

Variable-Length Coding for Binary-Input Channels With Limited Stop Feedback

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Abstract—This paper focuses on the numerical evaluation of the maximal achievable rate of variable-length stop-feedback (VLSF) codes with m decoding times at a given message size and error probability for binary-input additive white Gaussian noise channel, binary symmetric channel, and binary erasure channel (BEC). Leveraging the Edgeworth and Petrov expansions, we develop tight approximations to the tail probability of length- n cumulative information density that are accurate for any blocklength n . We reduce Yavas *et al.*'s non-asymptotic achievability bound on VLSF codes with m decoding times to an integer program of minimizing the upper bound on the average blocklength subject to the average error probability, minimum gap, and integer constraints. We develop two distinct methods to solve this program. Numerical evaluations show that Polyanskiy's achievability bound for VLSF codes, which assumes $m = \infty$, can be approached with a relatively small m in all of the three channels. For BEC, we consider systematic transmission followed by random linear fountain coding. This allows us to obtain a new achievability bound stronger than a previously known bound and new VLSF codes whose rate further outperforms Polyanskiy's bound.

Index Terms—Limited stop feedback, non-asymptotic analysis, random linear fountain coding, sequential differential optimization, variable-length coding,

I. INTRODUCTION

FEEDBACK has been shown to be useful both in the variable-length and fixed-length regimes, even though it does not improve the capacity of a memoryless, point-to-point channel [2]. In the variable-length regime, feedback has been shown to simplify the construction of coding schemes [3]–[8], to significantly improve the optimal error exponent [9], and to achieve universality [10]–[12]. In the fixed-length regime, feedback is recently shown to improve the second-order coding rate for the compound-dispersion discrete memoryless channels [13].

In [14], Polyanskiy *et al.* demonstrated the advantage of feedback in the non-asymptotic regime for the discrete memoryless channel (DMC) by means of the variable-length stop-feedback (VLSF) code and an information density decoder. Under their code construction, M infinite-length VLSF codewords are fixed before the start of transmission. The

stop feedback only affects the portion of a codeword being transmitted rather than the codeword symbol. During transmission, a stop-feedback symbol “0” indicates that the decoder is not ready to decode and transmission should continue, whereas a “1” signifies that the decoder is ready to decode and the transmitter must stop. In Polyanskiy's framework, the decoder stops transmission and sends out a stop-feedback symbol “1” whenever the cumulative information density of a certain message passes a given threshold γ for the first time; otherwise, a stop-feedback symbol “0” is sent. Using the VLSF code, the information density decoder, and a probabilistic pre-transmission stopping rule, Polyanskiy *et al.* showed that the ϵ -capacity $\frac{C}{1-\epsilon}$ is achievable for a DMC with capacity C and target error probability ϵ .

The VLSF code defined in [14] can be thought of as a variable-length feedback (VLF) code with infinitely many decoding times, i.e., the number of decoding times $m = \infty$. However, in practical systems, the feedback opportunities are often sparse, i.e., $m < \infty$, and the decoder is only allowed to send a limited feedback signal such as acknowledgement (ACK) or negative acknowledgement (NACK) informing the transmitter of the decoding result at finite time instants n_1, n_2, \dots, n_m . A typical example is a system that employs hybrid automatic repeat request (ARQ) and incremental redundancy, e.g., [15]. In this regard, a theoretical characterization and an accurate numerical evaluation of the maximal achievable rate for VLSF codes with m decoding times are of practical importance.

We mention a few previous works on VLSF codes with finite decoding times. In [16], Kim *et al.* investigated VLSF codes with m periodic decoding times and derived a lower bound on throughput. To minimize the average blocklength, Vakili *et al.* [17] developed the *sequential differential optimization* (SDO) procedure that produces decoding time n_{i+1} based on the knowledge of n_i, n_{i-1} , and their successful decoding probabilities approximated by a differentiable function. The SDO procedure in [17] uses a Gaussian model to approximate the probability of successful decoding at each decoding time. Later, variations of the SDO procedure were developed to improve the Gaussian model accuracy [18], [19]. The SDO procedure has been utilized to optimize systems that employ incremental redundancy and hybrid ARQ [20], and to code for the binary erasure channel [21], [22].

However, for a given error probability, the Gaussian model developed in all previous works is an application of the central limit theorem (CLT), which typically requires a sufficiently large blocklength. In the non-asymptotic regime, however, this

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condition is often missed, rendering the Gaussian model inaccurate. This issue becomes especially prominent for decoding times less than 100. Hence, a refined approximation to tail probabilities is desired for the SDO procedure. In addition, the SDO procedure studied in all previous works assumes real-valued decoding times and can be seen as the solution to an *unconstrained* minimization of the upper bound on average blocklength. Thus, it fails to consider the inherent gap constraint that two decoding times must be separated by at least one.

In statistics, the *Edgeworth expansion* [23], [24] and *Petrov expansion* [25] have been known as powerful tools to approximate the distribution of a sum of independent and identically distributed (i.i.d.) random variables. A fascinating feature of these expansions is that they only require the knowledge of higher-order *cumulants* of each individual random variable. We refer the reader to [24, Chapter 2] for a detailed introduction to Edgeworth expansion and its applications. While the original Edgeworth expansion considers non-lattice random variables (e.g., any continuous random variable), for lattice random variables, Kolassa [26] provided the continuity-corrected Edgeworth series that can be used to approximate the tail probability. In this paper, we apply these tools to approximate the tail probability of a length- n cumulative information density, a quantity that is crucial in the analysis of random fixed-length or VLF codes [14], [27]. Numerical evaluations show that these approximations remain accurate at blocklengths as short as 20.

In a recent work [28], Yavas *et al.* developed an achievability bound for VLSF codes with m decoding times for the additive white Gaussian noise (AWGN) channel under maximal power constraint P . This achievability bound is predicated upon Polyanskiy's information density decoder. By applying CLT for functions to their achievability bound and optimizing decoding times, Yavas *et al.* showed an asymptotic expansion of their bound on the maximum message size $M^*(l, m, \epsilon, P)$ for a given average blocklength l , number of decoding times m , target error probability ϵ , and maximal power P [28, Th. 1]. They showed that a slight increase in m can dramatically improve the achievable rate of VLSF codes. Unfortunately, due to the nested logarithm term, the expansion is only defined for very large l or small m . Yavas *et al.* only numerically show their approximation for $m \leq 4$, $\epsilon = 10^{-3}$, and $l \leq 2000$. Yavas *et al.* also demonstrated that the decoding times generated from the SDO procedure will yield the same second-order coding rates as attained by their construction of decoding times.

In this paper, we are mainly concerned with numerical evaluations of the maximal achievable rate of VLSF codes with m decoding times at a given message size M and target error probability ϵ for classical binary-input channels, including the binary-input AWGN (BI-AWGN) channel, the binary symmetric channel (BSC), and the binary erasure channel (BEC). A key problem is to assess whether approaching Polyanskiy's achievability bound for VLSF codes [14], which assumes $m = \infty$, requires a large number of decoding times for practically interesting target error probability ϵ .

Building upon Yavas *et al.*'s achievability bound, for a

fixed information density threshold γ , we formulate an integer program of minimizing the upper bound on average blocklength over all decoding times n_1, n_2, \dots, n_m subject to average error probability, minimum gap, and integer constraints. Eventually, minimization of locally minimum upper bounds over all information density thresholds γ yields the globally minimum upper bound, and this method is referred to as the *two-step minimization*. We develop two methods to numerically evaluate this integer program: the *gap-constrained SDO procedure* and the *discrete SDO procedure*. The former relies on approximating the tail probability with a monotone, differentiable function. The latter only requires an estimate of tail probability at each integer decoding times but comes with added search complexity.

For a given integer M' , we derive an error regime $\epsilon \leq \epsilon^*(M')$ in which Polyanskiy's stopping at zero technique does not improve the achievability bound for all message sizes $M \leq M'$. In this error regime, numerical evaluations show that Polyanskiy's achievability bound can be approached with a finite and relatively small m for classical binary-input channels, including the BI-AWGN channel, BSC, and BEC.

A particular attention is paid to the BEC in that the decoder has the ability to identify the correct message whenever only one codeword is compatible with the unerased received symbols. Motivated by this key observation, we construct a new random VLSF code by first transmitting the k -bit message systematically and then applying the random linear fountain coding (RLFC) [29], [30, Chapter 50]. Specifically, after systematic transmission, both the encoder and decoder select the same nonzero basis vector in $\{0, 1\}^k$ according to some common randomness. The encoder produces the transmitted symbol by linearly combining the message bits using the selected basis vector. The decoder, known as the *rank decoder*, keeps track of the rank of the generator matrix associated with unerased received symbols. As soon as the rank equals k , the decoder stops transmission and reproduces the transmitted k -bit message with zero error using the inverse of the generator matrix.

The systematic transmission followed by RLFC (ST-RLFC) allows us to develop a new VLSF achievability bound that outperforms the state-of-the-art VLSF achievability bound developed by Devassy *et al.* [31, Th. 9]. More importantly, our bound reduces the 23.4% backoff from capacity at information length $k = 3$ reported in [31]. We show that the gap to capacity at $k = 3$ diminishes to 0 as erasure probability decreases to 0. This gives a negative answer to the open problem in [31] whether the gap to capacity (or to the converse) at $k = 3$ is fundamental. In fact, the backoff percentage from capacity derived from Devassy *et al.*'s result is independent of the erasure probability and therefore must be loose. The ST-RLFC scheme facilitates a similar integer program that can be solved with the discrete SDO procedure. Numerical evaluations show that the achievable rate of VLSF codes constructed from ST-RLFC scheme significantly outperforms Polyanskiy's achievability bound. For 16 decoding times, the achievability bound even outperforms Devassy's bound at small values of k .

The remainder of this paper is organized as follows. In

Section II, we introduce the notation, classical binary-input channels and information density, the VLSF code with m decoding times, and previously known achievability bounds for VLSF codes. In Section III, we present tight approximations to the tail probability of length- n information density. In Section IV, we formulate the integer program, establish the properties for optimal decoding times, and present two methods to solve the integer program: gap-constrained SDO and discrete SDO procedures. In Section V, we numerically evaluate the achievability bound for VLSF codes with m decoding times for the BI-AWGN, BSC, and BEC, and compare our results with Polyanskiy's achievability bound. In Section VI, we present the ST-RLFC scheme, a new VLSF achievability bound for the BEC, and numerical evaluations. Section VII concludes the paper.

II. PRELIMINARIES

A. Notation

Let $\mathbb{N} = \{0, 1, \dots\}$ be the set of natural numbers. For $i \in \mathbb{N}$, $[i] \triangleq \{1, 2, \dots, i\}$. We use x_i^j to denote a sequence $(x_i, x_{i+1}, \dots, x_j)$, $1 \leq i \leq j$. When the context is clear, x_1^n is abbreviated to x^n . We use $\log(\cdot)$ and $\ln(\cdot)$ to denote the base-2 and natural logarithms, respectively. We denote by i the imaginary unit, and by $\mathbf{1}_E$ the indicator function for an event E . We use $\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$, $\Phi(x) = \int_{-\infty}^x \phi(z) dz$, and $Q(x) = \int_x^{\infty} \phi(z) dz$ to respectively denote the probability density function (PDF), cumulative distribution function (CDF), and the tail probability of a standard normal $\mathcal{N}(0, 1)$. For a finite, discrete set \mathcal{X} , we use $\text{Unif}(\mathcal{X})$ to denote the uniform distribution over \mathcal{X} . We denote the distribution of a random variable X by P_X .

B. Classical Binary-Input Channels and Information Density

In this paper, we investigate three memoryless, binary-input channels: the BI-AWGN channel, the BSC, and the BEC.

A BI-AWGN channel consists of input alphabet $\mathcal{X} = \{-\sqrt{P}, \sqrt{P}\}$, output alphabet $\mathcal{Y} = \mathbb{R}$, and conditional PDF

$$P_{Y|X}(y|x) = \phi(y-x), \quad x \in \mathcal{X}, y \in \mathcal{Y}, \quad (1)$$

where $P > 0$ denotes the signal-to-noise ratio (SNR). A BSC(p) consists of binary input and output alphabets $\mathcal{X} = \mathcal{Y} = \{0, 1\}$, and conditional probability

$$P_{Y|X}(y|x) = p\mathbf{1}_{\{y \neq x\}} + (1-p)\mathbf{1}_{\{y=x\}}, \quad x \in \mathcal{X}, y \in \mathcal{Y}, \quad (2)$$

where $p \in (0, 1/2)$ is called the crossover probability. A BEC(p) consists of a binary input alphabet $\mathcal{X} = \{0, 1\}$, a ternary output alphabet $\mathcal{Y} = \{0, 1, ?\}$, and conditional probability

$$P_{Y|X}(y|x) = p\mathbf{1}_{\{y=?\}} + (1-p)\mathbf{1}_{\{y=x\}}, \quad x \in \mathcal{X}, y \in \mathcal{Y}, \quad (3)$$

where $p \in [0, 1)$ is called the erasure probability. For the three channels described above, $\text{Unif}(\mathcal{X})$ is the capacity-achieving input distribution.

The information density of a channel $P_{Y^n|X^n}$ under input distribution P_{X^n} is defined as

$$i(x^n; y^n) \triangleq \log \frac{P_{Y^n|X^n}(y^n|x^n)}{P_{Y^n}(y^n)}, \quad (4)$$

where P_{Y^n} is the marginal of $P_{X^n}P_{Y^n|X^n}$. If $P_{X^n} = \prod_{i=1}^n P_{X_i}$ and the channel is memoryless, we have

$$i(x^n; y^n) = \sum_{i=1}^n i(x_i; y_i). \quad (5)$$

Furthermore, define

$$C \triangleq \mathbb{E}_{P_X^* P_{Y|X}}[i(X; Y)], \quad (6)$$

$$V \triangleq \mathbb{E}_{P_X^* P_{Y|X}}[i^2(X; Y)] - C^2, \quad (7)$$

as the capacity and the dispersion of the channel, respectively, where P_X^* is a capacity-achieving input distribution.

C. VLSF Codes with Finite Decoding Times

We consider variable-length coding for a memoryless channel $(\mathcal{X}, \mathcal{Y}, P_{Y|X})$ with m stop-feedback opportunities. Below, we formally define such codes.

Definition 1: An (l, n_1^m, M, ϵ) VLSF code for memoryless channel $(\mathcal{X}, \mathcal{Y}, P_{Y|X})$, where $l > 0$, $n_1^m \in \mathbb{N}^m$ satisfying $n_1 < n_2 < \dots < n_m$, $M \in \mathbb{N}_+$, and $\epsilon \in (0, 1)$, is defined by:

- 1) A finite alphabet \mathcal{U} and a probability distribution P_U on \mathcal{U} defining the common randomness random variable U that is revealed to both the transmitter and the receiver before the start of the transmission.
- 2) A sequence of encoders $f_n : \mathcal{U} \times [M] \rightarrow \mathcal{X}$, $n = 1, 2, \dots, n_m$, defining the channel inputs

$$X_n = f_n(U, W), \quad (8)$$

where $W \in [M]$ is the equiprobable message.

- 3) A non-negative integer-valued random stopping time $\tau \in \{n_1, n_2, \dots, n_m\}$ of the filtration generated by $\{U, Y^{n_i}\}_{i=1}^m$ that satisfies the average decoding time constraint

$$\mathbb{E}[\tau] \leq l. \quad (9)$$

- 4) m decoding functions $g_{n_i} : \mathcal{U} \times \mathcal{Y}^{n_i} \rightarrow [M]$, providing the best estimate of W at time n_i , $i \in [m]$. The final decision \hat{W} is computed at time instant τ , i.e., $\hat{W} = g_\tau(U, Y^\tau)$ and must satisfy the average error probability constraint

$$P_e \triangleq \mathbb{P}[\hat{W} \neq W] \leq \epsilon. \quad (10)$$

The rate of an (l, n_1^m, M, ϵ) VLSF code is defined by

$$R \triangleq \frac{\log M}{\mathbb{E}[\tau]}. \quad (11)$$

In Definition 1, the cardinality \mathcal{U} specifies the number of deterministic codes under consideration to construct the random code in Definition 1. In [12, Appendix D], Yavas *et al.* showed that $|\mathcal{U}| \leq 2$ suffices.

D. Previous Results on VLSF Codes

In [14], Polyanskiy *et al.* proved a general achievability bound on $(l, \mathbb{N}, M, \epsilon)$ VLSF codes. Namely, they considered $m = \infty$ and $n_i = i - 1$ for $i \in \mathbb{N}_+$.

Theorem 1 (Th. 3, [14]): Fix a scalar $\gamma > 0$ and a memoryless channel $(\mathcal{X}, \mathcal{Y}, P_{Y|X})$. Let X^n and \bar{X}^n be independent copies from the same process and let Y^n be the output of the channel when X^n is the input. Define a pair of hitting times

$$\psi \triangleq \min \{n \geq 0 : \iota(X^n; Y^n) \geq \gamma\}, \quad (12)$$

$$\bar{\psi} \triangleq \min \{n \geq 0 : \iota(\bar{X}^n; Y^n) \geq \gamma\}, \quad (13)$$

Then, for any $M \in \mathbb{N}$, there exists an $(l, \mathbb{N}, M, \epsilon')$ VLSF code satisfying

$$l \leq \mathbb{E}[\psi], \quad (14)$$

$$\epsilon' \leq (M - 1)\mathbb{P}[\bar{\psi} \leq \psi]. \quad (15)$$

The proof of Theorem 1 involves generating M infinite-length VLSF codewords at random and an information density decoder that seeks the smallest stopping time among M stopping times, one for each message.

In general, it is still difficult to compute $\mathbb{E}[\psi]$ and $\mathbb{P}[\bar{\psi} \leq \psi]$. Nonetheless, for memoryless channels with bounded information density $\iota(x; y) < \infty$, Polyanskiy *et al.* proved the following useful relaxations by drawing X^n i.i.d. from the capacity-achieving input distribution P_X^* :

$$\mathbb{E}[\psi] \leq \frac{\gamma + a_0}{C}, \quad (16)$$

$$\mathbb{P}[\bar{\psi} \leq \psi] \leq 2^{-\gamma}, \quad (17)$$

where $a_0 \triangleq \sup_{x \in \mathcal{X}, y \in \mathcal{Y}} \iota(x; y)$ for the channel. Given a target error probability $\epsilon \in (0, 1)$, by setting $\gamma = \log \frac{M-1}{\epsilon}$ in (16) and (17), (14) and (15) are further relaxed to

$$l \leq \frac{\log \frac{M-1}{\epsilon} + a_0}{C}, \quad (18)$$

$$\epsilon' \leq \epsilon. \quad (19)$$

In this paper, we use (18) and (19) to evaluate Theorem 1. We remark that in (18), the term a_0 is not tight, hence it is possible to outperform this bound at a finite number of decoding times.

Following the information density framework and a similar argument as in [14], Yavas *et al.* established an achievability bound for (l, n_1^n, M, ϵ) VLSF codes for the AWGN channel under maximal power constraint. With a slight modification of removing the maximal power constraint and the violation of power constraint term in the upper bound on error probability, their result holds for an arbitrary memoryless channel. We quote the modified result as follows.

Theorem 2 (Th. 3, [28]): Fix a constant $\gamma > 0$, integer-valued decoding times $n_1 < n_2 < \dots < n_m$, and a memoryless channel $(\mathcal{X}, \mathcal{Y}, P_{Y|X})$. For any $l > 0$ and $\epsilon \in (0, 1)$, there exists an (l, n_1^n, M, ϵ') VLSF code with

$$l \leq n_m + \sum_{i=1}^{m-1} (n_i - n_{i+1}) \mathbb{P} \left[\bigcup_{j=1}^i \{\iota(X^{n_j}; Y^{n_j}) \geq \gamma\} \right], \quad (20)$$

$$\epsilon' \leq 1 - \mathbb{P}[\iota(X^{n_m}; Y^{n_m}) \geq \gamma] + (M - 1)2^{-\gamma}, \quad (21)$$

where $P_{X^{n_m}}$ is the product of distributions of m subvectors of lengths $n_i - n_{i-1}$, $i \in [m]$, i.e.,

$$P_{X^{n_m}}(x^{n_m}) = \prod_{i=1}^m P_{X_{n_{i-1}+1}^{n_i}}(x_{n_{i-1}+1}^{n_i}). \quad (22)$$

The proof of Theorem 2 is analogous to that of Theorem 1, with the distinction that X^{n_m} is drawn according to (22) rather than i.i.d. from a fixed input distribution. In what follows, we assume that X^n is always drawn i.i.d. according to the capacity-achieving input distribution P_X^* unless otherwise specified. This particular choice clearly meets (22).

For the BEC, the decoder in fact has the ability to identify the correct message whenever only a single codeword is compatible with the unerased channel outputs up to that point. By exploiting this fact and utilizing the RLFC, Devassy *et al.* [31] obtained better achievability bound for zero-error VLSF codes whose message size M is a power of 2.

Theorem 3 (Th. 9, [31]): For each integer $k \geq 1$, there exists an $(l, \mathbb{N}, 2^k, 0)$ VLSF code for a BEC(p) with

$$l \leq \frac{1}{C} \left(k + \sum_{i=1}^{k-1} \frac{2^i - 1}{2^k - 2^i} \right), \quad (23)$$

where $C = 1 - p$.

In Section VI, we present a new upper bound on l (Theorem 12) using the ST-RLFC scheme that is tighter than (23).

III. TIGHT APPROXIMATIONS ON $\mathbb{P}[\iota(X^n; Y^n) \geq \gamma]$

In this section, we develop tight approximations to the tail probability $\mathbb{P}[\iota(X^n; Y^n) \geq \gamma]$. Under our construction that X^n are i.i.d., $\iota(X^n; Y^n)$ is a sum of independent random variables distributed the same as $\iota(X_1; Y_1)$. This fact facilitates the use of Edgeworth expansion, Petrov expansion, or Kolassa's continuity-corrected Edgeworth series, all of which can be seen as refined versions of the CLT. A fascinating feature of these expansions is that they only require the knowledge of higher-order cumulants of $\iota(X_1; Y_1)$.

We follow [25, Chapter I, §2] in introducing the cumulant of a random variable, which will play an important role in evaluating Edgeworth and Petrov expansions.

Definition 2: For a random variable W with distribution P_W , let $\chi_W(t) = \mathbb{E}[e^{itW}]$ be its characteristic function. The j th cumulant of W , $j \geq 1$, is defined by the equality

$$\kappa_j \triangleq \frac{1}{i^j} \left[\frac{d^j}{dt^j} \ln \chi_W(t) \right]_{t=0}. \quad (24)$$

The characteristic function $\chi_W(t)$ can be expressed in terms of the exponential of the power series of cumulants,

$$\chi_W(t) = \exp \left(\sum_{j=1}^{\infty} \frac{\kappa_j}{j!} (it)^j \right). \quad (25)$$

In general, the j th cumulant κ_j is a homogeneous polynomial in noncentral moments of degree j , given by

$$\kappa_j = j! \sum_{\{k_l\}} (-1)^{r-1} (r-1)! \prod_{l=1}^j \frac{1}{k_l!} \left(\frac{\mathbb{E}[W^{k_l}]}{k_l!} \right)^{k_l}, \quad (26)$$

where the set $\{k_l\}$ consists of all non-negative solutions to $\sum_{l=1}^j lk_l = j$ and $r = \sum_{l=1}^j k_l$.

Remark 1: Petrov provided the formula (26) and suggested an induction method as a proof. Blinnikov and Moessner [32, Appendix B] presented a direct proof of (26) and provided an efficient algorithm to compute the set $\{k_l\}$ in (26).

Theorem 4 (Edgeworth Expansion, Eq. (2.18), [24]): Let W_1, W_2, \dots, W_n be a sequence of i.i.d. random variables with zero mean and a finite variance σ^2 . Define $G_n(x) \triangleq \mathbb{P}[\sum_{i=1}^n W_i \leq x\sigma\sqrt{n}]$. Let $\chi_W(t) \triangleq \mathbb{E}[e^{itW}]$ be the characteristic function of W and let $\kappa = \{\kappa_i\}_{i=1}^\infty$ be the cumulants of W . If $\mathbb{E}[|W|^{s+2}] < \infty$ for some $s \in \mathbb{N}_+$ and $\limsup_{|t| \rightarrow \infty} |\chi_W(t)| < 1$ (known as Cramér's condition), then,

$$G_n(x) = \Phi(x) + \phi(x) \sum_{j=1}^s n^{-\frac{j}{2}} p_j(x) + o(n^{-\frac{s}{2}}), \quad (27)$$

where, letting $\bar{\kappa}_i = \sigma^{-i} \kappa_i$ be the order- i cumulant of the normalized random variable W/σ ,

$$p_j(x) = - \sum_{\{k_i\}} He_{j+2r-1}(x) \prod_{i=1}^j \frac{1}{k_i!} \left(\frac{\bar{\kappa}_{i+2}}{(i+2)!} \right)^{k_i}, \quad (28)$$

$$He_j(x) = j! \sum_{k=0}^{\lfloor j/2 \rfloor} \frac{(-1)^k x^{j-2k}}{k!(j-2k)!2^k}, \quad (29)$$

and in (28), the set $\{k_i\}$ consists of all non-negative solutions to $\sum_{i=1}^j ik_i = j$, $r \triangleq \sum_{i=1}^j k_i$. The polynomial $He_j(x)$ in (29) is known as the degree- j Hermite polynomial.

Proof: We follow the proof in [24] to derive the Edgeworth expansion. However, we derive explicit formula for the $p_j(x)$ polynomial. See Appendix A for the complete proof. ■

Remark 2: In Theorem 4, the Cramér's condition holds if the continuous random variable W has a proper density function. If W is a lattice random variable, then Cramér's condition in Theorem 4 is violated and Theorem 4 is no longer applicable. In [32, Appendix B], the authors presented a proof of (29).

We obtain an *order- s Edgeworth expansion* by ignoring the $o(n^{-s/2})$ term in (27). Meanwhile, (27) suggests that $\lim_{n \rightarrow \infty} G_n(x) = \Phi(x)$, which is exactly the CLT.

Theorem 5 (Petrov Expansion, Chapter VIII, Th. 1, [25]): Let W_1, W_2, \dots, W_n be a sequence of i.i.d. random variables with zero mean and a finite variance σ^2 . Define $G_n(x) \triangleq \mathbb{P}[\sum_{i=1}^n W_i \leq x\sigma\sqrt{n}]$. If $x \geq 0$, $x = o(\sqrt{n})$, and the moment generating function $\mathbb{E}[e^{tW}] < \infty$ for $|t| < H$ for some $H > 0$, then

$$G_n(x) = 1 - Q(x) \exp \left\{ \frac{x^3}{\sqrt{n}} \Lambda \left(\frac{x}{\sqrt{n}} \right) \right\} \left[1 + O \left(\frac{x+1}{\sqrt{n}} \right) \right], \quad (30)$$

$$G_n(-x) = Q(x) \exp \left\{ \frac{-x^3}{\sqrt{n}} \Lambda \left(\frac{-x}{\sqrt{n}} \right) \right\} \left[1 + O \left(\frac{x+1}{\sqrt{n}} \right) \right], \quad (31)$$

where $\Lambda(t) = \sum_{k=0}^\infty a_k t^k$ is called the Cramér series. Details on Cramér series can be found in the proof of [25, Chapter VIII, Theorem 2]. In particular, Petrov provided the order-3 Cramér series $\Lambda^{[3]}(t)$

$$\Lambda^{[3]}(t)$$

$$= \frac{\kappa_3}{6\kappa_2^{3/2}} + \frac{\kappa_4\kappa_2 - 3\kappa_3^2}{24\kappa_2^3} t + \frac{\kappa_5\kappa_2^2 - 10\kappa_4\kappa_3\kappa_2 + 15\kappa_3^3}{120\kappa_2^{9/2}} t^2, \quad (32)$$

where $\{\kappa_i\}_{i=1}^\infty$ denotes the cumulants of random variable W .

The use of κ_5 in (32) results in an order-3 Petrov expansion in Theorem 5, as can be seen in (28), where the order of the truncated Edgeworth expansion is determined by the highest order of cumulant minus 2. Note that at the 0th order, both Edgeworth and Petrov expansions reduce to $\Psi(x)$.

Remark 3: Both finite-order Edgeworth and Petrov expansions are approximations that are obtained by truncating an infinite series. Edgeworth expansion assumes a constant target probability compared to n , whereas Petrov expansion assumes that the target probability decays sub-exponentially to 0 as $n \rightarrow \infty$, defining a moderate deviation sequence in n . Therefore, the former performs better in the large n regime, while the latter performs better in small n regime.

For lattice random variables, though Theorem 4 becomes unavailable, Kolassa [26] provided the *continuity-corrected Edgeworth series* that guarantees the same order of error as usually obtained in an Edgeworth approximation for continuous random variables.

Theorem 6 (Continuity-Corrected Edgeworth Series, Chapter 3.15, [26]): Assume that $\{W_i\}_{i=1}^n$ is a sequence of i.i.d. random variables with zero mean, variance σ^2 , and cumulants $\kappa = (\kappa_j, j \in \mathbb{N})$. Suppose that $\{W_i\}_{i=1}^n$ are confined to the lattice $a + u\Delta$, $u \in \mathbb{N}$, $\Delta > 0$, almost surely. Let $Z = \sum_{i=1}^n W_i/(\sigma\sqrt{n})$ and let $F_Z(z)$ be its CDF. Let $\lambda_j^n = \kappa_j - \epsilon_j n^{-1}$ be the adjusted cumulants known as the Sheppard-adjusted cumulants, where $\epsilon_j = (\Delta/\sigma)^j B_j/j$, with B_j being the j th Bernoulli number [33, Chapter 15]. Let λ^n denote the infinite vector $(0, \lambda_2^n, \lambda_3^n, \dots)$. Let

$$E_s(z; \kappa) \triangleq \Phi(z) + \phi(z) \sum_{j=1}^s n^{-\frac{j}{2}} p_j(z) \quad (33)$$

be the order- s Edgeworth expansion evaluated at z , with $p_j(\cdot)$ polynomials computed from the first $s+2$ cumulants of κ . Then,

$$F_Z(z^+) = E_s(z^+; \lambda^n) + o(n^{-\frac{s}{2}}), \quad (34)$$

where $z^+ = z + \frac{\Delta/2}{\sigma\sqrt{n}}$ denotes the continuity-corrected lattice point for any lattice point z of Z , and the $p_j(\cdot)$ polynomials in $E_s(z^+; \lambda^n)$ are computed from the first $s+2$ Sheppard-adjusted cumulants in λ^n rather than κ .

Note that for any lattice point z , $F_Z(z') = F_Z(z^+)$ for all $z' \in [z, z + \frac{\Delta}{\sigma\sqrt{n}})$. Therefore, (34) provides an accurate approximation for the entire real line.

Next, we discuss the approximation to $\mathbb{P}[\iota(X^n; Y^n) \geq \gamma]$ for BI-AWGN channel, the BSC, and the BEC. Due to the distinct nature of these channels, we choose different strategies to approximate or exactly evaluate $\mathbb{P}[\iota(X^n; Y^n) \geq \gamma]$ for each of these channels. For brevity, denote by $F_\gamma(n)$ the function we use to estimate $\mathbb{P}[\iota(X^n; Y^n) \geq \gamma]$. The domain of n for $F_\gamma(n)$ also depends on the type of channel.

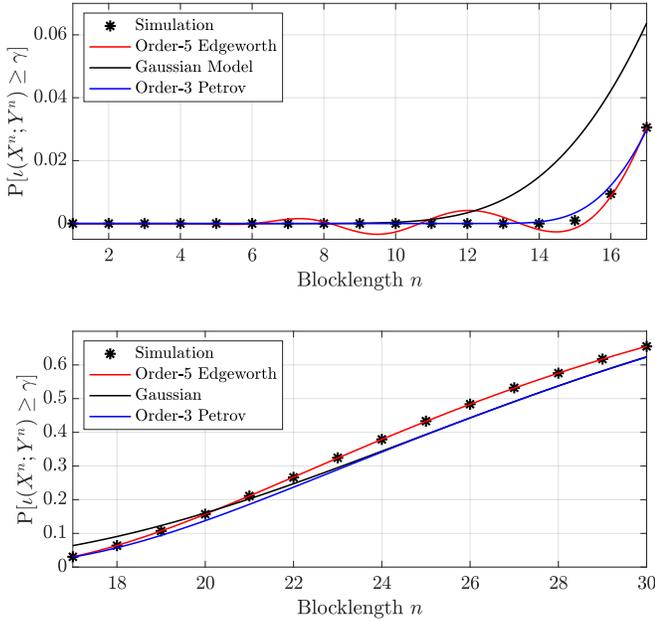


Fig. 1. Comparison of various approximation models for $\mathbb{P}[\iota(X^n; Y^n) \geq \gamma]$ with a fixed $\gamma = 13.62$ for BI-AWGN channel at 0.2 dB.

A. BI-AWGN Channel

For the BI-AWGN channel, the information density $\iota(X; Y)$ under $\text{Unif}(\mathcal{X})$ is given by

$$\iota(X; Y) = 1 - \log(1 + e^{-2XY}). \quad (35)$$

Clearly, $\iota(X; Y)$ is a continuous random variable with a proper density function. Hence, the order- s Edgeworth expansion in Theorem 4 is applicable in this case. In our experimentation, we identify that $s = 5$ meets our desired approximation accuracy at large n .

However, a caveat of the order- s Edgeworth expansion is that for small values of n , the order- s Edgeworth expansion oscillates around 0 due to the truncation of an infinite series. To illustrate this issue, Fig. 1 shows the order-5 Edgeworth expansion depicted in solid red curve to approximate $\mathbb{P}[\iota(X^n; Y^n) \geq \gamma]$, where $\gamma = 13.62$ and the BI-AWGN channel is at 0.2 dB. We use Monte Carlo simulation to obtain the tail probability at each time instant. As can be seen, the order-5 Edgeworth expansion oscillates around 0 for $n < 16$. Yet beyond this range, the order-5 Edgeworth expansion seamlessly matches the simulated tail probability.

To circumvent the oscillation issue, we resort to the order-3 Petrov expansion in Theorem 5 for small n satisfying $n < \gamma/C$. More specifically, the $F_\gamma(n)$ we use is given by

$$F_\gamma(n) = \begin{cases} Q(x(n)) - \phi(x(n)) \sum_{j=1}^5 n^{-\frac{j}{2}} p_j(x(n)), & n > n^* \\ Q(x(n)) \exp\left\{\frac{x^3(n)}{\sqrt{n}} \Lambda^{[3]}\left(\frac{x(n)}{\sqrt{n}}\right)\right\}, & 0 \leq n \leq n^*, \end{cases} \quad (36)$$

where $x(n) \triangleq \frac{\gamma - nC}{\sqrt{nV}}$ and $n^* < \gamma/C$ is the largest n value for which two expansions are equal with a common value less than 1/2.

Fig. 1 illustrates the order-3 Petrov expansion as depicted in solid blue curve. We see that the order-3 Petrov expansion

provides a good approximation for $n < 16$ yet starts to deviate from the simulated tail probability as n increases. Thus, combining both expansions at switching threshold $n^* = 16.84$ will provide a good approximation over the entire range of blocklength. Fig. 1 also shows the Gaussian model $Q(x(n))$ considered in [18], which corresponds to the order-0 Edgeworth expansion. As can be seen, the Gaussian model is inaccurate over the entire range of blocklength.

B. BSC

For the BSC(p), $p \in (0, 1/2)$, the information density $\iota(X; Y)$ under $\text{Unif}(\mathcal{X})$ is given by

$$\iota(X; Y) = \log(2 - 2p) - (X \oplus Y) \log \frac{1-p}{p} \quad (37)$$

$$= \log(2 - 2p) - Z \log \frac{1-p}{p}, \quad (38)$$

where $Z \sim \text{Bern}(p)$. Hence, $\iota(X; Y)$ is a random walk taking steps $\log(2p)$ and $\log(2-2p)$ with probabilities p and $1-p$. The length- n information density is thus given by

$$\iota(X^n; Y^n) = n \log(2 - 2p) - \left(\log \frac{1-p}{p}\right) \sum_{i=1}^n Z_i. \quad (39)$$

The tail probability $\mathbb{P}[\iota(X^n; Y^n) \geq \gamma]$ can be computed from the CDF of the binomial distribution. Hence,

$$\begin{aligned} \mathbb{P}[\iota(X^n; Y^n) \geq \gamma] &= \mathbb{P}\left[\sum_{i=1}^n Z_i \leq \frac{n \log(2 - 2p) - \gamma}{\log((1-p)/p)}\right] \\ &= \sum_{c=0}^{\lfloor \frac{n \log(2 - 2p) - \gamma}{\log((1-p)/p)} \rfloor} \binom{n}{c} p^c (1-p)^{n-c}. \end{aligned} \quad (40)$$

Thus, we choose $F_\gamma(n) = \mathbb{P}[\iota(X^n; Y^n) \geq \gamma]$ which is given by (40), where $n \in \mathbb{N}$. Next, we show that for a fixed $\gamma > 0$, $F_\gamma(n)$ exhibits a zig-zag shape as n increases.

Theorem 7: Fix $\gamma > 0$ and $p \in (0, 1/2)$. Define $\alpha_i \triangleq \left\lceil \frac{\gamma + i \log((1-p)/p)}{\log(2-2p)} \right\rceil$, $i \in \mathbb{N}$. Then, if $n = \alpha_i - 1$, $F_\gamma(n) < F_\gamma(n+1)$; if $n \in [\alpha_i, \alpha_{i+1} - 1)$, $F_\gamma(n) > F_\gamma(n+1)$, where $i \in \mathbb{N}$.

Proof: First, we show that the interval $[\alpha_i, \alpha_{i+1} - 1)$ contains at least one integer. This is because

$$\begin{aligned} \alpha_{i+1} - \alpha_i &\geq \frac{\gamma + (i+1) \log \frac{1-p}{p}}{\log(2-2p)} - \left\lceil \frac{\gamma + i \log \frac{1-p}{p}}{\log(2-2p)} \right\rceil \\ &> \frac{\log \frac{1}{p} + \log(1-p)}{1 + \log(1-p)} - 1 \\ &> 1. \end{aligned} \quad (41)$$

It follows that $(\alpha_{i+1} - 1) - \alpha_i \geq 1$, implying that the interval $[\alpha_i, \alpha_{i+1} - 1)$ contains at least one integer. Next, it suffices to show the result for a fixed $i \in \mathbb{N}$. If $n = \alpha_i - 1$,

$$\begin{aligned} F_\gamma(n) &= \mathbb{P}\left[\sum_{j=1}^n Z_j \leq i - 1\right] < \mathbb{P}\left[\sum_{j=1}^n Z_j \leq i - Z_{n+1}\right] \\ &= F_\gamma(n+1), \end{aligned} \quad (42)$$

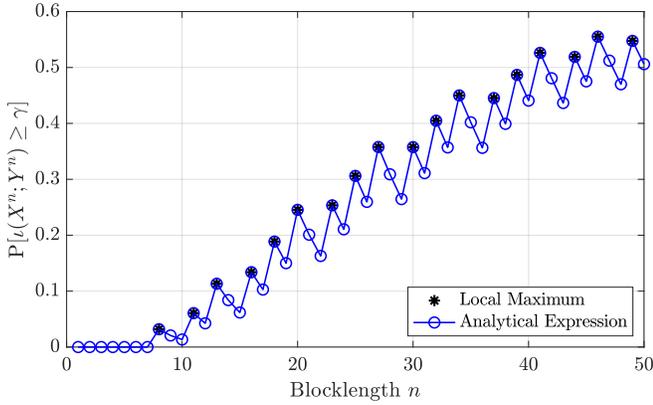


Fig. 2. Tail probability $\mathbb{P}[\iota(X^n; Y^n) \geq \gamma]$ for BSC(0.35) with $\gamma = 3$.

where (42) follows from $Z_{n+1} \in \{0, 1\}$. If $n \in [\alpha_i, \alpha_{i+1} - 1)$,

$$F_\gamma(n) = \mathbb{P}\left[\sum_{j=1}^n Z_j \leq i\right] > \mathbb{P}\left[\sum_{j=1}^n Z_j \leq i - Z_{n+1}\right] \quad (43)$$

$$= F_\gamma(n+1),$$

where (43) follows from $Z_{n+1} \in \{0, 1\}$. ■

Theorem 7 implies that the sequence $\{\alpha_i\}_{i=0}^\infty$ corresponds to the set of local maximizers, whereas the sequence $\{\alpha_i - 1\}_{i=0}^\infty$ corresponds to the set of local minimizers. As an example, Fig. 2 shows the tail probability $F_\gamma(n)$ as a function of blocklength n , which exhibits a zig-zag behavior. The local maximum values correspond to the tail probability at local maximizers $\{\alpha_i\}_{i=0}^\infty$. We see that the tail probabilities at local maximizers may not be a monotonically increasing sequence.

C. BEC

For BEC(p), $p \in (0, 1)$, the information density $\iota(X; Y)$ under $\text{Unif}(\mathcal{X})$ is given by

$$\iota(X; Y) = 1 - \mathbf{1}_{\{Y \neq X\}} = 1 - Z, \quad (44)$$

where $Z \sim \text{Bern}(p)$. Thus, the tail probability can be computed from the CDF of binomial distribution.

$$\mathbb{P}[\iota(X^n; Y^n) \geq \gamma] = \mathbb{P}\left[\sum_{i=1}^n Z_i \leq n - \gamma\right] \quad (45)$$

$$= \sum_{c=0}^{\lfloor n-\gamma \rfloor} \binom{n}{c} p^c (1-p)^{n-c}. \quad (46)$$

Note that $\iota(X; Y) \in \{0, 1\}$, it follows that $\mathbb{P}[\iota(X^n; Y^n) \geq \gamma]$ is a strictly increasing function of n .

Since $\iota(X; Y) = 1 - Z$ is a lattice random variable with span 1, Theorem 6 is readily available for approximating the tail probability $\mathbb{P}[\iota(X^n; Y^n) \geq \gamma]$. Numerical experiments show that with an order- s continuity-corrected Edgeworth expansion, the oscillation issue observed in the BI-AWGN channel also persists in the BEC case. The severity of oscillation is observed to depend on the order of Edgeworth expansion s , the erasure probability p , and the choice of γ . For example, consider the continuity-corrected point $\gamma = 10.5$. Fig. 3 shows the order-5 continuity-corrected Edgeworth expansions

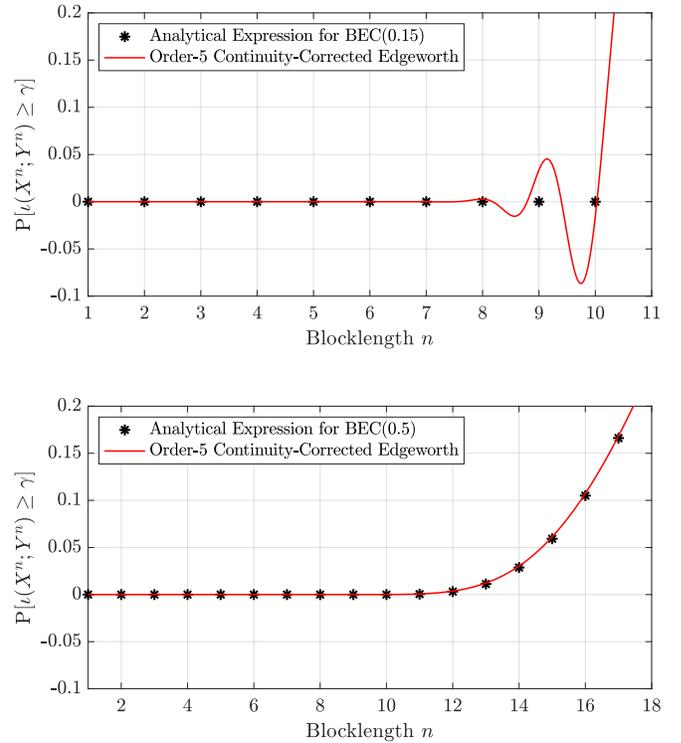


Fig. 3. Comparison of $\mathbb{P}[\iota(X^n; Y^n) \geq \gamma]$ between BEC(0.15) and BEC(0.5) with $\gamma = 10.5$.

to $\mathbb{P}[\iota(X^n; Y^n) \geq \gamma]$ for BEC(0.15) and for BEC(0.5). We see that for BEC(0.15), the approximation curve has a visible oscillation around 0, whereas for BEC(0.5), one can barely see the oscillation.

In our implementation, we choose $s = 5$ as the order for the truncated continuity-corrected Edgeworth series to approximate $\mathbb{P}[\iota(X^n; Y^n) \geq \gamma]$. For a target error probability $\epsilon = 10^{-3}$, we numerically found a sufficiently good erasure probability threshold $p^* = 0.2$, beyond which the oscillation issue becomes negligible for any continuity-corrected point $\gamma^+ \geq \log \frac{1}{\epsilon} = 10$. Hence, for target error probability $\epsilon = 10^{-3}$, the $F_\gamma(n)$ we choose for the BEC is given by

$$F_\gamma(n) = \begin{cases} \sum_{c=0}^{\lfloor n-\gamma \rfloor} \binom{n}{c} p^c (1-p)^{n-c}, & n \in \mathbb{N}, \quad \text{if } p < 0.2 \\ 1 - E_5(x(n); \lambda_n), & n \in \mathbb{R}_+, \quad \text{if } p \geq 0.2, \end{cases} \quad (47)$$

where $x(n) = \frac{n(1-p) - \gamma^+}{\sqrt{np(1-p)}}$, with $\gamma^+ = \lceil \gamma \rceil - 1/2$ being the continuity-corrected point for γ . For $p < 0.2$, we apply the discrete SDO procedure. For $p \geq 0.2$, we apply the gap-constrained SDO procedure. Both procedures are introduced in Section IV below.

IV. AN INTEGER PROGRAM AND TWO ALGORITHMS

In this section, we formulate an integer program of minimizing the upper bound on $\mathbb{E}[\tau]$ based on Theorem 2 and derive an error regime where Polyanskiy's stopping at zero scheme does not improve the achievability bound. Next, we provide two methods to solve this integer program: the gap-constrained SDO and the discrete SDO procedures.

By lower bounding $\mathbb{P}[\bigcup_{j=1}^i \{\iota(X^{n_j}; Y^{n_j}) \geq \gamma\}]$ to the marginal tail probability $\mathbb{P}[\iota(X^{n_i}; Y^{n_i}) \geq \gamma]$ in Theorem 2, we write the relaxed upper bound on $\mathbb{E}[\tau]$ as

$$N(\gamma, n_1^m) \triangleq n_m + \sum_{i=1}^{m-1} (n_i - n_{i+1}) \mathbb{P}[\iota(X^{n_i}; Y^{n_i}) \geq \gamma]. \quad (48)$$

Define the feasible region by

$$\begin{aligned} \mathcal{F}_m(\gamma, M, \epsilon) \triangleq \{n_1^m \in \mathbb{R}_+^m : n_{i+1} - n_i \geq 1, \forall i \in [m-1]; \\ \mathbb{P}[\iota(X^{n_m}; Y^{n_m}) \geq \gamma] \geq 1 - \epsilon + (M-1)2^{-\gamma}\}. \end{aligned} \quad (49)$$

As can be seen, the feasible region (49) consists of real-valued decoding times n_1^m such that two consecutive decoding times are separated by at least one and the target error probability is guaranteed at decoding time n_m .

Theorem 2 motivates the following integer program: for a given $m \in \mathbb{N}_+$, $M \in \mathbb{N}_+$, $\epsilon \in (0, 1)$, and $\gamma \geq \log \frac{M-1}{\epsilon}$,

$$\begin{aligned} \min_{n_1^m} \quad & N(\gamma, n_1^m) \\ \text{s. t.} \quad & n_1^m \in \mathcal{F}_m(\gamma, M, \epsilon) \\ & n_1^m \in \mathbb{N}_+^m. \end{aligned} \quad (50)$$

Let $\tilde{N}(\gamma)$ denote the minimum upper bound over n_1^m after solving the integer program (50). Then, $\min_{\gamma} \tilde{N}(\gamma)$ yields the globally minimum upper bound $N^*(\gamma, n_1^m)$. This method is called the *two-step minimization*. In this paper, we apply the two-step minimization to identify $N^*(\gamma, n_1^m)$. The key problem is to develop efficient algorithms for integer program (50).

In integer program (50), we consider $n_1 \geq 1$ and integer-valued decoding times. That is, we do not allow stopping the VLSF code at $\tau = 0$. In what follows, we identify an error regime where stopping a VLSF code at $\tau = 0$ does not improve the achievability bound.

In [14], Polyanskiy *et al.* showed that the ϵ -capacity $\frac{C}{1-\epsilon}$ is achievable in the non-vanishing error setting. This result is obtained by constructing a new code that stops an $(l', \mathbb{N}_+, M, \epsilon')$ VLSF code satisfying $\log M = Cl' + \log \epsilon' - a_0$, with $a_0 \triangleq \max_{x,y} \iota(x; y)$, at $\tau = 0$ with probability $p = \frac{\epsilon - \epsilon'}{1 - \epsilon'}$ and employs this VLSF code with probability $1 - p$, where $\epsilon' \leq \epsilon$. Such a new code has probability of error ϵ , average length $l = l'(1 - p)$, and message size $M^*(l, \epsilon) \geq M$. However, the following theorem shows that for sufficiently small ϵ , choosing a VLSF code with $\epsilon' = \epsilon$ yields the best average length l' achieved by this strategy.

Theorem 8: Let the aforementioned notation prevail. Fix $M \in \mathbb{N}_+$. Define

$$\epsilon^* \triangleq \arg \min_{x \in (0, 1)} \frac{\log M + a_0 - \log x}{1 - x}. \quad (51)$$

Then, if $\epsilon \in (0, \epsilon^*]$, $\epsilon' = \epsilon$ is the minimizer that yields a minimum average length $l = l'$.

Proof: The minimization problem we intend to solve is stated as below.

$$\begin{aligned} \min_{\epsilon'} \quad & l' \left(1 - \frac{\epsilon - \epsilon'}{1 - \epsilon'} \right) \\ \text{s. t.} \quad & \log M = Cl' + \log \epsilon' - a_0 \\ & \epsilon' \in (0, \epsilon]. \end{aligned} \quad (52)$$

This program is equivalent to the following program

$$\begin{aligned} \min_{\epsilon'} \quad & \left(\frac{1 - \epsilon}{C} \right) \frac{\log M + a_0 - \log \epsilon'}{1 - \epsilon'} \\ \text{s. t.} \quad & \epsilon' \in (0, \epsilon]. \end{aligned} \quad (53)$$

Define function $f(x) \triangleq \frac{\log M + a_0 - \log x}{1 - x}$. Since $f(x)$ is convex in $(0, 1)$, there exists a unique minimizer $\epsilon^* \in (0, 1)$ for $f(x)$. Hence, if $\epsilon \leq \epsilon^*$ and is fixed, the minimizer $\epsilon' = \epsilon$, giving the minimum average length $l = l'$. ■

Aiming to identify optimal decoding times that minimize $N(\gamma, n_1^m)$ for a given γ , we first establish a necessary condition for optimal decoding times that will aid the search for decoding times.

Theorem 9: Fix a memoryless channel $(\mathcal{X}, \mathcal{Y}, P_{Y|X})$ and scalars $m \in \mathbb{N}_+$, $M \in \mathbb{N}_+$, $\epsilon \in (0, 1)$, and $\gamma \geq \log \frac{M-1}{\epsilon}$. Let $n_0 \triangleq 0$. If n_1^m are optimal decoding times for integer program (50), then

$$\mathbb{P}[\iota(X^{n_i}; Y^{n_i}) \geq \gamma] \geq \max_{n_{i-1} \leq j < n_i} \mathbb{P}[\iota(X^j; Y^j) \geq \gamma] \quad (54)$$

for $i \in [m]$.

Proof: For brevity, define $S_n \triangleq \iota(X^n; Y^n)$. Since the target error probability is ensured by decoding time n_m in the feasible region $\mathcal{F}_m(\gamma, M, \epsilon)$, it follows that the optimal n_m corresponds to the minimum integer such that $\mathbb{P}[S_{n_m} \geq \gamma] \geq 1 - \epsilon + (M-1)2^{-\gamma}$, which clearly satisfies (54).

Assume there exist a set of decoding times $n_1^m \in \mathcal{F}_m(\gamma, M, \epsilon)$ for which (54) holds. For an appropriate integer $1 < l < m$, assume there exists \tilde{n}_l such that $n_l \neq \tilde{n}_l$, $n_{l-1} < \tilde{n}_l < n_{l+1}$. There are two ways to violate (54) for choosing \tilde{n}_l . If $\mathbb{P}[S_{\tilde{n}_l} \geq \gamma] < \mathbb{P}[S_{n_{l-1}} \geq \gamma]$, then

$$\begin{aligned} & N(\gamma, n_1^{l-1}, n_l, n_{l+1}^m) - N(\gamma, n_1^{l-1}, \tilde{n}_l, n_{l+1}^m) \\ &= (\tilde{n}_l - n_l) \mathbb{P}[S_{n_{l-1}} \geq \gamma] + (n_l - n_{l+1}) \mathbb{P}[S_{n_l} \geq \gamma] \\ &\quad - (\tilde{n}_l - n_{l+1}) \mathbb{P}[S_{\tilde{n}_l} \geq \gamma] \\ &< (\tilde{n}_l - n_l) \mathbb{P}[S_{n_{l-1}} \geq \gamma] + (n_l - n_{l+1}) \mathbb{P}[S_{n_l} \geq \gamma] \\ &\quad - (\tilde{n}_l - n_{l+1}) \mathbb{P}[S_{n_{l-1}} \geq \gamma] \\ &= (n_{l+1} - n_l) (\mathbb{P}[S_{n_{l-1}} \geq \gamma] - \mathbb{P}[S_{n_l} \geq \gamma]) \\ &\leq 0. \end{aligned} \quad (55)$$

If $n_l < \tilde{n}_l < n_{l+1}$ and $\mathbb{P}[S_{n_{l-1}} \geq \gamma] \leq \mathbb{P}[S_{\tilde{n}_l} \geq \gamma] < \mathbb{P}[S_{n_l} \geq \gamma]$, then

$$\begin{aligned} & N(\gamma, n_1^{l-1}, n_l, n_{l+1}^m) - N(\gamma, n_1^{l-1}, \tilde{n}_l, n_{l+1}^m) \\ &= (\tilde{n}_l - n_l) \mathbb{P}[S_{n_{l-1}} \geq \gamma] + (n_l - n_{l+1}) \mathbb{P}[S_{n_l} \geq \gamma] \\ &\quad - (\tilde{n}_l - n_{l+1}) \mathbb{P}[S_{\tilde{n}_l} \geq \gamma] \\ &< (\tilde{n}_l - n_l) \mathbb{P}[S_{n_{l-1}} \geq \gamma] + (n_l - n_{l+1}) \mathbb{P}[S_{n_l} \geq \gamma] \\ &\quad - (\tilde{n}_l - n_{l+1}) \mathbb{P}[S_{n_l} \geq \gamma] \\ &= (\tilde{n}_l - n_l) (\mathbb{P}[S_{n_{l-1}} \geq \gamma] - \mathbb{P}[S_{n_l} \geq \gamma]) \end{aligned}$$

$$\leq 0. \quad (56)$$

This implies that choosing \tilde{n}_l in violation of (54) incurs a penalty to the upper bound $N(\gamma, n_1^m)$. Hence, (54) is a necessary condition for optimal decoding times n_1^m . ■

For the BSC, Theorems 7 and 9 imply the following useful corollary.

Corollary 1: Fix a BSC(p), $p \in (0, 1/2)$, and scalars $m \in \mathbb{N}_+$, $M \in \mathbb{N}_+$, $\epsilon \in (0, 1)$ and $\gamma \geq \log \frac{M-1}{\epsilon}$. The optimal decoding times n_1^m for the integer program (50) are among the local maximizers $\{\alpha_i\}_{i=0}^\infty$.

Proof: For brevity, let $S_n \triangleq \iota(X^n; Y^n)$. By Theorem 9, it follows that no two optimal decoding times belong to the same interval $[\alpha_i, \alpha_{i+1})$. Otherwise, by Theorem 7, their tail probabilities violate (54).

Let us assume that there exists a sequence of decoding times $n_1^m \in \mathcal{F}_m(\gamma, M, \epsilon)$ for BSC(p) satisfying $n_i \in [\alpha_{b_i}, \alpha_{b_i+1})$, $i \in [m]$, where $b_1 < b_2 < \dots < b_m$. If $n_i > \alpha_{b_i}$, then by Theorem 7, $\mathbb{P}[S_{\alpha_{b_i}} \geq \gamma] > \mathbb{P}[S_{n_i} \geq \gamma]$. This violates the necessary condition in Theorem 9. Hence, if n_1^m are optimal, we must have $n_i = \alpha_{b_i}$. ■

Finally, we present two search algorithms to numerically solve integer program (50): the gap-constrained SDO procedure and the discrete SDO procedure. The first algorithm relies on a monotone, differentiable function $F_\gamma(n)$, $n \geq 0$, to approximate $\mathbb{P}[\iota(X^n; Y^n) \geq \gamma]$. In contrast, the second algorithm only relies on a good estimate of $\mathbb{P}[\iota(X^n; Y^n) \geq \gamma]$, $n \in \mathbb{N}_+$, at a cost of increased search complexity.

A. The Gap-Constrained SDO Procedure

To facilitate a program that is computationally tractable, we consider the relaxed program of (50) by allowing $n_1^m \in \mathbb{R}_+^m$. For a given $m \in \mathbb{N}_+$, $M \in \mathbb{N}_+$, $\epsilon \in (0, 1)$, and $\gamma \geq \log \frac{M-1}{\epsilon}$,

$$\begin{aligned} \min_{n_1^m} \quad & N(\gamma, n_1^m) \\ \text{s. t.} \quad & n_1^m \in \mathcal{F}_m(\gamma, M, \epsilon). \end{aligned} \quad (57)$$

In the relaxed program (57), the tail probability $\mathbb{P}[\iota(X^n; Y^n) \geq \gamma]$ is approximated by a monotonically increasing and differentiable function $F_\gamma(n)$ satisfying $F_\gamma(0) = 0$ and $F_\gamma(\infty) = 1$. Define

$$f_\gamma(n) \triangleq \frac{dF_\gamma(n)}{dn}. \quad (58)$$

The next theorem gives the analytical solution to the relaxed program (57).

Theorem 10: Fix a memoryless channel $(\mathcal{X}, \mathcal{Y}, P_{Y|X})$ for which $\iota(X; Y)$ is continuous and $\mathbb{P}[\iota(X^n; Y^n) \geq \gamma]$ is monotone. For a given $m \in \mathbb{N}_+$, $M \in \mathbb{N}_+$, $\epsilon \in (0, 1)$, and $\gamma \geq \log \frac{M-1}{\epsilon}$, the optimal real-valued decoding times $n_1^*, n_2^*, \dots, n_m^*$ for the relaxed program (57) satisfy

$$n_m^* = F_\gamma^{-1}(1 - \epsilon + (M-1)2^{-\gamma}), \quad (59)$$

$$n_{i+1}^* = n_i^* + \max \left\{ 1, \frac{F_\gamma(n_i^*) - F_\gamma(n_{i-1}^*) - \lambda_{i-1}}{f_\gamma(n_i^*)} \right\}, \quad (60)$$

$$\lambda_i = \max \{ \lambda_{i-1} + f_\gamma(n_i^*) - F_\gamma(n_i^*) + F_\gamma(n_{i-1}^*), 0 \}, \quad (61)$$

where $i \in [m-1]$, $\lambda_0 \triangleq 0$, and $n_0^* \triangleq 0$.

Proof: For brevity, denote by $\mathbf{n} \triangleq (n_1, n_2, \dots, n_m)$ the vector of decoding times and by $\mathbf{n}^* \triangleq (n_1^*, n_2^*, \dots, n_m^*)$ the vector of optimal decoding times. By introducing the Lagrangian multipliers ν, λ_1^{m-1} , the Lagrangian of the relaxed program (57) is given by

$$\begin{aligned} \mathcal{L}(\mathbf{n}, \nu, \lambda_1^{m-1}) &= n_1 + \nu(1 - F_\gamma(n_m)) - \epsilon + (M-1)2^{-\gamma} \\ &+ \sum_{i=1}^{m-1} (n_{i+1} - n_i)(1 - F_\gamma(n_i)) + \sum_{i=1}^{m-1} \lambda_i(n_i - n_{i+1} + 1). \end{aligned}$$

The optimal decoding times $\mathbf{n}^* = (n_1^*, n_2^*, \dots, n_m^*)$ must satisfy the Karush-Kuhn-Tucker (KKT) conditions [34, Sec. 5.5.3],

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial n_i} \Big|_{\mathbf{n}=\mathbf{n}^*} &= F_\gamma(n_i^*) - F_\gamma(n_{i-1}^*) - (n_{i+1}^* - n_i^*)f_\gamma(n_i^*) \\ &+ \lambda_i - \lambda_{i-1} = 0, \quad i \in [m-1], \end{aligned} \quad (62)$$

$$\frac{\partial \mathcal{L}}{\partial n_m} \Big|_{\mathbf{n}=\mathbf{n}^*} = 1 - F_\gamma(n_{m-1}^*) - \nu f_\gamma(n_m^*) = 0, \quad (63)$$

$$\nu(1 - F_\gamma(n_m^*) - \epsilon + (M-1)2^{-\gamma}) = 0, \quad (64)$$

$$\lambda_i(n_i^* - n_{i+1}^* + 1) = 0, \quad i \in [m-1]. \quad (65)$$

Since $F_\gamma(n) \in (0, 1)$ and $f_\gamma(n) > 0$ for $n > 0$, (63) implies that $\nu > 0$. Hence, we obtain $n_m^* = F_\gamma^{-1}(1 - \epsilon + (M-1)2^{-\gamma})$ from (64).

Next, we analyze (65). There are two cases. If $\lambda_i > 0$, then $n_{i+1}^* = n_i^* + 1$. By (62), we obtain

$$\lambda_i = \lambda_{i-1} + f_\gamma(n_i^*) - F_\gamma(n_i^*) + F_\gamma(n_{i-1}^*). \quad (66)$$

If $n_{i+1}^* > n_i^* + 1$, then $\lambda_i = 0$. By (62), we obtain

$$n_{i+1}^* = n_i^* + \frac{F_\gamma(n_i^*) - F_\gamma(n_{i-1}^*) - \lambda_{i-1}}{f_\gamma(n_i^*)}. \quad (67)$$

Rewriting the above two cases in a compact form yields (60) and (61). ■

The procedures (59) – (61) are called the *gap-constrained SDO procedure*. The name indicates that the solution ensures two consecutive decoding times are separated by at least one. In contrast, the *unconstrained SDO procedure* considered in previous works [17]–[22] does not consider the gap constraint and admits a simple recursion

$$n_{i+1}^* = n_i^* + \frac{F_\gamma(n_i^*) - F_\gamma(n_{i-1}^*)}{f_\gamma(n_i^*)}, \quad i \in [m-1] \quad (68)$$

with n_m^* determined by (59).

To illustrate the distinction between the gap-constrained and unconstrained SDO procedures, Fig. 4 shows how the optimal real-valued decoding times n_1^m evolves as m increases using these two algorithms for the BI-AWGN channel at 0.2 dB, $M = 2^{20}$, and $\epsilon = 10^{-2}$. Together, these parameters uniquely determine $n_m^* = 101.91$ via (59). For $m \leq 20$, the gap-constrained SDO procedure behaves indistinguishably from the unconstrained SDO procedure, since the SDO solution naturally has a minimum gap larger than one. For large values of m , the unconstrained SDO procedure avoids early decoding times and instead adds later decoding times so densely that their separation is less than one. In contrast,

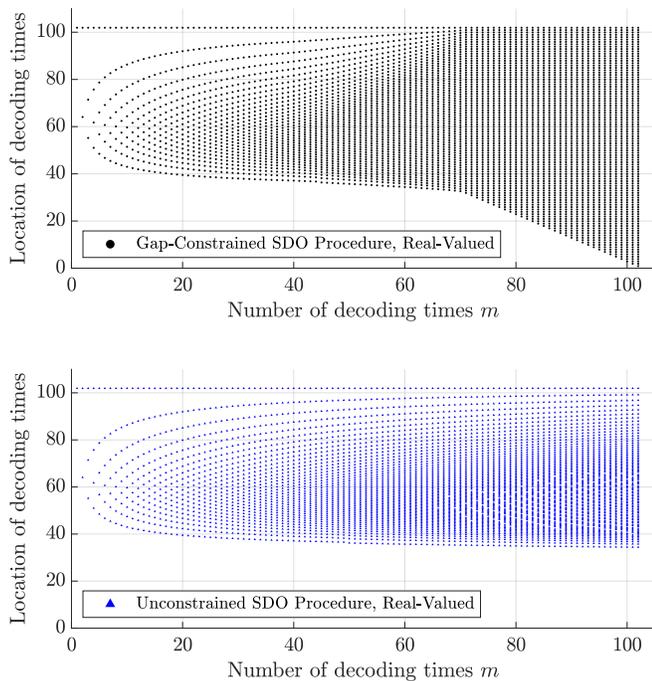


Fig. 4. Comparison of the optimal real-valued decoding times between the gap-constrained and unconstrained SDO procedures for the BI-AWGN channel at 0.2 dB. In this example, we choose $\epsilon = 10^{-2}$, $(M-1)2^{-\gamma} = \epsilon/2$, where $M = 2^{20}$. This produces $\gamma = 27.64$ and $n_m^* = 101.91$ using (59). We consider number of decoding times m ranging from 1 to $\lceil n_m^* \rceil = 102$.

the gap-constrained SDO procedure is forced to add early decoding times when all existing gaps become one.

We remark that the form of the gap-constrained SDO procedure naturally calls for a bisection search to identify n_1^* that subsequently determines n_2^*, \dots, n_{m-1}^* . When evaluating at small values of n , both $F_\gamma(n)$ and $f_\gamma(n)$ will become infinitesimally small. In this case, a direct numerical computation using (60) and (61) may cause a precision issue. Fortunately, the gap-constrained SDO procedures also admit a ratio form. Define $\lambda_k^{(r)} \triangleq \lambda_k / f_\gamma(n_k^*)$. Then, (60) and (61) can be equivalently written as

$$\begin{aligned}
 n_{i+1}^* &= n_i^* \\
 &+ \max \left\{ 1, \frac{F_\gamma(n_i^*)}{f_\gamma(n_i^*)} - \frac{F_\gamma(n_{i-1}^*)}{f_\gamma(n_{i-1}^*)} - \lambda_{i-1}^{(r)} \frac{f_\gamma(n_{i-1}^*)}{f_\gamma(n_i^*)} \right\}, \quad (69) \\
 \lambda_i^{(r)} &= \max \left\{ \lambda_{i-1}^{(r)} \frac{f_\gamma(n_{i-1}^*)}{f_\gamma(n_i^*)} + 1 - \frac{F_\gamma(n_i^*)}{f_\gamma(n_i^*)} + \frac{F_\gamma(n_{i-1}^*)}{f_\gamma(n_i^*)}, 0 \right\}. \quad (70)
 \end{aligned}$$

The purpose of using $F_\gamma(\tilde{n})/f_\gamma(n)$, $f_\gamma(\tilde{n})/f_\gamma(n)$, and $\lambda_k^{(r)}$ is that they have a closed-form expression that cancels out the common infinitesimal factor in both the numerator and denominator. In our implementation for the BI-AWGN channel, we applied the ratio form of the gap-constrained SDO procedure.

B. The Discrete SDO Procedure

The gap-constrained SDO procedure in Theorem 10 hinges on the existence of a monotonically increasing and differ-

entiable function $F_\gamma(n)$ to approximate $\mathbb{P}[\iota(X^n; Y^n) \geq \gamma]$. In general, however, such a function may not exist. For example, for the BSC(p), $p \in (0, 1/2)$, the tail probability $\mathbb{P}[\iota(X^n; Y^n) \geq \gamma]$ as a function of n cannot be approximated by a monotone and differentiable function, as seen in Fig. 2.

As a general solution to the integer program (50), we develop the *discrete SDO procedure* that only relies on a good estimate of $\mathbb{P}[\iota(X^n; Y^n) \geq \gamma]$ at $n \in \mathbb{N}_+$.

Theorem 11: Fix a memoryless channel $(\mathcal{X}, \mathcal{Y}, P_{Y|X})$ and scalars $m \in \mathbb{N}_+$, $M \in \mathbb{N}_+$, $\epsilon \in (0, 1)$, and $\gamma \geq \log \frac{M-1}{\epsilon}$. Define $S_n \triangleq \iota(X^n; Y^n)$. The optimal integer-valued decoding times $n_1^*, n_2^*, \dots, n_m^*$ for the integer program (50) satisfy

$$n_1^* + \max(1, g_-^{(1)}(n_1^*)) \leq n_2^* \leq n_1^* + g_+^{(1)}(n_1^*), \quad (71)$$

$$n_i^* + \max(1, g_-^{(i)}(n_i^*, n_{i-1}^*)) \leq n_{i+1}^* \leq n_i^* + g_+^{(i)}(n_i^*, n_{i-1}^*), \quad (72)$$

for $i \in \{2, 3, \dots, m-1\}$,

where n_m^* is the smallest integer n_m at which $\mathbb{P}[S_{n_m} \geq \gamma] \geq 1 - \epsilon + (M-1)2^{-\gamma}$. For n_1 , the g functions associated with n_1 are defined by

$$g_-^{(1)}(n_1) \triangleq \max_{\substack{n \in [1, n_m^* - m + 1] \\ \mathbb{P}[S_n \geq \gamma] < \mathbb{P}[S_{n_1} \geq \gamma]}} \frac{\mathbb{P}[S_n \geq \gamma](n_1 - n)}{\mathbb{P}[S_{n_1} \geq \gamma] - \mathbb{P}[S_n \geq \gamma]}, \quad (73)$$

$$g_+^{(1)}(n_1) \triangleq \min_{\substack{n \in [1, n_m^* - m + 1] \\ \mathbb{P}[S_n \geq \gamma] > \mathbb{P}[S_{n_1} \geq \gamma]}} \frac{\mathbb{P}[S_n \geq \gamma](n_1 - n)}{\mathbb{P}[S_{n_1} \geq \gamma] - \mathbb{P}[S_n \geq \gamma]}. \quad (74)$$

For $2 \leq i \leq m-1$, the g functions associated with the pair (n_i, n_{i-1}) are defined by

$$\begin{aligned}
 g_-^{(i)}(n_i, n_{i-1}) &\triangleq \max_{\substack{n \in [n_{i-1} + 1, n_m^* - m + i] \\ \mathbb{P}[S_n \geq \gamma] < \mathbb{P}[S_{n_i} \geq \gamma]}} \frac{\mathbb{P}[S_n \geq \gamma] - \mathbb{P}[S_{n_{i-1}} \geq \gamma]}{\mathbb{P}[S_{n_i} \geq \gamma] - \mathbb{P}[S_n \geq \gamma]} (n_i - n), \quad (75)
 \end{aligned}$$

$$\begin{aligned}
 g_+^{(i)}(n_i, n_{i-1}) &\triangleq \min_{\substack{n \in [n_{i-1} + 1, n_m^* - m + i] \\ \mathbb{P}[S_n \geq \gamma] > \mathbb{P}[S_{n_i} \geq \gamma]}} \frac{\mathbb{P}[S_n \geq \gamma] - \mathbb{P}[S_{n_{i-1}} \geq \gamma]}{\mathbb{P}[S_{n_i} \geq \gamma] - \mathbb{P}[S_n \geq \gamma]} (n_i - n). \quad (76)
 \end{aligned}$$

For g_- functions defined above, if the maximizer is empty, $g_-^{(i)}(\cdot) = -\infty$, $i \in [m-1]$. For g_+ functions defined above, if the minimizer is empty, $g_+^{(i)}(\cdot) = \infty$, $i \in [m-1]$.

Proof: Since the m th decoding time is used to meet the target error probability, it follows that the optimal n_m^* corresponds to the smallest integer n_m at which $\mathbb{P}[S_{n_m} \geq \gamma] \geq 1 - \epsilon + (M-1)2^{-\gamma}$.

Assume that $n_1^*, n_2^*, \dots, n_m^*$ are optimal decoding times. This means that for any other n_1 , we have

$$N(\gamma, n_1^*, n_2^*, \dots, n_m^*) \leq N(\gamma, n_1, n_2^*, \dots, n_m^*). \quad (77)$$

Inequality (77) is equivalent to the following

$$(n_1^* - n_2^*)\mathbb{P}[S_{n_1^*} \geq \gamma] - (n_1 - n_2^*)\mathbb{P}[S_{n_1} \geq \gamma] \leq 0. \quad (78)$$

We distinguish two cases. If $\mathbb{P}[S_{n_1} \geq \gamma] < \mathbb{P}[S_{n_1^*} \geq \gamma]$, (78) is equivalent to

$$n_2^* \geq n_1^* + \frac{\mathbb{P}[S_{n_1} \geq \gamma](n_1^* - n_1)}{\mathbb{P}[S_{n_1^*} \geq \gamma] - \mathbb{P}[S_{n_1} \geq \gamma]}. \quad (79)$$

If $\mathbb{P}[S_{n_1^*} \geq \gamma] < \mathbb{P}[S_{n_1} \geq \gamma]$, (78) is equivalent to

$$n_2^* \leq n_1^* + \frac{\mathbb{P}[S_{n_1} \geq \gamma](n_1^* - n_1)}{\mathbb{P}[S_{n_1^*} \geq \gamma] - \mathbb{P}[S_{n_1} \geq \gamma]}. \quad (80)$$

Note that n_2^* should satisfy (79) for all $n_1 \in [1, n_1^* - m + 1]$ with $\mathbb{P}[S_{n_1} \geq \gamma] < \mathbb{P}[S_{n_1^*} \geq \gamma]$. Using the $g_-^{(1)}$ function defined earlier and noting that $n_2^* \geq n_1^* + 1$, (79) can be compactly written as

$$n_2^* \geq n_1^* + \max(1, g_-^{(1)}(n_1^*)). \quad (81)$$

In a similar fashion, (80) can be written as

$$n_2^* \leq n_1^* + g_+^{(1)}(n_1^*). \quad (82)$$

The necessary conditions for n_3^*, \dots, n_m^* can be derived analogously. ■

Inequalities (71) to (72) are called the *discrete SDO procedure* due to their resemblance to the unconstrained SDO update rule (68) and the discrete nature. Note that the discrete SDO procedure automatically meets the desired gap constraint. Furthermore, the discrete SDO procedure is in fact a depth-first search which may produce a collection of decoding times n_1^m that satisfy inequalities (71) to (72). In this case, the optimal decoding times are among these finalists that yield the minimum upper bound $\tilde{N}(\gamma)$ for a fixed γ .

In general, the complexity of the discrete SDO procedure is much higher than that of the gap-constrained SDO procedure when $\mathbb{P}[\iota(X^n; Y^n) \geq \gamma]$ is an increasing function of n . Nevertheless, the discrete SDO procedure could become much more efficient if the search for optimal decoding times is restricted to a sparse set that meets Theorem 9, for instance, the set of local maximizers $\{\alpha_i\}_{i=1}^\infty$ in the BSC case.

V. NUMERICAL EVALUATIONS

This section numerically evaluate the achievability bound of VLSF codes with finite decoding times for three important channels: the BI-AWGN channel, the BSC, and the BEC. For each channel, we apply distinct computational methods, but we always use the two-step minimization in Sec. IV to obtain the globally minimum upper bound $N^*(\gamma, n_1^m)$. We then use $N^*(\gamma, n_1^m)$ to obtain the achievability bound on rate given by $\frac{\log M}{N^*(\gamma, n_1^m)}$, where M is the message size.

We consider the error regime in which Polyanskiy's stopping-at-zero scheme does not improve the achievability bound, which is identified by Theorem 8. Denote by $k \triangleq \log M$ the information length. Note that the three binary-input channels have maximum information density $a_0 \in \{1, \log 2(1-p)\}$. Thus, for $k \leq 1000$, numerical evaluation of (51) shows that $\epsilon \leq 1.4 \times 10^{-3}$ is the error regime in which Polyanskiy's stopping-at-zero scheme does not improve the achievability bound for any of the three binary-input channels. Throughout this section, we consider a fixed target error probability $\epsilon = 10^{-3}$ which falls into the above error regime for $k \leq 1000$. In this section, we use (18) and (19) to numerically evaluate Polyanskiy's achievability bound on VLSF codes in Theorem 1 for the three binary-input channels.

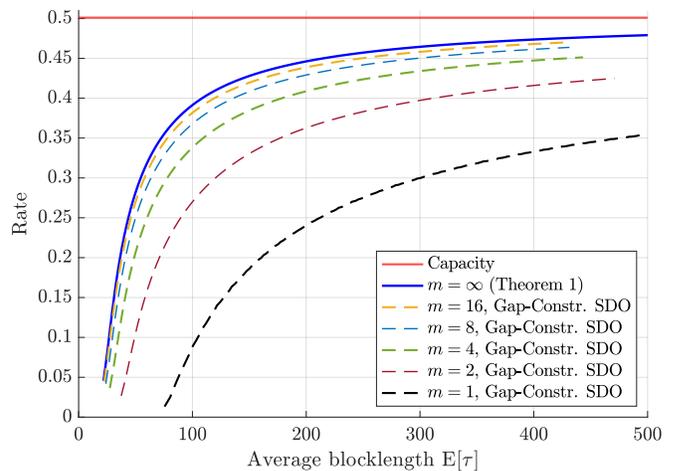


Fig. 5. Rate vs. average blocklength $\mathbb{E}[\tau]$ for BI-AWGN channel at 0.2 dB and $\epsilon = 10^{-3}$. In this example, k ranges from 1 to 200.

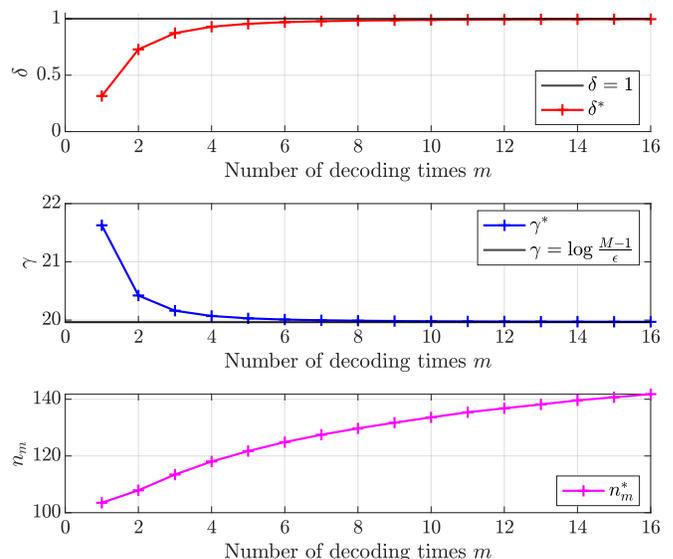


Fig. 6. Globally optimal δ^* , γ^* , and n_m^* as a function of the number of decoding times m for BI-AWGN channel at 0.2 dB, $k = 10$, and $\epsilon = 10^{-3}$.

A. BI-AWGN Channel

We consider the BI-AWGN channel at SNR 0.2 dB, so that the capacity $C = 0.5$ bits/channel use. The approximation function $F_\gamma(n)$ that we use to approximate $\mathbb{P}[\iota(X^n; Y^n) \geq \gamma]$ is given by (36), namely, a combination between the order-5 Edgeworth expansion and the order-3 Petrov expansion. Although the derivative at n^* in (36) is unspecified, one can define its derivative as its left or right derivative and this does not affect the SDO performance. We apply the gap-constrained SDO procedure to solve the relaxed program (57).

For $\epsilon = 10^{-3}$ and the BI-AWGN channel at 0.2 dB, Fig. 5 shows achievability bounds estimated by the gap-constrained SDO procedure and two-step minimization for $m = 1, 2, 4, 8, 16$. When m is small, a slight increase in m dramatically improves the achievability bound of the VLSF code. However, this improvement is diminishing as m gets large enough. We see that Polyanskiy's achievability bound

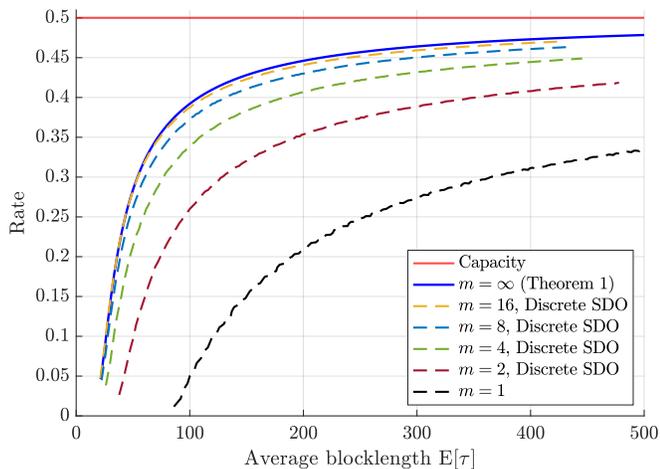


Fig. 7. Rate vs. average blocklength $\mathbb{E}[\tau]$ for the BSC(0.11) and $\epsilon = 10^{-3}$. In this example, k ranges from 1 to 200.

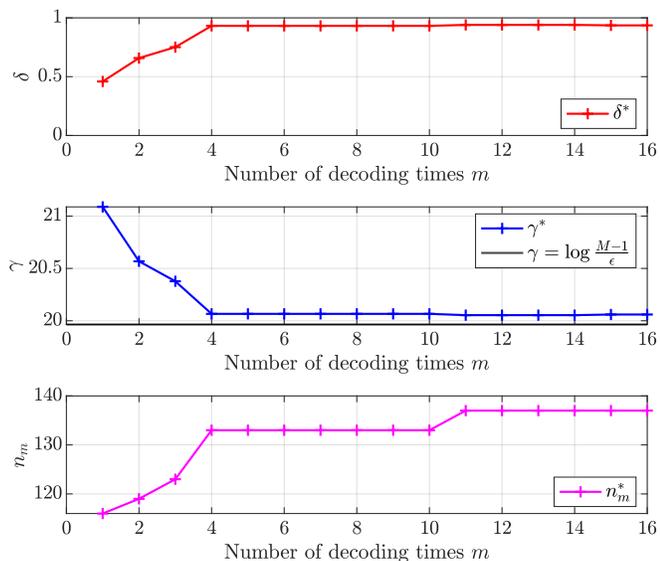


Fig. 8. Globally optimal δ^* , γ^* , and n_m^* as a function of the number of decoding times m for BSC(0.11) $k = 10$, and $\epsilon = 10^{-3}$.

can be closely approached with $m = 16$ for a wide range of average blocklength (or k).

One may wonder the following problem: for a fixed k , how do the optimal γ^* and n_m^* evolve as m increases? By introducing a new parameter $\delta \in (0, 1)$ to (59), we assign $\delta\epsilon$ error probability to the term $(M-1)2^{-\gamma}$ so that

$$\gamma(\delta) = \log \frac{M-1}{\delta\epsilon}, \quad (83)$$

$$n_m(\delta) = F_{\gamma(\delta)}^{-1}(1 - \epsilon + \delta\epsilon). \quad (84)$$

Fig. 6 shows how the optimal δ^* evolves as m increases for $k = 10$ and $\epsilon = 10^{-3}$ for the BI-AWGN channel at 0.2 dB. Using (83) and (84), γ^* and n_m^* are uniquely determined. We observe that for $m = 1$, $\delta^* < 1/2$. As m increases, δ^* quickly approaches one. This drives γ to approach $\log \frac{M-1}{\epsilon}$, and n_m to ∞ . This trend matches Polyanskiy's setup for an $(l, \mathbb{N}, M, \epsilon)$ VLSF code.

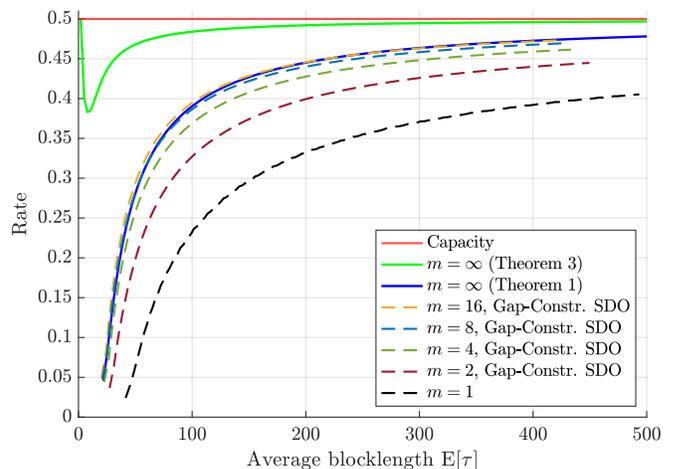


Fig. 9. Rate vs. average blocklength $\mathbb{E}[\tau]$ for the BEC(0.5) and $\epsilon = 10^{-3}$. In this example, k ranges from 1 to 200.

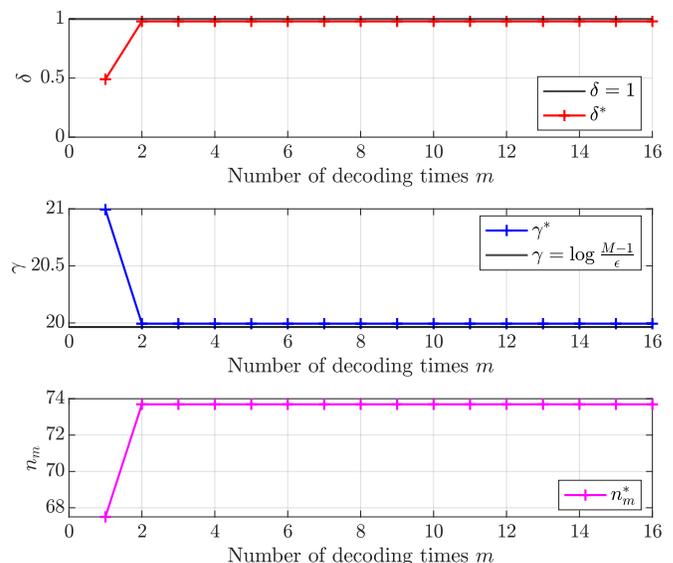


Fig. 10. Globally optimal δ^* , γ^* , and n_m^* as a function of the number of decoding times m for BEC(0.5) $k = 10$, and $\epsilon = 10^{-3}$.

B. BSC

For ease of comparison with Sec. V-A, we consider the BSC with capacity $C = 0.5$ bits/channel use, which is BSC(0.11). By Theorem 7, the approximation function $F_\gamma(n)$ we use to estimate $\mathbb{P}[\iota(X^n; Y^n) \geq \gamma]$ is given by (40). We apply the discrete SDO procedure restricted to the set of local maximizers $\{\alpha_i\}_{i=0}^\infty$ to solve the integer program (50).

For $\epsilon = 10^{-3}$ and BSC(0.11), Fig. 7 shows achievability bounds estimated by the discrete SDO procedure and the two-step minimization, along with Polyanskiy's achievability bound. We observe a similar trend as in the BI-AWGN channel case. Once again, Polyanskiy's achievability bound can be approached closely with $m = 16$ for a wide range of average blocklength (or k).

Fig. 8 shows the behavior of δ^* , γ^* and n_m^* as a function of the number of decoding times m for $k = 10$, $\epsilon = 10^{-3}$ and BSC(0.11). We observed a similar trend as in the BI-

AWGN channel case. However, due to the discreteness of the information density, we see a non-smooth variation in the three parameters.

C. BEC

We consider the BEC(0.5) with capacity $C = 0.5$ bits/channel use. By (47), we use the order-5 continuity-corrected Edgeworth series in Theorem 6 as $F_\gamma(n)$, $n \in \mathbb{R}_+$, to approximate $\mathbb{P}[\iota(X^n; Y^n) \geq \gamma]$. We apply the gap-constrained SDO procedure along with the two-step minimization to solve the relaxed program (57).

For $\epsilon = 10^{-3}$ and BEC(0.5), Fig. 9 shows achievability bounds estimated by the gap-constrained SDO procedure and the two-step minimization. Previous achievability bounds for VLSF codes obtained by Polyanskiy *et al.* and Devassy *et al.* are also displayed. Polyanskiy's achievability bound can be closely approached or exceeded with $m = 8$ for a wide range of average blocklength (or information length k). With $m = 16$, the achievability bound estimated by the gap-constrained SDO procedure and the two-step minimization exceeds Polyanskiy's achievability bound for average blocklengths below 240. Note that it is not surprising that Polyanskiy's achievability bound can be exceeded since the constant term is not tight as discussed in Sec. II.

As shown by the green curve in Fig. 9, there is a significant gap between Polyanskiy's VLSF achievability bound for information density decoder and Devassy's achievability bound for RLFC which achieves zero-error transmission (Theorem 3). This suggests that information density decoding is in fact a suboptimal use of the BEC. We apply SDO to an improved version of RLFC below in Sec. VI.

Fig. 10 shows the behavior of δ^* , γ^* and n_m^* as a function of the number of decoding times m for $k = 10$, $\epsilon = 10^{-3}$ and BEC(0.5). We see that δ^* quickly approaches 1 as m increases from 1 to 2, and then remains roughly constant as m further increases. This trend again matches Polyanskiy's setting.

VI. VLSF CODES UNDER ST-RLFC FOR BEC

Previous sections have been focused on Polyanskiy's framework of utilizing a random VLSF code and an information density decoder. However for the BEC, the decoder has the ability to identify the correct transmitted message whenever only a single codeword is compatible with the unerased received symbols. Motivated by this key observation, we propose a new random VLSF code using the systematic transmission followed by random linear fountain coding (ST-RLFC). The ST-RLFC scheme also facilitates a new $(l, n_1^m, 2^k, \epsilon)$ VLSF code at finite blocklength.

A. The ST-RLFC Scheme

Consider transmitting a k -bit message

$$\mathbf{b} = (b_1, b_2, \dots, b_k) \in \{0, 1\}^k. \quad (85)$$

Let us define the set of nonzero basis vectors in $\{0, 1\}^k$ by

$$\mathcal{G}_k \triangleq \{\mathbf{v} \in \{0, 1\}^k : \mathbf{v}^\top \mathbf{1} > 0\}. \quad (86)$$

We construct a random linear fountain code. Specifically, the channel input at time n for message \mathbf{b} is given by

$$X_n = \begin{cases} b_n, & \text{if } n \leq k \\ \bigoplus_{i=1}^k g_{n,i} b_i & \text{if } n > k, \end{cases} \quad (87)$$

where \oplus denotes bit-wise exclusive-or (XOR) operator, and $\mathbf{g}_n = (g_{n,1}, g_{n,2}, \dots, g_{n,k})^\top \in \mathcal{G}_k$ is generated at time n according to a uniformly distributed random variable $\tilde{U} \in \mathcal{G}_k$ defined in Definition 1. Note that the encoder and decoder share the same common random variable \tilde{U} at time $n > k$ so that the decoder can produce the same \mathbf{g}_n at time n . For $1 \leq n \leq k$, both the encoder and decoder simply use the natural basis vector $\mathbf{e}_n \in \mathbb{R}^{k \times 1}$. For all $\mathbf{b} \in \{0, 1\}^k$, the procedure (87) specifies the common codebook before the start of transmission, i.e., the random variable U in Definition 1.

Let Y_n be the received symbol after transmitting X_n over a BEC(p), $p \in [0, 1)$. We consider a *rank decoder* which keeps track of the rank of generator matrix G associated with received symbols Y^n . Let $G(n)$ denote the n th column of G . If $Y_n = ?$, $G(n) = \mathbf{0}$; otherwise, $G(n) = \mathbf{g}_n$. Define the stopping time

$$\tau \triangleq \inf\{n \in \mathbb{N} : G(1:n) \text{ has rank } k\}, \quad (88)$$

where $G(i:j)$ denotes the column vectors from time i to j , $1 \leq i \leq j$. Thus, the rank decoder stops transmission at time τ and reproduces the k -bit message \mathbf{b} using Y^τ and the inverse of $G(1:\tau)$. Clearly, the probability of error associated with the ST-RLFC scheme is zero.

Using ST-RLFC scheme, we obtain the a new achievability bound for zero-error VLSF codes over BEC(p) in the following theorem.

Theorem 12: For a given integer $k \geq 1$, there exists an $(l, \mathbb{N}, 2^k, 0)$ VLSF code for BEC(p), $p \in [0, 1)$, with

$$l \leq k + \frac{1}{C} \sum_{i=0}^{k-1} \frac{2^k - 1}{2^k - 2^i} F(i; k, 1 - p). \quad (89)$$

where $C = 1 - p$ and

$$F(i; k, 1 - p) \triangleq \sum_{j=0}^i \binom{k}{j} (1 - p)^j p^{k-j} \quad (90)$$

denotes the CDF evaluated at i , $0 \leq i \leq k$, of a binomial distribution with k trials and success probability $1 - p$.

Proof: See Appendix B. ■

For non-vanishing error probability $\epsilon > 0$, using Polyanskiy's scheme by stopping the zero-error VLSF code at $\tau = 0$ with probability ϵ , the corresponding achievability bound can be readily obtained by multiplying the right-hand side (RHS) of (89) by a factor $(1 - \epsilon)$.

We remark that the new achievability bound (89) is tighter than Devassy's bound in Theorem 3 and two bounds are equal if $p = 1$ or $k = 1$. This is stated in the following corollary.

Corollary 2: For a given $k \in \mathbb{N}_+$ and BEC(p), $p \in [0, 1]$, it holds that

$$kC + \sum_{i=0}^{k-1} \frac{2^k - 1}{2^k - 2^i} F(i; k, 1 - p) \leq k + \sum_{i=1}^{k-1} \frac{2^i - 1}{2^k - 2^i}, \quad (91)$$

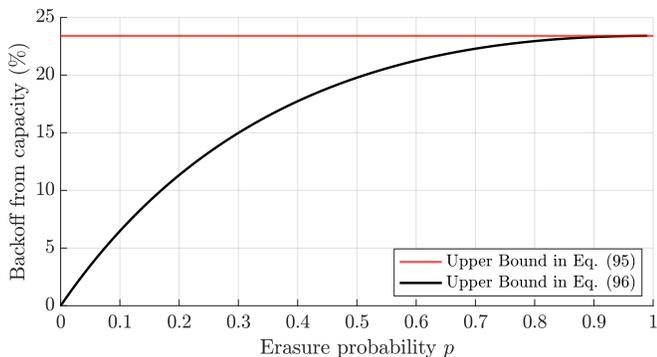


Fig. 11. Percentage of backoff from the capacity of BEC for $k = 3$. The red curve corresponds to a backoff percentage 23.4%.

where $C = 1 - p$ and $F(i; k, 1 - p)$ is given by (90). Equality holds if $p = 1$ or $k = 1$.

Proof: Fix $k \in \mathbb{N}_+$ and $p \in [0, 1]$. First, note that

$$k + \sum_{i=1}^{k-1} \frac{2^i - 1}{2^k - 2^i} = \sum_{i=0}^{k-1} \frac{2^k - 1}{2^k - 2^i}. \quad (92)$$

Hence,

$$\begin{aligned} & \sum_{i=0}^{k-1} \frac{2^k - 1}{2^k - 2^i} - \sum_{i=0}^{k-1} \frac{2^k - 1}{2^k - 2^i} F(i; k, 1 - p) - k(1 - p) \\ &= \sum_{i=0}^{k-1} \frac{2^k - 1}{2^k - 2^i} F^c(i; k, 1 - p) - k(1 - p) \end{aligned} \quad (93)$$

$$\begin{aligned} & \geq \sum_{i=0}^{k-1} F^c(i; k, 1 - p) - k(1 - p) \\ &= 0, \end{aligned} \quad (94)$$

where in (94), $F^c(\cdot) \triangleq 1 - F(\cdot)$ denotes the tail probability and the sum of tail probability equals the expectation $k(1 - p)$. Note that (93) equals 0 if $p = 1$ or $k = 1$. This completes the proof of Corollary 2. ■

A straightforward case is BEC(0) and $k \geq 2$, in which the RHS of (89) reduces to k , whereas the RHS of (23) is still larger than k . Moreover, (23) also implies an upper bound independent of p on the backoff percentage from capacity,

$$1 - \frac{R}{C} \leq \frac{\sum_{i=1}^{k-1} \frac{2^i - 1}{2^k - 2^i}}{k + \sum_{i=1}^{k-1} \frac{2^i - 1}{2^k - 2^i}}. \quad (95)$$

Devassy *et al.* reported in [31] that this upper bound attains its maximum 23.4% at $k = 3$, thus raising the question whether this backoff percentage is fundamental. In contrast, our result in (89) implies a refined upper bound dependent on p ,

$$1 - \frac{R}{C} \leq \frac{\sum_{i=0}^{k-1} \frac{2^k - 1}{2^k - 2^i} F(i; k, 1 - p) - kp}{\sum_{i=0}^{k-1} \frac{2^k - 1}{2^k - 2^i} F(i; k, 1 - p) + k(1 - p)}. \quad (96)$$

Fig. 11 shows the comparison of these two upper bounds at $k = 3$. We see that for $k = 3$, the upper bound in (96) is a strictly increasing function of p . As $p \rightarrow 0$, this upper bound converges to 0, which closes the backoff from capacity at $k = 3$. As $p \rightarrow 1$, the upper bound in (96) converges to the backoff percentage in (95), as shown in Corollary 2.

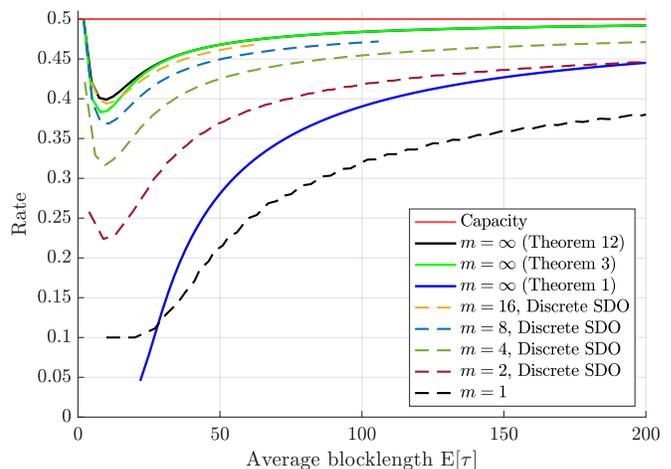


Fig. 12. Rate vs. average blocklength $\mathbb{E}[\tau]$ for the BEC(0.5) and $\epsilon = 10^{-3}$ using the ST-RLFC scheme. In this example, k ranges from 1 to 100 for $m = 1, 2, 4$; k ranges from 1 to 50 for $m = 8$; and k ranges from 2 to 30 for $m = 16$.

B. New VLSF Codes With Finite Decoding Times for BECs

The ST-RLFC scheme also facilitates a new $(l, n_1^m, 2^k, \epsilon')$ VLSF code at finite blocklength for BEC. We first present a general non-asymptotic achievability bound for such a code.

Theorem 13: Fix $n_1^m \in \mathbb{N}_+^m$ satisfying $n_1 < n_2 < \dots < n_m$. For any positive integer $k \in \mathbb{N}_+$ and $\epsilon' \in (0, 1)$, there exists an $(l, n_1^m, 2^k, \epsilon')$ VLSF code for the BEC(p) with

$$l \leq n_m + \sum_{i=1}^{m-1} (n_i - n_{i+1}) \mathbb{P}[S_{n_i} = k], \quad (97)$$

$$\epsilon' \leq 1 - \mathbb{P}[S_{n_m} = k], \quad (98)$$

where the random variable S_n denote the rank of the generator matrix $G(1 : n)$ observed by the rank decoder. Specifically, $\mathbb{P}[S_n = k]$ is given by

$$\mathbb{P}[S_n = k] = \begin{cases} 0, & \text{if } n < k \\ 1 - \boldsymbol{\alpha}^\top T^{n-k} \mathbf{1}, & \text{if } n \geq k, \end{cases} \quad (99)$$

where $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \dots, \alpha_k]^\top \in \mathbb{R}^{k \times 1}$ with $\alpha_i = F(i; k, 1 - p)$, $0 \leq i \leq k - 1$, where $F(i; k, 1 - p)$ is given by (90), $T \in \mathbb{R}^{k \times k}$ with entries given by

$$T_{i,i} = p + \frac{(1-p)(2^{i-1} - 1)}{2^k - 1}, \quad (100)$$

$$T_{i,i+1} = \frac{(1-p)(2^k - 2^{i-1})}{2^k - 1}, \quad (101)$$

$$T_{i,j} = 0, \text{ for } j \neq i \text{ and } j \neq i + 1. \quad (102)$$

Proof: See Appendix C. ■

Theorem 13 facilitates a similar but a much simpler integer program. Define

$$N(n_1^m) \triangleq n_m + \sum_{i=1}^{m-1} (n_i - n_{i+1}) \mathbb{P}[S_{n_i} = k]. \quad (103)$$

For a given $k \in \mathbb{N}_+$ and a target error probability $\epsilon \in (0, 1)$,

$$\begin{aligned} & \min_{n_1^m} N(n_1^m) \\ & \text{s.t. } 1 - \mathbb{P}[S_{n_m} = k] \leq \epsilon \end{aligned} \quad (104)$$

Unlike the double minimization in the information density framework, the integer program (104) only involves a single minimization over n_1^m . Using (99), we solve integer program (104) with the discrete SDO procedure.

Under the ST-RLFC framework, Fig. 12 shows the achievability bounds estimated by the discrete SDO procedure for BEC(0.5) and $\epsilon = 10^{-3}$. The new achievability bound in Theorem 12 along with the ones developed by Polyanskiy (Theorem 1) and Devassy (Theorem 3) are also shown. We see that the maximal achievable rate for $(l, n_1^m, 2^k, \epsilon)$ VLSF codes operated over a BEC(p) is significantly improved, compared to the information density framework in Fig. 9. In particular, achievability bounds for $m \geq 2$ outperform Polyanskiy's achievability bound by a wide margin. The achievability bound for $m = 16$ even exceeds Devassy's bound at small values of k . This demonstrates that the ST-RLFC scheme further improves the VLSF code performance for BEC.

We remark that Polyanskiy obtained a much better VLF achievability bound for the BEC [14, Th. 7] by simply retransmitting each of the k bits until it gets through the BEC. However, for $k \geq 2$, this particular code construction yields a VLF code rather than a VLSF code. Therefore, the VLF achievability bound for the BEC in [14, Th. 7] is omitted from discussion.

VII. CONCLUSION

In this paper, we provided both theoretical and numerical tools that enable us to evaluate the achievability bound for VLSF codes with m decoding times for the three classical binary-input channels. Numerical evaluations for the three channels all confirm that Polyanskiy's achievability bound, which assumes $m = \infty$, can be approached with a finite and relatively small m . Especially, for $\epsilon = 10^{-3}$, we show that $m = 16$ suffices to approach Polyanskiy's achievability bound. This result has an exciting implication that a small number of stop-feedback actions suffices to dramatically improve the achievable rate at a given message size and target error probability. For BEC, using the ST-RLFC scheme, one can further improve the VLSF code performance beyond Polyanskiy's achievability bound.

This paper gives rise to several interesting problems that are worth future investigation. For example, we lack a proof that shows that as $m \rightarrow \infty$, the VLSF codes with m optimal decoding times converges to Polyanskiy's setting. Moreover, the behavior of optimal δ^* , γ^* , and n_m^* is still less understood. From a practical point of view, the problem of designing a deterministic VLSF code with m decoding times that approaches achievability bounds demonstrated in this paper still remains open.

APPENDIX A

DERIVATION OF THE EDGEWORTH EXPANSION

Our derivation is analogous to the one in [24], with the distinction that we provide explicit expression for the polynomial involved in the Edgeworth expansion.

Let W_1, W_2, \dots, W_n be i.i.d. random variables with zero mean and variance σ^2 . Let $\chi_W(t) = \mathbb{E}[e^{itW}]$ be the characteristic function of W and let $\{\kappa_j\}_{j=1}^\infty$ be the cumulants for W . Note that $\kappa_1 = 0$, $\kappa_2 = \sigma^2$.

Let $Y = W/\sigma$ be the normalized random variable. The characteristic function $\chi_Y(t) = \chi_W(t/\sigma)$. $\chi_Y(t)$ can also be expressed in terms of the exponential of a power series involving cumulants $\{\bar{\kappa}_j\}_{j=1}^\infty$, where $\bar{\kappa}_j = \sigma^{-j}\kappa_j$ denotes the j th cumulant of Y . Namely,

$$\chi_Y(t) = \exp\left(\sum_{j=1}^{\infty} \frac{\bar{\kappa}_j}{j!} (it)^j\right), \quad (105)$$

where $\bar{\kappa}_1 = 0$ and $\bar{\kappa}_2 = 1$.

Consider the standardized sum

$$S \triangleq \frac{1}{\sqrt{n}} \sum_{i=1}^n Y_i. \quad (106)$$

The characteristic function $\chi_S(t)$ for S is given by

$$\chi_S(t) = \mathbb{E}[\exp(itS)] = \mathbb{E}\left[\exp\left(i\frac{t}{\sqrt{n}} \sum_{i=1}^n Y_i\right)\right] \quad (107)$$

$$= \left(\chi_Y\left(tn^{-\frac{1}{2}}\right)\right)^n. \quad (108)$$

Substituting (105) into (108) and invoking $\bar{\kappa}_1 = 0$ and $\bar{\kappa}_2 = 1$ yields

$$\begin{aligned} \chi_S(t) &= \exp\left(\sum_{j=1}^{\infty} n^{-\frac{j-2}{2}} \frac{\bar{\kappa}_j}{j!} (it)^j\right) \\ &= \exp\left(-\frac{1}{2}t^2 + \sum_{j=3}^{\infty} \frac{\bar{\kappa}_j (it)^j}{j!} n^{-\frac{j-2}{2}}\right) \\ &= e^{-\frac{t^2}{2}} \exp\left(\sum_{j=1}^{\infty} \frac{\bar{\kappa}_{j+2} (it)^{j+2}}{(j+2)!} n^{-\frac{j}{2}}\right) \end{aligned} \quad (109)$$

Our goal is to represent (109) as a power series. Namely,

$$\chi_S(t) = e^{-\frac{t^2}{2}} \left(1 + \sum_{j=1}^{\infty} n^{-\frac{j}{2}} r_j(it)\right) \quad (110)$$

for some polynomial $r_j(\cdot)$, $j \geq 1$.

In [32, Appendix A], Blinnikov and Moessner proved the following useful lemma regarding the n th derivative of a composite function $f \circ g(x) \equiv f(g(x))$.

Lemma 1: Let $f(x)$ and $g(x)$ be two differentiable functions with up to n th derivatives. Let $f^{(r)}(x)$ represent the r th derivative of $f(x)$ evaluated at x , $1 \leq r \leq n$. Then,

$$\frac{d^n}{dx^n} f(g(x)) = n! \sum_{\{k_l\}} f^{(r)}(y) \Big|_{y=g(x)} \prod_{l=1}^n \frac{1}{k_l!} \left(\frac{1}{l!} g^{(l)}(x)\right)^{k_l}, \quad (111)$$

where $r \triangleq \sum_{l=1}^n k_l$, and the set $\{k_l\}$ consists of all non-negative integer solutions to the Diophantine equation

$$k_1 + 2k_2 + \dots + nk_n = n. \quad (112)$$

As an application of Lemma 1, with $f \equiv \exp(x)$ and $g \equiv \sum_{j=1}^{\infty} \frac{\bar{\kappa}_{j+2} u^{j+2}}{(j+2)!} x^j$, we obtain

$$\begin{aligned} r_j(u) &= \frac{1}{j!} \frac{d^j}{dx^j} f(g(x)) \Big|_{x=0} \\ &= \frac{1}{j!} \cdot j! \sum_{\{k_i\}} \prod_{i=1}^j \frac{1}{k_i!} \left(\frac{1}{i!} \cdot \frac{\bar{\kappa}_{i+2} u^{i+2}}{(i+2)!} i! \right)^{k_i} \\ &= \sum_{\{k_i\}} u^{j+2r} \prod_{i=1}^j \frac{1}{k_i!} \left(\frac{\bar{\kappa}_{i+2}}{(i+2)!} \right)^{k_i}. \end{aligned} \quad (113)$$

Thus, (113) gives the polynomial $r_j(\cdot)$ that we are seeking.

Since the characteristic function for a standard normal $\phi(x)$ is exactly $e^{-t^2/2}$, the form of (110) suggests the following ‘‘inverse’’ expansion

$$\mathbb{P}[S \leq x] = \Phi(x) + \sum_{j=1}^{\infty} n^{-\frac{j}{2}} R_j(x), \quad (114)$$

where $R_j(x)$ denotes the function whose Fourier transform equals $r_j(it)e^{-t^2/2}$. Our next step is to find $R_j(x)$.

Repeated integration by parts gives

$$e^{-t^2/2} = (-it)^{-j} \int_{-\infty}^{\infty} e^{itx} d\Phi^{(j)}(x), \quad (115)$$

where $\Phi^{(j)}(x) = (d/dx)^j \Phi(x)$. Let $D = d/dx$ denote the differential operator. Then, (115) is equivalent to

$$\int_{-\infty}^{\infty} e^{itx} d [(-D)^j \Phi(x)] = (it)^j e^{-t^2/2}. \quad (116)$$

Interpreting $r_j(-D)$ as a polynomial in D so that $r_j(-D)$ itself is a differential operator. By (116), we obtain

$$\int_{-\infty}^{\infty} e^{itx} d [r_j(-D)\Phi(x)] = r_j(it)e^{-t^2/2}. \quad (117)$$

Hence, it follows that

$$R_j(x) = r_j(-D)\Phi(x). \quad (118)$$

For $j \geq 1$, we have the relation

$$(-D)^j \Phi(x) = -H e_{j-1}(x)\phi(x), \quad (119)$$

where $H e_i(x)$ denotes the degree- i Hermite polynomial, $i \geq 0$. In [32, Eq. (13)], the authors provided an explicit formula for the degree- i Hermite polynomial

$$H e_i(x) = i! \sum_{k=0}^{\lfloor i/2 \rfloor} \frac{(-1)^k x^{j-2k}}{k!(j-2k)!2^k}. \quad (120)$$

Combining (113), (118), and (119), we obtain

$$\begin{aligned} R_j(x) &= r_j(-D)\Phi(x) \\ &= \sum_{\{k_i\}} (-D)^{j+2r} \Phi(x) \prod_{i=1}^j \frac{1}{k_i!} \left(\frac{\bar{\kappa}_{i+2}}{(i+2)!} \right)^{k_i} \\ &= - \sum_{\{k_i\}} H e_{j+2r-1}(x)\phi(x) \prod_{i=1}^j \frac{1}{k_i!} \left(\frac{\bar{\kappa}_{i+2}}{(i+2)!} \right)^{k_i}. \end{aligned} \quad (121)$$

Hence, (121) gives the polynomial $R_j(x)$ we are seeking.

Finally, let us define

$$p_j(x) \triangleq - \sum_{\{k_i\}} H e_{j+2r-1}(x) \prod_{i=1}^j \frac{1}{k_i!} \left(\frac{\bar{\kappa}_{i+2}}{(i+2)!} \right)^{k_i}, \quad (122)$$

which is exactly (28). Hence, $R_j(x) = p_j(x)\phi(x)$ for $j \geq 1$. Substituting this into (114) yields

$$\mathbb{P}[S \leq x] = \Phi(x) + \phi(x) \sum_{j=1}^{\infty} n^{-\frac{j}{2}} p_j(x). \quad (123)$$

In [24], it is argued that under the sufficient regularity conditions $\mathbb{E}[|W|^{s+2}] < \infty$, $s \in \mathbb{N}_+$ and $\limsup_{|t| \rightarrow \infty} |\chi_W(t)| < 1$, for all $x \in \mathbb{R}$,

$$\mathbb{P}[S \leq x] = \Phi(x) + \phi(x) \sum_{j=1}^s n^{-\frac{j}{2}} p_j(x) + o(n^{-\frac{s}{2}}), \quad (124)$$

which is exactly (27). This concludes the derivation of the Edgeworth expansion.

APPENDIX B

PROOF OF THEOREM 12

Let random variable S_n denote the rank of generator matrix $G(1:n)$. According to the ST-RLFC scheme, the probability mass function (PMF) of S_k at time k is given by

$$\mathbb{P}[S_k = r] = \binom{k}{r} (1-p)^r p^{k-r}, \quad 0 \leq r \leq k. \quad (125)$$

For $n \geq k$, due to the BEC(p) and our RLFC scheme, $S_{n+1} = S_n = r$ occurs if $Y_{n+1} = ?$ or if $Y_{n+1} \neq ?$ and \mathbf{g}_{n+1} is a linear combination of previous r independent basis vectors. Otherwise, $S_{n+1} = r + 1$. Hence, the behavior of S_n , $n \geq k$, is characterized by the following discrete-time homogeneous Markov chain with $k + 1$ states.

$$\mathbb{P}[S_{n+1} = r | S_n = r] = p + \frac{(1-p)(2^r - 1)}{2^k - 1}, \quad (126)$$

$$\mathbb{P}[S_{n+1} = r + 1 | S_n = r] = \frac{(1-p)(2^k - 2^r)}{2^k - 1}, \quad (127)$$

where $0 \leq r \leq k-1$, and $\mathbb{P}[S_{n+1} = k | S_n = k] = 1$. Note that this Markov chain has a single absorbing state $S_n = k$. The time to absorption for this Markov chain follows a discrete phase-type distribution [35, Chapter 2]. More specifically, the one-step transfer matrix $P \in \mathbb{R}^{(k+1) \times (k+1)}$ of this Markov chain can be written as

$$P = \begin{bmatrix} T & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}, \quad (128)$$

where the entries of $T \in \mathbb{R}^{k \times k}$ are given by

$$T_{i,i} = p + \frac{(1-p)(2^{i-1} - 1)}{2^k - 1}, \quad (129)$$

$$T_{i,i+1} = \frac{(1-p)(2^k - 2^{i-1})}{2^k - 1}, \quad (130)$$

and $T_{i,j} = 0$ for any other pair (i, j) , $1 \leq i, j \leq k$. Since P is a stochastic matrix, it follows that

$$\mathbf{t} = (I - T)\mathbf{1}. \quad (131)$$

The initial probability distribution is given by $[\alpha^\top, \alpha_k]$, where

$$\alpha^\top \triangleq [\mathbb{P}[S_k = 0] \quad \mathbb{P}[S_k = 1] \quad \cdots \quad \mathbb{P}[S_k = k-1]], \quad (132)$$

with $\mathbb{P}[S_k = r]$ given by (125), and $\alpha_k = 1 - \alpha^\top \mathbf{1}$. Let random variable $X \in \mathbb{N}$ denote the time to absorbing state k with initial distribution $[\alpha^\top, \alpha_k]$. Hence, it follows that X has PMF

$$\mathbb{P}[X = n] = \alpha^\top T^{n-1} \mathbf{t}, \quad n \in \mathbb{N}_+, \quad (133)$$

and $\mathbb{P}[X = 0] = \alpha_k$. Define the generating function of X by

$$\begin{aligned} H_X(z) &\triangleq \mathbb{E}[z^X] = \sum_{n=0}^{\infty} z^n \mathbb{P}[X = n] \\ &= \alpha_k + \sum_{n=1}^{\infty} z^n \alpha^\top T^{n-1} \mathbf{t} \\ &= \alpha_k + z \alpha^\top \left(\sum_{n=0}^{\infty} (zT)^n \right) \mathbf{t} \\ &= \alpha_k + z \alpha^\top (I - zT)^{-1} (I - T) \mathbf{1}, \end{aligned} \quad (134)$$

where in (134), we have used $\sum_{n=0}^{\infty} A^n = (I - A)^{-1}$ whenever $|\lambda_i| < 1$ for all $i \in [k]$, where $\{\lambda_i\}_{i=1}^k$ denotes the eigenvalues of a square matrix $A \in \mathbb{R}^{k \times k}$. Hence, the expected time to absorbing state k is given by

$$\mathbb{E}[X] = \left. \frac{dH_X(z)}{dz} \right|_{z=1} = \alpha^\top (I - T)^{-1} \mathbf{1}. \quad (136)$$

Therefore, the expected stopping time $\mathbb{E}[\tau]$, with τ defined in (88), is given by

$$\begin{aligned} \mathbb{E}[\tau] &= k + \mathbb{E}[X] \\ &= k + \alpha^\top (I - T)^{-1} \mathbf{1} \end{aligned} \quad (137)$$

Note that

$$\begin{aligned} I - T &= (1-p) \operatorname{diag} \left(1, \frac{2^k - 2^1}{2^k - 1}, \frac{2^k - 2^2}{2^k - 1}, \dots, \frac{2^k - 2^{k-1}}{2^k - 1} \right) \\ &= \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}. \end{aligned} \quad (138)$$

Hence,

$$\begin{aligned} (I - T)^{-1} &= (1-p)^{-1} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \\ &\cdot \operatorname{diag} \left(1, \frac{2^k - 1}{2^k - 2^1}, \frac{2^k - 1}{2^k - 2^2}, \dots, \frac{2^k - 1}{2^k - 2^{k-1}} \right) \\ &= (1-p)^{-1} \begin{bmatrix} 1 & \frac{2^k - 1}{2^k - 2^1} & \frac{2^k - 1}{2^k - 2^2} & \cdots & \frac{2^k - 1}{2^k - 2^{k-1}} \\ 0 & \frac{2^k - 1}{2^k - 2^1} & \frac{2^k - 1}{2^k - 2^2} & \cdots & \frac{2^k - 1}{2^k - 2^{k-1}} \\ 0 & 0 & \frac{2^k - 1}{2^k - 2^2} & \cdots & \frac{2^k - 1}{2^k - 2^{k-1}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{2^k - 1}{2^k - 2^{k-1}} \end{bmatrix}. \end{aligned} \quad (139)$$

Substituting (132) and (139) into (137), we finally obtain

$$\mathbb{E}[\tau] = k + (1-p)^{-1} \sum_{i=0}^{k-1} \frac{2^k - 1}{2^k - 2^i} \sum_{j=0}^i \mathbb{P}[S_k = j] \quad (140)$$

$$= k + \frac{1}{C} \sum_{i=0}^{k-1} \frac{2^k - 1}{2^k - 2^i} F(i; k, 1-p), \quad (141)$$

where $C = 1-p$ and $F(i; k, 1-p) \triangleq \sum_{j=0}^i \mathbb{P}[S_k = j]$ denotes the CDF evaluated at i of a binomial distribution with k trials and success probability $1-p$. Since (141) is the expected stopping time for an ensemble of zero-error VLSF codes, there exists an $(l, \mathbb{N}, 2^k, 0)$ VLSF code with

$$l \leq k + \frac{1}{C} \sum_{i=0}^{k-1} \frac{2^k - 1}{2^k - 2^i} F(i; k, 1-p). \quad (142)$$

This concludes the proof of Theorem 12.

APPENDIX C PROOF OF THEOREM 13

The proof builds upon the proof of Theorem 12 with the distinction that we need to specify the rank decoder for a given set of decoding times n_1, n_2, \dots, n_m .

Fix $n_1^m \in \mathbb{N}_+$ with $n_1 < n_2 < \dots < n_m$. For a given $k \in \mathbb{N}_+$ and $\epsilon' \in (0, 1)$, the encoder of a random $(l, n_1^m, 2^k, \epsilon')$ VLSF code is the same as described in (87). The rank decoder still shares the same common randomness with the encoder in selecting the basis vector \mathbf{g}_n , except that it now adopts the following stopping time:

$$\tau^* \triangleq \inf \{ n \in \{n_i\}_{i=1}^m : G(1:n) \text{ has rank } k \text{ or } n = n_m \}. \quad (143)$$

If $\tau \leq n_m$ and $G(1:\tau)$ is full rank, the rank decoder reproduces the transmitted message using Y^τ and the inverse of $G(1:\tau)$. If $\tau = n_m$ and $G(1:n_m)$ is rank deficient, then the rank decoder outputs an arbitrary message.

Let S_n denote the rank of the generator matrix $G(1:n)$ observed at the rank decoder. The expected stopping time $\mathbb{E}[\tau^*]$ is written as

$$\mathbb{E}[\tau^*] = \sum_{n=0}^{\infty} \mathbb{P}[\tau^* > n]$$

$$= n_1 + \sum_{i=1}^{m-1} (n_{i+1} - n_i) \mathbb{P}[\tau^* > n_i] \quad (144)$$

$$= n_1 + \sum_{i=1}^{m-1} (n_{i+1} - n_i) \mathbb{P}[S_{n_i} < k] \quad (145)$$

$$= n_m + \sum_{i=1}^{m-1} (n_i - n_{i+1}) \mathbb{P}[S_{n_i} = k], \quad (146)$$

which is equal to the upper bound in (97).

Note that at finite blocklength, the error only occurs when the rank of generator matrix $G(1:n_m)$ is still less than k . Hence,

$$\epsilon' \leq \mathbb{P}[S_{n_m} < k] \quad (147)$$

$$= 1 - \mathbb{P}[S_{n_m} = k], \quad (148)$$

which is equal to the upper bound in (98).

At time $n < k$, due to the systematic transmission, $\mathbb{P}[S_n = k] = 0$. At time $n \geq k$, as discussed in Appendix B, the behavior of S_n is characterized by a discrete-time homogeneous Markov chain with $k + 1$ states whose one-step transfer matrix is given by (128), and whose initial probability distribution is $[\alpha^\top, \alpha_k]$, where α^\top is given by (132). Hence, for $n \geq k$,

$$\begin{aligned} \mathbb{P}[S_n = k] &= 1 - \mathbb{P}[S_n < k] \\ &= 1 - \alpha^\top T^{n-k} \mathbf{1}. \end{aligned} \quad (149)$$

$$(150)$$

This completes the proof of Theorem 13.

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