

Beyond Capacity: The Joint Time-Rate Region

Michael Langberg

Michelle Effros

Abstract—The traditional notion of capacity studied in the context of memoryless network communication builds on the concept of block-codes and requires that, for sufficiently large blocklength n , all receiver nodes simultaneously decode their required information after n channel uses. In this work, we generalize the traditional capacity region by exploring communication rates achievable when some receivers are required to decode their information before others, at different predetermined times; referred here as the *time-rate* region. Through a reduction to the standard notion of capacity, we present an inner-bound on the time-rate region. The time-rate region has been previously studied and characterized for the memoryless broadcast channel (with a sole common message) under the name *static broadcasting*.

I. INTRODUCTION

In the context of communication over multi-source multi-terminal memoryless channels (i.e., networks), one traditionally seeks the design of communication schemes that, for a given blocklength n , allow the successful decoding of source information at receiver nodes after n channel uses. Roughly speaking,¹ rate vector $\underline{R} = (R_1, \dots, R_k)$, is said to be achievable with blocklength n and decoding error $\varepsilon > 0$ over a given k -source network, if for uniformly distributed $R_i n$ -bit messages m_i (for $i = 1, \dots, k$) there exists a communication scheme that after n channel uses allows all receivers to decode their required source information with success probability at least $1 - \varepsilon$. The capacity region of the communication problem at hand describes the closure of all rate vectors \underline{R} achievable with asymptotic blocklength and vanishing error (see, e.g., [1]).

In this work, we generalize the notion of capacity and study communication in the setting in which some network nodes are required to decode their information before others. More precisely, we study communication schemes of blocklength n in which receiver v_j is required to decode the $R_i n$ -bit message m_i after $\sigma_{ij} n$ channel uses, where σ_{ij} is a predetermined *time constraint* less than or equal to 1. The traditional capacity region is captured when all constraints σ_{ij} equal 1.

M. Langberg is with the Department of Electrical Engineering at the University at Buffalo (State University of New York). Email: mikel@buffalo.edu

M. Effros is with the Department of Electrical Engineering at the California Institute of Technology. Email: effros@caltech.edu

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¹All concepts mentioned in this section are defined in detail in Section II.

Different values for parameters σ_{ij} represent settings in which certain receivers are required to decode earlier than others due to, e.g., time-sensitive information, physical receiver constraints such as battery life, or computational receiver constraints that are due to parallel communication or processing tasks. For example, in the setting of IoT, a base station with side information including the remaining battery life of the sensors under its control may design broadcast codes that allow earlier decoding at low-battery sensors; in disaster areas, a base station with event information may design broadcast codes that allow earlier decoding at receivers close to a critical event.

To represent the achievable rates $\underline{R} = (R_1, \dots, R_k)$ under 0/1 demand matrix S and time constraints $\underline{\sigma} = (\sigma_{ij} : s_{ij} = 1)$, in this work we define and study the joint *time-rate* region \mathcal{T} which is the closure of vector pairs $(\underline{\sigma}, \underline{R})$ for which rate \underline{R} is achievable with time constraints $\underline{\sigma}$. Here, for the communication problem at hand, demand matrix $S = [s_{ij}]$ sets $s_{ij} = 1$ if receiver v_j wants message m_i and 0 otherwise. Rate \underline{R} is achievable with time constraints $\underline{\sigma}$ on network \mathcal{N} , if there exists a blocklength- n communication scheme for \mathcal{N} such that for $s_{ij} = 1$, receiver v_j can decode the $R_i n$ -bit message m_i (with high probability) after $\sigma_{ij} n$ channel uses (see Section II, and in particular Definition 2.1, for formal details).

It is convenient to represent the time-rate region by considering its *slices* with respect to $\underline{\sigma}$ or alternatively with respect to \underline{R} . For the former, consider expressing \mathcal{T} by the collection $\{\mathcal{R}_{\underline{\sigma}}\}_{\underline{\sigma}}$. Here, for any setting of time constraints $\underline{\sigma}$, $\mathcal{R}_{\underline{\sigma}} = \{\underline{R} \mid (\underline{\sigma}, \underline{R}) \in \mathcal{T}\}$ is the closure of the set of rate vectors \underline{R} that are achievable with blocklength n . Achievability implies that for (i, j) with $s_{ij} = 1$ receiver v_j can decode message m_i (with high probability) after $\sigma_{ij} n$ channel uses. Using this notation, the standard capacity region, in which all time constraints σ_{ij} equal 1, is denoted by $\mathcal{R}_{\underline{1}}$. Given $\underline{\sigma}$, the region $\mathcal{R}_{\underline{\sigma}}$ captures the *tradeoff in message rates* \underline{R} achievable given a collection of decoding time-constraints implied by $\underline{\sigma}$.

For the latter, one may express \mathcal{T} by the collection $\{\sigma_{\underline{R}}\}_{\underline{R}}$, where for any rate vector $\underline{R} = (R_1, \dots, R_k)$, $\sigma_{\underline{R}} = \{\underline{\sigma} \mid (\underline{\sigma}, \underline{R}) \in \mathcal{T}\}$ is the closure of the set of time-constraints $\underline{\sigma}$ that allow the communication of rate- \underline{R} messages. Given a fixed rate vector \underline{R} , the region $\sigma_{\underline{R}}$ represents the *tradeoff in decoding times* $\underline{\sigma}$ supporting

the communication of messages of predetermined rate represented by \underline{R} .

The characterizations $\mathcal{R}_{\underline{\sigma}}$ and $\sigma_{\underline{R}}$ are equivalent in the sense that each suffices to recover the time-rate region \mathcal{T} . While the perspectives represented by $\mathcal{R}_{\underline{\sigma}}$ and $\sigma_{\underline{R}}$ both have operational significance, in this work, we focus mainly on the study of $\mathcal{R}_{\underline{\sigma}}$. Like the standard capacity region \mathcal{R}_1 , for any $\underline{\sigma}$, the region $\mathcal{R}_{\underline{\sigma}}$ is convex by the usual time-sharing argument (here, codes should be interleaved). As a result, $\mathcal{R}_{\underline{\sigma}}$ lends itself more naturally to our analysis. This is in contrast to the region $\sigma_{\underline{R}}$, which is not necessarily convex (since time sharing between a code that delivers rate \underline{R} at time constraints $\underline{\sigma}$ and a code that delivers rate \underline{R} at time constraints $\underline{\sigma}'$ does not always create a code that delivers rate \underline{R} at time constraints $\alpha\underline{\sigma} + (1 - \alpha)\underline{\sigma}'$). The latter is shown, e.g., in [2], [3], for the two-terminal broadcast channel.

In this work, we study the joint time-rate region for general multiple-source multiple-terminal memoryless networks \mathcal{N} . This manuscript is structured as follows. In Section II, we define our model in detail. In Section III, we outline prior related works, focusing on *static broadcasting* (initiated in [2], [3]) which studies the time-rate region in the single-source broadcast setting (i.e., $k = 1$), and rateless codes, e.g., [4], [5], [6]. Our main result appears in Section IV, where, given any network \mathcal{N} and set of time constraints $\underline{\sigma}$, we design an inner bound on $\mathcal{R}_{\underline{\sigma}}(\mathcal{N})$ through a reduction to the study of *traditional* capacity regions on related networks. To put our results in perspective, in Section V, we present a single-message network for which our inner bound is not tight. Finally, we conclude in Section VI.

II. MODEL

We use the following notational conventions. Scalars are represented by lowercase letters, e.g., v ; vectors by underlined lowercase letters, e.g., \underline{v} ; matrices and sets are represented by uppercase letters; script letters typically denote alphabets, e.g., \mathcal{X} , or more complex structures; and random variables are denoted in bold. For a positive real value r , $[r]$ represents the set $\{1, 2, \dots, [r]\}$.

A. Memoryless Communication Channel

Our model for a given discrete memoryless communication channel \mathcal{W} comprises:

- **Nodes:** a collection of ℓ network communication nodes $V = (v_j : j \in [\ell])$.
- **Alphabets:** each $v_j \in V$ observes channel outputs from alphabet \mathcal{Y}_j and transmits channel inputs in alphabet \mathcal{X}_j .
- **Channel:** Let $W(y|x)$ be a conditional probability distribution of $\underline{y} = (y_j : j \in [\ell]) \in \prod_{j \in [\ell]} \mathcal{Y}_j$ given $\underline{x} = (x_j : j \in [\ell]) \in \prod_{j \in [\ell]} \mathcal{X}_j$.

Thus, the channel \mathcal{W} is represented by the tuple

$$\mathcal{W} = \left(\prod_{j \in [\ell]} \mathcal{X}_j, W(\underline{y}|\underline{x}), \prod_{j \in [\ell]} \mathcal{Y}_j \right).$$

B. The Message Set, Message Side-information, and Requirements

We denote the collection of source messages to be communicated over channel \mathcal{W} by $\underline{M} = (m_i : i \in [k])$. The message side-information is defined by the $k \times \ell$ binary matrix $H = [h_{ij}]$ for which $h_{ij} = 1$ if and only if message m_i is available to node v_j at the start of the communication process. Similarly demand matrix $S = [s_{ij}]$ is a $k \times \ell$ binary matrix for which $s_{ij} = 1$ if and only if node v_j requires message m_i .

C. Network Communication Problem

Combining the elements above, a network communication problem \mathcal{N} is defined by the tuple $(\mathcal{W}, \underline{M}, H, S)$.

D. Network code

Let $\mathcal{N} = (\mathcal{W}, \underline{M}, H, S)$ be a network communication problem as above. Let n be an integer and $\underline{R} = (R_1, \dots, R_k)$ be a rate vector. For $i \in [k]$, let $\mathcal{M}_i = [2^{R_i n}]$ be the message alphabet of $m_i \in \underline{M}$. A (\underline{R}, n) code \mathcal{C} for communication problem \mathcal{N} consists of the following components:

- **Encoders:** with each node $v_j \in V$ we associate a time-varying encoder, which at time τ is defined as

$$E_j^\tau : \prod_{h_{ij}=1} \mathcal{M}_i \times \mathcal{Y}_j^{[\tau-1]} \rightarrow \mathcal{X}_j.$$

- **Decoders:** with each node $v_j \in V$ we associate a time-varying decoder, which at time τ is defined as

$$D_j^\tau : \prod_{h_{ij}=1} \mathcal{M}_i \times \mathcal{Y}_j^{[\tau]} \rightarrow \prod_{s_{ij}=1} \mathcal{M}_i.$$

Thus a code \mathcal{C} for \mathcal{N} is defined by the tuple

$$(\mathcal{E}, \mathcal{D}) \triangleq ((E_j^\tau : j \in [\ell], \tau \in \mathbb{N}), (D_j^\tau : j \in [\ell], \tau \in \mathbb{N})).$$

Achievability: Let $\underline{\sigma} = (\sigma_{ij} : s_{ij} = 1)$ for $\sigma_{ij} > 0$ and let $\varepsilon > 0$. A (\underline{R}, n) code \mathcal{C} for network communication problem \mathcal{N} is said to be an $(\varepsilon, \underline{\sigma}, \underline{R}, n)$ code if it allows successful decoding with probability at least $1 - \varepsilon$. Specifically, given independent messages \mathbf{m}_i , $i \in [k]$, uniformly distributed over $\mathcal{M}_i = [2^{R_i n}]$, operating code \mathcal{C} over channel \mathcal{W} yields a time- τ channel output \underline{y}^τ for which

$$\Pr[\forall(i, j) \text{ s.t. } s_{ij} = 1, D_j^{\sigma_{ij} n}((\mathbf{m}_i : h_{ij} = 1), \underline{y}_j^{[\sigma_{ij} n]}) = \mathbf{m}_i],$$

is at least $1 - \varepsilon$. Here, the probability is taken over the randomness of the messages and the channel W .

Remark 2.1: In our definition above, we consider time parameters $\underline{\sigma} = (\sigma_{ij} : s_{ij} = 1)$ for any positive values of σ_{ij} . Although this seemingly generalizes the

discussion in Section I in which the collection of time-parameters σ_{ij} has a maximum value of 1 (i.e., the setting in which some receiver nodes decode at time n , and others earlier), it is not hard to verify that the two definitions are equivalent, and we use the former to simplify our analysis. More precisely, our definitions imply the following tradeoff between the blocklength n and the pair $(\underline{\sigma}, \underline{R})$.

Claim 2.1: Let \mathcal{N} be a network communication problem. If \mathcal{C} is an $(\varepsilon, \underline{\sigma}, \underline{R}, n)$ code for \mathcal{N} , then for any $\alpha > 0$, \mathcal{C} is also an $(\varepsilon, \alpha \underline{\sigma}, \alpha \underline{R}, \frac{n}{\alpha})$ code for \mathcal{N} .

E. Time-rate region

We now define the time-rate region $\mathcal{T}(\mathcal{N})$ of network communication problem \mathcal{N} .

Definition 2.1 (Joint time-rate region): The joint time-rate region $\mathcal{T}(\mathcal{N})$ of communication problem \mathcal{N} is the collection of $(\underline{\sigma}, \underline{R})$ such that for every $\varepsilon > 0$ and $\delta > 0$, for all n sufficiently large there exists an $(\varepsilon, \underline{\sigma} + \delta, \underline{R} - \delta, n)$ code for \mathcal{N} . Here, for a vector \underline{v} and a scalar δ , $\underline{v} - \delta$ represents the vector with entries $v_i - \delta$. Equivalently, one can express the joint time-rate region \mathcal{T} by the collection $\{\sigma_{\underline{R}}\}_{\underline{R} \in \mathbb{R}^k}$ where for each $\underline{R} \in \mathbb{R}^k$,

$$\sigma_{\underline{R}}(\mathcal{N}) = \{\underline{\sigma} \mid (\underline{\sigma}, \underline{R}) \in \mathcal{T}(\mathcal{N})\},$$

or the collection $\{\mathcal{R}_{\underline{\sigma}}\}_{\underline{\sigma}}$ where for each $\underline{\sigma} \in \mathbb{R}_{>0}^{\|S\|_0}$

$$\mathcal{R}_{\underline{\sigma}}(\mathcal{N}) = \{\underline{R} \mid (\underline{\sigma}, \underline{R}) \in \mathcal{T}(\mathcal{N})\}.$$

Here $\|S\|_0$ represents the number of entries in S that equal 1, and $\mathbb{R}_{>0}$ represents the positive reals. Using the latter notation, the traditional network capacity region, e.g., [1], of \mathcal{N} equals $\mathcal{R}_{\underline{1}}(\mathcal{N})$ for $\underline{1} = (1, 1, 1, \dots, 1)$.

III. RELATED WORK

A. Static Broadcasting

Previous work on the time-rate region (defined in a different but equivalent manner) under the name *static broadcasting* treats the broadcast channel with a common message [2], [3] and single-source multicast network coding [7]. In [7], the optimal decoding time σ_j at terminal v_j is characterized by the corresponding min-cut to the source. In [2], [3], a single transmitter transmits a single message m to a pair² of terminal nodes, here denoted by t_1 and t_2 , over a broadcast channel $(\mathcal{X}, W(y_1, y_2|x), \mathcal{Y}_1 \times \mathcal{Y}_2)$. Time parameter $\underline{\sigma} = (\sigma_1, \sigma_2)$ represents the decoding times for message m at terminals t_1 and t_2 , respectively.

The time-rate region $\mathcal{T}(\mathcal{N})$ is characterized in [2], [3] by the collection of all (σ_1, σ_2, R) for which there exists

²For simplicity of presentation, we consider only two terminal nodes. In [2], [3], the broadcast setting with multiple terminal nodes (not necessarily two) is studied. Our discussion generalizes to multiple terminals as well.

an n and a collection of distributions $(\mathbf{x}^\tau \sim p_x^\tau : \tau \in \mathbb{N})$ over alphabet \mathcal{X} such that

$$Rn \leq \sum_{\tau=1}^{\sigma_1 n} I(\mathbf{x}^\tau; \mathbf{y}_1^\tau),$$

$$Rn \leq \sum_{\tau=1}^{\sigma_2 n} I(\mathbf{x}^\tau; \mathbf{y}_2^\tau).$$

To simplify the representation of $\mathcal{T}(\mathcal{N})$ in [2], [3], let $W(y_1, y_2|x)$ equal the channel pair $W_1(y_1|x), W_2(y_2|x)$. Let C_1 and C_2 be the point-to-point capacities of W_1 and W_2 , respectively. Using the concavity of mutual information (over the distributions $\{p_x^\tau\}_\tau$), the region $\mathcal{T}(\mathcal{N})$ can be described by the collection of all (σ_1, σ_2, R) for which there exists a distribution $\mathbf{x} \sim p_x$ over alphabet \mathcal{X} such that for $\sigma_1 \leq \sigma_2$,

$$R \leq \sigma_1 I(\mathbf{x}; \mathbf{y}_1),$$

$$R \leq \sigma_1 I(\mathbf{x}; \mathbf{y}_2) + (\sigma_2 - \sigma_1) C_2$$

and for $\sigma_1 \geq \sigma_2$,

$$R \leq \sigma_2 I(\mathbf{x}; \mathbf{y}_2),$$

$$R \leq \sigma_2 I(\mathbf{x}; \mathbf{y}_1) + (\sigma_1 - \sigma_2) C_1$$

This *two-phase* representation of $\mathcal{T}(\mathcal{N})$ resembles the inner-bound methodology presented in the main result of this work (Theorem 4.2) in which we concatenate block-codes corresponding to the different phases of communication. For the case at hand, e.g., for $\sigma_1 \leq \sigma_2$, during the first phase of time-steps up to $\sigma_1 n$, terminal t_1 is required to decode the message m while terminal t_2 receives *partial* information on m . During the second phase of the remaining time steps up to $\sigma_2 n$, terminal t_2 is required to obtain additional information that allows successful decoding of m .

B. Rateless codes

Rateless codes, in which reliable decoding does not occur at a predetermined time (i.e., blocklength) n but may vary depending on the channel realization, are somewhat related to our problem. See, for example, [4], [5], [6] in the context of the erasure channel, [8], [9], [10], [11], [12], [13] in the context of adaptive routing protocols for networks, and [14], [15], [16], [17] in the context of discrete memoryless networks such as multiple access, relay, and broadcast channels.

The model and measure of quality in rateless codes, however, differ significantly from our model. Specifically, we assume decoding times $\underline{\sigma}$ that are fixed for each demand (described by a message-receiver pair (i, j)) and characterize message rates \underline{R} achievable with high probability over the channel realization.

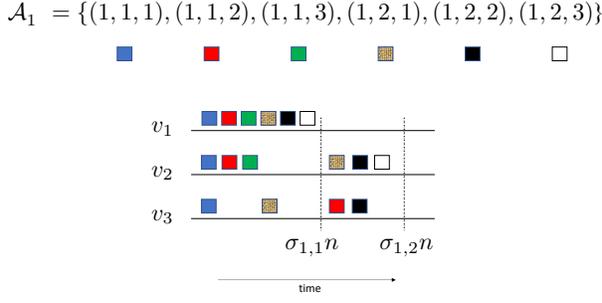


Fig. 1: A depiction of the message set \underline{M}_0 of \mathcal{N}_0 from Example 4.1.

IV. AN INNER BOUND ON THE TIME-RATE REGION

Given a network problem \mathcal{N} , the discussion that follows proposes a *time-expansion* $\mathcal{N}_1, \dots, \mathcal{N}_\Lambda$ of \mathcal{N} and then uses the standard capacities of networks in the time expansion (i.e., $\mathcal{R}_\perp(\mathcal{N}_\lambda)$ for $\lambda \in [\Lambda]$) to derive an inner bound on the time-rate region $\mathcal{T}(\mathcal{N})$.

Consider any $(\underline{\sigma}, \underline{R})$. We start by defining a network $\mathcal{N}_0 = (\mathcal{W}_0, \underline{M}_0, H_0, S_0)$ and a corresponding pair $(\underline{\sigma}_0, \underline{R}_0)$ such that

$$\underline{R}_0 \in \mathcal{R}_{\underline{\sigma}_0}(\mathcal{N}_0) \Leftrightarrow \underline{R} \in \mathcal{R}_{\underline{\sigma}}(\mathcal{N}).$$

We then expand \mathcal{N}_0 to $\mathcal{N}_1, \dots, \mathcal{N}_\Lambda$ such that one can express an inner bound on $\mathcal{R}_{\underline{\sigma}_0}(\mathcal{N}_0)$ by the sequence of standard capacity regions $(\mathcal{R}_\perp(\mathcal{N}_\lambda) : \lambda \in [\Lambda])$. Details on our reductions follow.

• **The network \mathcal{N}_0 :** Let $\mathcal{N} = (\mathcal{W}, \underline{M}, H, S)$ and let $\underline{\sigma} = (\sigma_{ij} : s_{ij} = 1)$ be a collection of time-parameters. We design network communication problem $\mathcal{N}_0 = (\mathcal{W}_0, \underline{M}_0, H_0, S_0)$ using $\underline{\sigma}$. Let Λ be the number of distinct decoding times in vector $\underline{\sigma}$. The major difference between \mathcal{N} and \mathcal{N}_0 is in the message sets \underline{M} and \underline{M}_0 and in the corresponding decoding times.

For each message $m_i \in \underline{M}$ in \mathcal{N} , we design a partition $(m_{(i,\underline{a})} : \underline{a} \in \mathcal{A}_i)$ of m_i into sub-messages $m_{(i,\underline{a})}$ to be transmitted in \mathcal{N}_0 . For each i , the vector $\underline{a} = (a_1, \dots, a_\ell)$ falls in the set $\mathcal{A}_i \subseteq [\Lambda + 1]^\ell$ to be defined shortly. The message set \underline{M}_0 of \mathcal{N}_0 consists of messages $(m_{(i,\underline{a})} : i \in [k], \underline{a} \in \mathcal{A}_i)$.

Roughly speaking, in \mathcal{N}_0 we require the sub-messages $m_{(i,\underline{a})}$ of message m_i to be decoded at or before the decoding times for m_i in \mathcal{N} . Specifically, for $i \in [k]$, $\underline{a} = (a_1, \dots, a_\ell) \in \mathcal{A}_i$, and $j \in [\ell]$, the parameter a_j represents the updated decoding time for $m_{(i,\underline{a})}$ at v_j . The partition of m_i and the updated decoding times, govern our inner-bound on $\mathcal{T}(\mathcal{N})$ through \mathcal{N}_0 and $\mathcal{N}_1, \dots, \mathcal{N}_\Lambda$. Details follow.

Consider sorting the distinct values of time-parameters in $\underline{\sigma}$ in increasing order. Let $\sigma_1 < \sigma_2 < \dots < \sigma_\Lambda$ be

these distinct values. For $\lambda \in [\Lambda]$, we define the set $\Delta(\lambda)$ to include all pairs (i, j) such that $\sigma_{ij} = \sigma_\lambda$. We set $\Delta(\Lambda + 1) = \{(i, j) : s_{ij} = 0\}$ to be all remaining pairs (not included in $\Delta(\lambda)$ for $\lambda \in [\Lambda]$).

For $i \in [k]$, the set \mathcal{A}_i is defined to be all vectors $\underline{a} = (a_1, \dots, a_\ell)$ that satisfy for $j \in [\ell]$:

- If $s_{ij} = 1$ and $\sigma_{ij} = \sigma_\lambda$, then $a_j \leq \lambda$; or
- If $s_{ij} = 0$ then $a_j \leq \Lambda + 1$.

In message $m_{(i,\underline{a})}$, the j 'th entry a_j of \underline{a} corresponds to the decoding time of $m_{(i,\underline{a})}$ at receiver v_j in \mathcal{N}_0 . If $a_j = \lambda$ we require decoding time σ_λ at v_j . To guarantee that a code for \mathcal{N}_0 will also imply one for \mathcal{N} , we require that all parts $m_{(i,\underline{a})}$ of m_i will be decoded at v_j at or before their required time σ_{ij} in \mathcal{N} . This latter requirement is met by setting $a_j \leq \lambda$ when $\sigma_{ij} = \sigma_\lambda$ (according to the first bullet above). Moreover, in \mathcal{N}_0 , we may require that part $m_{(i,\underline{a})}$ of message m_i be decoded at v_j , even when message m_i is not required at all by v_j in \mathcal{N} (i.e., $s_{ij} = 0$). Namely, for $s_{ij} = 0$ and a given message $m_{(i,\underline{a})}$, we may set a_j in \underline{a} to be equal to $\lambda \in [\Lambda]$ to represent a decoding time of σ_λ . The critical point here is, that for $s_{ij} = 0$, we can choose $m_{(i,\underline{a})}$ with $a_j = \Lambda + 1$ to represent that v_j does not require $m_{(i,\underline{a})}$ or with $a_j \in [\Lambda]$ to allow for the possibility that it might be useful for v_j to learn some or all of m_i (for example to be used as side-information).

We now formalize the discussion above. We define $\mathcal{W}_0, \underline{M}_0, H_0, S_0$ and $\underline{\sigma}_0$.

- The network \mathcal{W}_0 is identical to \mathcal{W} .
- The message set \underline{M}_0 is equal to $(m_{(i,\underline{a})} : i \in [k], \underline{a} \in \mathcal{A}_i)$.
- Define matrix S_0 with entries $s_{(i,\underline{a}),j}^{(0)}$ and vector $\underline{\sigma}_0$ with entries $\sigma_{(i,\underline{a}),j}^{(0)}$ as follows. For $i \in [k]$, $\underline{a} = (a_1, \dots, a_\ell) \in \mathcal{A}_i$, and $j \in [\ell]$, if $a_j = \lambda \in [\Lambda]$ set $s_{(i,\underline{a}),j}^{(0)} = 1$ and $\sigma_{(i,\underline{a}),j}^{(0)} = \sigma_\lambda$; otherwise, set $s_{(i,\underline{a}),j}^{(0)} = 0$.
- Define matrix H_0 with entries $h_{(i,\underline{a}),j}^{(0)}$ as follows. For $i \in [k]$, $\underline{a} \in \mathcal{A}_i$, and $j \in [\ell]$, define $h_{(i,\underline{a}),j}^{(0)} = 1$ if and only if $h_{ij} = 1$ and $h_{(i,\underline{a}),j}^{(0)} = 0$ otherwise.

An example of our definitions on a 3-node network are given below and depicted in Figure 1.

Example 4.1: Consider a network \mathcal{N} with nodes v_1, v_2 , and v_3 , and a single message m_1 . Suppose that m_1 is required by node v_1 at time-parameter $\sigma_{1,1}$ and by node v_2 at time-parameter $\sigma_{1,2}$; m_1 is not required by node v_3 . Let $\sigma_{1,1} < \sigma_{1,2}$; then $\Lambda = 2$ and $\sigma_1 = \sigma_{1,1}$, $\sigma_2 = \sigma_{1,2}$. Message m_1 is partitioned into sub-messages $m_{1,\underline{a}}$ for $\underline{a} \in \mathcal{A}_1 \subseteq [\Lambda + 1]^\ell = [3]^3$. The set \mathcal{A}_1 includes all $\underline{a} = (a_1, a_2, a_3)$ such that $a_1 \leq 1$ (as $\sigma_1 = \sigma_{1,1}$), $a_2 \leq 2$ (as $\sigma_2 = \sigma_{1,2}$), and $a_3 \leq \Lambda + 1 = 3$ (as m_1 is not required by node v_3 in \mathcal{N}). In Figure 1, each sub-message $m_{1,\underline{a}}$ is depicted by a colored box. For example,

the blue box represents the message $m_{1,(1,1,1)}$. For each message $m_{1,\underline{a}}$ the vector \underline{a} specifies the decoding time of message $m_{1,\underline{a}}$ at nodes v_1, v_2, v_3 . That is, if $\underline{a} = (a_1, a_2, a_3)$ then for each j with $a_j \leq \Lambda$, $m_{1,\underline{a}}$ is to be decoded by node v_j by time $\sigma_{a_j}n$ while for $a_j = \Lambda + 1 = 3$, $m_{1,\underline{a}}$ is not required by node v_j . For example, message $m_{1,(1,2,3)}$, represented by a white box, must be decoded by time $\sigma_1n = \sigma_{1,1}n$ at node v_1 , by time $\sigma_2n = \sigma_{1,2}n$ at node v_2 , and is not required by v_3 . Thus a white box is placed before $\sigma_{1,1}n$ on the horizontal *time-line* for v_1 and before $\sigma_{1,2}n$ on the time-line for v_2 (but is absent from the time-line for v_3). Similarly, message $m_{1,(1,2,1)}$, represented by a brown box, is to be decoded by time $\sigma_1n = \sigma_{1,1}n$ at node v_1 , by time $\sigma_2n = \sigma_{1,2}n$ at node v_2 , and by time $\sigma_3n = \sigma_{1,3}n$ at node v_3 . The locations of the brown box on the time-lines of nodes v_1, v_2, v_3 appear accordingly.

We now have the following theorem:

Theorem 4.1: Let $\mathcal{N} = (\mathcal{W}, \underline{M}, H, S)$. Let $\underline{\sigma} = (\sigma_{ij} : s_{ij} = 1)$ be a collection of time parameters. Let Λ be the number of distinct values in $\underline{\sigma}$. Let $\mathcal{N}_0 = (\mathcal{W}_0, \underline{M}_0, H_0, S_0)$ and $\underline{\sigma}_0$ be defined as above. For rate vector $\underline{R}_0 = (R_{(i,\underline{a})}^{(0)} : i \in [k], \underline{a} \in \mathcal{A}_i)$, let $\underline{R} = (R_i : i \in [k])$ be defined by $R_i = \sum_{\underline{a}} R_{(i,\underline{a})}^{(0)}$ for all $i \in [k]$. Then,

$$\underline{R}_0 \in \mathcal{R}_{\underline{\sigma}_0}(\mathcal{N}_0) \Rightarrow \underline{R} \in \mathcal{R}_{\underline{\sigma}}(\mathcal{N}).$$

Moreover, for rate vector $\underline{R} = (R_i : i \in [k])$, let $\underline{R}_0 = (R_{(i,\underline{a})}^{(0)} : i \in [k], \underline{a} \in \mathcal{A}_i)$ satisfy $R_{(i,\underline{a})}^{(0)} = R_i$ for all $i \in [k]$ and the unique $\underline{a} = (a_1, \dots, a_\ell)$ in which $a_j = \lambda$ if $s_{ij} = \sigma_\lambda$ and $a_j = \Lambda + 1$ otherwise, and let $R_{(i,\underline{a})}^{(0)} = 0$ otherwise, then

$$\underline{R} \in \mathcal{R}_{\underline{\sigma}}(\mathcal{N}) \Rightarrow \underline{R}_0 \in \mathcal{R}_{\underline{\sigma}_0}(\mathcal{N}_0).$$

Proof: Let $\mathcal{R}_0 = (R_{(i,\underline{a})}^{(0)} : i \in [k], \underline{a} \in \mathcal{A}_i)$ satisfy $R_i = \sum_{\underline{a}} R_{(i,\underline{a})}^{(0)}$. To prove that $\underline{R}_0 \in \mathcal{R}_{\underline{\sigma}_0}(\mathcal{N}_0)$ implies $\underline{R} \in \mathcal{R}_{\underline{\sigma}}(\mathcal{N})$, consider any $(\varepsilon, \underline{\sigma}_0, \underline{R}_0, n)$ code \mathcal{C}_0 for \mathcal{N}_0 . Using the exact same code on \mathcal{N} where each message m_i of \mathcal{N} is taken to be the concatenation of the messages $(m_{(i,\underline{a})} : \underline{a} \in \mathcal{A}_i)$ of \mathcal{N}_0 results in an $(\varepsilon, \underline{\sigma}, \underline{R}, n)$ code \mathcal{C} for \mathcal{N} . Specifically, by our definition of \mathcal{A}_i and $\underline{\sigma}_0$, if $s_{ij} = 1$ then each part $m_{(i,\underline{a})}$ of message m_i is decoded by v_j at or before time $\sigma_{ij}n$.

For the other direction, consider any $(\varepsilon, \underline{\sigma}, \underline{R}, n)$ code \mathcal{C} for \mathcal{N} . Let \underline{R}_0 be as defined in the theorem statement. For all $R_{(i,\underline{a})}^{(0)}$ that equal R_i , message $m_{(i,\underline{a})}$ has decoding times identical to those of m_i in \mathcal{N} , and thus can be communicated in \mathcal{N}_0 using \mathcal{C} . Moreover, all other $R_{(i,\underline{a})}^{(0)}$ equal 0. Thus, the exact same code \mathcal{C} is an $(\varepsilon, \underline{\sigma}_0, \underline{R}_0, n)$ code for \mathcal{N}_0 as well. ■

• **Expanding \mathcal{N}_0 :** Let $\mathcal{N} = (\mathcal{W}, \underline{M}, H, S)$ and $\mathcal{N}_0 = (\mathcal{W}_0, \underline{M}_0, H_0, S_0)$ be as defined above. We now present

an expansion of \mathcal{N}_0 to network problems $\mathcal{N}_1, \dots, \mathcal{N}_\Lambda$. As with the definition of \mathcal{N}_0 , the expansion depends on $\underline{\sigma}$. Let Λ be the number of distinct values in $\underline{\sigma}$, and let $\sigma_1, \dots, \sigma_\Lambda$ and $\Delta(1), \dots, \Delta(\Lambda + 1)$ be defined as before. Our goal in defining networks $\mathcal{N}_1, \dots, \mathcal{N}_\Lambda$ is to better capture the communication process over \mathcal{N}_0 . The networks \mathcal{N}_λ for $\lambda \in [\Lambda]$ differ from \mathcal{N}_0 only in the requirements S_λ and in the side-information H_λ .

Namely,

- For $\lambda \in [\Lambda]$, $\mathcal{N}_\lambda = (\mathcal{W}_0, \underline{M}_0, H_\lambda, S_\lambda)$.
- Define matrix S_λ with entries $s_{(i,\underline{a}),j}^{(\lambda)}$ as follows. For $\lambda \in [\Lambda]$, $i \in [k]$, $\underline{a} \in \mathcal{A}_i$, and $j \in [\ell]$, let $s_{(i,\underline{a}),j}^{(\lambda)} = 1$ for all pairs $((i,\underline{a}), j)$ with $\underline{a} = (a_1, \dots, a_\ell)$ such that $a_j = \lambda$. Otherwise set $s_{(i,\underline{a}),j}^{(\lambda)} = 0$. Notice that

$$\sum_{\lambda=1}^{\Lambda} S_\lambda = S_0.$$

- Define matrix H_λ with entries $h_{(i,\underline{a}),j}^{(\lambda)}$ as follows. For $\lambda \in [\Lambda]$, $i \in [k]$, $\underline{a} \in \mathcal{A}_i$, and $j \in [\ell]$,

$$h_{(i,\underline{a}),j}^{(\lambda)} = h_{(i,\underline{a}),j}^{(0)} \cup \sum_{\lambda'=1}^{\lambda-1} s_{(i,\underline{a}),j}^{(\lambda')}.$$

Here, ‘ \cup ’, represents binary logical OR. Namely, in \mathcal{N}_λ , node v_j holds all sub-messages $m_{i,\underline{a}}$ that it holds in \mathcal{N}_0 and, in addition, it holds all sub-messages $m_{i,\underline{a}}$ that are required to be decoded at v_j in \mathcal{N}_0 before time σ_λ .

Theorem 4.2: Let $\mathcal{N} = (\mathcal{W}, \underline{M}, H, S)$. Let $\underline{\sigma} = (\sigma_{ij} : s_{ij} = 1)$ be a collection of time parameters. Let $\mathcal{N}_0 = (\mathcal{W}_0, \underline{M}_0, H_0, S_0)$ and $\underline{\sigma}_0$ be defined as above. Let Λ be the number of distinct values $\sigma_1 < \sigma_2 < \dots < \sigma_\Lambda$ in $\underline{\sigma}$. Let $\sigma_0 = 0$. For $\lambda \in [\Lambda]$, let $\mathcal{N}_\lambda = (\mathcal{W}_0, \underline{M}_0, H_\lambda, S_\lambda)$ be defined as above. Let $\underline{R}_0 = (R_{(i,\underline{a})}^{(0)} : i \in [k], \underline{a} \in \mathcal{A}_i)$ be a rate vector. For $\lambda \in [\Lambda]$, let $\underline{R}_\lambda = (R_{(i,\underline{a})}^{(\lambda)} : i \in [k], \underline{a} \in \mathcal{A}_i)$ be defined by $R_{(i,\underline{a})}^{(\lambda)} = R_{(i,\underline{a})}^{(0)}$ if there exists $j \in [\ell]$ such that $s_{(i,\underline{a}),j}^{(\lambda)} = 1$ and $R_{(i,\underline{a})}^{(0)} = 0$ otherwise. If for $\lambda \in [\Lambda]$ it holds that

$$\frac{\underline{R}_\lambda}{\sigma_\lambda - \sigma_{\lambda-1}} \in \mathcal{R}_{\underline{\sigma}_\lambda}(\mathcal{N}_\lambda),$$

then

$$\underline{R}_0 \in \mathcal{R}_{\underline{\sigma}_0}(\mathcal{N}_0).$$

Proof: Let n be an integer to be determined later. Let $n_\lambda = n(\sigma_\lambda - \sigma_{\lambda-1})$. For $\lambda \in [\Lambda]$, consider any collection of $(\varepsilon, \underline{\mathbf{1}}, \frac{\underline{R}_\lambda}{\sigma_\lambda - \sigma_{\lambda-1}}, n_\lambda)$ codes $\mathcal{C}_\lambda = (\mathcal{E}_\lambda, \mathcal{D}_\lambda)$ for \mathcal{N}_λ . Here, we take n to be as large as needed to ensure that n_λ for each $\lambda \in [\Lambda]$ in turn is sufficiently large to allow the existence of codes $\mathcal{C}_1, \dots, \mathcal{C}_\Lambda$. By Claim 2.1, our definitions imply for $\lambda \in [\Lambda]$ that \mathcal{C}_λ is also an $(\varepsilon, (\sigma_\lambda - \sigma_{\lambda-1})\underline{\mathbf{1}}, \underline{R}_\lambda, n)$ code for \mathcal{N}_λ .

For $\lambda \in [\Lambda]$, let $\mathcal{E}_\lambda^* = (\mathcal{E}_\lambda^\tau : \tau \in [(\sigma_\lambda - \sigma_{\lambda-1})n])$ represent the first $(\sigma_\lambda - \sigma_{\lambda-1})n$ encoders in \mathcal{C}_λ . Similarly for \mathcal{D}_λ^* . The code $\mathcal{C}_0 = (\mathcal{E}_0, \mathcal{D}_0)$ for \mathcal{N}_0 is defined to consist of the concatenations $\mathcal{E}_0 = \mathcal{E}_1^* \circ \mathcal{E}_2^* \circ \dots \circ \mathcal{E}_\Lambda^*$ and $\mathcal{D}_0 = \mathcal{D}_1^* \circ \mathcal{D}_2^* \circ \dots \circ \mathcal{D}_\Lambda^*$.

We now show by induction that \mathcal{C}_0 is an $(\Lambda\varepsilon, \underline{\sigma}_0, \underline{R}_0, n)$ code for \mathcal{N}_0 . First consider all messages $m_{(i, \underline{a})}$ and nodes v_j such that for $\lambda = 1$, $s_{(i, \underline{a}), j}^{(\lambda)} = 1$ and thus $\sigma_{(i, \underline{a}), j}^{(0)} = \sigma_\lambda = \sigma_1$. By our definitions of \mathcal{C}_1 and \mathcal{N}_1 , with error probability at most ε , for all such pairs $((i, \underline{a}), j)$, message $m_{(i, \underline{a})}$ is decoded successfully by v_j after $\sigma_1 n = (\sigma_1 - \sigma_0)n$ time steps of \mathcal{C}_0 . This implies, for $\lambda = 1$, that at time $\sigma_\lambda n$, if $h_{(i, \underline{a}), j}^{(\lambda+1)} = 1$ then v_j holds message $m_{(i, \underline{a})}$. Recall that,

$$h_{(i, \underline{a}), j}^{(\lambda+1)} = h_{(i, \underline{a}), j}^{(0)} \cup \sum_{\lambda'=1}^{\lambda} s_{(i, \underline{a}), j}^{(\lambda')}.$$

We continue by induction. Assume, for $\lambda - 1$, that after $\sigma_{\lambda-1}n$ time steps of \mathcal{C}_0 , with error probability at most $(\lambda - 1)\varepsilon$ if $h_{(i, \underline{a}), j}^{(\lambda)} = 1$ then v_j holds message $m_{(i, \underline{a})}$. We wish to prove the corresponding statement for λ . As with the base case of $\lambda = 1$, consider all messages $m_{(i, \underline{a})}$ and nodes v_j such that $s_{(i, \underline{a}), j}^{(\lambda)} = 1$ and thus $\sigma_{(i, \underline{a}), j}^{(0)} = \sigma_\lambda$. By induction, after $\sigma_{\lambda-1}n$ time steps of \mathcal{C}_0 , with error probability at most $(\lambda - 1)\varepsilon$, if $h_{(i, \underline{a}), j}^{(\lambda)} = 1$ then node v_j holds message $m_{(i, \underline{a})}$. Code \mathcal{C}_0 in time steps $(\sigma_\lambda - \sigma_{\lambda-1})n$ employs the encoder and decoder of \mathcal{C}_λ , which is an $(\varepsilon, (\sigma_\lambda - \sigma_{\lambda-1})\mathbf{1}, \underline{R}_\lambda, n)$ code for \mathcal{N}_λ . By our definitions, during those time steps of \mathcal{C}_0 , with error probability at most ε , for all pairs $((i, \underline{a}), j)$ for which $s_{(i, \underline{a}), j}^{(\lambda)} = 1$ message $m_{(i, \underline{a})}$ is decoded by v_j . Thus by the union bound, with overall error probability of at most $\lambda\varepsilon$, after the first $\sigma_\lambda n$ time steps of \mathcal{C}_0 , if $h_{(i, \underline{a}), j}^{(\lambda+1)} = 1$ then v_j holds message $m_{(i, \underline{a})}$.

Continuing this process until $\lambda = \Lambda$, and using our definitions of $S_0, S_\lambda, H_\lambda$, and \underline{R}_λ for $\lambda \in [\Lambda]$, we conclude that, employing \mathcal{C}_0 , with probability at least $1 - \Lambda\varepsilon$ all nodes v_j decode their required information according to $\underline{\sigma}_0$. Thus, \mathcal{C}_0 is an $(\Lambda\varepsilon, \underline{\sigma}_0, \underline{R}_0, n)$ code for \mathcal{N}_0 . ■

Combining Theorems 4.1 and 4.2 now implies an inner bound to $\mathcal{R}_{\underline{\sigma}}(\mathcal{N})$ derived from $(\mathcal{R}_\lambda(\mathcal{N}_\lambda) : \lambda \in [\Lambda])$.

Remark 4.1: In the design of $\mathcal{N}_0, \mathcal{N}_1, \dots, \mathcal{N}_\Lambda$, the message set \underline{M}_0 included messages $(m_{i, \underline{a}} : i \in [k], \underline{a} \in \mathcal{A}_i)$, where for each $i \in [k]$, in the proof of Theorem 4.1, the collection $(m_{i, \underline{a}} : \underline{a} \in \mathcal{A}_i)$ is considered to be a partition of the original message $m_i \in \underline{M}$ of \mathcal{N} . We note that partitioning the message m_i into more sub-messages does not imply a stronger inner-bound on $\mathcal{T}(\mathcal{N})$. Specifically, consider a partition $(m_{i, b} : b \in \mathcal{B}_i)$ for some index set \mathcal{B}_i for which $|\mathcal{B}_i| > |\mathcal{A}_i|$. By the pigeonhole principle, no matter how we set the

decoding times for messages $(m_{i, b} : b \in \mathcal{B}_i)$, there exist at least two messages m_{i, b_1} and m_{i, b_2} with identical corresponding decoding times (at each of the nodes $v_j \in V$). For the purposes of the proofs of Theorems 4.1 and 4.2, these messages can be merged into a single one. Continuing in this manner, one can formally show that any rate achievable using the proofs of Theorems 4.1 and 4.2 with the message set $(m_{i, b} : i \in [k], b \in \mathcal{B}_i)$ can also be achieved using the original message set $(m_{i, \underline{a}} : i \in [k], \underline{a} \in \mathcal{A}_i)$.

Moreover, in the design of $\mathcal{N}_0, \mathcal{N}_1, \dots, \mathcal{N}_\Lambda$ and in the proofs of Theorems 4.1 and 4.2, for a given $\underline{\sigma}$, we consider Λ rounds of communication in which round $\lambda \in [\Lambda]$ is done over \mathcal{N}_λ . Here, Λ equals the number of distinct time-parameters in $\underline{\sigma}$. One could consider increasing the number of rounds of communication beyond the number of distinct time-parameters in $\underline{\sigma}$ (and modifying the message set \underline{M}_0 and the networks $\mathcal{N}_0, \mathcal{N}_1, \dots$ accordingly). We note, similar to the discussion above, that modifying Theorems 4.1 and 4.2 in this manner also does not yield a better inner-bound on $\mathcal{T}(\mathcal{N})$.

Specifically, using the notation of Theorems 4.1 and 4.2 and their proofs, consider, for example, partitioning a specific round λ into two phases, denoted $\lambda^{(1)}$ and $\lambda^{(2)}$, the first phase taking place during time-steps $n(\sigma_{\lambda^{(1)}} - \sigma_{\lambda-1})$ of \mathcal{C}_0 and the second during time steps $n(\sigma_\lambda - \sigma_{\lambda^{(1)}})$ for a new parameter $\sigma_{\lambda^{(1)}} \in (\sigma_{\lambda-1}, \sigma_\lambda)$. Consider also the corresponding modifications needed in the proofs of Theorems 4.1 and 4.2, including refining the message set \underline{M}_0 , the sets \mathcal{A}_i for $i \in [k]$, and replacing \mathcal{N}_λ by two network communication problems $\mathcal{N}_{\lambda^{(1)}}$ and $\mathcal{N}_{\lambda^{(2)}}$. By merging messages $m_{i, \underline{a}}$ and $m_{i, \underline{a}'}$ with $\underline{a} = (a_1, \dots, a_\ell)$ and $\underline{a}' = (a'_1, \dots, a'_\ell)$ that differ only in locations $j \in [\ell]$ for which $a_j = \lambda^{(1)}$ and $a'_j = \lambda^{(2)}$ or vice versa, it can be shown that any achievable rate for \mathcal{N}_0 in the modified proof of Theorem 4.2 can also be obtained in the original proof.

V. THEOREM 4.2 IS NOT TIGHT

It may not come as a surprise to the reader that Theorem 4.2 is not tight (even for *single-message* networks \mathcal{N}). For completeness, we present a single-message network communication problem \mathcal{N} for which Theorem 4.2 is not tight. Consider the memoryless degraded 2-terminal broadcast channel characterized by $W(y_1, y_2|x)$ where $y_1 = x$ is the binary identity channel, and $y_2 = BEC_{0.5}(x)$ corresponds to the binary erasure channel in which $y_2 = x$ with probability 0.5 and y_2 is an erasure symbol \perp otherwise. Namely, \mathcal{W} is defined by three nodes that are denoted here as the encoder u and two receivers t_1 and t_2 . The message set \underline{M} includes a single message m , available at node u and required by nodes t_1 and t_2 . Consider the setting in which $\underline{\sigma} = (\sigma_1, \sigma_2)$, where $\sigma_1 = 0.5$ is the decoding

time required at t_1 and $\sigma_2 = 1$ the decoding time at t_2 . As in Section III-A, for clarity of presentation, we use the common notation for the broadcast channel, instead of that given in Section II.

On one hand, using the characterization of [2], [3] presented in Section III-A, it holds that $\mathcal{R}_{\sigma}(\mathcal{N}) = [0, 0.5]$. The optimal rate $R = 0.5$ is obtained, for example, using a codebook chosen uniformly at random, i.e., using the uniform distribution \mathbf{x} over $\{0, 1\}$ in the characterization

$$R \leq \sigma_1 I(\mathbf{x}; \mathbf{y}_1) = \sigma_1 \cdot 1 = 0.5,$$

$$R \leq \sigma_1 I(\mathbf{x}; \mathbf{y}_2) + (\sigma_2 - \sigma_1) C_2 = \sigma_1 \cdot 0.5 + 0.25 = 0.5$$

On the other hand, applying Theorems 4.1 and 4.2, the message set \underline{M}_0 of \mathcal{N}_0 includes two messages $m_{(1,1)}$ and $m_{(1,2)}$ with corresponding decoding times $\underline{\sigma}_0$ of $\sigma_{(1,1),1}^{(0)} = \sigma_{(1,1),2}^{(0)} = \sigma_{(1,2),1}^{(0)} = \sigma_1 = 0.5$ and $\sigma_{(1,2),2}^{(0)} = \sigma_2 = 1$. Consider any $\underline{R}_0 = (R_{(1,1)}^{(0)}, R_{(1,2)}^{(0)}) \in \mathcal{R}_{\sigma_0}(\mathcal{N}_0)$. By Theorem 4.1, it holds that the sum-rate $R_{(1,1)}^{(0)} + R_{(1,2)}^{(0)}$ can be as large as $R = 0.5$. We now show that for any $\underline{R}_0^* = (R_{(1,1)}^*, R_{(1,2)}^*)$ satisfying the conditions of Theorem 4.2 it holds that $R_{(1,1)}^* + R_{(1,2)}^* < 0.5$, implying that Theorem 4.2 is not tight.

Consider the network communication problem \mathcal{N}_1 corresponding to the first phase of communication in \mathcal{N}_0 . Namely, \mathcal{N}_1 takes into consideration the requirements $s_{(1,1),1}^{(1)} = s_{(1,1),2}^{(1)} = s_{(1,2),1}^{(1)} = 1$, and $s_{(1,2),2}^{(1)} = 0$. It can be seen that $2R_{(1,1)}^{(1)} + R_{(1,2)}^{(1)} \leq 0.5$, for any $\frac{\underline{R}_1}{\sigma_1} = 2\underline{R}_1 = (2R_{(1,1)}^{(1)}, R_{(1,2)}^{(1)}) \in \mathcal{R}_{\underline{1}}(\mathcal{N}_1)$. This follows from studying the capacity of the (degraded) broadcast channel W with common message $m_{(1,1)}$ and private message $m_{(1,2)}$ at terminal t_1 . By our definitions in Theorem 4.2 it holds that $(R_{(1,1)}^{(1)}, R_{(1,2)}^{(1)}) = (R_{(1,1)}^*, R_{(1,2)}^*)$. Thus, either $R_{(1,1)}^* = 0$, or alternatively, $R_{(1,1)}^* + R_{(1,2)}^* < 0.5$. In the latter, we conclude that Theorem 4.2 is not tight. For the former, we further study $R_{(1,2)}^*$. This time we consider \mathcal{N}_2 and see that $R_{(1,2)}^{(2)} = R_{(1,2)}^* \in (\sigma_2 - \sigma_1) \mathcal{R}_{\underline{1}}(\mathcal{N}_2) = \frac{1}{2} \mathcal{R}_{\underline{1}}(\mathcal{N}_2)$ implies $R_{(1,2)}^* \leq 0.25$. Which in turn implies that $R_{(1,1)}^* + R_{(1,2)}^* \leq 0 + 0.25 < 0.5$ and thus that Theorem 4.2 is not tight in this case as well.

VI. CONCLUSIONS

In this work we generalize the standard notion of capacity by studying the time-rate region \mathcal{T} of discrete memoryless networks \mathcal{N} . We present an inner bound on \mathcal{T} based on the concatenation of a series of block-codes corresponding to a time-expansion of \mathcal{N} . Improving on these bounds, for general networks \mathcal{N} , or for specific network components or network communication settings, is the subject of future work.

REFERENCES

- [1] A. El Gamal and Y-H. Kim. *Network information theory*. Cambridge university press, 2011.
- [2] Nadav Shulman and Meir Feder. Static broadcasting. In *IEEE International Symposium on Information Theory (ISIT)*, page 23, 2000.
- [3] N. Shulman. Communication over an unknown channel via common broadcasting. *PhD thesis. Tel-Aviv University.*, 2003.
- [4] M. Luby. LT codes. In *Proceedings of the 43rd Annual IEEE Symposium on Foundations of Computer Science*, pages 271–280, 2002.
- [5] D. J. C. MacKay. Fountain codes. *IEEE Proceedings-Communications*, 152(6):1062–1068, 2005.
- [6] A. Shokrollahi. Raptor codes. *IEEE transactions on Information Theory*, 52(6):2551–2567, 2006.
- [7] D. S. Lun, M. Médard, R. Koetter, and M. Effros. On coding for reliable communication over packet networks. *Physical Communication*, 1(1):3–20, 2008.
- [8] D. S. Lun, M. Médard, R. Koetter, and M. Effros. The DoD internet architecture model. *Computer Networks*, 7(5):307–318, 1983.
- [9] V. G. Cerf and R. E. Kahn. A protocol for packet network intercommunication. *IEEE Trans. on Comms.*, 22(5):71–82, 1974.
- [10] J. Postel. User datagram protocol (No. RFC 768). Technical report, 1980.
- [11] V. D. Park and M. S. Corson. A highly adaptive distributed routing algorithm for mobile wireless networks. In *Proceedings of Sixteenth Annual Joint Conference of the IEEE Computer and Communications Societies (INFOCOM)*, volume 3, pages 1405–1413, 1997.
- [12] B. Y. Zhao, J. Kubiawicz, and A. D. Joseph. Tapestry: An infrastructure for fault-tolerant wide-area location and routing. *Report No. UCB/CSD-01-1141, Computer Science Division, University of California Berkeley*, 2001.
- [13] K. R. Fall and W. R. Stevens. *TCP/IP illustrated, volume 1: The protocols*. addison-Wesley, 2011.
- [14] J. Castura and Y. Mao. Rateless coding for wireless relay channels. In *Proceedings of IEEE International Symposium on Information Theory*, pages 810–814, 2005.
- [15] M. Uppal, A. Host-Madsen, and Z. Xiong. Practical rateless cooperation in multiple access channels using multiplexed raptor codes. In *Proceedings of IEEE International Symposium on Information Theory*, pages 671–675, 2007.
- [16] A. F. Molisch, N. B. Mehta, J. S. Yedidia, and J. Zhang. Performance of fountain codes in collaborative relay networks. *IEEE Transactions on Wireless Communications*, 6(11), 2007.
- [17] X. Liu and T. J. Lim. Fountain codes over fading relay channels. *IEEE Transactions on Wireless Communications*, 8(6), 2009.