v^{-1}$ , which has conductor exponent 1, as does  $\omega_{p,2} \chi_v$ . Hence the conductor exponent of  $\pi_v \otimes \chi_v$  is  $c(\omega_{p,1} \chi_v) + c(\omega_{p,2} \chi_v) = 2$ , which means that  $f \otimes \bar{\psi}$  has conductor  $p^2$ .
  - (b) If  $d = p$ , then  $\pi_v$  has conductor exponent  $c(\pi_v) = 2$ . All three possibilities for  $\pi_v$  are open and one can check in each case that twisting leaves its conductor exponent unchanged.

This establishes Theorem B.1. □

#### ACKNOWLEDGEMENTS

The author expresses his sincere gratitude to Eren Mehmet Kiral for supervising this work and proposing ideas when difficulties have arisen in his calculations. He thanks Valentin Blomer, Jack Buttcane, Paul Nelson and Peter Sarnak for having an interest in some earlier versions of this article. Appendix B is attributed to Peter Humphries, and the author acknowledges his diverse comments that improved the readability of the article. Special thanks are owed to Alexandre Perozim de Faveri, Yoichi Motohashi, Maksym Radziwiłł and Will Sawin for their assistance.

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