Universal Manifold Pairings and Positivity

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Abstract: Gluing two manifolds M_1 and M_2 with a common boundary S yields a closed manifold M. Extending to formal linear combinations $x = \Sigma a_i M_i$ yields a sesquilinear pairing $p = \langle\ ,\ \rangle$ with values in (formal linear combinations of) closed manifolds. Topological quantum field theory (TQFT) represents this universal pairing p onto a finite dimensional quotient pairing q with values in $\mathbb{C}$ which in physically motivated cases is positive definite. To see if such a "unitary" TQFT can potentially detect any nontrivial x, we ask if $\langle x, x\rangle \neq 0$ whenever $x \neq 0$. If this is the case, we call the pairing p positive. The question arises for each dimension $d = 0, 1, 2, \ldots$. We find $p(d)$ positive for $d = 0, 1,$ and 2 and not positive for $d \geq 4$. We conjecture that $p(3)$ is also positive. Similar questions may be phrased for (manifold, submanifold) pairs and manifolds with other additional structure. The negative result in dimension $= 4$ may illuminate the difficulties that have been met by several authors in their attempts to formulate unitary TQFTs for $d = 3 + 1$.

1 Introduction

We begin by establishing notation. We will work with oriented, compact, possibly disconnected, smooth manifolds, although some comments will also be made concerning the unoriented case. The choice of smooth category is not essential: the general statements in the abstract remain true in $P.L.$ and Top (finer details do depend on the chosen category).

Let S be a $d-1$ dimensional manifold and let $\mathcal{M}_S$ be the $\mathbb{C}-$vector space of (finite) formal combinations of manifolds M_i with $\partial M_i = S$, so $x = \sum_i a_i M_i \in \mathcal{M}_S$. (Note: If S does not bound, dimension $(\mathcal{M}_S) = 0$). If we denote S with the opposite orientation by $\overline{S}$, then we have a bilinear pairing

$$\mathcal{M}_S \times \mathcal{M}_{\overline{S}} \longrightarrow \mathcal{M} \tag{1}$$

given by $(\sum_i a_i M_i, \sum_j b_j N_j) \longrightarrow \sum_{i,j} a_i b_j M_i \cup_S N_j$, where $\mathcal{M} = \mathcal{M}_\emptyset$ is the vector space of formal linear combinations of closed $d-$manifolds. To fit better with the role of Hilbert space in physics we choose to rewrite (1) as a sesquilinear pairing

$$\langle\ ,\ \rangle : \mathcal{M}_S \times \mathcal{M}_S \longrightarrow \mathcal{M},\ \langle \sum_i a_i M_i, \sum_j b_j N_j\rangle = \sum_{i,j} a_i \overline{b}_j M_i \cup_S \overline{N}_j \tag{2}$$

which is linear in the first entry and conjugate linear in the second. The map from $\mathcal{M}_S \times \mathcal{M}_{\overline{S}}$ to $\mathcal{M}_S \times \mathcal{M}_S$ which intertwines between the pairings is just the conjugate linear extension of orientation reversal on the second factor.

We need to be perfectly clear about when two boundary manifolds M_i and M_j are considered the same element of $\mathcal{M}_S$. A basis element M_i of $\mathcal{M}_S$ is a manifold M_i together with a diffeomorphism f_i of ∂M_i to S. We say (M_i, f_i) and (M_j, f_j) are equivalent if there is a diffeomorphism $\phi : M_i \longrightarrow M_j$ such that:

$$f_j \circ \phi \mid_{\partial M_i} = f_i. \tag{3}$$

With this definition, we have examples where the manifolds M_i, M_j are the same, but attached differently to the boundary and hence not equivalent. Perhaps the simplest of these is shown in figure 1, where the manifolds both consist of two line segments, attached to the four boundary points in different ways. Less trivially, a surface bounds infinitely many distinct handle bodies parameterized by the cosets: MC_g/HC_g; the genus $= g$ mapping class group modulo the subgroup which extends over a fixed handlebody.

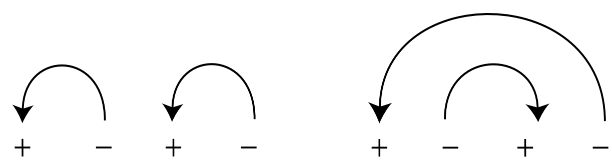

Figure 1: Two inequivalent examples of 1-manifolds with boundary. In both cases, the manifold consists of two oriented line segments and the boundary of two positively and two negatively oriented points, but the attaching maps are different

Occasionally we consider simply the set of bounded manifolds up to equivalence (i.e. the basis vectors of $\mathcal{M}_S$) and denote this set by $\dot{\mathcal{M}}_S$. We reserve the dot to mean "unlinearize".

Our definitions easily extend to (manifold, submanifold) pairs (if $K \subset M$ is a submanifold we always assume $\partial K \subset \partial M$). Let $\mathcal{M}^{d,k}$ be the space of formal combinations of (d-manifold, k-submanifold) pairs. If (S, L) is a fixed (($d-1$)-manifold, $(k-1)$-submanifold) pair, we may define $\mathcal{M}^{d,k}_{(S,L)}$ to be formal combinations of bounding (d-manifold, k-submanifold) pairs with an equivalence relation analogous to (3) and a sesquilinear pairing:

$$\mathcal{M}^{d,k}_{S,L} \times \mathcal{M}^{d,k}_{S,L} \longrightarrow \mathcal{M}^{d,k} \tag{4}$$

by a formula like (2).

A variant on gluing pairs is to require the outer manifolds to be as simple as possible, spheres and disks. This gives sesquilinear "tangle pairings":

$$\Im^{d,k}_L \times \Im^{d,k}_L \longrightarrow \mathcal{L}^{d,k} \tag{5}$$

where L is a fixed $(k-1)$-submanifold of S^{d-1} and $\Im^{d,k}_L$ is the span of k-submanifolds in D^d bounded by L. The target $\mathcal{L}^{d,k}$ is the span of k-submanifolds in S^d.

For all the sesquilinear pairings above we may ask if they are positive, that is, whether $\langle x, x\rangle = 0$ implies $x = 0$. The motivation is to understand how much of manifold topology can potentially be detected by unitary topological quantum field theories (UTQFTs. See [A] for a definition). To touch on only the most elementary aspect of this structure, a UTQFT should assign a scalar to a closed d-manifold and a finite dimensional Hilbert space V_S to each $(d-1)$-manifold S. For X with $\partial X = S$, a vector $\widetilde{X} \in V_S$ is assigned and if X' also satisfies $\partial X' = S$ then $\langle \widetilde{X}, \widetilde{X}'\rangle$ must equal the closed manifold invariant of $X\overline{X}' := X \cup_S \overline{X}'$. Clearly if one of our pairings is not formally positive, there will be an $x = \Sigma a_i X_i \neq 0$ for which $\langle x, x\rangle = 0$, and no unitary TQFT will be able to distinguish the combination x from zero. This question is (roughly) in the same spirit as asking if the Jones polynomial detects all knots.

To make the connection to TQFTs more exact one might choose to enhance our manifolds with framings, spin structure, p_1-structures, etc..., the necessary input for certain TQFTs. But the investigation is at such a preliminary stage that this level of detail is not yet warranted. Also, one may note that the invariants for closed manifolds often depend only weakly on the extra structures in the definition of TQFTs, so our results for closed manifolds may already be useful in such cases. With the definition complete, let us do what is easy.

2 Lowest Dimensions

Theorem 2.1. *The following pairings are positive:*

$$\mathcal{M}^d_Y \times \mathcal{M}^d_Y \longrightarrow \mathcal{M}^d,$$

for $d = 0, 1, 2$*,*

$$\mathcal{M}^{d,k}_{Y,L} \times \mathcal{M}^{d,k}_{Y,L} \longrightarrow \mathcal{M}^{d,k},$$

for $d = 0, 1, 2$*;* $k < d$ *and*

$$\Im^{d,k}_L \times \Im^{d,k}_L \longrightarrow \mathcal{L}^{d,k},$$

for $d = 0, 1, 2$*;* $k < d$*.*

Conjecture 2.2. *Theorem 2.1 extends to* $d = 3$ *in the above cases.*

Lemma 2.3. *Suppose there exists a function (the "complexity function")* $C : \dot{\mathcal{M}}^d \to \mathcal{V}$*, where* $\mathcal{V}$ *is some partially ordered set, such that for all* $M, N \in \dot{\mathcal{M}}^d_Y$*,* $M \neq N$ *implies* $C(M \cup \overline{N}) < C(M \cup \overline{M})$ *or* $C(M \cup \overline{N}) < C(N \cup \overline{N})$*. Then the pairing for* $\mathcal{M}^{d,k}_{Y,L}$ *is positive. Similar statements hold for* $\mathcal{M}^{d,k}$ *and* $\mathcal{L}^{d,k}$*.*

Proof: The hypothesis of the lemma implies that the terms of maximal complixity in the right-hand side of Equation 2 all lie on the diagonal. Since all coefficients on the diagonal are positive, there can be no cancellation among these terms. □

Proof of Theorem 2.1: By the previous Lemma, it suffices in each case to define an appropriate complexity function C.

We ignore $\mathcal{L}^{d,k}$, since these cases are implied by the $\mathcal{M}^{d,k}$ cases.

For $\mathcal{M}^1$, define $C(M)$ to be the number of components of M. Let Y be the 0-manifold with j positive and j negative points. Let $M, N \in \mathcal{M}^1_Y$, and assume for the moment that neither M nor N contain closed components. Then $C(M \cup \overline{M}) = C(N \cup \overline{N}) = j$. If $M \neq N$ then at least one component of $M \cup \overline{N}$ contains 4 or more arcs, and so $C(M \cup \overline{N}) < j$. Then general case (where M and N might have closed components) is similar.

Next consider $\mathcal{M}^2_Y \times \mathcal{M}^2_Y \longrightarrow \mathcal{M}^2$, where Y is the disjoint union of j circles. Let $M \in \dot{\mathcal{M}}^2$ (i.e. M is a closed oriented 2-manifold). Let n be the number of connected components of M, let χ be the Euler characteristic of M, and let $\chi_1, \ldots, \chi_n$ be the Euler characteristics of the components of M, listed in increasing order. Define the complexity of M to be the lexicographic tuple

$$C(M) = (n, -\chi, -\chi_1, \ldots, -\chi_n).$$

The smallest integer that can appear in the tuple is -2 so we formally pad tuples by adding a list of -3's at the end so that tuples of different lengths can be compared.

Now let $M, N \in \dot{\mathcal{M}}^2_Y$ and assume $M \neq N$. For simplicity, assume that neither M nor N contain any closed components. M determines a partition of the components of Y: two components of Y are in the same part of the partition if they and connected by a component of M. In the same way N also determines a partition of the components of Y. If these two partions differ then $M \cup \overline{N}$ has fewer components than at least one of $M \cup \overline{M}$ and $N \cup \overline{N}$, so the hypothesis of the lemma is satisfied. We assume from now on that the partitions associated to M and N are the same. If $\chi(M) < \chi(N)$, then $C(M \cup \overline{N}) < C(M \cup \overline{M})$ (consider the second component of the complexity tuple), so we assume from now on that $\chi(M) = \chi(N)$. The components of M and N are paired according to their common partition of the components of Y. Since $M \neq N$, there must be at least one pair with differing Euler characteristics. Amongst such components in non-matching pairs, choose the one with lowest Euler characteristic, and assume WLOG that this extremal component belongs to M. It follow that $C(M \cup \overline{N}) < C(M \cup \overline{M})$, and we are done.

The remaining cases of Theorem 2.1 are proved similarly. The most complicated case is $\mathcal{M}^{2,1}$. For $(M, K) \in \dot{\mathcal{M}}^{2,1}$ define $C(M, K)$ to be the lexicographic triple $(C(M), C(K), C(M \setminus K))$, where C is as above for plain 1- and 2-manifolds. □

Remark 2.4. *Because the Turaev-Viro TQFTs do not require orientations it is reasonable to also investigate the universal pairings in the context of unoriented manifolds. In this context, define* $\langle \sum_i a_i M_i, \sum_j b_j N_j \rangle = \sum_{i,j} a_i \overline{b}_j M_i N_j \rangle$. *Theorem 2.1 continues to hold in the unoriented context.*

3 Three dimensional pairings.

This is the most interesting case and we hope it will be the subject of future research. We establish positivity only in a few rather easy cases where all the work is contained in old theorems.

Theorem 3.1. *The pairings* $\mathcal{M}^3_S \times \mathcal{M}^3_S \longrightarrow \mathcal{M}^3$*, S a fixed (possibly empty) union of 2-spheres, and* $\Im^{3,1}_2 \times \Im^{3,1}_2 \longrightarrow \mathcal{L}^{3,1}$*, where 2 denotes two points with opposite orientations, are positive.*

Proof: The essential ingredient in both arguments is the existence and uniqueness of prime decompositions of 3-manifolds [M] and knots [S]. Using this, both cases reduce to the following lemma:

Lemma 3.2. *Consider the polynomial rings $\mathbb{C}[p_i \dots p_n]$ on indeterminates $p_1, \dots, p_n$ and with a fixed antilinear involution* $^-$ *which sends indeterminates to indeterminates. The natural sesquilinear pairing on these rings,*

$$\begin{array}{ccccc} \mathbb{C}[p_1 \dots p_n] & \otimes & \mathbb{C}[p_1 \dots p_n] & \longrightarrow & \mathbb{C}[p_1 \dots p_n] \\ a & \otimes & b & \mapsto & a\overline{b}, \end{array}$$

is formally positive.

Proof: We define the complexity of a monomial as some kind of list of prime powers it contains. If two distinct primes are related by involution we form a lexicographic pair: (sum of the two exponents, the smaller of the two exponents). For primes paired with themselves the pair is simply (exponent, zero). Note that the latter case will arise in the proof of theorem 3.1, since some prime knots (3-manifolds) are diffeomorphic to their arrow reversed (orientation reversed) mirror image.

Now list the pairs in order (padded by $(0,0)$'s) and use this list of pairs to lexicographically order monomials. Suppose $X_i = \prod_k p_{i,k}^{d_{i,k}}$ are monomials, $x = \Sigma a_i X_i$, then the monomials of greatest complexity in $\langle x, x \rangle$ are among the diagonal terms $a_i \overline{a}_i X_i \overline{X}_i = a_i \overline{a}_i \prod_k p_{i,k}^{d_{i,k}} \overline{p}_{i,k}^{d_{i,k}}$. This is easily checked. ☐

We make one further observation:

Theorem 3.3. *Consider the pairing $\mathcal{M}_S^3 \otimes \mathcal{M}_S^3 \longrightarrow \mathcal{M}^3$ where S is a connected* genus $= g$ *surface. Suppose x is a linear combination $\Sigma a_i X_i$, where each X_i is a handle body of* genus $= g$ *(but each attached to S differently), then $\langle x, x \rangle = 0$ implies each $a_i = 0$.*

Proof: The diagonal terms yield $\sum_i a_i \overline{a}_i \left(S^1 \times S^2 \# \cdots \# S^1 \times S^2\right)$, a positive multiple of the double of the genus $= g$ handle body. Could an off diagonal term cancel this contribution? Such an off diagonal term must have $X_i \overline{X}_j$ diffeomorphic to $N_g := \underset{g \text{ copies}}{\#} S^1 \times S^2$, and in fact constitute an *exotic* minimal Heegaard decomposition of N_g (if not exotic, our equivalence relation on bounding manifolds (3) - allows us to write the $X_i \overline{X}_j$ term as an untwisted double). This contradicts a theorem of Haken's [H] which shows that any two Heegaard decompositions of N_g are related by an (orientation preserving) diffeomorphism. ☐

Remark 3.4. *3-manifold topology, in practice, is often two subjects, "compression-theory" and hyperbolic geometry, patched together. Some geometric arguments (about volumes, lengths, and Ricci flow) offer hope for positivity on the hyperbolic side of the subject and Theorem 3.3 offers hope on the compression side. For these reasons we conjecture that the three dimensional pairings are positive.*

4 4-manifold pairings and high dimensions.

For many 3-manifolds S, we can exhibit vectors $x = M - M' \in \mathcal{M}_S^4$ such that

$$\langle x, x \rangle = M\overline{M} - M\overline{M}' + \overline{M}M' + M'\overline{M}' = 0 \in \mathcal{M}^4 = \mathcal{M}_\emptyset^4 \tag{6}$$

In some cases the difference between M and M' is a matter of differentiable structure. (In one prominent example, which has its roots in [Ak], M and M' are both copies of the "Mazur manifold" and $(S = \partial M)$, but M is attached by the "identity" and M' is attached by a diffeomorphism of the boundary which does not extend to a diffeomorphism of the interior (but does extend as a homeomorphism). Line (3) declares M and M' distinct.) However, the central phenomenon is actually less delicate than this example suggests — it does not depend on an unstable phenomenon in dimension $= 4$. For the most natural choice: $\{$3-manifold$= S$, $\partial M = S = \partial M'$, $x = M - M' \neq 0$, yet $\langle x, x \rangle = 0\}$, we will find that $M \times T^k - M' \times T^k =: x \times T^k \neq 0$ and of course $\langle x \times T^k, x \times T^k \rangle = 0$, where T^k is the k-torus for all $k \geq 0$. We begin with this natural example.

Fundamental Construction: Let N be the result of plumbing two copies of $S^2 \times D^2$ along two positive and one negative points. (In coordinates z_i near a point of S_i^2 and w_i on D_i^2, $i = 1$ and 2, $(+)$-plumbing identifies $(z_1, w_1) \longleftrightarrow (w_2, z_2)$ and $(-)$-plumbing identifies $(z_1, w_1) \longleftrightarrow (w_2, \overline{z}_2)$). A schematic picture of N is shown in figure 2.

A more detailed graphical description of N is given in figure 3, using the Kirby [K] calculus notation. In this notation, 1-handles are denoted as loops with dots, while undotted loops denote 2-handles.

We may draw the 2-handles in Figure 3 round and obtain figure 4(a). From there, we get Figure 4(b) by drawing each 1-handle curve as the boundary of a punctured torus created by plumbing two cylinders. For this plumbing, deform one cylinder along the arc, twist 90° to the right until it

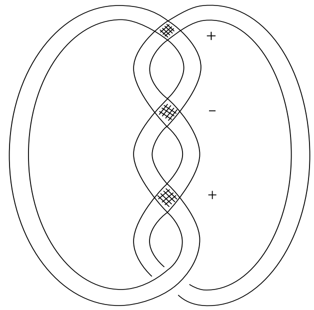

Figure 2: A schematic picture of N

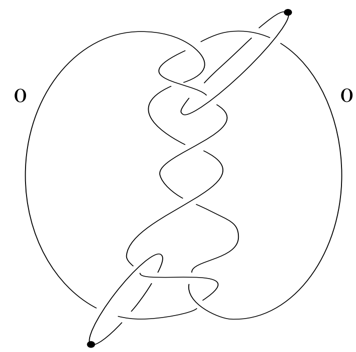

Figure 3: A picture of N in Kirby [K] calculus notation. The dotted loops are 1-handles, the other loops are 0-framed 2-handles.

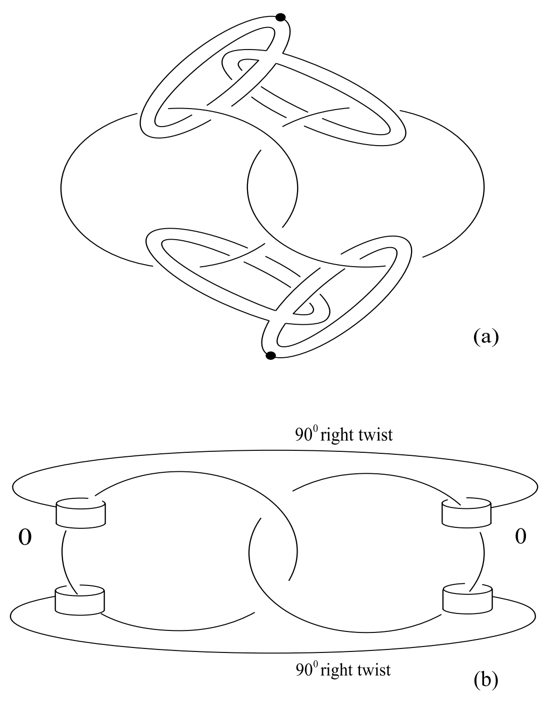

Figure 4: **(a)** Another picture of N in Kirby calculus notation. **(b)** The same with each 1-handle drawn as the boundary of a punctured torus.

reaches the other cylinder. Of course the picture is and remains symmetric under a $180°$-rotation about the center.

Define $M(M')$ as the result of surgery on the left (right) 2-sphere in N. By the symmetry, M is diffeomorphic to M' but this diffeomorphism when restricted to the boundary (an involution ϕ) is *not* isotopic to the identity (as we will prove) so M and M' are (by (3)) distinct basis elements of $\mathcal{M}^4_S$, $S = \partial M$.

Lemma 4.1. *The following four* 4*-manifolds are diffeomorphic to* $S^3 \times S^1 \# S^3 \times S^1$*:*

$$M \cup \overline{M},\ M \cup \overline{M}',\ M \cup \overline{M},\ \overline{M}' \cup \overline{M}'.$$

We have the immediate corollary:

Corollary 4.2. $\langle M - M', M - M' \rangle = 0 \in \mathcal{M}^4$ *(we will prove soon that* $M \neq M' \in \mathcal{M}^4_s$*).*

Proof of Lemma 4.1. Doubling N yields $D(N)$ with a handle decomposition: one 0-handle, two 1-handles, four 2-handles, two 3-handles and one 4-handle. The 3-handles and 4-handles are canonical and do not need to be drawn; the interesting part of $D(N)$ is the 1-handles and 2-handles, drawn in Figure 5 below.

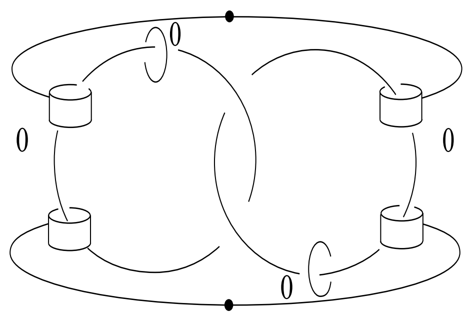

Figure 5: Kirby Calculus picture of the 3-handles and 4-handles of $D(N)$.

After two 2-handle slides which algebraically yield:

$$\begin{array}{c|cccc} & a & b & \widehat{a} & \widehat{b} \\ a & 0 & 1 & 1 & 0 \\ b & 1 & 0 & 0 & 1 \\ \widehat{a} & 1 & 0 & 0 & 0 \\ \widehat{b} & 0 & 1 & 0 & 0 \end{array} \rightsquigarrow \begin{array}{cccc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{array} \rightsquigarrow \begin{array}{c|cccc} & a & b & c & d \\ a & 0 & 1 & 0 & 0 \\ b & 1 & 0 & 0 & 0 \\ c & 0 & 0 & 0 & -1 \\ d & 0 & 0 & -1 & 0 \end{array}$$

we arrive at an alternative handle decomposition for $D(N)$, shown in Figure 6.

The four manifolds listed in Lemma 4.1. are created from $D(N)$ by surgery on $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, and $\{b, d\}$ respectively. In all 4 cases the 2-handles disappear from the diagram and the left and right columns of cylinders are linked twice: once positively and once negatively (note the differing sign of the two Hopf links in Figure 6 and how it is consistent with the final intersection matrix above). The two linking operations cancel (geometrically) and the resulting four handle decompositions are identical and contain no 2-handles (see figure 7 for this decomposition). They describe $S^3 \times S^1 \# S^3 \times S^1$.

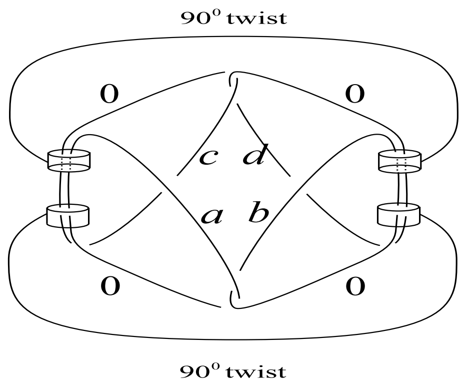

Figure 6: Alternative handle decomposition for $D(N)$

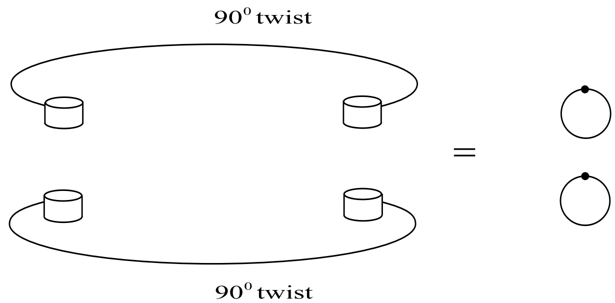

Figure 7: Handle diagram after surgery.

Now we come to the question: How do we know $M \neq M' \in \mathcal{M}^4_{\partial M}$? One answer is that simply connected h-cobordant 4-manifolds are known to differ by cutting out M and gluing back M' (or by such a replacement of their generalization coming from an N made from more hyperbolic pairs and more double points). From Donaldson theory we know many distinct but h-cobordant 4-manifolds. This line of thought has a certain merit and we will pursue it briefly even though it obscures the important point that the underlying phenomenon is stable and will persist into higher dimensions (even though these 5-dimensional h-cobordisms become products when crossed with a circle).

From [CFWS] and [Ma] we know that in the middle level of any 1-connected h-cobordism $\left(W^5; P^4, Q^4\right)$, the ascending and descending 2-spheres can be engulfed in a compact manifold $\widetilde{N}$ with homology sphere boundary $S = \partial \widetilde{N}$, so that surgery on the ascending (descending) spheres produces a contractible $\widetilde{M}$ $\left(\widetilde{M}'\right)$. Thus $P = C \cup_S \widetilde{M}$ and $Q = C \cup_S \widetilde{M}'$, $C = \overline{\text{mid level} \setminus \widetilde{N}}$. It can further be arranged (Kirby, private communication 1996 and [Ma]) that $\widetilde{M} \cup \widetilde{M} \cong \widetilde{M} \cup \widetilde{M}' \cong \widetilde{M}' \cup \widetilde{M} \cong \widetilde{M}' \cup \widetilde{M}' \cong S^4$. (We have dropped the reverse of orientation from our notation.) We conclude:

Theorem 4.3. UTQFTs cannot distinguish h-cobordant 4-manifolds. In fact even more general unitary "theories" where Atiyah's gluing axiom is only enforced along homology 3-spheres will likewise be unable to distinguish h-cobordant 4-manifolds.

Proof: In the proceeding notation, if $P = C \cup_S \widetilde{M}$, $Q = C \cup_S \widetilde{M}'$ then the partition functions: $Z(P) = \langle Z(C), Z(\widetilde{M})\rangle = \langle Z(C), Z(\widetilde{M}')\rangle = Z(Q)$, where $Z(C), Z(\widetilde{M}), Z(\widetilde{M}') \in V(S)$, the physical Hilbert space of S. This is because $Z(\widetilde{M}) = Z(\widetilde{M}') \in V(S)$ since $\widetilde{M} \cong \widetilde{M}'$ in the universal space $\mathcal{M}^4_S$ of which $V(S)$ is a quotient. □

If one returns to the context of full UTQFTs, where gluing must hold along any 3-manifold, then we have the following refinement.

Theorem 4.4. If $(W; P, Q)$ is an integral homology h-cobordism in which $\pi_1 P \longrightarrow \pi_1 W$ and $\pi_1 Q \longrightarrow \pi_1 W$ are surjective then P and Q cannot be distinguished by a UTQFT.

Proof: Standard 5-dimensional Morse theory reduces the problem to the study of $(N; M, M')$ as originally defined and their generalizations involving more hyperbolic pairs of ascending - descending spheres with numerous additional double points which cancel over $\mathbb{Z}$. Computations similar to those above show:

$$M\overline{M} \cong M\overline{M}' \cong M'\overline{M} \cong M'\overline{M}' \cong S^3 \times S^1 \# \ldots \# S^3 \times S^1.$$

□

We conclude this section with an example of $(W; P, Q)$ as above with $P \neq Q$, and where P and Q can be distinguished by fundamental group so that the distinction will be stable under product with the k-torus T^k.

Example: Mid level =: $L = S^3 \times S^1 \# S^2 \times S^2$ let a and b be the factor 2-spheres. Let a be the ascending sphere of W and let the descending sphere d be an imbedded representative of $(2t - 1)b \in \pi_2(L)$, where t is a generator of $\pi_1(L) \cong \mathbb{Z}$. So $P \cong L \diagdown \text{surger}(a)$ and $Q \cong L \diagdown \text{surger}(d)$. Since $2t - 1$ is not a unit of $\mathbb{Z}[t, t^{-1}]$, d has no (immersed) dual sphere. To see this, note that the intersection form on L is a single hyperbolic pair $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ over $\mathbb{Z}[\mathbb{Z}]$ in the $\{a, b\}$-basis. The intersection number of d with the general class $\alpha a + \beta b$ is $\langle d, \alpha a + \beta b \rangle = (2t - 1)\beta \neq 1$, so d has no dual sphere. Thus c, the small linking circle is essential in $L \diagdown d$ and $\pi_1(L \diagdown d) \cong \pi_1(Q)$. Although c is essential it bounds an interesting complex J = 3-punctured sphere with boundary components identified (with 2 plus and 1 minus orientation), see Figure 8.

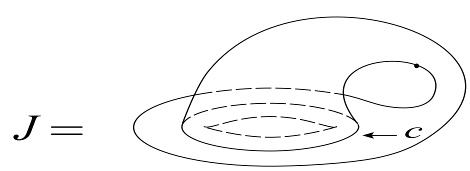

Figure 8.

J has an infinite cyclic cover homeomorphic to the space depicted in Figure 9.

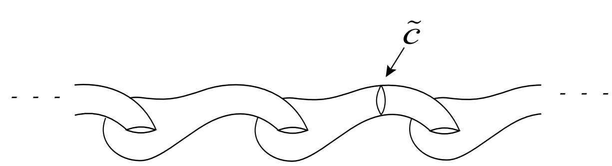

Figure 9: Infinite cyclic cover of J

From this figure, it is clear that $c \in (\pi_1 Q)_\alpha$, the α-term of the lower central series for every ordinal α. On the other hand $P \cong S^3 \times S^1$ so $\pi_1 P \cong \mathbb{Z}$ and $(\pi_1 P)_2 = \{e\}$. Thus $P \ncong Q$ and for all k. If we cross with the k-torus T^k, $P \times T^k \neq Q \times T^k$, the former manifold having trivial commutator subgroup of π_1 and the latter having nontrivial α-term all α. This proves:

Theorem 4.5. For N as in Figure 3 and 4 let $S = \partial N$. For $k = 0, 1, 2, 3, \ldots$ the manifold pairings

$$\mathcal{M}^{4+k}_{S\times T^k} \times \mathcal{M}^{4+k}_{S+T^k} \longrightarrow \mathcal{M}^{4+k}$$

are non positive.

5 Problems

Problem 1a: Given $x \in \mathcal{M}^d_S$ with $\langle x, x\rangle \neq 0$, is it possible to construct a UTQFT which assigns to x a nonzero vector $\tilde{x} \in V_S$?
Problem 1b: Similarly, given $x \in \mathcal{M}^d_S$ such that all d-dimensional UTQFTs assign the zero vector to x, does it follow that $\langle x, x\rangle = 0$?

Problem 2: Is there a 3-manifold S and a nonzero vector $x \in \mathcal{M}^4_S$ such that $\langle x, y\rangle = 0$ for all $y \in \mathcal{M}^4_S$?
As motivation for this question, note that $\langle x, x\rangle = 0 \in \mathcal{M}^4$ implies that any UTQFT will assign the value 0 to $\langle x, y\rangle$ for all y.

Problem 3: Find out if $\langle x, x\rangle = 0$ for $x = M - M'$ with M, M' homology equivalent elements of $\mathcal{M}^d_S,\ (d \geq 4)$.

Problem 4: In dimensions ≥ 5 characterize the zero loci of these universal manifold quadratic forms.
Problem 4a: Are there any elements in these loci that have an odd number of manifolds with non-zero coefficients?
Even numbers of nonzero coefficients may be obtained using manifolds which are disjoint unions of the examples given already

Problem 5: Analyze the 3-dimensional pairings.

Problem 6: Consider other coefficients besides $\mathbb{C}$.

This is almost certainly of interest, because there exist classes of UTQFTs whose invariants take values in rings other than $\mathbb{C}$. Clearly the pairings will never be positive in any dimension for coefficient rings with elements x with satisfy $x\bar{x} = 0$, but nevertheless, even for such rings, a characterisation of the nullity may be interesting. For example, in the ring $(\mathbb{Z}/7\mathbb{Z})[\omega]$ with $\omega = e^{2\pi i/7}$ and the involution given by extension of $\bar{1} = 1$ and $\bar{\omega} = \omega^{-1}$, there are elements x for which $x\bar{x} = 0$, but using the Milnor sphere of order 7, one can construct an example of a linear combination of manifolds in $\mathcal{M}_{S^6}$ which does not have such elements as coefficients and which does pair to $0 \in \mathcal{M}^7$.

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