

COMPATIBILITY OF SPECIAL VALUE CONJECTURES WITH THE FUNCTIONAL EQUATION OF ZETA FUNCTIONS

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ABSTRACT. We prove that the special value conjecture for the Zeta function $\zeta(\mathcal{X}, s)$ of a proper, regular arithmetic scheme \mathcal{X} that we formulated in [8][Conj. 5.12] is compatible with the functional equation of $\zeta(\mathcal{X}, s)$ provided that the factor $C(\mathcal{X}, n)$ we were not able to compute in loc. cit. has the simple explicit form suggested in [9].

1. INTRODUCTION

This article is a continuation of our previous article [8] in which we formulated a conjecture describing the leading Taylor coefficient of the Zeta function $\zeta(\mathcal{X}, s)$ of a proper regular arithmetic scheme \mathcal{X} at integer arguments $n \in \mathbb{Z}$. Our conjecture involved a rather inexplicit correction factor $C(\mathcal{X}, n) \in \mathbb{Q}^\times$ which we could only compute for $\mathcal{X} = \text{Spec}(\mathcal{O}_F)$ where F is a number field all of whose completions F_v are absolutely abelian. Based on this example a general formula for $C(\mathcal{X}, n)$ in terms of factorials

$$(1) \quad C(\mathcal{X}, n)^{-1} = \prod_{i \leq n-1; j} (n-1-i)!^{(-1)^{i+j} \dim_{\mathbb{Q}} H^j(\mathcal{X}_{\mathbb{Q}}, \Omega^i)}$$

was suggested in [9] and proven for $n = 1$ in [10]. For $n \leq 0$ one has $C(\mathcal{X}, n) = 1$ by definition. In this article we prove that our special value conjecture is compatible with the functional equation of the Zeta function if $C(\mathcal{X}, n)$ is given by (1), see Thm. 1.4 below in this introduction. We regard this as convincing evidence that (1) is indeed the right factor, even though we cannot yet prove that (1) equals the original definition of $C(\mathcal{X}, n)$ in terms of p -adic Hodge theory. The original definition was made in such a way that our conjecture is compatible with the Tamagawa number conjecture of Bloch, Kato, Fontaine and Perrin-Riou [6], [11]. By restating our conjecture with the explicit factor (1) we are in effect making a special value conjecture that is independent of p -adic Hodge theory and that is compatible with the functional equation of $\zeta(\mathcal{X}, s)$. Note that compatibility with the functional equation of motivic L-functions is not in general known for the Tamagawa number conjecture. Even for Tate motives over number fields F it is only known if all F_v are absolutely abelian.

A precise statement of our special value conjecture is given in Conjecture 1.1 in subsection 1.1 of this introduction but not needed for this article. Neither do any of the results in sections 2-4 of this article depend on unproven conjectures. Our main result Thm. 1.4 follows from an unconditional theorem, Thm. 1.2, which is stated in subsection 1.2 of this introduction.

1.1. Cyclic homology and $C(\mathcal{X}, n)$. We make a few remarks which do not pertain to the content of this paper but put formula (1) in perspective. One might wonder about a more conceptual origin of the numerical factor $C(\mathcal{X}, n)$ and in order to say something about this

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question we first recall our special value conjecture in more detail. Let \mathcal{X} be a regular scheme of dimension d , proper over $\mathrm{Spec}(\mathbb{Z})$. Associated to \mathcal{X} and $n \in \mathbb{Z}$ is an invertible \mathbb{Z} -module ("fundamental line")

$$\Delta(\mathcal{X}/\mathbb{Z}, n) := \det_{\mathbb{Z}} R\Gamma_{W,c}(\mathcal{X}, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \det_{\mathbb{Z}} R\Gamma(\mathcal{X}_{Zar}, L\Omega_{\mathcal{X}/\mathbb{Z}}^{\leq n})$$

where $L\Omega_{\mathcal{X}/\mathbb{Z}}^{\leq n}$ is the derived de Rham complex [14] modulo the n -th step in the Hodge filtration and $R\Gamma_{W,c}(\mathcal{X}, \mathbb{Z}(n))$ is a perfect complex of abelian groups whose definition is dependent on assumptions (finite generation of étale motivic cohomology, Artin-Verdier duality for torsion motivic cohomology) denoted by $\mathbf{L}(\overline{\mathcal{X}}_{et}, n)$, $\mathbf{L}(\overline{\mathcal{X}}_{et}, d-n)$, $\mathbf{AV}(\overline{\mathcal{X}}_{et}, n)$ in [8]. Also assuming the Beilinson conjectures in the form of conjecture $\mathbf{B}(\mathcal{X}, n)$ of [8] one can construct a natural trivialization

$$(2) \quad \lambda_{\infty} : \mathbb{R} \xrightarrow{\sim} \Delta(\mathcal{X}/\mathbb{Z}, n) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Let $\zeta(\mathcal{X}, s)$ be the Zeta function of \mathcal{X} and $\zeta^*(\mathcal{X}, n) \in \mathbb{R}^{\times}$ its leading Taylor coefficient at $s = n$.

Conjecture 1.1.

$$\lambda_{\infty}(\zeta^*(\mathcal{X}, n)^{-1} \cdot C(\mathcal{X}, n) \cdot \mathbb{Z}) = \Delta(\mathcal{X}/\mathbb{Z}, n)$$

This conjecture determines the real number $\zeta^*(\mathcal{X}, n) \in \mathbb{R}$ up to sign. As noted above, the factor $C(\mathcal{X}, n)$ was originally defined in [8][Conj. 5.12] as the product over its p -primary parts where the definition of each p -part [8][Def. 5.6] involves p -adic Hodge theory and yet another assumption $\mathbf{D}_p(\mathcal{X}, n)$ as well as assumption $\mathbf{R}(\mathbb{F}_p, \dim(\mathcal{X}_{\mathbb{F}_p}))$ borrowed from [12]. We now prefer to simply restate Conjecture 1.1 with $C(\mathcal{X}, n)$ given by (1). Theorem 1.4 will then show that Conjecture 1.1 holds for (\mathcal{X}, n) if and only if it holds for $(\mathcal{X}, d-n)$, provided that $\zeta(\mathcal{X}, s)$ satisfies the expected functional equation (see Conjecture 1.3 in subsection 1.2).

It was shown in [9][Remark5.2] that one can define a fairly natural modification $\tilde{L}\Omega_{\mathcal{X}/\mathbb{Z}}^{\leq n}$ of the derived deRham complex such that

$$\det_{\mathbb{Z}} R\Gamma(\mathcal{X}_{Zar}, L\Omega_{\mathcal{X}/\mathbb{Z}}^{\leq n}) = C(\mathcal{X}, n) \cdot \det_{\mathbb{Z}} R\Gamma(\mathcal{X}_{Zar}, \tilde{L}\Omega_{\mathcal{X}/\mathbb{Z}}^{\leq n})$$

inside $\det_{\mathbb{Q}} R\Gamma(\mathcal{X}_{Zar}, L\Omega_{\mathcal{X}/\mathbb{Z}}^{\leq n})_{\mathbb{Q}} \cong \det_{\mathbb{Q}} R\Gamma(\mathcal{X}_{\mathbb{Q}}, \Omega_{\mathcal{X}_{\mathbb{Q}}/\mathbb{Q}}^{\leq n})$, leading to a version of Conjecture 1.1 without any correction factor. We would like to point out another modification of derived deRham cohomology that is perhaps even more natural than the definition of $\tilde{L}\Omega_{\mathcal{X}/\mathbb{Z}}^{\leq n}$ and should also explain formula (1). Recall from [1] that there is a motivic filtration on cyclic homology $\mathrm{Fil}_{Mot}^* HC(\mathcal{X})$ with graded pieces given by derived deRham cohomology modulo the n -th step in the Hodge filtration

$$gr_{Mot}^n HC(\mathcal{X}) \cong R\Gamma(\mathcal{X}_{Zar}, L\Omega_{\mathcal{X}/\mathbb{Z}}^{\leq n})[2n-2].$$

The corresponding spectral sequence already appears in [19]. Cyclic homology arises as S^1 -homotopy-coinvariants on Hochschild homology $HC(\mathcal{X}) \cong HH(\mathcal{X})_{S^1}$. One can consider the topological analogue and define

$$TC^+(\mathcal{X}) := THH(\mathcal{X})_{S^1}$$

where THH denotes topological Hochschild homology (see for example [20] for a review). Note that $TC^+(\mathcal{X})$ is *not* what is usually called topological cyclic homology. We conjecture that there is a motivic filtration $\mathrm{Fil}_{Mot}^* TC^+(\mathcal{X})$ that maps to $\mathrm{Fil}_{Mot}^* HC(\mathcal{X})$ inducing an isomorphism

$$\mathrm{Fil}_{Mot}^* TC^+(\mathcal{X})_{\mathbb{Q}} \cong \mathrm{Fil}_{Mot}^* HC(\mathcal{X})_{\mathbb{Q}}.$$

Defining

$$R\Gamma(\mathcal{X}_{Zar}, L\Omega_{\mathcal{X}/\mathbb{S}}^{\leq n}) := gr_{Mot}^n TC^+(\mathcal{X})[-2n + 2]$$

we expect

$$\det_{\mathbb{Z}} R\Gamma(\mathcal{X}_{Zar}, L\Omega_{\mathcal{X}/\mathbb{Z}}^{\leq n}) = C(\mathcal{X}, n) \cdot \det_{\mathbb{Z}} R\Gamma(\mathcal{X}_{Zar}, L\Omega_{\mathcal{X}/\mathbb{S}}^{\leq n})$$

and therefore a version of Conjecture 1.1

$$\lambda_{\infty}(\zeta^*(\mathcal{X}, n)^{-1} \cdot \mathbb{Z}) = \det_{\mathbb{Z}} R\Gamma_{W,c}(\mathcal{X}, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} R\Gamma(\mathcal{X}_{Zar}, L\Omega_{\mathcal{X}/\mathbb{S}}^{\leq n})$$

without correction factor. Here we may define $\det_{\mathbb{Z}} R\Gamma(\mathcal{X}_{Zar}, L\Omega_{\mathcal{X}/\mathbb{S}}^{\leq n})$ as the alternating determinant of the homotopy groups although we in fact expect $R\Gamma(\mathcal{X}_{Zar}, L\Omega_{\mathcal{X}/\mathbb{S}}^{\leq n})$ to be a $H\mathbb{Z}$ -module spectrum.

We can currently only define the motivic filtration if \mathcal{X} is smooth proper over \mathbb{F}_p , or for $\mathcal{X} = \text{Spec}(\mathcal{O}_F)$. In the first case the motivic filtration was defined in [3], one can verify that both $R\Gamma(\mathcal{X}_{Zar}, L\Omega_{\mathcal{X}/\mathbb{Z}}^{\leq n})$ and $R\Gamma(\mathcal{X}_{Zar}, L\Omega_{\mathcal{X}/\mathbb{S}}^{\leq n})$ have finite multiplicative Euler characteristic given by Milne's correction factor [8][Def. 5.4], and indeed one has $C(\mathcal{X}, n) = 1$ by formula (1). For $\mathcal{X} = \text{Spec}(\mathcal{O}_F)$ the motivic filtration on $HC(\mathcal{O}_F)$ is given by

$$\text{Fil}_{Mot}^n HC(\mathcal{O}_F) = \tau_{\geq 2n-3} HC(\mathcal{O}_F)$$

and one may define

$$\text{Fil}_{Mot}^n TC^+(\mathcal{O}_F) := \tau_{\geq 2n-3} TC^+(\mathcal{O}_F)$$

by the same formula. Denote by \mathcal{D}_F the different ideal of \mathcal{O}_F and by $|D_F| = N\mathcal{D}_F$ the absolute value of the discriminant. As was shown in [8][1.6] there is an exact sequence

$$0 \rightarrow HC_{2n-2}(\mathcal{O}_F) \rightarrow \mathcal{O}_F \rightarrow \Omega_{\mathcal{O}_F/\mathbb{Z}}(n) \rightarrow HC_{2n-3}(\mathcal{O}_F) \rightarrow 0$$

where $\Omega_{\mathcal{O}_F/\mathbb{Z}}(n)$ is a finite abelian group of cardinality $|D_F|^{n-1}$, i.e. we have

$$|HC_{2n-3}(\mathcal{O}_F)| \cdot [\mathcal{O}_F : HC_{2n-2}(\mathcal{O}_F)] = |D_F|^{n-1}.$$

By a theorem of Lindenstrauss and Madsen [18] one has

$$THH_i(\mathcal{O}_F) = \begin{cases} \mathcal{O}_F & i = 0 \\ \mathcal{D}_F^{-1}/j \cdot \mathcal{O}_F & i = 2j - 1 \\ 0 & \text{else.} \end{cases}$$

An easy analysis of the spectral sequence

$$H_i(BS^1, THH_j(\mathcal{O}_F)) \Rightarrow TC_{i+j}^+(\mathcal{O}_F)$$

then shows that $TC_{2n-3}^+(\mathcal{O}_F)$ is finite and $TC_{2n-2}^+(\mathcal{O}_F) \subseteq \mathcal{O}_F$ is a sublattice so that

$$|TC_{2n-3}^+(\mathcal{O}_F)| \cdot [\mathcal{O}_F : TC_{2n-2}^+(\mathcal{O}_F)] = (n-1)!^{[F:\mathbb{Q}]} \cdot |D_F|^{n-1}.$$

And indeed one has $C(\text{Spec}(\mathcal{O}_F), n) = (n-1)!^{-[F:\mathbb{Q}]}$ by (1).

1.2. Statement of the main result. Under our standard assumptions $\mathbf{L}(\overline{\mathcal{X}}_{et}, n)$, $\mathbf{L}(\overline{\mathcal{X}}_{et}, d-n)$ and $\mathbf{AV}(\overline{\mathcal{X}}_{et}, n)$, we defined in [8] an exact triangle of perfect complexes of abelian groups

$$(3) \quad R\Gamma_{W,c}(\mathcal{X}, \mathbb{Z}(n)) \rightarrow R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z}(n)) \rightarrow R\Gamma_W(\mathcal{X}_\infty, \mathbb{Z}(n)).$$

Here $\overline{\mathcal{X}}$ is an Artin-Verdier compactification, \mathcal{X}_∞ is the quotient topological space $\mathcal{X}(\mathbb{C})/G_{\mathbb{R}}$ and

$$R\Gamma_W(\mathcal{X}_\infty, \mathbb{Z}(n)) := R\Gamma(\mathcal{X}_\infty, i_\infty^* \mathbb{Z}(n))$$

where $i_\infty^* \mathbb{Z}(n)$ is a certain complex of sheaves on \mathcal{X}_∞ , which is unconditionally defined.

In [8][5.7] we defined the invertible \mathbb{Z} -module

$$\begin{aligned} \Xi_\infty(\mathcal{X}/\mathbb{Z}, n) &:= \det_{\mathbb{Z}} R\Gamma_W(\mathcal{X}_\infty, \mathbb{Z}(n)) \otimes \det_{\mathbb{Z}}^{-1} R\Gamma(\mathcal{X}_{Zar}, L\Omega_{\mathcal{X}/\mathbb{Z}}^{<n}) \\ &\quad \otimes \det_{\mathbb{Z}}^{-1} R\Gamma_W(\mathcal{X}_\infty, \mathbb{Z}(d-n)) \otimes \det_{\mathbb{Z}} R\Gamma(\mathcal{X}_{Zar}, L\Omega_{\mathcal{X}/\mathbb{Z}}^{<d-n}) \end{aligned}$$

and a canonical trivialization

$$\xi_\infty : \mathbb{R} \xrightarrow{\sim} \Xi_\infty(\mathcal{X}/\mathbb{Z}, n) \otimes \mathbb{R}.$$

We denote by

$$x_\infty(\mathcal{X}, n)^2 \in \mathbb{R}_{>0}$$

the strictly positive real number such that

$$\xi_\infty(x_\infty(\mathcal{X}, n)^{-2} \cdot \mathbb{Z}) = \Xi_\infty(\mathcal{X}/\mathbb{Z}, n)$$

and prove the following unconditional

Theorem 1.2. *Let \mathcal{X} be a regular scheme of dimension d , proper and flat over $\text{Spec}(\mathbb{Z})$. We have*

$$x_\infty(\mathcal{X}, n)^2 = \pm A(\mathcal{X})^{n-d/2} \cdot \frac{\zeta^*(\mathcal{X}_\infty, n)}{\zeta^*(\mathcal{X}_\infty, d-n)} \cdot \frac{C(\mathcal{X}, n)}{C(\mathcal{X}, d-n)}$$

where $A(\mathcal{X})$ is the Bloch conductor (see Definition 3.2) and $\zeta(\mathcal{X}_\infty, s)$ is the archimedean Euler factor of \mathcal{X} (see Section 4).

We now explain the significance of this result. Let $\zeta(\overline{\mathcal{X}}, s) := \zeta(\mathcal{X}, s) \cdot \zeta(\mathcal{X}_\infty, s)$ be the completed Zeta-function of \mathcal{X} .

Conjecture 1.3. *(Functional Equation) Let \mathcal{X} be a regular scheme of dimension d , proper and flat over $\text{Spec}(\mathbb{Z})$. Then $\zeta(\mathcal{X}, s)$ has a meromorphic continuation to all $s \in \mathbb{C}$ and*

$$A(\mathcal{X})^{(d-s)/2} \cdot \zeta(\overline{\mathcal{X}}, d-s) = \pm A(\mathcal{X})^{s/2} \cdot \zeta(\overline{\mathcal{X}}, s).$$

Assume that $\mathbf{L}(\overline{\mathcal{X}}_{et}, n)$, $\mathbf{L}(\overline{\mathcal{X}}_{et}, d-n)$, $\mathbf{AV}(\overline{\mathcal{X}}_{et}, n)$, $\mathbf{B}(\mathcal{X}, n)$ and $\mathbf{B}(\mathcal{X}, d-n)$ hold, so that Conjecture 1.1 for (\mathcal{X}, n) and $(\mathcal{X}, d-n)$ makes sense. By [8][Prop. 5.29], the exact triangle (3) and Weil-étale duality [8][Thm. 3.22] induce a canonical isomorphism

$$\Delta(\mathcal{X}/\mathbb{Z}, n) \otimes \Xi_\infty(\mathcal{X}/\mathbb{Z}, n) \xrightarrow{\sim} \Delta(\mathcal{X}/\mathbb{Z}, d-n)$$

compatible with ξ_∞ and the trivializations (2) of $\Delta(\mathcal{X}/\mathbb{Z}, n)$ and $\Delta(\mathcal{X}/\mathbb{Z}, d-n)$. We obtain

Theorem 1.4. *Assume that $\zeta(\overline{\mathcal{X}}, s)$ satisfies Conjecture 1.3. Then Conjecture 1.1 for (\mathcal{X}, n) is equivalent to Conjecture 1.1 for $(\mathcal{X}, d-n)$.*

1.3. Outline of this article. In section 2 we study Verdier duality on the locally compact space $\mathcal{X}_\infty := \mathcal{X}(\mathbb{C})/G_{\mathbb{R}}$ and how it applies to the complexes of sheaves $i_\infty^* \mathbb{Z}(n)$ introduced in [8][Def. 3.23]. The key result in terms of relevance for the following sections is Prop. 2.23 which provides the correct power of 2 appearing in the functional equation.

In section 3 we review duality results for the exterior powers of the cotangent complex $L_{\mathcal{X}/\mathbb{Z}}$ due to T. Saito [23] and deduce duality for derived de Rham cohomology of \mathcal{X} . It turns out that the Bloch conductor $A(\mathcal{X})$ of \mathcal{X} introduced in [4] measures the failure of a perfect duality for these theories, see Thm. 3.3 and Prop. 3.5. Corollary 3.9 then provides the correct power of $A(\mathcal{X})$ appearing in the functional equation.

In section 4 we recall the archimedean Euler factors for $\zeta(\mathcal{X}, s)$ and make some preliminary computations towards the main result.

Finally, in section 5 we prove Thm. 1.2 and Thm. 1.4, also employing the results already established in [8][Cor. 5.31] towards compatibility with the functional equation.

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We would also like to thank Spencer Bloch for interesting discussions related to $C(\mathcal{X}, n)$.

2. VERDIER DUALITY ON $\mathcal{X}_\infty = \mathcal{X}(\mathbb{C})/G_{\mathbb{R}}$

2.1. Statement of the duality theorem. Let \mathcal{X} be a regular, flat and proper scheme over $\text{Spec}(\mathbb{Z})$. Assume that \mathcal{X} is connected of dimension d . We denote by $\mathcal{X}_\infty := \mathcal{X}(\mathbb{C})/G_{\mathbb{R}}$ the quotient topological space, where $\mathcal{X}(\mathbb{C})$ is endowed with the complex topology. Let

$$p : \mathcal{X}(\mathbb{C}) \rightarrow \mathcal{X}_\infty$$

be the quotient map and let

$$\pi : \text{Sh}(G_{\mathbb{R}}, \mathcal{X}(\mathbb{C})) \rightarrow \text{Sh}(\mathcal{X}_\infty)$$

be the canonical morphism of topoi, where $\text{Sh}(G_{\mathbb{R}}, \mathcal{X}(\mathbb{C}))$ is the category of $G_{\mathbb{R}}$ -equivariant sheaves on $\mathcal{X}(\mathbb{C})$. We have the formula

$$\pi_*(\mathcal{F}) \simeq (p_*\mathcal{F})^{G_{\mathbb{R}}}.$$

Let $\mathbb{Z}(n) := (2i\pi)^n \cdot \mathbb{Z}$ be the abelian sheaf on $\text{Sh}(G_{\mathbb{R}}, \mathcal{X}(\mathbb{C}))$ defined by the obvious $G_{\mathbb{R}}$ -action on $(2i\pi)^n \cdot \mathbb{Z}$. In [8][Def. 3.23], we defined the complex of sheaves on \mathcal{X}_∞

$$i_\infty^* \mathbb{Z}(n) := \text{Fib}(R\pi_* \mathbb{Z}(n) \rightarrow \tau^{>n} R\hat{\pi}_* \mathbb{Z}(n))$$

for any $n \in \mathbb{Z}$. We define similarly

$$Ri_\infty^! \mathbb{Z}(n+1)[3] := \mathbb{Z}'(n) := \text{Fib}(R\pi_* \mathbb{Z}(n) \rightarrow \tau^{\geq n} R\hat{\pi}_* \mathbb{Z}(n))$$

and we set

$$e := d - 1.$$

If Z is a locally compact topological space, we denote by $\mathcal{D}_Z := Rf^! \mathbb{Z}$ the dualizing complex, where $f : Z \rightarrow \{*\}$ is the map from Z to the point.

Theorem 2.1. *There is an equivalence $\mathbb{Z}'(e) \xrightarrow{\sim} \mathcal{D}_{\mathcal{X}_\infty}[-2e]$ and a perfect pairing*

$$i_\infty^* \mathbb{Z}(n) \otimes^L \mathbb{Z}'(e-n) \longrightarrow \mathbb{Z}'(e) \xrightarrow{\sim} \mathcal{D}_{\mathcal{X}_\infty}[-2e]$$

in the derived category of abelian sheaves over \mathcal{X}_∞ , for any $n \in \mathbb{Z}$.

Proof. We set $\mathbb{Z}(n) := i_\infty^* \mathbb{Z}(n)$, we denote by $\iota : \mathcal{X}(\mathbb{R}) \rightarrow \mathcal{X}_\infty$ the closed immersion and by j the complementary open immersion. By Proposition 2.5 there is a product map

$$\mathbb{Z}(n) \otimes^L \mathbb{Z}'(e-n) \rightarrow \mathcal{D}_{\mathcal{X}_\infty}[-2e]$$

inducing

$$(4) \quad \mathbb{Z}(n) \rightarrow \underline{R\mathrm{Hom}}(\mathbb{Z}'(e-n), \mathcal{D}_{\mathcal{X}_\infty}[-2e]).$$

Then (4) induces an equivalence

$$j^* \mathbb{Z}(n) \xrightarrow{\sim} j^* \underline{R\mathrm{Hom}}(\mathbb{Z}'(e-n), \mathcal{D}_{\mathcal{X}_\infty}[-2e])$$

by Proposition 2.7. Similarly, (4) induces an equivalence

$$R\iota^! \mathbb{Z}(n) \xrightarrow{\sim} R\iota^! \underline{R\mathrm{Hom}}(\mathbb{Z}'(e-n), \mathcal{D}_{\mathcal{X}_\infty}[-2e])$$

by Proposition 2.17. It follows that (4) is an equivalence. Applying $\underline{R\mathrm{Hom}}(-, \mathcal{D}_{\mathcal{X}_\infty}[-2e])$, we get an equivalence

$$\mathbb{Z}'(e-n) \xrightarrow{\sim} \underline{R\mathrm{Hom}}(\mathbb{Z}(n), \mathcal{D}_{\mathcal{X}_\infty}[-2e]).$$

Since $\mathbb{Z}(0)$ is the constant sheaf \mathbb{Z} , we have

$$\mathbb{Z}'(e) \xrightarrow{\sim} \mathcal{D}_{\mathcal{X}_\infty}[-2e].$$

□

We immediately obtain

Corollary 2.2. *There is a trace map $R\Gamma(\mathcal{X}_\infty, \mathbb{Z}'(e)) \rightarrow \mathbb{Z}[-2e]$ and a perfect pairing*

$$R\Gamma(\mathcal{X}_\infty, i_\infty^* \mathbb{Z}(n)) \otimes^L R\Gamma(\mathcal{X}_\infty, \mathbb{Z}'(e-n)) \rightarrow R\Gamma(\mathcal{X}_\infty, \mathbb{Z}'(e)) \rightarrow \mathbb{Z}[-2e]$$

of perfect complexes of abelian groups, for any $n \in \mathbb{Z}$.

The following corollaries also follow easily from Theorem 2.1. We state them in order to justify the notation $Ri_\infty^! \mathbb{Z}(n)$.

Corollary 2.3. *There is a trace map*

$$R\Gamma(\mathcal{X}_\infty, Ri_\infty^! \mathbb{Z}(d)) \rightarrow \mathbb{Z}[-2d-1]$$

and a perfect pairing

$$R\Gamma(\mathcal{X}_\infty, i_\infty^* \mathbb{Z}(n)) \otimes^L R\Gamma(\mathcal{X}_\infty, Ri_\infty^! \mathbb{Z}(d-n)) \rightarrow R\Gamma(\mathcal{X}_\infty, Ri_\infty^! \mathbb{Z}(d)) \rightarrow \mathbb{Z}[-2d-1]$$

of perfect complexes of abelian groups, for any $n \in \mathbb{Z}$.

Corollary 2.4. *Assume that \mathcal{X} satisfies the assumptions $\mathbf{L}(\overline{\mathcal{X}}_{et}, n)$, $\mathbf{L}(\overline{\mathcal{X}}_{et}, d-n)$ and $\mathbf{AV}(\overline{\mathcal{X}}_{et}, n)$ of [8]/3.2]. We define*

$$R\Gamma_W(\mathcal{X}, \mathbb{Z}(n)) := R\mathrm{Hom}(R\Gamma_{W,c}(\mathcal{X}, \mathbb{Z}(d-n)), \mathbb{Z}[-2d-1]).$$

Then we have an exact triangle

$$R\Gamma(\mathcal{X}_\infty, Ri_\infty^! \mathbb{Z}(n)) \rightarrow R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z}(n)) \rightarrow R\Gamma_W(\mathcal{X}, \mathbb{Z}(n)).$$

2.2. Proof of the duality theorem. The proof of Theorem 2.1 relies on the results proven below.

2.2.1. Notations. We denote by $\iota : \mathcal{X}(\mathbb{R}) \rightarrow \mathcal{X}_\infty$ the closed immersion and by $j : \mathcal{X}_\infty^\circ \rightarrow \mathcal{X}_\infty$ the complementary open immersion, where $\mathcal{X}_\infty^\circ := \mathcal{X}_\infty - \mathcal{X}(\mathbb{R})$. We set $\mathcal{X}(\mathbb{C})^\circ := \mathcal{X}(\mathbb{C}) - \mathcal{X}(\mathbb{R})$. We denote by

$$p^\circ : \mathcal{X}(\mathbb{C})^\circ \rightarrow \mathcal{X}_\infty^\circ$$

the quotient map, and by

$$\pi^\circ : \mathrm{Sh}(G_{\mathbb{R}}, \mathcal{X}(\mathbb{C})^\circ) \xrightarrow{\sim} \mathrm{Sh}(\mathcal{X}_\infty^\circ)$$

the morphism of topoi induced by π , which is an equivalence since $G_{\mathbb{R}}$ has no fixed point on $\mathcal{X}(\mathbb{C})^\circ$. If $x \in \mathcal{X}(\mathbb{R})$ then we denote $\iota_x : x \rightarrow \mathcal{X}(\mathbb{R})$ (or $\iota_x : x \rightarrow \mathcal{X}_\infty$) the inclusion. The complex of sheaves over \mathcal{X}_∞ denoted by $\mathbb{Z}(n)$ always refers to $i_\infty^* \mathbb{Z}(n)$.

We denote by $C^*(G_{\mathbb{R}}, \mathbb{Z}(n)) := R\Gamma(G_{\mathbb{R}}, \mathbb{Z}(n))$ the cohomology of $G_{\mathbb{R}}$ with coefficients in $(2i\pi)^n \mathbb{Z}$, by $\widehat{C}^*(G_{\mathbb{R}}, \mathbb{Z}(n)) := R\widehat{\Gamma}(G_{\mathbb{R}}, \mathbb{Z}(n))$ Tate cohomology, and by $C_*(G_{\mathbb{R}}, \mathbb{Z}(n))$ the homology of $G_{\mathbb{R}}$ with coefficients in $(2i\pi)^n \mathbb{Z}$. We have a fiber sequence

$$C_*(G_{\mathbb{R}}, \mathbb{Z}(n)) \rightarrow C^*(G_{\mathbb{R}}, \mathbb{Z}(n)) \rightarrow \widehat{C}^*(G_{\mathbb{R}}, \mathbb{Z}(n)).$$

Recall that, if Z is a locally compact topological space, we denote by $\mathcal{D}_Z := Rf^! \mathbb{Z}$ the dualizing complex, where $f : Z \rightarrow \{*\}$ is the map from Z to the point.

2.2.2. The duality map.

Proposition 2.5. *For any $n \in \mathbb{Z}$, there is a canonical map*

$$i_\infty^* \mathbb{Z}(n) \otimes^L \mathbb{Z}'(e - n) \rightarrow \mathcal{D}_{\mathcal{X}_\infty}[-2e]$$

in the derived category of abelian sheaves over \mathcal{X}_∞ .

Proof. Let f be the map from \mathcal{X}_∞ to the point. We start with the morphism

$$i_\infty^* \mathbb{Z}(n) \otimes^L \mathbb{Z}'(e - n) \rightarrow R\pi_*((2i\pi)^n \mathbb{Z}) \otimes^L R\pi_*((2i\pi)^{e-n} \mathbb{Z}) \rightarrow R\pi_*((2i\pi)^e \mathbb{Z}).$$

Then the map

$$R\pi_*((2i\pi)^e \mathbb{Z}) \rightarrow \mathcal{D}_{\mathcal{X}_\infty}[-2e] := f^! \mathbb{Z}[-2e]$$

is given by

$$Rf_! R\pi_*((2i\pi)^e \mathbb{Z}) \simeq R\Gamma(G_{\mathbb{R}}, \mathcal{X}(\mathbb{C}), (2i\pi)^e \mathbb{Z}) \rightarrow \mathbb{Z}[-2e]$$

where the last map is

$$R\Gamma(G_{\mathbb{R}}, \mathcal{X}(\mathbb{C}), (2i\pi)^e \mathbb{Z}) \rightarrow R\Gamma(\mathcal{X}(\mathbb{C}), (2i\pi)^e \mathbb{Z}) \rightarrow \tau^{\geq 2e} R\Gamma(\mathcal{X}(\mathbb{C}), (2i\pi)^e \mathbb{Z}) \rightarrow \mathbb{Z}[-2e].$$

Note that $Rf_! = Rf_*$ since \mathcal{X}_∞ is compact. □

Definition 2.6. *For any $n \in \mathbb{Z}$, we consider the morphism*

$$(5) \quad \mathbb{Z}(n) \rightarrow R\mathrm{Hom}(\mathbb{Z}'(e - n), \mathcal{D}_{\mathcal{X}_\infty}[-2e])$$

induced by the product map above.

2.2.3. The map $j^*\mathbb{Z}(n) \rightarrow j^*R\mathbf{H}\mathbf{om}(\mathbb{Z}'(e-n), \mathcal{D}_{\mathcal{X}_\infty}[-2e])$.

Proposition 2.7. *The canonical map*

$$j^*\mathbb{Z}(n) \rightarrow j^*R\mathbf{H}\mathbf{om}(\mathbb{Z}'(e-n), \mathcal{D}_{\mathcal{X}_\infty}[-2e])$$

is an equivalence.

Proof. We replace n by $e-n$. We have

$$\begin{aligned} j^*R\mathbf{H}\mathbf{om}(\mathbb{Z}'(n), \mathcal{D}_{\mathcal{X}_\infty}[-2e]) &\simeq R\mathbf{H}\mathbf{om}_{\mathrm{Sh}(\mathcal{X}_\infty)}(j^*\mathbb{Z}'(n), \mathcal{D}_{\mathcal{X}_\infty}[-2e]) \\ &\simeq R\mathbf{H}\mathbf{om}_{\mathrm{Sh}(\mathcal{X}_\infty)}(\pi_*^\circ(2i\pi)^n\mathbb{Z}, \mathcal{D}_{\mathcal{X}_\infty}[-2e]). \end{aligned}$$

Similarly, we have $j^*\mathbb{Z}(e-n) = \pi_*^\circ(2i\pi)^{e-n}\mathbb{Z}$. So we need to check that the map

$$\pi_*^\circ(2i\pi)^{e-n}\mathbb{Z} \rightarrow R\mathbf{H}\mathbf{om}_{\mathrm{Sh}(\mathcal{X}_\infty)}(\pi_*^\circ(2i\pi)^n\mathbb{Z}, \mathcal{D}_{\mathcal{X}_\infty}[-2e])$$

is an equivalence. The map $p^\circ : \mathcal{X}(\mathbb{C})^\circ \rightarrow \mathcal{X}_\infty^\circ$ is a finite étale Galois cover, hence $p^{\circ,*}$ is conservative. Hence it is enough to check that

$$p^{\circ,*}\pi_*^\circ(2i\pi)^{e-n}\mathbb{Z} \rightarrow R\mathbf{H}\mathbf{om}_{\mathrm{Sh}(\mathcal{X}(\mathbb{C})^\circ)}(p^{\circ,*}\pi_*^\circ(2i\pi)^n\mathbb{Z}, \mathcal{D}_{\mathcal{X}(\mathbb{C})^\circ}[-2e])$$

is an equivalence. But we have

$$p^{\circ,*}\pi_*^\circ(2i\pi)^n\mathbb{Z} \simeq (2i\pi)^n\mathbb{Z},$$

hence one is reduced to observe that

$$(2i\pi)^{e-n}\mathbb{Z} \rightarrow R\mathbf{H}\mathbf{om}_{\mathrm{Sh}(\mathcal{X}(\mathbb{C})^\circ)}((2i\pi)^n\mathbb{Z}, \mathcal{D}_{\mathcal{X}(\mathbb{C})^\circ}[-2e])$$

is an equivalence by Verdier duality on the complex (hence orientable) manifold $\mathcal{X}(\mathbb{C})^\circ$. \square

2.2.4. The complex $\iota_x^*R\iota^!\mathbb{Z}(n)$.

Lemma 2.8. *For any $n \in \mathbb{Z}$ and any $x \in \mathcal{X}(\mathbb{R})$, we have a fiber sequence*

$$R\Gamma(G_{\mathbb{R}}, \mathbb{Z}(n)) \rightarrow \iota_x^*Rj_*j^*\mathbb{Z}(n) \rightarrow R\Gamma(G_{\mathbb{R}}, \mathbb{Z}(n-e))[-(e-1)]$$

and $\iota_x^*Rj_*j^*\mathbb{Z}(n)$ is cohomologically concentrated in degrees $\in [0, e-1]$.

Proof. For $e=0$, the map j is both a closed and an open immersion hence $\iota_x^*Rj_*j^*\mathbb{Z}(n) = 0$. So the result is obvious in that case, hence we may assume $e \geq 1$.

Note first that $j^*\mathbb{Z}(n) \simeq R\pi_*^\circ((2i\pi)^n\mathbb{Z})$. Let $x \in \mathcal{X}(\mathbb{R}) \subset \mathcal{X}(\mathbb{C})$. For a point $z \in \mathcal{X}(\mathbb{C})$ in the neighbourhood of x , we have

$$z = (a_1, b_1, \dots, a_e, b_e) \in \mathbb{C}^e = (\mathbb{R} \oplus i \cdot \mathbb{R})^e$$

where σ acts as follows

$$(a_1, \dots, a_e, b_1, \dots, b_e) \mapsto (a_1, \dots, a_e, -b_1, \dots, -b_e) \in \mathbb{R}^e \oplus i \cdot \mathbb{R}^e.$$

So a basic open neighborhood of $x \in \mathcal{X}(\mathbb{R})$ in $\mathcal{X}(\mathbb{C})$ is of the form $B^e \times B^e$ where B^e denotes an open ball in \mathbb{R}^e , and σ acts trivially on the first ball and by multiplication by -1 on the second ball. We have

$$\mathcal{X}(\mathbb{R}) \cap (B^e \times B^e) = B^e \times 0$$

and a $G_{\mathbb{R}}$ -equivariant homotopy equivalence

$$\mathcal{X}(\mathbb{C})^\circ \cap (B^e \times B^e) = B^e \times (B^e - 0) \simeq B^e \times \mathbf{S}^{e-1} \simeq \mathbf{S}^{e-1}$$

where $G_{\mathbb{R}}$ acts by its antipodal action on the $(e-1)$ -sphere \mathbf{S}^{e-1} . We obtain

$$\begin{aligned} \iota_x^* Rj_* j^* \mathbb{Z}(n) &\simeq \operatorname{colim}_{x \in U \subset \mathcal{X}_{\infty}} R\Gamma(U - \mathcal{X}(\mathbb{R}), \mathbb{Z}(n)) \\ &\simeq \operatorname{colim}_{x \in U \subset \mathcal{X}_{\infty}} R\Gamma(G_{\mathbb{R}}, p^{-1}(U - \mathcal{X}(\mathbb{R})), \mathbb{Z}(n)) \\ &\simeq R\Gamma(G_{\mathbb{R}}, \mathbf{S}^{e-1}, \mathbb{Z}(n)) \end{aligned}$$

where $G_{\mathbb{R}}$ acts both on \mathbf{S}^{e-1} and $\mathbb{Z}(n) := (2i\pi)^n \mathbb{Z}$. But we have a fiber sequence in the derived category of $\mathbb{Z}[G_{\mathbb{R}}]$ -modules

$$\mathbb{Z}(n) \rightarrow R\Gamma(\mathbf{S}^{e-1}, \mathbb{Z}(n)) \rightarrow \mathbb{Z}(n-e)[-e]$$

where the boundary map $\mathbb{Z}(n-e)[-e] \rightarrow \mathbb{Z}(n)[1]$ is the non-trivial class in

$$\operatorname{Hom}_{\mathbb{Z}[G_{\mathbb{R}}]}(\mathbb{Z}(n-e)[-e], \mathbb{Z}(n)[1]) \simeq \operatorname{Hom}_{\mathbb{Z}[G_{\mathbb{R}}]}(\mathbb{Z}, \mathbb{Z}(e)[e]) \simeq H^e(G_{\mathbb{R}}, \mathbb{Z}(e)) \simeq \mathbb{Z}/2\mathbb{Z}.$$

Indeed, it must be the non-trivial class because

$$R\Gamma(G_{\mathbb{R}}, \mathbf{S}^{e-1}, \mathbb{Z}(n)) \simeq R\Gamma(\mathbf{S}^{e-1}/\{\pm 1\}, \mathbb{Z}(n))$$

is cohomologically concentrated in degrees $\in [0, e-1]$ since $\mathbf{S}^{e-1}/\{\pm 1\}$ is a $(e-1)$ -manifold. \square

Lemma 2.9. *For any $n \in \mathbb{Z}$, we have*

$$\iota_x^* R\iota^! \mathbb{Z}(n) \simeq \operatorname{Fib} \left(R\Gamma(G_{\mathbb{R}}, \mathbb{Z}(n-e))[-e] \rightarrow \tau^{>n} \widehat{R\Gamma}(G_{\mathbb{R}}, \mathbb{Z}(n)) \right).$$

Proof. First we assume $n \geq 0$, so that $\iota_x^* \mathbb{Z}(n) \simeq \tau^{\leq n} R\Gamma(G_{\mathbb{R}}, \mathbb{Z}(n))$. Then we have the following diagram with exact rows and columns:

$$\begin{array}{ccccc} \tau^{>n} R\Gamma(G_{\mathbb{R}}, \mathbb{Z}(n))[-1] & \longrightarrow & \tau^{\leq n} R\Gamma(G_{\mathbb{R}}, \mathbb{Z}(n)) & \longrightarrow & R\Gamma(G_{\mathbb{R}}, \mathbb{Z}(n)) \\ \downarrow & & \downarrow & & \downarrow \\ \iota_x^* R\iota^! \mathbb{Z}(n) & \longrightarrow & \iota_x^* \mathbb{Z}(n) & \longrightarrow & \iota_x^* Rj_* j^* \mathbb{Z}(n) \\ \downarrow & & \downarrow & & \downarrow \\ R\Gamma(G_{\mathbb{R}}, \mathbb{Z}(n-e))[-e] & \longrightarrow & 0 & \longrightarrow & R\Gamma(G_{\mathbb{R}}, \mathbb{Z}(n-e))[-(e-1)] \end{array}$$

Now we assume $n < 0$. By Lemma 2.11, we have an equivalence

$$\iota_x^* \mathbb{Z}(n) \simeq \tau_{\leq -n-2} C_*(G_{\mathbb{R}}, \mathbb{Z}(n))$$

where both sides vanish for $n = -1$. We obtain the following diagram with exact rows and columns:

$$\begin{array}{ccccc} (\tau^{>n} \widehat{C}(G_{\mathbb{R}}, \mathbb{Z}(n)))[-1] & \longrightarrow & \tau_{\leq -n-2} C_*(G_{\mathbb{R}}, \mathbb{Z}(n)) & \longrightarrow & C^*(G_{\mathbb{R}}, \mathbb{Z}(n)) \\ \downarrow & & \downarrow & & \downarrow \\ \iota_x^* R\iota^! \mathbb{Z}(n) & \longrightarrow & \iota_x^* \mathbb{Z}(n) & \longrightarrow & \iota_x^* Rj_* j^* \mathbb{Z}(n) \\ \downarrow & & \downarrow & & \downarrow \\ C^*(G_{\mathbb{R}}, \mathbb{Z}(n-e))[-e] & \longrightarrow & 0 & \longrightarrow & C^*(G_{\mathbb{R}}, \mathbb{Z}(n-e))[-(e-1)] \end{array}$$

\square

Proposition 2.10. *For $n < e$, we have*

$$\iota_x^* R\iota^! \mathbb{Z}(n) \simeq (\tau_{\leq e-n-2} C_*(G_{\mathbb{R}}, \mathbb{Z}(n-e)))[-e].$$

For $n \geq e$, we have

$$\iota_x^* R\iota^! \mathbb{Z}(n) \simeq (\tau^{\leq n-e} R\Gamma(G_{\mathbb{R}}, \mathbb{Z}(n-e)))[-e].$$

Proof. We have

$$\tau^{>n} R\widehat{\Gamma}(G_{\mathbb{R}}, \mathbb{Z}(n)) \simeq (\tau^{>n-e} R\widehat{\Gamma}(G_{\mathbb{R}}, \mathbb{Z}(n-e)))[-e]$$

and an equivalence

$$\iota_x^* R\iota^! \mathbb{Z}(n) \simeq \text{Fib} \left(R\Gamma(G_{\mathbb{R}}, \mathbb{Z}(n-e)) \rightarrow \tau^{>n-e} R\widehat{\Gamma}(G_{\mathbb{R}}, \mathbb{Z}(n-e)) \right) [-e].$$

Hence the result follows from Lemma 2.11 below. \square

Lemma 2.11. *For any $m \geq 0$, we have an equivalence*

$$\tau^{\leq m} R\Gamma(G_{\mathbb{R}}, \mathbb{Z}(m)) \simeq \text{Fib} \left(R\Gamma(G_{\mathbb{R}}, \mathbb{Z}(m)) \rightarrow \tau^{>m} R\widehat{\Gamma}(G_{\mathbb{R}}, \mathbb{Z}(m)) \right).$$

Similarly, for any $m < 0$, we have

$$\tau_{\leq -m-2} C_*(G_{\mathbb{R}}, \mathbb{Z}(m)) \simeq \text{Fib} \left(R\Gamma(G_{\mathbb{R}}, \mathbb{Z}(m)) \rightarrow \tau^{>m} R\widehat{\Gamma}(G_{\mathbb{R}}, \mathbb{Z}(m)) \right).$$

Proof. The first assertion is obvious. The second equivalence holds for $m = -1$ since both side vanish. It remains to show that the second equivalence holds for $m \leq -2$. We have the following exact diagram

$$\begin{array}{ccccc} (\tau^{\leq m} \widehat{C}(G_{\mathbb{R}}, \mathbb{Z}(m)))[-1] & \longrightarrow & 0 & \longrightarrow & \tau^{\leq m} \widehat{C}^*(G_{\mathbb{R}}, \mathbb{Z}(m)) \\ \downarrow & & \downarrow & & \downarrow \\ C_*(G_{\mathbb{R}}, \mathbb{Z}(m)) & \longrightarrow & C^*(G_{\mathbb{R}}, \mathbb{Z}(m)) & \longrightarrow & \widehat{C}^*(G_{\mathbb{R}}, \mathbb{Z}(m)) \\ \downarrow & & \downarrow & & \downarrow \\ F & \longrightarrow & C^*(G_{\mathbb{R}}, \mathbb{Z}(m)) & \longrightarrow & \tau^{>m} \widehat{C}^*(G_{\mathbb{R}}, \mathbb{Z}(m)) \end{array}$$

hence a cofiber sequence

$$(\tau^{\leq m} \widehat{C}(G_{\mathbb{R}}, \mathbb{Z}(m)))[-1] \rightarrow C_*(G_{\mathbb{R}}, \mathbb{Z}(m)) \rightarrow F.$$

In view of the equivalences

$$(\tau^{\leq m} \widehat{C}(G_{\mathbb{R}}, \mathbb{Z}(m)))[-1] \simeq \tau^{\leq m+1} (\widehat{C}(G_{\mathbb{R}}, \mathbb{Z}(m)))[-1] \simeq \tau^{\leq m+1} C_*(G_{\mathbb{R}}, \mathbb{Z}(m))$$

we obtain

$$F \simeq \tau^{>m+1} C_*(G_{\mathbb{R}}, \mathbb{Z}(m)) = \tau^{\geq m+2} C_*(G_{\mathbb{R}}, \mathbb{Z}(m)) = \tau_{\leq -m-2} C_*(G_{\mathbb{R}}, \mathbb{Z}(m)).$$

\square

Lemma 2.12. *For any $m > 0$, we have an equivalence*

$$\tau^{<m} R\Gamma(G_{\mathbb{R}}, \mathbb{Z}(m)) \simeq \text{Fib} \left(R\Gamma(G_{\mathbb{R}}, \mathbb{Z}(m)) \rightarrow \tau^{\geq m} R\widehat{\Gamma}(G_{\mathbb{R}}, \mathbb{Z}(m)) \right).$$

Similarly, for any $m \leq 0$, we have

$$\tau_{\leq -m} C_*(G_{\mathbb{R}}, \mathbb{Z}(m)) \simeq \text{Fib} \left(R\Gamma(G_{\mathbb{R}}, \mathbb{Z}(m)) \rightarrow \tau^{\geq m} R\widehat{\Gamma}(G_{\mathbb{R}}, \mathbb{Z}(m)) \right).$$

Proof. The first assertion is obvious. The second equivalence for $m = 0$ follows from the exact sequence

$$0 = \widehat{H}^{-1}(G_{\mathbb{R}}, \mathbb{Z}) \rightarrow H_0(G_{\mathbb{R}}, \mathbb{Z}) \rightarrow H^0(G_{\mathbb{R}}, \mathbb{Z}) \rightarrow \widehat{H}^0(G_{\mathbb{R}}, \mathbb{Z}) \rightarrow 0$$

and the isomorphism $H^i(G_{\mathbb{R}}, \mathbb{Z}) \xrightarrow{\sim} \widehat{H}^i(G_{\mathbb{R}}, \mathbb{Z})$ for $i > 0$.

It remains to show that the second equivalence holds for $m \leq -1$. We have the following exact diagram

$$\begin{array}{ccccc} (\tau^{<m}\widehat{C}(G_{\mathbb{R}}, \mathbb{Z}(m)))[-1] & \longrightarrow & 0 & \longrightarrow & \tau^{<m}\widehat{C}^*(G_{\mathbb{R}}, \mathbb{Z}(m)) \\ \downarrow & & \downarrow & & \downarrow \\ C_*(G_{\mathbb{R}}, \mathbb{Z}(m)) & \longrightarrow & C^*(G_{\mathbb{R}}, \mathbb{Z}(m)) & \longrightarrow & \widehat{C}^*(G_{\mathbb{R}}, \mathbb{Z}(m)) \\ \downarrow & & \downarrow & & \downarrow \\ F & \longrightarrow & C^*(G_{\mathbb{R}}, \mathbb{Z}(m)) & \longrightarrow & \tau^{\geq m}\widehat{C}^*(G_{\mathbb{R}}, \mathbb{Z}(m)) \end{array}$$

hence a cofiber sequence

$$(\tau^{<m}\widehat{C}(G_{\mathbb{R}}, \mathbb{Z}(m)))[-1] \rightarrow C_*(G_{\mathbb{R}}, \mathbb{Z}(m)) \rightarrow F.$$

In view of the equivalences

$$(\tau^{<m}\widehat{C}(G_{\mathbb{R}}, \mathbb{Z}(m)))[-1] \simeq \tau^{<m+1}(\widehat{C}(G_{\mathbb{R}}, \mathbb{Z}(m)))[-1]) \simeq \tau^{<m+1}C_*(G_{\mathbb{R}}, \mathbb{Z}(m))$$

we obtain

$$F \simeq \tau^{\geq m+1}C_*(G_{\mathbb{R}}, \mathbb{Z}(m)) = \tau_{\leq -m-1}C_*(G_{\mathbb{R}}, \mathbb{Z}(m)) \simeq \tau_{\leq -m}C_*(G_{\mathbb{R}}, \mathbb{Z}(m))$$

since

$$H_{-m}(G_{\mathbb{R}}, \mathbb{Z}(m)) = \widehat{H}^{m-1}(G_{\mathbb{R}}, \mathbb{Z}(m)) = 0.$$

□

2.2.5. *The complex $Ri^1R\mathbf{H}\mathbf{om}(\mathbb{Z}'(e-n), \mathcal{D}_{\mathcal{X}_{\infty}}[-2e])$.* We denote by $f : \mathcal{X}(\mathbb{R}) \rightarrow \{*\}$ the map from $\mathcal{X}(\mathbb{R})$ to the point and we denote by $\omega_{\mathcal{X}(\mathbb{R})}$ the orientation sheaf on the e -manifold $\mathcal{X}(\mathbb{R})$. We have

$$\mathcal{D}_{\mathcal{X}(\mathbb{R})} := f^!\mathbb{Z} \simeq \omega_{\mathcal{X}(\mathbb{R})}[e].$$

Proposition 2.13. *For $e - n > 0$ we have*

$$Ri^1R\mathbf{H}\mathbf{om}(\mathbb{Z}'(e-n), \mathcal{D}_{\mathcal{X}_{\infty}}[-2e]) \simeq f^*(\tau_{\leq e-n-2}C_*(G_{\mathbb{R}}, \mathbb{Z}(e-n))) \otimes^L \omega_{\mathcal{X}(\mathbb{R})}[-e].$$

Proof. Using Lemma 2.12 and Lemma 2.15, we obtain

$$\begin{aligned} Ri^1R\mathbf{H}\mathbf{om}(\mathbb{Z}'(e-n), \mathcal{D}_{\mathcal{X}_{\infty}})[-2e] &\simeq R\mathbf{H}\mathbf{om}(i^*\mathbb{Z}'(e-n), \mathcal{D}_{\mathcal{X}(\mathbb{R})})[-2e] \\ &\simeq R\mathbf{H}\mathbf{om}(f^*\tau^{<e-n}R\Gamma(G_{\mathbb{R}}, \mathbb{Z}(e-n)), \mathcal{D}_{\mathcal{X}(\mathbb{R})})[-2e] \\ &\simeq f^!R\mathbf{H}\mathbf{om}(\tau^{<e-n}R\Gamma(G_{\mathbb{R}}, \mathbb{Z}(e-n)), \mathbb{Z})[-2e] \\ &\simeq f^!R\mathbf{H}\mathbf{om}(\tau^{\leq e-n-2}R\Gamma(G_{\mathbb{R}}, \mathbb{Z}(e-n)), \mathbb{Z})[-2e] \\ &\simeq f^!(\tau_{\leq e-n-2}C_*(G_{\mathbb{R}}, \mathbb{Z}(e-n)))[-2e] \\ &\simeq f^*(\tau_{\leq e-n-2}C_*(G_{\mathbb{R}}, \mathbb{Z}(e-n))) \otimes^L \omega_{\mathcal{X}(\mathbb{R})}[-e]. \end{aligned}$$

□

Proposition 2.14. *For $e - n \leq 0$ we have*

$$R\iota^1 R\mathbf{H}\mathbf{om}(\mathbb{Z}'(e - n), \mathcal{D}_{\mathcal{X}_\infty}[-2e]) \simeq f^*(\tau^{\leq n-e} R\Gamma(G_{\mathbb{R}}, \mathbb{Z}(n - e))) \otimes^L \omega_{\mathcal{X}(\mathbb{R})}[-e].$$

Proof. Using Lemma 2.12 and Lemma 2.15, we obtain

$$\begin{aligned} R\iota^1 R\mathbf{H}\mathbf{om}(\mathbb{Z}'(e - n), \mathcal{D}_{\mathcal{X}_\infty})[-2e] &\simeq R\mathbf{H}\mathbf{om}(\iota^* \mathbb{Z}'(e - n), \mathcal{D}_{\mathcal{X}(\mathbb{R})})[-2e] \\ &\simeq R\mathbf{H}\mathbf{om}(f^* \tau_{\leq n-e} C_*(G_{\mathbb{R}}, \mathbb{Z}(e - n)), \mathcal{D}_{\mathcal{X}(\mathbb{R})})[-2e] \\ &\simeq f^! R\mathbf{H}\mathbf{om}(\tau_{\leq n-e} C_*(G_{\mathbb{R}}, \mathbb{Z}(e - n)), \mathbb{Z})[-2e] \\ &\simeq f^!(\tau^{\leq n-e} R\Gamma(G_{\mathbb{R}}, \mathbb{Z}(n - e)))[-2e] \\ &\simeq f^*(\tau^{\leq n-e} R\Gamma(G_{\mathbb{R}}, \mathbb{Z}(n - e))) \otimes^L \omega_{\mathcal{X}(\mathbb{R})}[-e]. \end{aligned}$$

□

Lemma 2.15. *For any $n \in \mathbb{Z}$, the pairing*

$$C_*(G_{\mathbb{R}}, \mathbb{Z}(-n)) \otimes_{\mathbb{Z}}^L C^*(G_{\mathbb{R}}, \mathbb{Z}(n)) \rightarrow C_*(G_{\mathbb{R}}, \mathbb{Z}(0)) \rightarrow \mathbb{Z}[0]$$

induces a perfect pairing

$$\tau_{\leq n} C_*(G_{\mathbb{R}}, \mathbb{Z}(-n)) \otimes_{\mathbb{Z}}^L \tau^{\leq n} C^*(G_{\mathbb{R}}, \mathbb{Z}(n)) \rightarrow \mathbb{Z}[0]$$

of perfect complexes of abelian groups.

Proof. The result is trivial for $n < 0$ and clear for $n = 0$. So we assume $n > 0$. The pairing induces an equivalence

$$C^*(G_{\mathbb{R}}, \mathbb{Z}(n)) \rightarrow R\mathbf{H}\mathbf{om}(C_*(G_{\mathbb{R}}, \mathbb{Z}(-n)), \mathbb{Z})$$

hence it is enough to observe that

$$\tau^{\leq n} R\mathbf{H}\mathbf{om}(C_*(G_{\mathbb{R}}, \mathbb{Z}(-n)), \mathbb{Z}) \simeq R\mathbf{H}\mathbf{om}(\tau_{\leq n} C_*(G_{\mathbb{R}}, \mathbb{Z}(-n)), \mathbb{Z}).$$

For any cohomological complex A^* , we have a short exact sequence

$$0 \rightarrow \text{Ext}(H^{-i+1}(A^*), \mathbb{Z}) \rightarrow H^i(R\mathbf{H}\mathbf{om}(A^*, \mathbb{Z})) \rightarrow \text{Hom}(H^{-i}(A^*), \mathbb{Z}) \rightarrow 0.$$

We obtain

$$H^i(R\mathbf{H}\mathbf{om}(\tau_{\leq n} C_*(G_{\mathbb{R}}, \mathbb{Z}(-n)), \mathbb{Z})) = H^i(R\mathbf{H}\mathbf{om}(C_*(G_{\mathbb{R}}, \mathbb{Z}(-n)), \mathbb{Z}))$$

for $i \leq n$ and $i > n + 1$. Since we have $H_n(G_{\mathbb{R}}, \mathbb{Z}(-n)) = 0$ for any $n > 0$, we get

$$H^{n+1}(R\mathbf{H}\mathbf{om}(\tau_{\leq n} C_*(G_{\mathbb{R}}, \mathbb{Z}(-n)), \mathbb{Z})) = 0.$$

□

Remark 2.16. *For $n > 0$, we have $H_n(G_{\mathbb{R}}, \mathbb{Z}(-n)) = 0$ hence*

$$\tau_{\leq n} C_*(G_{\mathbb{R}}, \mathbb{Z}(-n)) \simeq \tau_{< n} C_*(G_{\mathbb{R}}, \mathbb{Z}(-n)).$$

2.2.6. The map $R\iota^!\mathbb{Z}(n) \rightarrow R\iota^!R\mathbf{H}\mathbf{om}(\mathbb{Z}'(e-n), \mathcal{D}_{\mathcal{X}_\infty}[-2e])$.

Proposition 2.17. *The map*

$$R\iota^!\mathbb{Z}(n) \rightarrow R\iota^!R\mathbf{H}\mathbf{om}(\mathbb{Z}'(e-n), \mathcal{D}_{\mathcal{X}_\infty}[-2e])$$

is an equivalence.

Proof. For $e-n > 0$ and any $x \in \mathcal{X}(\mathbb{R})$, the map

$$\iota_x^*R\iota^!\mathbb{Z}(n) \rightarrow \iota_x^*R\iota^!R\mathbf{H}\mathbf{om}(\mathbb{Z}'(e-n), \mathcal{D}_{\mathcal{X}_\infty}[-2e])$$

can be identified with the identity

$$\tau_{\leq e-n-2}C_*(G_{\mathbb{R}}, \mathbb{Z}(e-n))[-e] = \tau_{\leq e-n-2}C_*(G_{\mathbb{R}}, \mathbb{Z}(e-n))[-e]$$

by Prop. 2.10 and Prop. 2.13.

For $e-n \leq 0$ and any $x \in \mathcal{X}(\mathbb{R})$, the map

$$\iota_x^*R\iota^!\mathbb{Z}(n) \rightarrow \iota_x^*R\iota^!R\mathbf{H}\mathbf{om}(\mathbb{Z}'(e-n), \mathcal{D}_{\mathcal{X}_\infty}[-2e])$$

can be identified with the identity

$$\tau^{\leq n-e}R\Gamma(G_{\mathbb{R}}, \mathbb{Z}(n-e))[-e] = \tau^{\leq n-e}R\Gamma(G_{\mathbb{R}}, \mathbb{Z}(n-e))[-e]$$

by Prop. 2.10 and Prop. 2.14.

The result follows since the family of functors $\{\iota_x^*, x \in \mathcal{X}(\mathbb{R})\}$ is conservative. \square

2.3. Comparison with $R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{Z}(n))$. Recall that we define $G_{\mathbb{R}}$ -equivariant sheaves

$$\mathbb{Z}(n) := (2i\pi)^n\mathbb{Z} \subset \mathbb{Q}(n) := (2i\pi)^n\mathbb{Q} \subset \mathbb{R}(n) := (2i\pi)^n\mathbb{R} \subset \mathbb{C}$$

on $\mathcal{X}(\mathbb{C})$. We abbreviate $C^* := R\mathbf{H}\mathbf{om}(C, \mathbb{Q})$ for a complex of \mathbb{Q} -vector spaces C and let C^\pm be the image of the idempotent $(\sigma \pm 1)/2$ if C carries a $G_{\mathbb{R}} = \{1, \sigma\}$ -action. Recall that

$$R\Gamma_W(\mathcal{X}_\infty, \mathbb{Z}(n)) := R\Gamma(\mathcal{X}_\infty, i_\infty^*\mathbb{Z}(n))$$

and that $i_\infty^*\mathbb{Z}(n) \otimes \mathbb{Q} \cong \pi_*\mathbb{Q}(n) \cong R\pi_*\mathbb{Q}(n)$ in $\mathbf{Sh}(\mathcal{X}_\infty)$. We therefore have isomorphisms

$$R\Gamma_W(\mathcal{X}_\infty, \mathbb{Z}(n))_{\mathbb{Q}} \simeq R\Gamma(\mathcal{X}_\infty, R\pi_*\mathbb{Q}(n)) \simeq R\Gamma(G_{\mathbb{R}}; \mathcal{X}(\mathbb{C}), \mathbb{Q}(n)) \simeq R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{Q}(n))^+$$

and combining this with Poincaré duality

$$(6) \quad R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{Q}(r)) \otimes R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{Q}(e-r)) \xrightarrow{\cup} R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{Q}(e)) \xrightarrow{\mathrm{Tr}} \mathbb{Q}[-2e]$$

on the $2e$ -manifold $\mathcal{X}(\mathbb{C})$ we obtain an isomorphism

$$R\Gamma_W(\mathcal{X}_\infty, \mathbb{Z}(d-n))_{\mathbb{Q}}^* \simeq R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{Q}(d-n))^*,+ \simeq R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{Q}(n-1))^+[-2e]$$

using $e = d - 1$. There is also a tautological isomorphism τ induced by multiplication by $2\pi i$ in the sense that the diagram

$$(7) \quad \begin{array}{ccc} R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{Q}(n-1))^+ & \longrightarrow & R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{C}) \\ \sim \downarrow \tau & & \sim \downarrow \cdot 2\pi i \\ R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{Q}(n))^- & \longrightarrow & R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{C}) \end{array}$$

commutes. Combining the previous isomorphisms we obtain an isomorphism

$$\begin{aligned}
(8) \quad & (\det_{\mathbb{Z}} R\Gamma_W(\mathcal{X}_{\infty}, \mathbb{Z}(n)) \otimes \det_{\mathbb{Z}}^{-1} R\Gamma_W(\mathcal{X}_{\infty}, \mathbb{Z}(d-n)))_{\mathbb{Q}} \\
& \simeq \det_{\mathbb{Q}} (R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{Q}(n))^+ \oplus R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{Q}(n-1))^+) \\
& \simeq \det_{\mathbb{Q}} (R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{Q}(n))^+ \oplus R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{Q}(n))^-) \\
& \simeq \det_{\mathbb{Q}} R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{Q}(n)) \\
& \simeq (\det_{\mathbb{Z}} R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{Z}(n)))_{\mathbb{Q}}
\end{aligned}$$

which we denote by λ_B .

Corollary 2.18. *We have*

$$\begin{aligned}
\lambda_B (\det_{\mathbb{Z}} R\Gamma_W(\mathcal{X}_{\infty}, \mathbb{Z}(n)) \otimes \det_{\mathbb{Z}}^{-1} R\Gamma_W(\mathcal{X}_{\infty}, \mathbb{Z}(d-n))) \\
= \det_{\mathbb{Z}} R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{Z}(n)) \otimes \det_{\mathbb{Z}}^{(-1)^n} R\Gamma(\mathcal{X}(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})
\end{aligned}$$

Proof. We write $G_{\mathbb{R}} = \{1, \sigma\}$. We have an exact sequence of $\mathbb{Z}[G_{\mathbb{R}}]$ -modules

$$0 \rightarrow \mathbb{Z} \cdot (\sigma - 1) \rightarrow \mathbb{Z}[G_{\mathbb{R}}] \xrightarrow{\epsilon} \mathbb{Z} \rightarrow 0$$

where ϵ is the augmentation map. We have an isomorphism of $\mathbb{Z}[G_{\mathbb{R}}]$ -modules $(\sigma - 1) \cdot \mathbb{Z} \simeq (2i\pi)\mathbb{Z}$ which maps $(\sigma - 1)$ to $(2i\pi)$. We write $\mathbb{Z}(n) := (2i\pi)^n \mathbb{Z}$, so that we have an exact sequence of $\mathbb{Z}[G_{\mathbb{R}}]$ -modules

$$(9) \quad 0 \rightarrow \mathbb{Z}(1) \rightarrow \mathbb{Z}[G_{\mathbb{R}}] \xrightarrow{\epsilon} \mathbb{Z} \rightarrow 0.$$

We denote by

$$p : \mathrm{Sh}(G_{\mathbb{R}}, \mathcal{X}(\mathbb{C})) \rightarrow \mathrm{Sh}(G_{\mathbb{R}}, \mathcal{X}_{\infty})$$

the morphism of topoi induced by the equivariant continuous map $p : \mathcal{X}(\mathbb{C}) \rightarrow \mathcal{X}_{\infty}$, where $G_{\mathbb{R}}$ acts trivially on \mathcal{X}_{∞} . The category of abelian sheaves on $\mathrm{Sh}(G_{\mathbb{R}}, \mathcal{X}_{\infty})$ is equivalent to the category of sheaves of $\mathbb{Z}[G_{\mathbb{R}}]$ -modules over \mathcal{X}_{∞} . For any sheaf \mathcal{F} of $\mathbb{Z}[G_{\mathbb{R}}]$ -modules over \mathcal{X}_{∞} , and any $\mathbb{Z}[G_{\mathbb{R}}]$ -module M , we define

$$R\mathrm{Hom}_{\mathrm{Sh}(G_{\mathbb{R}}, \mathcal{X}_{\infty})}(M, \mathcal{F})$$

where M is seen as a constant sheaf of $\mathbb{Z}[G_{\mathbb{R}}]$ -modules over \mathcal{X}_{∞} . We have

$$R\pi_* \mathbb{Z}(n) \simeq R\mathrm{Hom}_{\mathrm{Sh}(G_{\mathbb{R}}, \mathcal{X}_{\infty})}(\mathbb{Z}, Rp_* \mathbb{Z}(n)).$$

Moreover the functor

$$\begin{array}{ccc}
\mathrm{Ab}(G_{\mathbb{R}}, \mathcal{X}_{\infty}) & \longrightarrow & \mathrm{Ab}(G_{\mathbb{R}}, \mathcal{X}_{\infty}) \\
\mathcal{F} & \longmapsto & \mathcal{F}(1) := \mathcal{F} \otimes_{\mathbb{Z}} \mathbb{Z}(1)
\end{array}$$

is an equivalence of abelian categories with quasi-inverse $(-)\otimes_{\mathbb{Z}} \mathbb{Z}(-1)$. In particular we have

$$\begin{aligned}
R\pi_* \mathbb{Z}(n-1) & \simeq R\mathrm{Hom}_{\mathrm{Sh}(G_{\mathbb{R}}, \mathcal{X}_{\infty})}(\mathbb{Z}, Rp_* \mathbb{Z}(n-1)) \\
& \simeq R\mathrm{Hom}_{\mathrm{Sh}(G_{\mathbb{R}}, \mathcal{X}_{\infty})}(\mathbb{Z}(1), (Rp_* \mathbb{Z}(n-1))(1)) \\
& \simeq R\mathrm{Hom}_{\mathrm{Sh}(G_{\mathbb{R}}, \mathcal{X}_{\infty})}(\mathbb{Z}(1), Rp_* \mathbb{Z}(n)).
\end{aligned}$$

Finally, we have

$$p_* \mathbb{Z}(n) \simeq Rp_* \mathbb{Z}(n) \simeq R\mathrm{Hom}_{\mathrm{Sh}(G_{\mathbb{R}}, \mathcal{X}_{\infty})}(\mathbb{Z}[G_{\mathbb{R}}], Rp_* \mathbb{Z}(n)).$$

Therefore, (9) induces an exact triangle

$$R\pi_*\mathbb{Z}(n) \rightarrow Rp_*\mathbb{Z}(n) \rightarrow R\pi_*\mathbb{Z}(n-1)$$

and an exact diagram:

$$\begin{array}{ccccc} Rp_*\mathbb{Z}(n) & \longrightarrow & i_\infty^*\mathbb{Z}(n-1) & \longrightarrow & i_\infty^*\mathbb{Z}(n)[1] \\ \downarrow & & \downarrow & & \downarrow \\ Rp_*\mathbb{Z}(n) & \longrightarrow & R\pi_*\mathbb{Z}(n-1) & \longrightarrow & R\pi_*\mathbb{Z}(n)[1] \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \tau^{>n-1}R\hat{\pi}_*\mathbb{Z}(n-1) & \longrightarrow & (\tau^{>n}R\hat{\pi}_*\mathbb{Z}(n))[1] \end{array}$$

In particular, there is an exact triangle

$$i_\infty^*\mathbb{Z}(n) \rightarrow Rp_*\mathbb{Z}(n) \rightarrow i_\infty^*\mathbb{Z}(n-1)$$

hence

$$(10) \quad R\Gamma(\mathcal{X}_\infty, \mathbb{Z}(n)) \rightarrow R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{Z}(n)) \rightarrow R\Gamma(\mathcal{X}_\infty, \mathbb{Z}(n-1)).$$

Moreover, we have the duality equivalence

$$(11) \quad R\Gamma(\mathcal{X}_\infty, \mathbb{Z}(d-n)) \xrightarrow{\sim} R\mathrm{Hom}(R\Gamma(\mathcal{X}_\infty, \mathbb{Z}'(n-1)), \mathbb{Z}[-2e]).$$

Finally, we have the following exact diagram

$$\begin{array}{ccccc} \iota_*\mathbb{Z}/2\mathbb{Z}_{\mathcal{X}(\mathbb{R})}[-n] & \longrightarrow & \mathbb{Z}'(n-1) & \longrightarrow & i_\infty^*\mathbb{Z}(n-1) \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & R\pi_*\mathbb{Z}(n-1) & \longrightarrow & R\pi_*\mathbb{Z}(n-1) \\ \downarrow & & \downarrow & & \downarrow \\ \iota_*\mathbb{Z}/2\mathbb{Z}_{\mathcal{X}(\mathbb{R})}[-n+1] & \longrightarrow & \tau^{\geq n-1}R\hat{\pi}_*\mathbb{Z}(n-1) & \longrightarrow & \tau^{>n-1}R\hat{\pi}_*\mathbb{Z}(n-1) \end{array}$$

where $\mathbb{Z}/2\mathbb{Z}_{\mathcal{X}(\mathbb{R})}$ is the constant sheaf $\mathbb{Z}/2\mathbb{Z}$ on $\mathcal{X}(\mathbb{R})$, hence an exact triangle

$$(12) \quad R\Gamma(\mathcal{X}(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})[-n] \rightarrow R\Gamma(\mathcal{X}_\infty, \mathbb{Z}'(n-1)) \rightarrow R\Gamma(\mathcal{X}_\infty, \mathbb{Z}(n-1)).$$

Then (10), (11) and (12) induce the following canonical isomorphisms:

$$\begin{aligned} & \det_{\mathbb{Z}}R\Gamma(\mathcal{X}_\infty, \mathbb{Z}(n)) \otimes \det_{\mathbb{Z}}^{-1}R\Gamma(\mathcal{X}_\infty, \mathbb{Z}(d-n)) \\ & \simeq \det_{\mathbb{Z}}R\Gamma(\mathcal{X}_\infty, \mathbb{Z}(n)) \otimes \det_{\mathbb{Z}}R\Gamma(\mathcal{X}_\infty, \mathbb{Z}'(n-1)) \\ & \simeq \det_{\mathbb{Z}}R\Gamma(\mathcal{X}_\infty, \mathbb{Z}(n)) \otimes \det_{\mathbb{Z}}R\Gamma(\mathcal{X}_\infty, \mathbb{Z}(n-1)) \otimes \det_{\mathbb{Z}}R\Gamma(\mathcal{X}(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})[-n] \\ & \simeq \det_{\mathbb{Z}}R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{Z}(n)) \otimes \det_{\mathbb{Z}}R\Gamma(\mathcal{X}(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})[-n] \\ & \simeq \det_{\mathbb{Z}}R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{Z}(n)) \otimes \det_{\mathbb{Z}}^{(-1)^n}R\Gamma(\mathcal{X}(\mathbb{R}), \mathbb{Z}/2\mathbb{Z}). \end{aligned}$$

□

We now introduce some notation: we set

$$d_+(\mathcal{X}, n) := \sum_{i \in \mathbb{Z}} (-1)^i \dim_{\mathbb{Q}} H^i(\mathcal{X}(\mathbb{C}), \mathbb{Q}(n))^+$$

and

$$d_-(\mathcal{X}, n) := \sum_{i \in \mathbb{Z}} (-1)^i \dim_{\mathbb{Q}} H^i(\mathcal{X}(\mathbb{C}), \mathbb{Q}(n))^-.$$

If Z is a manifold and F a field, we set

$$\chi(Z, F) := \sum_{i \in \mathbb{Z}} (-1)^i \dim_F H^i(Z, F).$$

Definition 2.19. For a perfect complex of abelian groups C with finite cohomology groups we denote by

$$\chi^\times(C) = \prod_{i \in \mathbb{Z}} |H^i(C)|^{(-1)^i}$$

its multiplicative Euler characteristic.

Proposition 2.20. We have

$$\chi^\times(R\Gamma(\mathcal{X}(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})[-n]) = 2^{d_+(\mathcal{X}, n) - d_-(\mathcal{X}, n)}.$$

Proof. We have

$$d_+(\mathcal{X}, n) = d_-(\mathcal{X}, n-1) = d_+(\mathcal{X}, n-2)$$

hence

$$d_\pm(\mathcal{X}, n) = (-1)^n \cdot d_\pm(\mathcal{X}, 0).$$

We obtain

$$2^{d_+(\mathcal{X}, n) - d_-(\mathcal{X}, n)} = (2^{d_+(\mathcal{X}, 0) - d_-(\mathcal{X}, 0)})^{(-1)^n}.$$

Similarly, we have

$$\chi^\times(R\Gamma(\mathcal{X}(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})[-n]) := \chi^\times(R\Gamma(\mathcal{X}(\mathbb{R}), \mathbb{Z}/2\mathbb{Z}))^{(-1)^n},$$

hence it is enough to show the result for $n = 0$. In view of Lemma 2.21 and Lemma 2.22, we have

$$\begin{aligned} d_+(\mathcal{X}, 0) - d_-(\mathcal{X}, 0) &= \sum_{i \in \mathbb{Z}} (-1)^i \cdot (\dim_{\mathbb{Q}} H^i(\mathcal{X}(\mathbb{C}), \mathbb{Q})^+ - \dim_{\mathbb{Q}} H^i(\mathcal{X}(\mathbb{C}), \mathbb{Q})^-) \\ &= \sum_{i \in \mathbb{Z}} (-1)^i \cdot \text{Tr}(\sigma | H^i(\mathcal{X}(\mathbb{C}), \mathbb{Q})) \\ &= \chi(\mathcal{X}(\mathbb{R}), \mathbb{Q}) \\ &= \chi(\mathcal{X}(\mathbb{R}), \mathbb{F}_2). \end{aligned}$$

Hence the result follows from

$$\chi^\times(R\Gamma(\mathcal{X}(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})) = 2^{\chi(\mathcal{X}(\mathbb{R}), \mathbb{F}_2)}.$$

□

Lemma 2.21. Let Y be a compact orientable manifold with an involution σ whose fixed points form a closed submanifold Z . Then we have

$$\sum_{i \in \mathbb{Z}} (-1)^i \cdot \text{Tr}(\sigma | H^i(Y, \mathbb{Q})) = \chi(Z, \mathbb{Q}).$$

Proof. Let $G_{\mathbb{R}} := \{1, \sigma\}$. If C is a perfect complex of \mathbb{Q} -vector spaces with $G_{\mathbb{R}}$ -action, we set

$$\mathrm{Tr}(\sigma | C) := \sum_{i \in \mathbb{Z}} (-1)^i \cdot \mathrm{Tr}(\sigma | H^i(C)).$$

Let $Y^\circ := Y - Z$. The exact triangle

$$R\Gamma_c(Y^\circ, \mathbb{Q}) \rightarrow R\Gamma(Y, \mathbb{Q}) \rightarrow R\Gamma(Z, \mathbb{Q})$$

gives

$$\begin{aligned} \mathrm{Tr}(\sigma | R\Gamma(Y, \mathbb{Q})) &= \mathrm{Tr}(\sigma | R\Gamma_c(Y^\circ, \mathbb{Q})) + \mathrm{Tr}(\sigma | R\Gamma(Z, \mathbb{Q})) \\ &= \mathrm{Tr}(\sigma | R\Gamma_c(Y^\circ, \mathbb{Q})) + \chi(Z, \mathbb{Q}) \end{aligned}$$

since σ acts trivially on Z hence on $R\Gamma(Z, \mathbb{Q})$. Therefore the result follows from

$$\begin{aligned} \mathrm{Tr}(\sigma | R\Gamma_c(Y^\circ, \mathbb{Q})) &:= \sum_{i \in \mathbb{Z}} (-1)^i \cdot \mathrm{Tr}(\sigma | H_c^i(Y^\circ, \mathbb{Q})) \\ &= \sum_{i \in \mathbb{Z}} (-1)^i \cdot \mathrm{Tr}(\sigma | H_c^i(Y^\circ, \mathbb{Q})^*) \\ &= \sum_{i \in \mathbb{Z}} (-1)^i \cdot \mathrm{Tr}(\sigma | H^{d-i}(Y^\circ, \mathbb{Q})) \\ &= (-1)^d \sum_{i \in \mathbb{Z}} (-1)^i \cdot \mathrm{Tr}(\sigma | H^i(Y^\circ, \mathbb{Q})) \\ &= 0. \end{aligned}$$

where we use Poincaré duality and the Lefschetz fixed point theorem. Here $d = \dim(Y)$. \square

Lemma 2.22. *Let Z be a topological space which is homotopy equivalent to a finite CW-complex. Then we have*

$$\chi(Z, F) = \chi(Z, F')$$

for any pair of fields F, F' .

Proof. The complex $R\Gamma(Z, \mathbb{Z})$ is quasi-isomorphic to a strictly perfect complex of abelian groups C^* and we have

$$\sum_{i \in \mathbb{Z}} (-1)^i \mathrm{rank}_{\mathbb{Z}} C^i = \sum_{i \in \mathbb{Z}} (-1)^i \dim_F(C^i \otimes_{\mathbb{Z}} F) = \sum_{i \in \mathbb{Z}} (-1)^i \dim_F H^i(C^* \otimes_{\mathbb{Z}} F) = \chi(Z, F)$$

for any field F . The result follows. \square

Combining Corollary 2.18 with Prop. 2.20 we obtain.

Proposition 2.23. *We have*

$$\begin{aligned} \lambda_B(\det_{\mathbb{Z}} R\Gamma_W(\mathcal{X}_\infty, \mathbb{Z}(n)) \otimes \det_{\mathbb{Z}}^{-1} R\Gamma_W(\mathcal{X}_\infty, \mathbb{Z}(d-n))) \\ = \det_{\mathbb{Z}} R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{Z}(n)) \cdot 2^{d_-(\mathcal{X}, n) - d_+(\mathcal{X}, n)} \end{aligned}$$

Proof. Note that if C is as in Definition 2.19 then

$$\det_{\mathbb{Z}} C = \mathbb{Z} \cdot \chi^\times(C)^{-1}$$

under the canonical isomorphism

$$\det_{\mathbb{Q}} C_{\mathbb{Q}} \cong \mathbb{Q}$$

arising from the acyclicity of $C_{\mathbb{Q}}$. □

3. DUALITY FOR DERIVED DE RHAM COHOMOLOGY AND THE BLOCH CONDUCTOR

In this section \mathcal{X} is a regular scheme of dimension d , proper and flat over $\text{Spec}(\mathbb{Z})$. We denote by

$$(13) \quad L_{\mathcal{X}/\mathbb{Z}} \cong \Omega_{\mathcal{X}/\mathbb{Z}}[0]$$

the cotangent complex of \mathcal{X} over \mathbb{Z} , a perfect complex of $\mathcal{O}_{\mathcal{X}}$ -modules cohomologically concentrated in degree 0. For any $r \in \mathbb{Z}$ we let

$$L \wedge^r L_{\mathcal{X}/\mathbb{Z}} \cong L \wedge^r \Omega_{\mathcal{X}/\mathbb{Z}}[0]$$

be the r -th derived exterior power of $L_{\mathcal{X}/\mathbb{Z}}$ [13][4.2.2.6] which is again a perfect complex of $\mathcal{O}_{\mathcal{X}}$ -modules. By definition $L \wedge^r L_{\mathcal{X}/\mathbb{Z}} = 0$ for $r < 0$ but $L \wedge^r L_{\mathcal{X}/\mathbb{Z}}$ is in general nonzero for $r > d - 1 = \text{rank}_{\mathcal{O}_{\mathcal{X}}} \Omega_{\mathcal{X}/\mathbb{Z}}$.

3.1. Coherent duality for $L \wedge^r L_{\mathcal{X}/\mathbb{Z}}$. This subsection is a review of material from [22], [15] and [23] in the context of our global arithmetic scheme \mathcal{X} . The key result is Thm. 3.3 which is an immediate translation of [23][Cor. 4.9] to our context.

Lemma 3.1. *There is a canonical map*

$$(14) \quad L \wedge^{d-1} L_{\mathcal{X}/\mathbb{Z}} \rightarrow \det_{\mathcal{O}_{\mathcal{X}}} L_{\mathcal{X}/\mathbb{Z}} \cong \omega_{\mathcal{X}/\mathbb{Z}}$$

where $\omega_{\mathcal{X}/\mathbb{Z}}$ is the relative dualizing sheaf. Hence we get induced maps

$$L \wedge^r L_{\mathcal{X}/\mathbb{Z}} \otimes^L L \wedge^{d-1-r} L_{\mathcal{X}/\mathbb{Z}} \rightarrow L \wedge^{d-1} L_{\mathcal{X}/\mathbb{Z}} \rightarrow \omega_{\mathcal{X}/\mathbb{Z}}$$

and

$$(15) \quad L \wedge^r L_{\mathcal{X}/\mathbb{Z}} \rightarrow \underline{\mathbf{RHom}}(L \wedge^{d-1-r} L_{\mathcal{X}/\mathbb{Z}}, \omega_{\mathcal{X}/\mathbb{Z}})$$

in the derived category of coherent sheaves on \mathcal{X} .

Proof. The multiplicative structure on derived exterior powers will be briefly recalled in the proof of Prop. 3.5 below, so it remains to show the existence of (14). Assume first there is a closed embedding $i : \mathcal{X} \rightarrow P$ of \mathcal{X} into a smooth \mathbb{Z} -scheme P with ideal sheaf \mathcal{I} . The exact sequence of coherent sheaves on \mathcal{X}

$$0 \rightarrow \mathcal{I}/\mathcal{I}^2 \rightarrow i^* \Omega_{P/\mathbb{Z}} \rightarrow \Omega_{\mathcal{X}/\mathbb{Z}} \rightarrow 0$$

can be viewed as a realization of (13) as a strictly perfect complex since $\mathcal{I}/\mathcal{I}^2$ and $i^* \Omega_{P/\mathbb{Z}}$ are locally free of ranks $n - d + 1$ and n , respectively, where n is the relative dimension of P over \mathbb{Z} . The natural map

$$\wedge^{d-1} \Omega_{\mathcal{X}/\mathbb{Z}} \otimes \wedge^{n-d+1} (\mathcal{I}/\mathcal{I}^2) \rightarrow \wedge^n i^* \Omega_{P/\mathbb{Z}}$$

has adjoint

$$\wedge^{d-1} \Omega_{\mathcal{X}/\mathbb{Z}} \rightarrow \underline{\mathbf{Hom}}(\wedge^{n-d+1} (\mathcal{I}/\mathcal{I}^2), \wedge^n i^* \Omega_{P/\mathbb{Z}}) =: \omega_{\mathcal{X}/\mathbb{Z}}^P$$

and combined with the natural map

$$L \wedge^{d-1} L_{\mathcal{X}/\mathbb{Z}} \rightarrow \mathcal{H}^0(L \wedge^{d-1} L_{\mathcal{X}/\mathbb{Z}}) \cong \wedge^{d-1} \Omega_{\mathcal{X}/\mathbb{Z}}$$

we obtain a morphism (14) ^{P} depending on $i : \mathcal{X} \rightarrow P$. If $i' : \mathcal{X} \rightarrow P'$ is another embedding into a smooth \mathbb{Z} -scheme P' an isomorphism

$$\epsilon^{P',P} : \omega_{\mathcal{X}/\mathbb{Z}}^P \xrightarrow{\sim} \omega_{\mathcal{X}/\mathbb{Z}}^{P'}$$

was constructed in [2][A.2] which satisfies the usual cocycle condition in the presence of a third embedding i'' . Since embeddings into smooth schemes always exist Zariski locally on \mathcal{X} the cocycle condition implies that one can define $\omega_{\mathcal{X}/\mathbb{Z}}$ by glueing the locally defined $\omega_{\mathcal{X}/\mathbb{Z}}^P$. It remains to show that likewise the locally obtained maps $(14)^P$ glue to a global map (14). By considering the fibre product $P'' := P \times_{\text{Spec}(\mathbb{Z})} P'$ the construction of $\epsilon^{P',P}$ can be reduced to the case where there exists a smooth morphism $u : P' \rightarrow P$ over $\text{Spec}(\mathbb{Z})$ and under \mathcal{X} . Namely one defines

$$\epsilon^{P',P} := \epsilon^{P'',P'}(q')^{-1} \circ \epsilon^{P'',P}(q)$$

where $q' : P'' \rightarrow P'$ and $q : P'' \rightarrow P$ are the projections and

$$\epsilon^{P',P}(u) : \omega_{\mathcal{X}/\mathbb{Z}}^P \xrightarrow{\sim} \omega_{\mathcal{X}/\mathbb{Z}}^{P'}$$

depends on u . More precisely, $\epsilon^{P',P}(u)$ is defined by the commutative diagram with exact rows and columns

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{I}/\mathcal{I}^2 & \longrightarrow & i^*\Omega_{P/\mathbb{Z}} & \longrightarrow & \Omega_{\mathcal{X}/\mathbb{Z}} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & \mathcal{I}'/(\mathcal{I}')^2 & \longrightarrow & i'^*\Omega_{P'/\mathbb{Z}} & \longrightarrow & \Omega_{\mathcal{X}/\mathbb{Z}} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & i'^*\Omega_{P'/P} & \xlongequal{\quad} & i'^*\Omega_{P'/P} & & \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

where the columns are the transitivity triangles of the cotangent complex for $\mathcal{X} \rightarrow P' \xrightarrow{u} P$ and $P' \xrightarrow{u} P \rightarrow \text{Spec}(\mathbb{Z})$, respectively, and we refer to [2][A.2.2] for the precise sign conventions. The above commutative diagram induces a commutative diagram

$$\begin{array}{ccc} \wedge^{d-1}\Omega_{\mathcal{X}/\mathbb{Z}} & \longrightarrow & \omega_{\mathcal{X}/\mathbb{Z}}^P \\ \parallel & & \downarrow \epsilon^{P',P}(u) \\ \wedge^{d-1}\Omega_{\mathcal{X}/\mathbb{Z}} & \longrightarrow & \omega_{\mathcal{X}/\mathbb{Z}}^{P'} \end{array}$$

so that $(14)^P$ is indeed compatible with the isomorphisms $\epsilon^{P',P}(u)$ and therefore also with the isomorphisms $\epsilon^{P',P}$. □

Definition 3.2. *The Bloch conductor of the arithmetic scheme \mathcal{X} is the positive integer*

$$A(\mathcal{X}) := \prod_p p^{(-1)^{d-1}d_p}$$

where the product is over all prime numbers p , $d_p := \deg c_{d,\mathcal{X}_{\mathbb{F}_p}}^{\mathcal{X}}(\Omega_{\mathcal{X}/\mathbb{Z}}) \in \mathbb{Z}$ and

$$c_{d,\mathcal{X}_{\mathbb{F}_p}}^{\mathcal{X}}(\Omega_{\mathcal{X}/\mathbb{Z}}) \in CH_0(\mathcal{X}_{\mathbb{F}_p})$$

is a localized Chern class introduced in [4].

The Bloch conductor was introduced in [4] and further studied in [5],[22],[15],[23]. The deepest result about the Bloch conductor is its equality with the Artin conductor, defined in terms of the l -adic cohomology of \mathcal{X} , in certain cases. This equality was proven for $d = 2$ in [4] and if \mathcal{X} has everywhere semistable reduction in [15]. For general regular \mathcal{X} it is conjectured but still open. The equality of the Bloch and the Artin conductor is important for establishing cases of Conjecture 1.3 via the Langlands correspondence but plays no role in this section. Here we only review the (slightly) more elementary results of [22] and [23] about $A(\mathcal{X})$. Also note that our normalization of $A(\mathcal{X})$ is different from these references so that $A(\mathcal{X})$ equals the Artin conductor rather than its inverse.

The following theorem was proven by T. Saito in [23][Cor. 4.9]. The case $d = 2, r = 1$ is due to Bloch [5][Thm. 2.3] and the case $r \geq d - 1$ can already be found in T. Saito's earlier article [22]. We give some details of Saito's proof since the exposition in [23] is rather short.

Theorem 3.3. *For any $r \in \mathbb{Z}$ let $C_{\mathcal{X}/\mathbb{Z}}^r$ be the mapping cone of (15), a perfect complex of $\mathcal{O}_{\mathcal{X}}$ -modules. Then $R\Gamma(\mathcal{X}, C_{\mathcal{X}/\mathbb{Z}}^r)$ has finite cohomology and*

$$\chi^\times R\Gamma(\mathcal{X}, C_{\mathcal{X}/\mathbb{Z}}^r) = A(\mathcal{X})^{(-1)^r}$$

where χ^\times is the multiplicative Euler characteristic (see Definition 2.19).

Proof. First note that over the open subset $\mathcal{X}^{sm} \subseteq \mathcal{X}$ where $\mathcal{X} \rightarrow \text{Spec}(\mathbb{Z})$ is smooth the complex $L \wedge^r L_{\mathcal{X}/\mathbb{Z}}$ is concentrated in degree 0 with cohomology the locally free sheaf $\Omega_{\mathcal{X}^{sm}/\mathbb{Z}}^r = \wedge^r \Omega_{\mathcal{X}^{sm}/\mathbb{Z}}$. The map (14) is also an isomorphism over \mathcal{X}^{sm} . Hence, by linear algebra, the map (15) is an isomorphism over \mathcal{X}^{sm} and $C_{\mathcal{X}/\mathbb{Z}}^r$ is supported in $\mathcal{X} \setminus \mathcal{X}^{sm}$. Since $\mathcal{X}_{\mathbb{Q}} \rightarrow \text{Spec}(\mathbb{Q})$ is smooth $\mathcal{X} \setminus \mathcal{X}^{sm}$ is contained in a finite union of closed fibres $\mathcal{X}_{\mathbb{F}_p}$. By [15][Lemma 5.1.1] any point $x \in \mathcal{X} \setminus \mathcal{X}^{sm}$ has a Zariski open neighborhood $U \subseteq \mathcal{X}$ such that there exists a closed embedding

$$U \rightarrow P$$

where $P \rightarrow \text{Spec}(\mathbb{Z})$ is smooth of relative dimension d . The exact sequence

$$(16) \quad 0 \rightarrow N_{U/P} \rightarrow \Omega_{P/\mathbb{Z}} \otimes_{\mathcal{O}_P} \mathcal{O}_U \rightarrow \Omega_{U/\mathbb{Z}} \rightarrow 0$$

then shows that $\Omega_{\mathcal{X}/\mathbb{Z}}$ can be locally generated by d sections and that $\wedge^d \Omega_{\mathcal{X}/\mathbb{Z}}$ is locally monogenic. Following [15][Lemma 5.1.3] let

$$i: Z \rightarrow \mathcal{X}$$

be the closed subscheme with support $\mathcal{X} \setminus \mathcal{X}^{sm}$ [15][Lemma 3.1.2] defined by the ideal sheaf

$$\text{Ann } \wedge^d \Omega_{\mathcal{X}/\mathbb{Z}}.$$

Then $i^* \wedge^d \Omega_{\mathcal{X}/\mathbb{Z}}$ is an invertible \mathcal{O}_Z -module by definition and hence $i^* \Omega_{\mathcal{X}/\mathbb{Z}}$ is locally free of rank d , as the d generating sections have no relation on Z . It follows that

$$Li^* \Omega_{\mathcal{X}/\mathbb{Z}}|_U = \left(N_{U/P} \otimes_{\mathcal{O}_U} \mathcal{O}_{U \cap Z} \xrightarrow{0} \Omega_{P/\mathbb{Z}} \otimes_{\mathcal{O}_P} \mathcal{O}_{U \cap Z} \right)$$

and hence that

$$\mathcal{L} := L^1 i^* \Omega_{\mathcal{X}/\mathbb{Z}}$$

is an invertible \mathcal{O}_Z -module.

Lemma 3.4. *The coherent sheaves $\mathcal{H}^i(C_{\mathcal{X}/\mathbb{Z}}^r)$ are \mathcal{O}_Z -modules and there are canonical isomorphisms*

$$(17) \quad \mathcal{L} \otimes_{\mathcal{O}_Z} \mathcal{H}^i(C_{\mathcal{X}/\mathbb{Z}}^r) \cong \mathcal{H}^{i-1}(C_{\mathcal{X}/\mathbb{Z}}^{r+1})$$

for any $i, r \in \mathbb{Z}$.

Proof. We follow the proof of [22][Prop. 1.7] where the case $r \geq d-1$ is treated, see also [15][Lemma 2.4.2]. Recall that

$$L_{\mathcal{X}/\mathbb{Z}}|_U \cong \Omega_{U/\mathbb{Z}}[0]$$

is represented by the strictly perfect complex (16) where the conormal bundle $N_U := N_{U/P}$ is invertible and $E_U := \Omega_{P/U} \otimes_{\mathcal{O}_P} \mathcal{O}_U$ is a vector bundle of rank d . For $r \geq 0$ we have isomorphisms

$$(18) \quad \begin{aligned} L \wedge^r L_{\mathcal{X}/\mathbb{Z}}|_U &\cong L \wedge^r \left(N_U \xrightarrow{v} E_U \right) \\ &\cong \Gamma^r N_U \rightarrow \Gamma^{r-1} N_U \otimes E_U \rightarrow \Gamma^{r-2} N_U \otimes \wedge^2 E_U \rightarrow \cdots \rightarrow \wedge^r E_U \\ &\cong N_U^{\otimes r} \rightarrow N_U^{\otimes r-1} \otimes E_U \rightarrow N_U^{\otimes r-2} \otimes \wedge^2 E_U \rightarrow \cdots \rightarrow \wedge^r E_U \end{aligned}$$

where Γ^i denotes the divided power functor and $\Gamma^i N_U \cong N_U^{\otimes i}$ since N_U is invertible. The differential is given by

$$(19) \quad x' \otimes x \otimes y \in N_U^{\otimes i-1} \otimes N_U \otimes \wedge^{r-i} E_U \mapsto x' \otimes v(x) \wedge y \in N_U^{\otimes i-1} \otimes \wedge^{r-i+1} E_U$$

on local sections. This computation of the derived exterior powers of a strictly perfect two-term complex goes back to Illusie [13][4.3.1.3] and is also recalled in [15][1.2.7.2]. From this description it is clear that there is an identity of complexes

$$(20) \quad N_U \otimes L \wedge^r \left(N_U \xrightarrow{v} E_U \right) = \left(\sigma^{<0} L \wedge^{r+1} \left(N_U \xrightarrow{v} E_U \right) \right) [-1]$$

where $\sigma^{<0}$ refers to the naive truncation. Similarly we find

$$(21) \quad \begin{aligned} \underline{R}\underline{\mathrm{Hom}}(L \wedge^{d-1-r} L_{\mathcal{X}/\mathbb{Z}}, \omega_{\mathcal{X}/\mathbb{Z}})|_U &\cong \underline{\mathrm{Hom}}(L \wedge^{d-1-r} \left(N_U \xrightarrow{v} E_U \right), K_U) \\ &\cong \underline{\mathrm{Hom}}(\wedge^{d-1-r} E_U, K_U) \rightarrow \cdots \rightarrow \underline{\mathrm{Hom}}(N_U^{\otimes i} \otimes \wedge^{d-1-r-i} E_U, K_U) \rightarrow \cdots \end{aligned}$$

where

$$K_U := N_U^{-1} \otimes \wedge^d E_U \cong \omega_{\mathcal{X}/\mathbb{Z}}|_U.$$

Using the canonical isomorphism

$$(22) \quad N_U \otimes \underline{\mathrm{Hom}}(N_U^{\otimes i} \otimes \wedge^{d-1-r-i} E_U, K_U) \cong \underline{\mathrm{Hom}}(N_U^{\otimes i-1} \otimes \wedge^{d-1-(r+1)-(i-1)} E_U, K_U)$$

we find a canonical isomorphism of complexes

$$(23) \quad \begin{aligned} \sigma^{>0} N_U \otimes \underline{\mathrm{Hom}}(L \wedge^{d-1-r} \left(N_U \xrightarrow{v} E_U \right), K_U) \\ \cong \underline{\mathrm{Hom}}(L \wedge^{d-1-(r+1)} \left(N_U \xrightarrow{v} E_U \right), K_U)[-1]. \end{aligned}$$

The complex $C_{\mathcal{X}/\mathbb{Z}}^r|_U$ is obtained by splicing together (18) placed in degrees ≤ -1 with (21) placed in degrees ≥ 0 via the map

$$\phi_r : \wedge^r E_U \rightarrow \underline{\mathrm{Hom}}(\wedge^{d-1-r} E_U, K_U) \cong \underline{\mathrm{Hom}}(N_U \otimes \wedge^{d-1-r} E_U, \wedge^d E_U)$$

dual to

$$\wedge^r E_U \otimes N_U \otimes \wedge^{d-1-r} E_U \rightarrow \wedge^d E_U; \quad y \otimes x \otimes y' \mapsto v(x) \wedge y \wedge y'.$$

Denoting by ψ the canonical isomorphism

$$\psi : \wedge^{r+1} E_U \cong \underline{\mathbf{Hom}}(\wedge^{d-1-r} E_U, \wedge^d E_U) \cong N_U \otimes \underline{\mathbf{Hom}}(\wedge^{d-1-r} E_U, K_U)$$

we have a commutative diagram

(24)

$$\begin{array}{ccccc} N_U \otimes \wedge^r E_U & \xrightarrow{(19)} & \wedge^{r+1} E_U & \xrightarrow{\phi_{r+1}} & \underline{\mathbf{Hom}}(\wedge^{d-2-r} E_U, K_U) \\ \parallel & & \sim \downarrow \psi & & \sim \downarrow (22)^{-1} \end{array}$$

$$N_U \otimes \wedge^r E_U \xrightarrow{\text{id} \otimes \phi_r} N_U \otimes \underline{\mathbf{Hom}}(\wedge^{d-1-r} E_U, K_U) \rightarrow N_U \otimes \underline{\mathbf{Hom}}(N_U \otimes \wedge^{d-2-r} E_U, K_U)$$

as one verifies easily on local sections. Combining (20), (23) and (24) we obtain a canonical isomorphism

$$(25) \quad N_U \otimes C_{\mathcal{X}/\mathbb{Z}}^r|_U \cong C_{\mathcal{X}/\mathbb{Z}}^{r+1}|_U[-1].$$

As in [22][(1.6.1)] one has an isomorphism

$$C_{\mathcal{X}/\mathbb{Z}}^{d-1}|_U \cong K_U \otimes \text{Kos}(E_U^* \otimes N_U \xrightarrow{v^* \otimes \text{id}} N_U^* \otimes N_U \cong \mathcal{O}_U)$$

where $\text{Kos}(P \rightarrow A)$ denotes the Koszul algebra associated to a A -module homomorphism $P \rightarrow A$ where P is finitely generated projective over A . Using the fact that $H^i(\text{Kos}(P \rightarrow A))$ is a module over the ring $H^0(\text{Kos}(P \rightarrow A))$ [24][15.28.6] one deduces that all coherent sheaves $\mathcal{H}^i(C_{\mathcal{X}/\mathbb{Z}}^{d-1}|_U)$ are modules over $H^0(\text{Kos}) \cong \mathcal{O}_{U \cap Z}$. Using (25) and the fact that N_U is invertible we deduce that all coherent sheaves $\mathcal{H}^i(C_{\mathcal{X}/\mathbb{Z}}^r|_U)$ are modules over $\mathcal{O}_{U \cap Z}$, and an isomorphism

$$(26) \quad (\mathcal{L} \otimes_{\mathcal{O}_Z} \mathcal{H}^i(C_{\mathcal{X}/\mathbb{Z}}^r))|_U \cong \mathcal{H}^{i-1}(C_{\mathcal{X}/\mathbb{Z}}^{r+1})|_U$$

whose construction a priori depends on the choice of $U \rightarrow P$. However, as in the proof of [22][(1.7.2)] one shows that for a different embedding $U \rightarrow P'$, leading to a different strictly perfect resolution $N'_U \rightarrow E'_U$ of $L_{\mathcal{X}/\mathbb{Z}}|_U$, one has a quasi-isomorphism

$$g : (N'_U \rightarrow E'_U) \rightarrow (N_U \rightarrow E_U)$$

unique up to homotopy, inducing quasi-isomorphisms

$$g^r : L \wedge^r (N'_U \rightarrow E'_U) \rightarrow L \wedge^r (N_U \rightarrow E_U)$$

for all r , unique up to homotopy, which commute with the isomorphisms (20), (23) and (24). Hence (26) is in fact independent of the choice of $U \rightarrow P$ which also implies that the local isomorphisms (26) glue to the global isomorphism (17). \square

Since $\mathcal{X} \rightarrow \text{Spec}(\mathbb{Z})$ is proper and $C_{\mathcal{X}/\mathbb{Z}}^r$ is a perfect complex of $\mathcal{O}_{\mathcal{X}}$ -modules $R\Gamma(\mathcal{X}, C_{\mathcal{X}/\mathbb{Z}}^r)$ is a perfect complex of abelian groups. It has a finite filtration with subquotients

$$R\Gamma(\mathcal{X}, \mathcal{H}^i(C_{\mathcal{X}/\mathbb{Z}}^r)[-i]) \cong R\Gamma(Z, \mathcal{H}^i(C_{\mathcal{X}/\mathbb{Z}}^r)[-i])$$

which are perfect complexes of abelian groups with torsion cohomology, as Z is supported in a finite union of closed fibres $\mathcal{X}_{\mathbb{F}_p}$. Hence $R\Gamma(\mathcal{X}, C_{\mathcal{X}/\mathbb{Z}}^r)$ has finite cohomology. We can view χ^\times as a homomorphism

$$\chi^\times : G(Z) \rightarrow K_0(\mathbb{Z}; \mathbb{Q}) \cong \mathbb{Q}^{\times, >0}; \quad [\mathcal{F}] \mapsto [R\Gamma(Z, \mathcal{F})]$$

where $G(Z)$ is the Grothendieck group of the category of coherent sheaves on Z and $K_0(\mathbb{Z}; \mathbb{Q})$ is the Grothendieck group of the category of finite abelian groups (which is also the relative K_0 for the ring homomorphism $\mathbb{Z} \rightarrow \mathbb{Q}$). By [15][Lemma 5.1.3.3] one has $[\mathcal{L} \otimes_{\mathcal{O}_Z} \mathcal{F}] = [\mathcal{F}]$ in $G(Z)$ for any coherent sheaf \mathcal{F} on Z . Hence (17) implies

$$\chi^\times R\Gamma(Z, \mathcal{H}^i(C_{\mathcal{X}/\mathbb{Z}}^r)) = \chi^\times R\Gamma(Z, \mathcal{H}^{i-1}(C_{\mathcal{X}/\mathbb{Z}}^{r+1}))$$

and therefore

$$\chi^\times R\Gamma(\mathcal{X}, C_{\mathcal{X}/\mathbb{Z}}^r) = \chi^\times R\Gamma(\mathcal{X}, C_{\mathcal{X}/\mathbb{Z}}^{r+1})^{-1}$$

for any $r \in \mathbb{Z}$. On the other hand we have

$$\chi^\times R\Gamma(\mathcal{X}, C_{\mathcal{X}/\mathbb{Z}}^d) = \chi^\times R\Gamma(\mathcal{X}, L \wedge^d L_{\mathcal{X}/\mathbb{Z}}[1]) = A(\mathcal{X})^{(-1)^d}$$

by [22][Prop. 2.3]. This finishes the proof of the theorem. \square

3.2. Duality for derived de Rham cohomology. Denote by

$$\cdots \rightarrow F^{r+1} \rightarrow F^r \rightarrow \cdots \rightarrow R\Gamma_{dR}(\mathcal{X}/\mathbb{Z}) = F^0 = F^{-1} = \cdots$$

the Hodge filtration of (Hodge completed) derived de Rham cohomology and by F^n/F^m the mapping cone of $F^m \rightarrow F^n$ for $m \geq n$. Since

$$(27) \quad F^r/F^{r+1} \cong R\Gamma(\mathcal{X}, L \wedge^r L_{\mathcal{X}/\mathbb{Z}}[-r])$$

is a perfect complex of abelian groups, so are all F^n/F^m for $m \geq n$. Denote by $C^* = R\mathrm{Hom}(C, \mathbb{Z})$ the \mathbb{Z} -dual of a perfect complex of abelian groups.

Proposition 3.5. *a) For $n \leq d$ there is a (Poincaré) duality map*

$$(28) \quad \epsilon_n : F^n/F^d \rightarrow (R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^{d-n})^*[-2d+2]$$

satisfying

$$(29) \quad \chi^\times \mathrm{Cone}(\epsilon_n) = A(\mathcal{X})^{d-n}.$$

b) In particular, the discriminant of the Poincaré duality pairing

$$(30) \quad R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^d \otimes_{\mathbb{Z}}^L R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^d \rightarrow \mathbb{Z}[-2d+2]$$

has absolute value $A(\mathcal{X})^d$.

Remark 3.6. *For $d = 1$ we have $\mathcal{X} = \mathrm{Spec}(\mathcal{O}_F)$ and $A(\mathcal{X}) = |D_F|$, and b) reduces to the fact that the trace pairing*

$$\mathcal{O}_F \times \mathcal{O}_F \rightarrow \mathbb{Z}; \quad (a, b) \mapsto \mathrm{Tr}(ab)$$

has discriminant D_F . For $d = 2$ it was shown by Bloch in [5][Thm. 2] that the Poincaré duality pairing on the complex

$$R\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}} \rightarrow \Omega_{\mathcal{X}/\mathbb{Z}}) \cong R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^2$$

has discriminant $\pm A(\mathcal{X})^2$. For $d \geq 3$ it seems harder to describe the complex $R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^d$ more explicitly.

Remark 3.7. *If P is a perfect complex of abelian groups and $P \otimes_{\mathbb{Z}}^L P \rightarrow \mathbb{Z}[2\delta]$ is a pairing which induces an isogeny $\phi : P \rightarrow P^*[2\delta]$ in the sense that $\text{Cone}(\phi)$ has finite cohomology groups, we obtain isomorphisms*

$$\det_{\mathbb{Z}} P^* \simeq \det_{\mathbb{Z}} P \otimes_{\mathbb{Z}} \det_{\mathbb{Z}} \text{Cone}(\phi)$$

and

$$\det_{\mathbb{Z}} P \otimes \det_{\mathbb{Z}} P \simeq \det_{\mathbb{Z}}^{-1} \text{Cone}(\phi)$$

and hence a duality pairing on determinants

$$\langle \cdot, \cdot \rangle : \det_{\mathbb{Q}} P_{\mathbb{Q}} \otimes \det_{\mathbb{Q}} P_{\mathbb{Q}} \simeq \mathbb{Q}.$$

The discriminant of the pairing is $\langle b, b \rangle \in \mathbb{Q}$ where b is a \mathbb{Z} -basis of $\det_{\mathbb{Z}} P$. Since

$$\langle -b, -b \rangle = (-1)^2 \langle b, b \rangle = \langle b, b \rangle$$

the discriminant is a well-defined rational number (of absolute value $\chi^{\times} \text{Cone}(\phi)$).

Proof. Poincaré duality for algebraic de Rham cohomology of $\mathcal{X}_{\mathbb{Q}}/\mathbb{Q}$ is discussed in [24][Prop. 50.20.4]. It turns out that one can lift the construction of the cup product pairing in loc. cit. to the derived de Rham complex on \mathcal{X} since we are truncating by F^d . More precisely, choose a simplicial resolution $P_{\bullet} \rightarrow \mathcal{O}_{\mathcal{X}}$ in \mathcal{X}_{Zar} where P_i is a free \mathbb{Z} -algebra in \mathcal{X}_{Zar} and denote by $\Omega_{P_{\bullet}/\mathbb{Z}}^{[n,m]}$ the complex (of simplicial modules)

$$\Omega_{P_{\bullet}/\mathbb{Z}}^n \rightarrow \cdots \rightarrow \Omega_{P_{\bullet}/\mathbb{Z}}^m$$

in degrees $[n, m]$, zero for $n > m$, where the differential is the de Rham differential. Define a complex of sheaves of abelian groups on \mathcal{X}_{Zar}

$$L\Omega_{\mathcal{X}/\mathbb{Z}}^{[n,m]} := \text{Tot}_{\bullet}^* \Omega_{P_{\bullet}/\mathbb{Z}}^{[n,m], \sim}$$

so that

$$L \wedge^n L_{\mathcal{X}/\mathbb{Z}}[-n] = L\Omega_{\mathcal{X}/\mathbb{Z}}^{[n,n]}; \quad R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^n = R\Gamma(\mathcal{X}, L\Omega_{\mathcal{X}/\mathbb{Z}}^{[0,n-1]}).$$

Here and in the following we denote by M_{\bullet}^{\sim} the $(n$ -tuple) chain complex associated to a $(n$ -tuple) simplicial module M_{\bullet} [13][1.1] and we decorate the (partial) totalization of an n -tuple complex with the indices that are contracted into one. We use the convention that totalization of an upper and a lower index leads to an upper index. As in [24][50.4.0.1] the wedge product on differential forms induces a map of bicomplexes

$$\begin{aligned} \text{Tot}^{*,*} \text{Tot}_{\bullet,\bullet} \Omega_{P_{\bullet}/\mathbb{Z}}^{*,\sim} \otimes_{\mathbb{Z}} \Omega_{P_{\bullet}/\mathbb{Z}}^{*,\sim} &= \text{Tot}^{*,*} \text{Tot}_{\bullet,\bullet} \left(\Omega_{P_{\bullet}/\mathbb{Z}}^* \otimes_{\mathbb{Z}} \Omega_{P_{\bullet}/\mathbb{Z}}^* \right)^{\sim} \\ &\xrightarrow{\sigma} \text{Tot}^{*,*} \left(\Delta \left(\Omega_{P_{\bullet}/\mathbb{Z}}^* \otimes_{\mathbb{Z}} \Omega_{P_{\bullet}/\mathbb{Z}}^* \right) \right)^{\sim} \\ &= \left(\text{Tot}^{*,*} \Delta \left(\Omega_{P_{\bullet}/\mathbb{Z}}^* \otimes_{\mathbb{Z}} \Omega_{P_{\bullet}/\mathbb{Z}}^* \right) \right)^{\sim} \\ &\xrightarrow{\cup} \Omega_{P_{\bullet}/\mathbb{Z}}^{*,\sim} \rightarrow \Omega_{P_{\bullet}/\mathbb{Z}}^{[0,d-1],\sim} \end{aligned}$$

where σ is induced by shuffle map $\text{Tot}_{\bullet,\bullet} (M_{\bullet} \otimes N_{\bullet})^{\sim} \rightarrow (\Delta (M_{\bullet} \otimes N_{\bullet}))^{\sim}$ of [13][1.2.2.1] and Δ denotes the diagonal simplicial object of a bisimplicial object. Since we have truncated to degrees $\leq d-1$ the above pairing factors through a pairing

$$\text{Tot}^{*,*} \text{Tot}_{\bullet,\bullet} \Omega_{P_{\bullet}/\mathbb{Z}}^{[n,d-1],\sim} \otimes_{\mathbb{Z}} \Omega_{P_{\bullet}/\mathbb{Z}}^{[0,d-1-n],\sim} \rightarrow \Omega_{P_{\bullet}/\mathbb{Z}}^{[0,d-1],\sim}$$

and hence we obtain a pairing

$$\begin{aligned}
L\Omega_{\mathcal{X}/\mathbb{Z}}^{[n,d-1]} \otimes_{\mathbb{Z}}^L L\Omega_{\mathcal{X}/\mathbb{Z}}^{[0,d-1-n]} &= \mathrm{Tot}^{*,*} \mathrm{Tot}_{\bullet}^* \Omega_{P_{\bullet}/\mathbb{Z}}^{[n,d-1],\sim} \otimes_{\mathbb{Z}} \mathrm{Tot}_{\bullet}^* \Omega_{P_{\bullet}/\mathbb{Z}}^{[0,d-1-n],\sim} \\
&\simeq \mathrm{Tot}^{*,*} \mathrm{Tot}_{\bullet}^* \mathrm{Tot}_{\bullet}^* \Omega_{P_{\bullet}/\mathbb{Z}}^{[n,d-1],\sim} \otimes_{\mathbb{Z}} \Omega_{P_{\bullet}/\mathbb{Z}}^{[0,d-1-n],\sim} \\
&\simeq \mathrm{Tot}_{\bullet,\bullet}^{*,*} \Omega_{P_{\bullet}/\mathbb{Z}}^{[n,d-1],\sim} \otimes_{\mathbb{Z}} \Omega_{P_{\bullet}/\mathbb{Z}}^{[0,d-1-n],\sim} \\
&\simeq \mathrm{Tot}_{\bullet}^* \mathrm{Tot}^{*,*} \mathrm{Tot}_{\bullet,\bullet} \Omega_{P_{\bullet}/\mathbb{Z}}^{[n,d-1],\sim} \otimes_{\mathbb{Z}} \Omega_{P_{\bullet}/\mathbb{Z}}^{[0,d-1-n],\sim} \\
&\rightarrow \mathrm{Tot}_{\bullet}^* \Omega_{P_{\bullet}/\mathbb{Z}}^{[0,d-1],\sim} = L\Omega_{\mathcal{X}/\mathbb{Z}}^{[0,d-1]}
\end{aligned}$$

and an induced pairing on cohomology

$$\begin{array}{ccc}
R\Gamma(\mathcal{X}, L\Omega_{\mathcal{X}/\mathbb{Z}}^{[n,d-1]}) \otimes_{\mathbb{Z}}^L R\Gamma(\mathcal{X}, L\Omega_{\mathcal{X}/\mathbb{Z}}^{[0,d-1-n]}) & \longrightarrow & R\Gamma(\mathcal{X}, L\Omega_{\mathcal{X}/\mathbb{Z}}^{[0,d-1]}) \\
\parallel & & \parallel \\
F^n/F^d \otimes_{\mathbb{Z}}^L R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^{d-n} & \longrightarrow & R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^d.
\end{array}$$

Lemma 3.8. *One has $H^i(\mathcal{X}, L\Omega_{\mathcal{X}/\mathbb{Z}}^{[0,d-1]}) = 0$ for $i > 2d - 2$. Moreover, the natural map*

$$H^{2d-2}(\mathcal{X}, L \wedge^{d-1} L_{\mathcal{X}/\mathbb{Z}}[-d+1]) \rightarrow H^{2d-2}(\mathcal{X}, L\Omega_{\mathcal{X}/\mathbb{Z}}^{[0,d-1]})$$

induces an isomorphism

$$g : H^{2d-2}(\mathcal{X}, L \wedge^{d-1} L_{\mathcal{X}/\mathbb{Z}}[-d+1])/\mathrm{tor} \cong H^{2d-2}(\mathcal{X}, L\Omega_{\mathcal{X}/\mathbb{Z}}^{[0,d-1]})/\mathrm{tor}$$

and therefore a trace map

$$\begin{aligned}
(31) \quad R\Gamma(\mathcal{X}, L\Omega_{\mathcal{X}/\mathbb{Z}}^{[0,d-1]})[2d-2] &\rightarrow H^{2d-2}(\mathcal{X}, L\Omega_{\mathcal{X}/\mathbb{Z}}^{[0,d-1]})/\mathrm{tor} \xrightarrow{g^{-1}} \\
&H^{2d-2}(\mathcal{X}, L \wedge^{d-1} L_{\mathcal{X}/\mathbb{Z}}[-d+1])/\mathrm{tor} \xrightarrow{(14)_*} H^{2d-2}(\mathcal{X}, \omega_{\mathcal{X}/\mathbb{Z}}[-d+1])/\mathrm{tor} \xrightarrow{\mathrm{Tr}} \mathbb{Z}
\end{aligned}$$

Proof. We first remark that $H^i(\mathcal{X}, \mathcal{F}) = 0$ for $i \geq d$ and any coherent sheaf \mathcal{F} on \mathcal{X} . Indeed, this is clear for $i > d$ since the cohomological dimension of \mathcal{X}_{Zar} is d . Duality for $f : \mathcal{X} \rightarrow \mathrm{Spec}(\mathbb{Z})$

$$R\mathrm{Hom}_{\mathbb{Z}}(Rf_*\mathcal{F}, \mathbb{Z}) \cong R\mathrm{Hom}_{\mathcal{X}}(\mathcal{F}, \omega_{\mathcal{X}/\mathbb{Z}}[d-1])$$

evaluated in degree $-d$

$$\mathrm{Hom}_{\mathbb{Z}}(H^d(\mathcal{X}, \mathcal{F}), \mathbb{Z}) \cong H^{-1}R\mathrm{Hom}_{\mathcal{X}}(\mathcal{F}, \omega_{\mathcal{X}/\mathbb{Z}}) = 0$$

shows that $H^d(\mathcal{X}, \mathcal{F})$ is torsion. Evaluation in degree $-d+1$

$$0 \rightarrow \mathrm{Ext}^1(H^d(\mathcal{X}, \mathcal{F}), \mathbb{Z}) \rightarrow \mathrm{Hom}_{\mathcal{X}}(\mathcal{F}, \omega_{\mathcal{X}/\mathbb{Z}}) \rightarrow \mathrm{Hom}_{\mathbb{Z}}(H^{d-1}(\mathcal{X}, \mathcal{F}), \mathbb{Z}) \rightarrow 0$$

shows that $H^d(\mathcal{X}, \mathcal{F}) = 0$ since $\omega_{\mathcal{X}/\mathbb{Z}}$ is a line bundle, f is flat, and therefore $\mathrm{Hom}_{\mathcal{X}}(\mathcal{F}, \omega_{\mathcal{X}/\mathbb{Z}})$ is torsion free.

Since $L \wedge^r L_{\mathcal{X}/\mathbb{Z}}$ is an object of the derived category of coherent sheaves concentrated in degrees ≤ 0 we also have $H^i(\mathcal{X}, L \wedge^r L_{\mathcal{X}/\mathbb{Z}}) = 0$ for $i \geq d$. The exact triangle

$$R\Gamma(\mathcal{X}, L \wedge^r L_{\mathcal{X}/\mathbb{Z}}[-r]) \rightarrow R\Gamma(\mathcal{X}, L\Omega_{\mathcal{X}/\mathbb{Z}}^{[n,r]}) \rightarrow R\Gamma(\mathcal{X}, L\Omega_{\mathcal{X}/\mathbb{Z}}^{[n,r-1]}) \rightarrow$$

and an easy induction then show that $H^i(\mathcal{X}, L\Omega_{\mathcal{X}/\mathbb{Z}}^{[n,m]}) = 0$ for $i \geq d + m$. In particular, the map

$$H^{2d-2}(\mathcal{X}, L \wedge^{d-1} L_{\mathcal{X}/\mathbb{Z}}[-d+1]) \rightarrow H^{2d-2}(\mathcal{X}, L\Omega_{\mathcal{X}/\mathbb{Z}}^{[0,d-1]})$$

is surjective and an isomorphism after tensoring with \mathbb{Q} (see the proof of [24][Prop. 50.20.4]), hence induces an isomorphism

$$g : H^{2d-2}(\mathcal{X}, L \wedge^{d-1} L_{\mathcal{X}/\mathbb{Z}}[-d+1])/\text{tor} \cong H^{2d-2}(\mathcal{X}, L\Omega_{\mathcal{X}/\mathbb{Z}}^{[0,d-1]})/\text{tor}.$$

□

We now prove (29) by downward induction on n starting with the trivial case $n = d$. The induction step is provided by the diagram with exact rows and columns

$$\begin{array}{ccccc} F^{n+1}/F^d & \xrightarrow{\epsilon_{n+1}} & (F^0/F^{d-n-1})^*[-2d+2] & \longrightarrow & \text{Cone}(\epsilon_{n+1}) \\ \downarrow & & \downarrow & & \downarrow \\ F^n/F^d & \xrightarrow{\epsilon_n} & (F^0/F^{d-n})^*[-2d+2] & \longrightarrow & \text{Cone}(\epsilon_n) \\ \downarrow & & \downarrow & & \downarrow \\ F^n/F^{n+1} & \longrightarrow & (F^{d-n-1}/F^{d-n})^*[-2d+2] & \longrightarrow & R\Gamma(\mathcal{X}, C_{\mathcal{X}/\mathbb{Z}}^n)[-n] \end{array}$$

where the bottom exact triangle is $R\Gamma(\mathcal{X}, -)[-n]$ applied to (15) in view of (27) and coherent sheaf duality for $f : \mathcal{X} \rightarrow \text{Spec}(\mathbb{Z})$:

$$\begin{aligned} (F^{d-n-1}/F^{d-n})^*[-2d+2] &= R\text{Hom}(Rf_*(L \wedge^{d-1-n} L_{\mathcal{X}/\mathbb{Z}})[-d+1+n], \mathbb{Z})[-2d+2] \\ &\cong R\text{Hom}_{\mathcal{X}}(L \wedge^{d-1-n} L_{\mathcal{X}/\mathbb{Z}}[-d+1+n], \omega_{\mathcal{X}/\mathbb{Z}})[-d+1] \\ &\cong R\Gamma(\mathcal{X}, \underline{R\text{Hom}}_{\mathcal{X}}(L \wedge^{d-1-n} L_{\mathcal{X}/\mathbb{Z}}, \omega_{\mathcal{X}/\mathbb{Z}}))[-n]. \end{aligned}$$

By Theorem 3.3 we have

$$\chi^\times(\text{Cone}(\epsilon_n)) = \chi^\times(\text{Cone}(\epsilon_{n+1})) \cdot A(\mathcal{X})$$

which gives $\chi^\times(\text{Cone}(\epsilon_n)) = A(\mathcal{X})^{d-n}$ by induction. □

For any $n \in \mathbb{Z}$ we have an exact triangle on the generic fibre $X = \mathcal{X}_{\mathbb{Q}}$

$$(32) \quad F^n \rightarrow R\Gamma_{dR}(X/\mathbb{Q}) \rightarrow R\Gamma_{dR}(X/\mathbb{Q})/F^n$$

and we also have a duality isomorphism (28) $_{\mathbb{Q}}$ for any $n \in \mathbb{Z}$ since $F^n = F^d = 0$ on the generic fibre for $n \geq d$.

Corollary 3.9. *Let $n \in \mathbb{Z}$ and denote by λ_{dR} the isomorphism*

$$\begin{aligned} &\left(\det_{\mathbb{Z}}^{-1} R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^n \otimes \det_{\mathbb{Z}} R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^{d-n} \right)_{\mathbb{Q}} \\ &\simeq \det_{\mathbb{Q}}^{-1} R\Gamma_{dR}(X/\mathbb{Q})/F^n \otimes \det_{\mathbb{Q}} R\Gamma_{dR}(X/\mathbb{Q})/F^{d-n} \\ &\stackrel{(28)_{\mathbb{Q}}}{\simeq} \det_{\mathbb{Q}}^{-1} R\Gamma_{dR}(X/\mathbb{Q})/F^n \otimes \det_{\mathbb{Q}}^{-1} F^n \\ &\stackrel{(32)}{\simeq} \det_{\mathbb{Q}}^{-1} R\Gamma_{dR}(X/\mathbb{Q}) \\ &\simeq \left(\det_{\mathbb{Z}}^{-1} R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^d \right)_{\mathbb{Q}}. \end{aligned}$$

Then

$$\lambda_{dR} \left(\det_{\mathbb{Z}}^{-1} R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^n \otimes \det_{\mathbb{Z}} R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^{d-n} \right) = A(\mathcal{X})^{d-n} \cdot \det_{\mathbb{Z}}^{-1} R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^d.$$

Proof. For $n \leq d$ this is clear from Prop. 3.5 and the fact that (32) is the scalar extension to \mathbb{Q} of the exact triangle

$$F^n/F^d \rightarrow R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^d \rightarrow R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^n.$$

For $n > d$ we have $R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^{d-n} = 0$ and an exact triangle

$$F^d/F^n \rightarrow R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^n \rightarrow R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^d$$

where

$$\begin{aligned} \chi^\times(F^d/F^n) &= \prod_{r=d}^{n-1} \chi^\times(R\Gamma(\mathcal{X}, L \wedge^r L_{\mathcal{X}/\mathbb{Z}}[-r])) \\ &= \prod_{r=d}^{n-1} \chi^\times(R\Gamma(\mathcal{X}, C_{\mathcal{X}/\mathbb{Z}}^r[-r-1])) \\ &= A(\mathcal{X})^{d-n} \end{aligned}$$

by (27) and Theorem 3.3. Hence

$$\det_{\mathbb{Z}}^{-1} R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^n = A(\mathcal{X})^{d-n} \cdot \det_{\mathbb{Z}}^{-1} R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^d$$

inside $\det_{\mathbb{Q}}^{-1} R\Gamma_{dR}(X/\mathbb{Q})$. □

4. THE ARCHIMEDEAN EULER FACTOR

Following [21], for any pure \mathbb{R} -Hodge structure M over \mathbb{R} of weight $w(M)$ we define

$$h_j(M) = \dim F^j/F^{j+1} = h^{j, w(M)-j}(M)$$

$$d_{\pm}(M) = \dim_{\mathbb{R}} M^{F_{\infty} = \pm 1}$$

$$t_H(M) = \sum_j j h_j(M) = \frac{w(M) \cdot \dim_{\mathbb{R}} M}{2} = \frac{w(\det(M))}{2}$$

$$L_{\infty}(M, s) = \prod_{p < q := w(M) - p} \Gamma_{\mathbb{C}}(s-p)^{h^{p,q}} \cdot \prod_{p = \frac{w(M)}{2}} \Gamma_{\mathbb{R}}(s-p)^{h^{p,+}} \Gamma_{\mathbb{R}}(s-p+1)^{h^{p,-}}$$

where

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2} \Gamma(s/2); \quad \Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s} \Gamma(s).$$

Note that the factorization of $L_{\infty}(M, s)$ corresponds to the decomposition of M into simple \mathbb{R} -Hodge structures over \mathbb{R} . Also recall the leading coefficient of the Γ -function at $j \in \mathbb{Z}$

$$(33) \quad \Gamma^*(j) = \begin{cases} (j-1)! & j \geq 1 \\ (-1)^j / (-j)! & j \leq 0 \end{cases}$$

Lemma 4.1. (see also [21][4.3.2, Lemme C.3.7]) For any pure \mathbb{R} -Hodge structure M over \mathbb{R} one has

$$\frac{L_\infty^*(M, 0)}{L_\infty^*(M^*(1), 0)} = \pm 2^{d_+(M) - d_-(M)} (2\pi)^{d_-(M) + t_H(M)} \prod_j \Gamma^*(-j)^{h_j(M)}$$

Proof. The functional equation of the Γ -function

$$(34) \quad \Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$$

implies

$$\Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(1-s) = \frac{2}{\sin(\pi s)}; \quad \Gamma_{\mathbb{R}}(1+s)\Gamma_{\mathbb{R}}(1-s) = \cos\left(\frac{\pi s}{2}\right)^{-1}.$$

Hence

$$(35) \quad \frac{\Gamma_{\mathbb{C}}(s-p)}{\Gamma_{\mathbb{C}}(-s - (-q-1))} = \frac{\Gamma_{\mathbb{C}}(s-p)}{\Gamma_{\mathbb{C}}(1 - (s-q))} = \Gamma_{\mathbb{C}}(s-p)\Gamma_{\mathbb{C}}(s-q) \frac{\sin(\pi(s-q))}{2}.$$

Using in addition the identity $\Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1) = \Gamma_{\mathbb{C}}(s)$ we find

$$(36) \quad \frac{\Gamma_{\mathbb{R}}(s-p)}{\Gamma_{\mathbb{R}}(-s - (-p-1))} = \frac{\Gamma_{\mathbb{R}}(s-p)\Gamma_{\mathbb{R}}(s-p+1)}{\Gamma_{\mathbb{R}}(1 - (s-p))\Gamma_{\mathbb{R}}(s-p+1)} = \Gamma_{\mathbb{C}}(s-p) \cos\left(\frac{\pi(s-p)}{2}\right)$$

and similarly

$$(37) \quad \frac{\Gamma_{\mathbb{R}}(s-p+1)}{\Gamma_{\mathbb{R}}(-s - (-p-1) + 1)} = \frac{\Gamma_{\mathbb{R}}(s-p+1)\Gamma_{\mathbb{R}}(s-p)}{\Gamma_{\mathbb{R}}(2 - (s-p))\Gamma_{\mathbb{R}}(s-p)} = \Gamma_{\mathbb{C}}(s-p) \cos\left(\frac{\pi(s-p-1)}{2}\right).$$

Every pure \mathbb{R} -Hodge structure M over \mathbb{R} is the direct sum of simple \mathbb{R} -Hodge structures. The simple \mathbb{R} -Hodge structures are $M_{p,q}$ of dimension 2 for integers $p < q$ and $M_{p,\pm}$ of dimension 1 for integers p (where F_∞ operates via $\pm(-1)^p$). From the above definition of $L_\infty(M, s)$ and (35), (36), (37) we obtain the following table

M	$M^*(1)$	$L_\infty(M, s)$	$\frac{L_\infty(M, s)}{L_\infty(M^*(1), -s)}$
$M_{p,q}$	$M_{-p-1, -q-1}$	$\Gamma_{\mathbb{C}}(s-p)$	$\Gamma_{\mathbb{C}}(s-p)\Gamma_{\mathbb{C}}(s-q) \cdot \frac{\sin(\pi(s-q))}{2}$
$M_{p,+}$	$M_{-p-1, +}$	$\Gamma_{\mathbb{R}}(s-p)$	$\Gamma_{\mathbb{C}}(s-p) \cdot \cos\left(\frac{\pi(s-p)}{2}\right)$
$M_{p,-}$	$M_{-p-1, -}$	$\Gamma_{\mathbb{R}}(s-p+1)$	$\Gamma_{\mathbb{C}}(s-p) \cdot \cos\left(\frac{\pi(s-p-1)}{2}\right)$

We have

$$\sin(\pi(s-q))|_{s=0}^* = (-1)^q \pi$$

and

$$\cos\left(\frac{\pi(s-p)}{2}\right)|_{s=0}^* = \begin{cases} (-1)^{p/2} & p \text{ even} \\ (-1)^{(p-1)/2} \frac{\pi}{2} & p \text{ odd} \end{cases}$$

It is now straightforward to verify the entries of the following table which confirm Lemma 4.1 for simple \mathbb{R} -Hodge structures. Since all quantities are additive in M the general case

follows by writing M as a sum of simple \mathbb{R} -Hodge structures.

M	$d_+(M)$	$d_-(M)$	$h_j(M)$	$t_H(M)$	$\frac{L_\infty^*(M,0)}{L_\infty^*(M^*(1),0)}$
$M_{p,q}$	1	1	1 for $j = p, q$ 0 else	$p + q$	$\pm(2\pi)^{p+q+1}\Gamma^*(-p)\Gamma^*(-q)$
$M_{p,+}$					
p even	1	0	1 for $j = p$	p	$\pm 2(2\pi)^p\Gamma^*(-p)$
p odd	0	1	1 for $j = p$	p	$\pm 2(2\pi)^p\Gamma^*(-p) \cdot \frac{\pi}{2}$
$M_{p,-}$					
p even	0	1	1 for $j = p$	p	$\pm 2(2\pi)^p\Gamma^*(-p) \cdot \frac{\pi}{2}$
p odd	1	0	1 for $j = p$	p	$\pm 2(2\pi)^p\Gamma^*(-p)$

□

Suppose now \mathcal{X} is a regular scheme, proper and flat over $\text{Spec}(\mathbb{Z})$ with generic fibre $X := \mathcal{X}_{\mathbb{Q}}$. The archimedean Euler factor of \mathcal{X} is defined as

$$(38) \quad \zeta(\mathcal{X}_\infty, s) = \prod_{i \in \mathbb{Z}} L_\infty(h^i(X), s)^{(-1)^i}$$

where $h^i(X)$ is the \mathbb{R} -Hodge structure on $H^i(\mathcal{X}(\mathbb{C}), \mathbb{R})$.

Corollary 4.2. *One has*

$$\frac{\zeta^*(\mathcal{X}_\infty, n)}{\zeta^*(\mathcal{X}_\infty, d-n)} = \pm 2^{d_+(\mathcal{X}, n) - d_-(\mathcal{X}, n)} (2\pi)^{d_-(\mathcal{X}, n) + t_H(\mathcal{X}, n)} \prod_{p,q} \Gamma^*(n-p)^{h^{p,q} \cdot (-1)^{p+q}}$$

where

$$d_\pm(\mathcal{X}, n) = \sum_i (-1)^i d_\pm(h^i(X)(n)), \quad t_H(\mathcal{X}, n) = \sum_i (-1)^i t_H(h^i(X)(n))$$

and $h^i(X)(n)$ denotes the \mathbb{R} -Hodge structure on $H^i(\mathcal{X}(\mathbb{C}), (2\pi i)^n \mathbb{R})$.

Proof. For $M = h^i(X)(n)$ one has $M^*(1) \cong h^{2d-2-i}(X)(d-n)$ and

$$h_j(M) = h^{j, i-2n-j}(M) = h^{p-n, i-p-n}(M) = h^{p, i-p} = h^{p,q}$$

with $p+q = i$, $j = p-n$. Therefore Lemma 4.1 implies

$$\begin{aligned} \frac{\zeta^*(\mathcal{X}_\infty, n)}{\zeta^*(\mathcal{X}_\infty, d-n)} &= \prod_i \frac{L_\infty^*(h^i(X)(n), 0)^{(-1)^i}}{L_\infty^*(h^{2d-2-i}(X)(d-n), 0)^{(-1)^{2d-2-i}}} \\ &= 2^{d_+(\mathcal{X}, n) - d_-(\mathcal{X}, n)} (2\pi)^{d_-(\mathcal{X}, n) + t_H(\mathcal{X}, n)} \prod_{p,q} \Gamma^*(n-p)^{h^{p,q} \cdot (-1)^{p+q}}. \end{aligned}$$

□

Lemma 4.3. *One has*

$$\frac{C(\mathcal{X}, n)}{C(\mathcal{X}, d-n)} = \pm \left(\prod_{p,q} \Gamma^*(n-p)^{h^{p,q} \cdot (-1)^{p+q}} \right)^{-1}$$

Proof. Since $\mathcal{X}(\mathbb{C})$ is smooth proper of dimension $d - 1$ the Hodge numbers $h^{p,q}$ are nonzero only for $0 \leq p \leq d - 1$. By definition (1)

$$(39) \quad \begin{aligned} C(\mathcal{X}, n)^{-1} &= \prod_{0 \leq p \leq n-1, q} (n-p-1)!^{h^{p,q} \cdot (-1)^{p+q}} \\ &= \prod_{0 \leq p \leq n-1, q} \Gamma^*(n-p)^{h^{p,q} \cdot (-1)^{p+q}}. \end{aligned}$$

On the other hand (34) implies

$$\Gamma^*(j)\Gamma^*(1-j) = \pm 1$$

and therefore we have

$$(40) \quad \begin{aligned} \prod_{n \leq p \leq d-1, q} \Gamma^*(n-p)^{h^{p,q} \cdot (-1)^{p+q}} &= \pm \prod_{n \leq p \leq d-1, q} \Gamma^*(1-(n-p))^{-h^{p,q} \cdot (-1)^{p+q}} \\ &= \pm \prod_{0 \leq p \leq d-n-1, q} \Gamma^*(d-n-p)^{-h^{p,q} \cdot (-1)^{p+q}} \\ &= \pm C(\mathcal{X}, d-n). \end{aligned}$$

Combining (39) and (40) gives the Lemma. \square

5. THE MAIN RESULT

Recall the definition of the completed Zeta-function of \mathcal{X}

$$\zeta(\overline{\mathcal{X}}, s) := \zeta(\mathcal{X}, s) \cdot \zeta(\mathcal{X}_\infty, s)$$

where $\zeta(\mathcal{X}_\infty, s)$ was defined in (38). We repeat Conjecture 1.3 from the introduction.

Conjecture 1.3. (*Functional Equation*) *Let \mathcal{X} be a regular scheme of dimension d , proper and flat over $\text{Spec}(\mathbb{Z})$. Then $\zeta(\mathcal{X}, s)$ has a meromorphic continuation to all $s \in \mathbb{C}$ and*

$$A(\mathcal{X})^{(d-s)/2} \cdot \zeta(\overline{\mathcal{X}}, d-s) = \pm A(\mathcal{X})^{s/2} \cdot \zeta(\overline{\mathcal{X}}, s).$$

This conjecture is true for $d = 1$ where it reduces to the functional equation of the Dedekind Zeta function. It is true for $d = 2$ by [5][Prop. 1.1] provided that the L-function $L(h^1(\mathcal{X}_{\mathbb{Q}}), s)$ satisfies the expected functional equation involving the Artin conductor of the l -adic representation $H^1(\mathcal{X}_{\overline{\mathbb{Q}}}, \mathbb{Q}_l)$. This is the case if \mathcal{X} is a regular model of a potentially modular elliptic curve over a number field F in view of the compatibility of the (local) Langlands correspondence for GL_2 with ϵ -factors and hence conductors. Potential modularity of elliptic curves is known if F is totally real or quadratic over a totally real field. We refer to [7][1.1] for a discussion of these results and for the original references. In [7] potential modularity is also shown for abelian surfaces over totally real fields F and hence Conjecture 1.3 should hold for regular models of genus 2 curves over totally real fields F (since this involves the local Langlands correspondence for GSp_4/F we are unsure whether the conductor in the functional equation is indeed the Artin conductor).

We repeat Theorem 1.4 from the introduction which is the main result of this paper.

Theorem 1.4. *Assume that $\zeta(\overline{\mathcal{X}}, s)$ satisfies Conjecture 1.3. Then Conjecture 1.1 for (\mathcal{X}, n) is equivalent to Conjecture 1.1 for $(\mathcal{X}, d - n)$.*

Proof. The reduction of this theorem to Theorem 1.2 was already made in [8][5.7]. More precisely, recall the invertible \mathbb{Z} -module

$$\begin{aligned} \Xi_\infty(\mathcal{X}/\mathbb{Z}, n) &:= \det_{\mathbb{Z}} R\Gamma_W(\mathcal{X}_\infty, \mathbb{Z}(n)) \otimes \det_{\mathbb{Z}}^{-1} R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^n \\ &\quad \otimes \det_{\mathbb{Z}}^{-1} R\Gamma_W(\mathcal{X}_\infty, \mathbb{Z}(d-n)) \otimes \det_{\mathbb{Z}} R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^{d-n}, \end{aligned}$$

the canonical isomorphism

$$\Delta(\mathcal{X}/\mathbb{Z}, n) \otimes \Xi_\infty(\mathcal{X}/\mathbb{Z}, n) \xrightarrow{\sim} \Delta(\mathcal{X}/\mathbb{Z}, d-n)$$

and the canonical trivialization

$$\xi_\infty : \mathbb{R} \xrightarrow{\sim} \Xi_\infty(\mathcal{X}/\mathbb{Z}, n) \otimes \mathbb{R}$$

which is compatible with the trivializations (2) of $\Delta(\mathcal{X}/\mathbb{Z}, n)$ and $\Delta(\mathcal{X}/\mathbb{Z}, d-n)$ (see [8][Prop. 5.29]). Denoting by

$$x_\infty(\mathcal{X}, n)^2 \in \mathbb{R}_{>0}$$

the strictly positive real number such that

$$\xi_\infty(x_\infty(\mathcal{X}, n)^{-2} \cdot \mathbb{Z}) = \Xi_\infty(\mathcal{X}/\mathbb{Z}, n)$$

the canonical isomorphism

$$\Xi_\infty(\mathcal{X}/\mathbb{Z}, n) \otimes \Xi_\infty(\mathcal{X}/\mathbb{Z}, d-n) \cong \mathbb{Z}$$

implies that

$$(41) \quad x_\infty(\mathcal{X}, n)^2 \cdot x_\infty(\mathcal{X}, d-n)^2 = 1.$$

It was then shown in [8][Cor. 5.31] that Theorem 1.4 is equivalent to the following identity (note that there is a typo in the statement of [8][Cor. 5.31] and $C(\mathcal{X}, n)$ and $C(\mathcal{X}, d-n)$ have to be replaced by their inverses)

$$\begin{aligned} &A(\mathcal{X})^{n/2} \cdot \zeta^*(\mathcal{X}_\infty, n) \cdot x_\infty(\mathcal{X}, n)^{-1} \cdot C(\mathcal{X}, n) \\ &= \pm A(\mathcal{X})^{(d-n)/2} \cdot \zeta^*(\mathcal{X}_\infty, d-n) \cdot x_\infty(\mathcal{X}, d-n)^{-1} \cdot C(\mathcal{X}, d-n). \end{aligned}$$

But using (41) this identity is equivalent to the identity of Thm. 1.2. This already concludes the proof of Thm. 1.4. \square

It remains to prove Theorem 1.2 which we repeat here for the convenience of the reader.

Theorem 1.2. *Let \mathcal{X} be a regular scheme of dimension d , proper and flat over $\text{Spec}(\mathbb{Z})$. Then we have*

$$x_\infty(\mathcal{X}, n)^2 = \pm A(\mathcal{X})^{n-d/2} \cdot \frac{\zeta^*(\mathcal{X}_\infty, n)}{\zeta^*(\mathcal{X}_\infty, d-n)} \cdot \frac{C(\mathcal{X}, n)}{C(\mathcal{X}, d-n)}.$$

Proof. By Corollary 4.2 and Lemma 4.3 this identity is equivalent to

$$(42) \quad x_\infty(\mathcal{X}, n)^2 = \pm A(\mathcal{X})^{n-d/2} \cdot 2^{d_+(\mathcal{X}, n) - d_-(\mathcal{X}, n)} \cdot (2\pi)^{d_-(\mathcal{X}, n) + t_H(\mathcal{X}, n)}.$$

Lemma 5.1. *The isomorphism ξ_∞ is induced by the sequence of isomorphisms*

$$\begin{aligned}
(43) \quad & \left(\det_{\mathbb{Z}} R\Gamma_W(\mathcal{X}_\infty, \mathbb{Z}(n)) \otimes \det_{\mathbb{Z}}^{-1} R\Gamma_W(\mathcal{X}_\infty, \mathbb{Z}(d-n)) \right)_{\mathbb{R}} \\
& \xrightarrow{(8)_{\mathbb{R}}} \det_{\mathbb{R}} (R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{R}(n))^+ \oplus R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{R}(n-1))^+) \\
& \xrightarrow{(45)} \det_{\mathbb{R}} R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{C})^+ \\
& \xrightarrow{(46)^+} \det_{\mathbb{R}} R\Gamma_{dR}(\mathcal{X}_{\mathbb{C}}/\mathbb{C})^+ \\
& \simeq \det_{\mathbb{R}} R\Gamma_{dR}(\mathcal{X}_{\mathbb{R}}/\mathbb{R}) \simeq \left(\det_{\mathbb{Z}} R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^d \right)_{\mathbb{R}} \\
& \xrightarrow{\lambda_{dR}^{-1}} \left(\det_{\mathbb{Z}} R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^n \otimes \det_{\mathbb{Z}}^{-1} R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^{d-n} \right)_{\mathbb{R}}
\end{aligned}$$

where λ_{dR} was defined in Cor. 3.9.

Proof. The isomorphism ξ_∞ was defined in [8][Prop. 5.29]

$$\begin{aligned}
& \left(\det_{\mathbb{Z}} R\Gamma_W(\mathcal{X}_\infty, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \det_{\mathbb{Z}}^{-1} R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^n \right)_{\mathbb{R}} \\
& \simeq \det_{\mathbb{R}} R\Gamma_{\mathcal{D}}(\mathcal{X}/\mathbb{R}, \mathbb{R}(n)) \\
& \simeq \det_{\mathbb{R}} R\Gamma_{\mathcal{D}}(\mathcal{X}/\mathbb{R}, \mathbb{R}(d-n))^*[-2d+1] \\
& \simeq \det_{\mathbb{R}} R\Gamma_{\mathcal{D}}(\mathcal{X}/\mathbb{R}, \mathbb{R}(d-n)) \\
& \simeq \left(\det_{\mathbb{Z}} R\Gamma_W(\mathcal{X}_\infty, \mathbb{Z}(d-n)) \otimes_{\mathbb{Z}} \det_{\mathbb{Z}}^{-1} R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^{d-n} \right)_{\mathbb{R}}
\end{aligned}$$

using the defining exact triangle

$$(R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^n)_{\mathbb{R}}[-1] \rightarrow R\Gamma_{\mathcal{D}}(\mathcal{X}/\mathbb{R}, \mathbb{R}(n)) \rightarrow R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{R}(n))^+$$

and duality

$$R\Gamma_{\mathcal{D}}(\mathcal{X}/\mathbb{R}, \mathbb{R}(n)) \simeq R\Gamma_{\mathcal{D}}(\mathcal{X}/\mathbb{R}, \mathbb{R}(d-n))^*[-2d+1]$$

for Deligne cohomology. This duality is constructed in [8][Lemma 2.3] by taking $G_{\mathbb{R}}$ -invariants in

$$(44) \quad R\Gamma_{\mathcal{D}}(\mathcal{X}/\mathbb{C}, \mathbb{R}(n)) \simeq R\Gamma_{\mathcal{D}}(\mathcal{X}/\mathbb{C}, \mathbb{R}(d-n))^*[-2d+1]$$

which is obtained as follows. Dualizing the defining exact triangle

$$(R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^{d-n})_{\mathbb{C}}[-1] \rightarrow R\Gamma_{\mathcal{D}}(\mathcal{X}/\mathbb{C}, \mathbb{R}(d-n)) \rightarrow R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{R}(d-n))$$

and using Poincaré duality (6) and (28)_C on $\mathcal{X}(\mathbb{C})$ we obtain the bottom exact triangle in the diagram

$$\begin{array}{ccccc}
R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{R}(n))[-1] & \xlongequal{\quad} & R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{R}(n))[-1] & & \\
\downarrow & & \downarrow & & \\
F_{\mathbb{C}}^n \longleftarrow (R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^n)_{\mathbb{C}}[-1] & \xleftarrow{\beta} & R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{C})[-1] & & \\
\parallel & & \downarrow & & \\
F_{\mathbb{C}}^n \longleftarrow R\Gamma_{\mathcal{D}}(\mathcal{X}/\mathbb{C}, \mathbb{R}(d-n))^*[-2d+1] & \longleftarrow & R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{R}(n-1))[-1] & &
\end{array}$$

The right hand column is induced by the decomposition

$$(45) \quad \mathbb{C} \cong \mathbb{R}(n) \oplus \mathbb{R}(n-1)$$

on coefficients, and the map β is the comparison isomorphism

$$(46) \quad R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{C}) \simeq R\Gamma_{dR}(\mathcal{X}_{\mathbb{C}}/\mathbb{C})$$

composed with the natural projection. It is then clear that all rows and columns in the diagram are exact, and the middle column is the defining exact triangle of $R\Gamma_{\mathcal{D}}(\mathcal{X}/\mathbb{C}, \mathbb{R}(n))$, giving (44). Recalling that (8) was also defined using Poincaré duality (6) we find that the isomorphisms used in (43) are precisely those used in the construction of (44)⁺. \square

We call the real line $\det_{\mathbb{R}} R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{C})^+$ the de Rham real structure of $\det_{\mathbb{C}} R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{C})$ and the real line $\det_{\mathbb{R}} R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{R}(n))$ the Betti real structure of $\det_{\mathbb{C}} R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{C})$. By (7) we have

$$(47) \quad \det_{\mathbb{R}} R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{C})^+ \cdot (2\pi i)^{d-(\mathcal{X}, n)} = \det_{\mathbb{R}} R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{R}(n)).$$

In the remaining computations of the proof of Theorem 1.2 all identities should be understood up to sign. We choose bases of the various \mathbb{Z} -structures of the de Rham real line appearing in (43)

$$\begin{aligned} \mathbb{Z} \cdot \tilde{b}_B &= \det_{\mathbb{Z}} R\Gamma_W(\mathcal{X}_{\infty}, \mathbb{Z}(n)) \otimes \det_{\mathbb{Z}}^{-1} R\Gamma_W(\mathcal{X}_{\infty}, \mathbb{Z}(d-n)) \\ \mathbb{Z} \cdot b_{dR} &= \det_{\mathbb{Z}} R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^d \\ \mathbb{Z} \cdot \tilde{b}_{dR} &= \det_{\mathbb{Z}} R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^n \otimes \det_{\mathbb{Z}}^{-1} R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^{d-n} \end{aligned}$$

and we also choose a basis

$$\mathbb{Z} \cdot b_B = \det_{\mathbb{Z}} R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{Z}(n))$$

of the natural \mathbb{Z} -structure in the Betti real structure. Let $P \in \mathbb{C}^{\times}$ be the Betti-de Rham period, i.e. we have

$$b_{dR} = P \cdot b_B$$

under the comparison isomorphism (46).

Lemma 5.2. *Let $\varepsilon_B \in \{\pm 1\}$ be the discriminant (see Remark 3.7) of the Poincaré duality pairing*

$$R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{Z}(n)) \otimes R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{Z}(n)) \xrightarrow{\text{Tr} \circ \cup} \mathbb{Z}[-2d+2]$$

and $\varepsilon_{dR} \cdot A(\mathcal{X})^d$ the discriminant of the deRham duality pairing (30). Then

$$P = \sqrt{\varepsilon_B \varepsilon_{dR}} \cdot (2\pi i)^{t_H(\mathcal{X}, n)} \cdot A(\mathcal{X})^{\frac{d}{2}}.$$

Moreover $P \cdot (2\pi i)^{d-(\mathcal{X}, n)}$ is real and hence we have

$$P \cdot (2\pi i)^{d-(\mathcal{X}, n)} = (2\pi)^{d-(\mathcal{X}, n)+t_H(\mathcal{X}, n)} \cdot A(\mathcal{X})^{\frac{d}{2}}.$$

Proof. We have a commutative diagram

$$\begin{array}{ccc} R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{Z}(n)) \otimes R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{Z}(n)) & \xrightarrow{\text{Tr} \circ \cup} & \mathbb{Z}(2n-d+1)[-2d+2] \\ \downarrow & & \downarrow \\ R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{C}) \otimes R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{C}) & \xrightarrow{\text{Tr} \circ \cup} & \mathbb{C}[-2d+2] \\ \uparrow & & \uparrow \\ R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^d \otimes_{\mathbb{Z}}^L R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^d & \xrightarrow{(30)} & \mathbb{Z}[-2d+2] \end{array}$$

where the bottom square commutes since the comparison isomorphism

$$R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{C}) \simeq R\Gamma_{dR}(\mathcal{X}_{\mathbb{C}}/\mathbb{C}) \simeq R\Gamma_{dR}(\mathcal{X}_{\mathbb{Q}}/\mathbb{Q})_{\mathbb{C}}$$

is compatible with cup product and cycle classes, and the trace map sends the cycle class of a closed point to its degree over the base field. We also use the fact that the trace map in algebraic de Rham cohomology

$$H_{dR}^{2d-2}(\mathcal{X}_{\mathbb{Q}}/\mathbb{Q}) \xleftarrow{\sim} H^{d-1}(\mathcal{X}_{\mathbb{Q}}, \Omega_{\mathcal{X}_{\mathbb{Q}}/\mathbb{Q}}^{d-1}) \xrightarrow{\text{Tr}} \mathbb{Q}$$

is the base change of the Trace map (31) under $\text{Spec}(\mathbb{Q}) \rightarrow \text{Spec}(\mathbb{Z})$ by [25]. Applying the construction of Remark 3.7 we then obtain a pairing

$$\langle \cdot, \cdot \rangle : \det_{\mathbb{C}} R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{C}) \otimes_{\mathbb{C}} \det_{\mathbb{C}} R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{C}) \simeq \mathbb{C}$$

which restricts to the corresponding \mathbb{Q} -valued pairing on $\det_{\mathbb{Z}} R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^d$ and a $\mathbb{Q} \cdot (2\pi i)^{(2n-d+1)\chi}$ -valued pairing on $\det_{\mathbb{Z}} R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{Z}(n))$. Here

$$\chi = \text{rank}_{\mathbb{Z}} R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{Z}(n)) = \dim_{\mathbb{R}} R\Gamma(\mathcal{X}(\mathbb{C}), \mathbb{R}(n)) = \sum_i (-1)^i \dim_{\mathbb{R}} H^i(\mathcal{X}(\mathbb{C}), \mathbb{R}(n)).$$

We then have

$$\varepsilon_{dR} \cdot A(\mathcal{X})^d = \langle b_{dR}, b_{dR} \rangle = P^2 \langle b_B, b_B \rangle = P^2 \varepsilon_B (2\pi i)^{(2n-d+1)\chi}$$

and moreover

$$\begin{aligned} -(2n-d+1)\chi &= \sum_{i < d-1} (-1)^i (i-2n+2d-2-i-2n) \dim_{\mathbb{R}} H^i(\mathcal{X}(\mathbb{C}), \mathbb{R}(n)) \\ &\quad + (-1)^{d-1} (d-1-2n) \dim_{\mathbb{R}} H^{d-1}(\mathcal{X}(\mathbb{C}), \mathbb{R}(n)) \\ &= 2t_H(\mathcal{X}, n). \end{aligned}$$

Hence

$$P^2 = \varepsilon_{dR} \varepsilon_B \cdot (2\pi i)^{2t_H(\mathcal{X}, n)} \cdot A(\mathcal{X})^d$$

which shows the first statement. Since both b_B and $b_{dR} \cdot (2\pi i)^{d_-(\mathcal{X}, n)} = P \cdot (2\pi i)^{d_-(\mathcal{X}, n)} \cdot b_B$ lie in the Betti real structure, the factor $P \cdot (2\pi i)^{d_-(\mathcal{X}, n)}$ is real. This proves the second statement. \square

Remark 5.3. *From the Lemma we deduce*

$$\varepsilon_B \varepsilon_{dR} = (-1)^{d_-(\mathcal{X}, n) + t_H(\mathcal{X}, n)} = (-1)^{d_-(\mathcal{X}, 0) + \frac{d-1}{2}\chi}.$$

This generalizes the classical fact that the sign of the discriminant of a number field F is $(-1)^{r_2}$ where $r_2 = d_-(\text{Spec}(\mathcal{O}_F), 0)$ is the number of complex places. In this case $\varepsilon_B = 1$.

We can now finish the proof of Theorem 1.2 by verifying (42). By Prop. 2.23 we have

$$\tilde{b}_B \cdot (2\pi i)^{d_-(\mathcal{X}, n)} = b_B \cdot 2^{d_-(\mathcal{X}, n) - d_+(\mathcal{X}, n)}$$

and by Corollary 3.9

$$\tilde{b}_{dR}^{-1} = A(\mathcal{X})^{d-n} \cdot b_{dR}^{-1}.$$

Therefore

$$\begin{aligned}
 x_\infty(\mathcal{X}, n)^{-2} &= \tilde{b}_B \cdot \tilde{b}_{dR}^{-1} = (2\pi i)^{-d_-(\mathcal{X}, n)} \cdot 2^{d_-(\mathcal{X}, n) - d_+(\mathcal{X}, n)} \cdot b_B \cdot A(\mathcal{X})^{d-n} \cdot b_{dR}^{-1} \\
 &= (2\pi i)^{-d_-(\mathcal{X}, n)} \cdot 2^{d_-(\mathcal{X}, n) - d_+(\mathcal{X}, n)} \cdot A(\mathcal{X})^{d-n} \cdot P^{-1} \\
 &= (2\pi i)^{-d_-(\mathcal{X}, n)} \cdot 2^{d_-(\mathcal{X}, n) - d_+(\mathcal{X}, n)} \cdot A(\mathcal{X})^{d-n} \cdot \sqrt{\varepsilon_B \varepsilon_{dR}} \cdot (2\pi i)^{-t_H(\mathcal{X}, n)} \cdot A(\mathcal{X})^{-\frac{d}{2}} \\
 &= A(\mathcal{X})^{\frac{d}{2} - n} \cdot 2^{d_-(\mathcal{X}, n) - d_+(\mathcal{X}, n)} \cdot (2\pi)^{-d_-(\mathcal{X}, n) - t_H(\mathcal{X}, n)}
 \end{aligned}$$

which is (42). □

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