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Constraining cold dark matter halo merger rates using the coagulation equations

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ABSTRACT

We place additional constraints on the three parameters of the dark matter halo merger rate function recently proposed by Parkinson, Cole & Helly by utilizing Smoluchowski's coagulation equation, which must be obeyed by any binary merging process which conserves mass. We find that the constraints from Smoluchowski's equation are degenerate, limiting to a thin plane in the three-dimensional parameter space. This constraint is consistent with those obtained from fitting to *N*-body measures of progenitor mass functions, and provides a better match to the evolution of the overall dark matter halo mass function, particularly for the most massive haloes. We demonstrate that the proposed merger rate function does not permit an exact solution of Smoluchowski's equation and, therefore, the choice of parameters must reflect a compromise between fitting various parts of the mass function. The techniques described herein are applicable to more general merger rate functions, which may permit a more accurate solution of Smoluchowski's equation. The current merger rate solutions are most probably sufficiently accurate for the vast majority of applications.

Key words: gravitation – cosmology: theory – dark matter – large-scale structure of Universe.

1 INTRODUCTION

In current cosmological theory, the mass density of the Universe is dominated by dark matter. The most successful model of structure formation is that based upon the concept of cold dark matter (CDM). In the CDM hypothesis, dark matter particles interact only via the gravitational force. Since the initial distribution of density perturbations in these models has greatest power on small scales, the first objects to collapse and form dark matter haloes are of low mass. Larger objects form through the merging of these smaller sub-units. Consequently, the entire process of structure formation is thought to proceed in a 'bottom-up', hierarchical manner.

Clearly then, the rate of dark matter halo mergers is an absolutely crucial ingredient in models of galaxy and large-scale-structure formation, from sub-galactic scales to galactic and galaxy-cluster scales. The Press–Schechter (PS) formalism (Press & Schechter 1974) has long provided a simple, intuitive and surprisingly accurate formula for the distribution of halo masses at a given redshift over a large range of mass scales and for a vast variety of initial power spectra.

An elegant paper by Lacey & Cole (1993) – and similar work by Bond et al. (1991) and Bower (1991) – extended the work of Press and Schechter to determine the rate at which haloes of a given mass merge with haloes of some other mass. In addition to providing valuable physical insight, these merger rates have extraordinary practical value, having been applied to galaxy-formation models, e.g. if galaxy morphologies are determined by the merger history (Gottlober, Klypin & Kravtsov 1991); active galactic nuclei (AGN) activity (Wyithe & Loeb 2003); models for Lyman-break galaxies (Kolatt et al. 1999); abundances of binary supermassive black holes (SMBHs) (Volonteri, Haardt & Madau 2002); rates for SMBH coalescence (Milosavljevic & Merritt 2001) and the resulting Laser Interferometer Space Antenna event rate (Haehnelt 1994; Menou, Haiman & Narayanan 2001); the first stars (Santos, Bromm & Kamionkowski 2002; Scannapieco, Schneider & Ferrara 2003); galactic-halo substructure (Kamionkowski & Liddle 2000; Bullock, Kravtsov & Weinberg 2000; Benson et al. 2002; Somerville 2002; Stiff, Widrow & Frieman 2001); halo angular momenta (Vivitska et al. 2002) and concentrations (Wechsler et al. 2002); galaxy clustering (Percival et al. 2003); particle acceleration in clusters (Gabici & Blasi 2003); and formation-redshift distributions for galaxies and clusters and thus their distributions in size, temperature, luminosity, mass and velocity (Verde et al. 2001; Verde, Haiman & Spergel 2002).

However, as first noted by Lacey & Cole (1993, see their section 3.1) and as we demonstrated in Benson, Kamionkowski & Hassani (2005, hereafter Paper I), these merger rates are fundamentally flawed. The extended-Press–Schechter (ePS) formulae for merger rates are mathematically self-inconsistent, providing *two*

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different results for the same merger rate, as was also pointed out by Santos et al. (2002), which are increasingly discrepant for larger mass ratios. This ambiguity will be particularly important for, for example, understanding galactic substructure and for SMBH-merger rates (Erickcek, Benson & Kamionkowski 2006). Even the smaller numerical inconsistency for mergers of nearly equal mass may be exponentially enhanced during the repeated application of the formula while constructing merger trees to high redshift. Moreover, the ambiguity calls into question the entire formalism, even when the two possibilities seem to give similar answers quantitatively.[1] Recently, Neistein & Dekel (2008b) have described an alternative derivation of merger rates from ePS theory and an implementation which appears to be self-consistent. Our goal here, however, is not to find a merger rate kernel which reproduces the results of ePS since that theory does not reproduce results found in *N*-body simulations. Instead, we aim to find a merger kernel which agrees well with *N*-body results and correctly evolves mass functions designed to fit the halo mass function measured from *N*-body simulations.

In Paper I, we discussed the mathematical requirements of a self-consistent theory of halo mergers. As already recognized (Silk & White 1978; Cavaliere, Colafrancesco & Menci 1992; Sheth 1995; Sheth & Pitman 1997), the merger process is described by Smoluchowski's coagulation equation. Indeed, several authors have employed Smoluchowski's equation in precisely this way (Cavaliere, Colafrancesco & Menci 1991, 1992; Cavaliere & Menci 1993; Menci et al. 2002). This equation simply says that the rate at which the abundance of haloes of a given mass changes is determined by the difference between the rate for creation of such haloes by mergers of lower-mass haloes and the rate for destruction of such haloes by mergers with other haloes. The correct expression for the merger rate must be one that yields the correct rate of evolution of the halo abundance when inserted into the coagulation equation. The problem is thus to find a merger rate, or 'kernel', that is consistent with the evolution of the halo abundance, either the PS abundance or one of its more recent *N*-body-inspired variants (Sheth, Mo & Tormen 2001; Jenkins et al. 2001).

The apparent simplicity of the mathematical problem is in fact quite deceptive. The Smoluchowski coagulation equation is in fact an infinite set of coupled non-linear differential equations. The equation appears in a variety of areas of science – e.g. aerosol physics, phase separation in liquid mixtures, polymerization, star formation theory (Silk & Takahashi 1979; Allen & Bastien 1995), planetesimals (Wetherill 1990; Lee 2000; Malyshkin & Goodman 2001), chemical engineering, biology and population genetics – so there is a vast but untidy literature on the subject (although see Leyvraz 2003 for an illuminating review). It has been studied a little by pure and applied mathematicians (Aldous 1999). Still, solutions to the coagulation equation are poorly understood. Furthermore, there is virtually no literature on the problem we face: i.e. how to find a merger kernel that, when inserted into the coagulation equation, yields the desired halo mass distribution and its evolution as a solution.

We will now discuss in more detail two approaches to solving Smoluchowski's equation in the cosmological context. The first, ultimately unsuccessful, approach follows from Paper I while the second is a new approach which forms the basis of the present work.

[1] ePS theory discusses the trajectories of points in the primordial density field as the smoothing scale is reduced. It is the association of such trajectories with halo masses, which is not necessarily well-defined (see e.g. Porciaini, Dekel & Hoffmann 2002 who find that only 40 per cent of proto-halo regions contain peaks of the density field), that leads to these problems with the derived merger rates.

1.1 Direct solution of Smoluchowski's equation

In Paper I, we solved Smoluchowski's equation numerically subjected to two additional physically motivated constraints (or regularization conditions). First, the merger kernel should be positive for all masses[2] and, second, the merger kernel should be a smooth function of its arguments. The second regularization condition was imposed by minimizing the second derivatives of the kernel with respect to the two arguments. This allowed us to find unique solutions to Smoluchowski's equation for several power-law power spectra. These solutions were in reasonable agreement with the limited *N*-body data available for comparison.

We have attempted to extend the work carried out in Paper I to the physically interesting case of a CDM power spectrum, using a much improved calculation able to span a wider dynamic range of halo masses. We have found this approach to be unsuccessful in finding a solution which is consistent with constraints from *N*-body simulations. However, we discuss in some detail our attempts to find a direct solution in Appendix B in the hope that they will provide a useful starting point for other researchers wishing to explore this problem further.

1.2 Parametric solutions to Smoluchowski's equation

Until such time as improved physical understanding of the physics governing the merger kernel is uncovered, we have adopted a different approach to finding a suitable merger kernel. Recently, several authors have used the Millennium Simulation (MS) to provide constraints on either the merger rate (Fakhouri & Ma 2008) or the progenitor mass functions of dark matter haloes (Neistein & Dekel 2008a). In particular, Parkinson, Cole & Helly (2008, hereafter PCH08) utilized progenitor mass functions to constrain the parameters of a function used to modify the standard ePS merger rate function and thereby constructed a merger-tree binary split algorithm which accurately matched those progenitor mass functions. This approach is successful, but is limited by the accuracy and extent of the *N*-body data. In this paper, we demonstrate how we can use Smoluchowski's equation to provide an additional constraint on such algorithms. This approach is powerful for the following reasons.

(i) The PCH08 merger-tree algorithm is a binary split algorithm and so should obey Smoluchowski's equation.

(ii) Constraining the three free parameters of the PCH08 algorithm is much easier than attempting to numerically invert Smoluchowski's equation subject to complicated regularization conditions.

(iii) Smoluchowski's equation applies over all halo masses, while even the MS has a limited dynamical range of masses that it can probe.

The disadvantage of this approach is that it assumes a particular functional form for the merger kernel (with free parameters that are to be fit). There is no guarantee that this functional form will permit a solution to Smoluchowski's equation for any values of its free

[2] A negative merger rate could be considered to describe a spontaneous fission process, but such processes should have a different dependence on halo abundances and so require the inclusion of additional terms in Smoluchowski's equation. We ignore such processes in this work.

parameters and, in fact, we will show that it does not. Nevertheless, we can find the best fit to Smoluchowski's equation. Further free parameters could be introduced, of course, which should result in improved agreement with Smoluchowski's equation.

The remainder of this paper is arranged as follows. In Section 2, we describe Smoluchowski's equation as used in this work and, in particular, how it is applied to the construction of merger trees. In Section 3, we present the results of fitting the parameters of the PCH08 model to *N*-body progenitor mass functions and to Smoluchowski's equation and examine how these influence the evolution of the dark matter halo mass function. Finally, in Section 4, we examine the implications of these results.

2 SMOLUCHOWSKI'S EQUATION AND MERGER TREES

In this section, we will describe the techniques used in this work and the specific implementations used to construct progenitor mass functions and evolving dark matter halo mass functions.

Smoluchowski's equation describes the changing distribution of 'masses'[3] of objects growing through coagulation. Given a distribution, $n(M;\, t)$, of object masses M at time t, Smoluchowski's equation gives the rate of change of this distribution:

$$\dot{n}(M;t) = \int_0^{M/2} Q(M-M', M';t)n(M-M';t)n(M';t)\mathrm{d}M' - \int_0^{\infty} Q(M, M';t)n(M;t)n(M';t)\mathrm{d}M', \quad (1)$$

where $Q(M_1, M_2; t)$ encodes the merger rate between objects of mass M_1 and M_2 at time t. It is, therefore, symmetric in its arguments, i.e. $Q(M_1, M_2;\, t) = Q(M_2, M_1;\, t)$. The first term in equation (1) represents creation events, while the second represents destruction events. Only three forms for $Q(M_1, M_2;\, t)$ are known to permit analytic solutions of Smoluchowski's equation: $Q(M_1, M_2;\, t) = k$ (where k is a constant), $Q(M_1, M_2;\, t) = k(M_1 + M_2)$ and $Q(M_1, M_2;\, t) = kM_1M_2$. In this work, the objects we consider are dark matter haloes, and the mass is the total mass of those haloes. The second analytic solution is of particular interest as it corresponds to the solution for dark matter haloes in a Universe with a $P(k) = k^n$ power spectrum when $n = 0$. The other two analytic solutions are not cosmologically interesting.

It is convenient to work with scaled variables for mass and time. As shown in Paper I, the natural time variable is $\tau = -\ln[\delta_c)(t)]$ where $\delta_c(t)$ is the extrapolated linear theory overdensity required for halo collapse in the spherical top-hat collapse model (as appears in the PS expression for the halo mass function). We also choose to express masses in units of the characteristic mass scale M_*, defined such that $\sigma(M_*[t]) = \delta_c(t) \equiv \exp(-\tau)$ where $\sigma^2(M)$ is the fractional mass variance of the linear density field extrapolated to $z = 0$. With these choices, Smoluchowski's equation becomes

$$y(\mu;\tau) = \int_0^{\mu/2} q(\mu-\mu', \mu';\tau)n(\mu-\mu';\tau)n(\mu';\tau)\mathrm{d}\mu' - \int_0^{\infty} q(\mu, \mu';\tau)n(\mu;\tau)n(\mu';\tau)\mathrm{d}\mu', \quad (2)$$

where $\mu = M/M_*(t)$, $n(\mu)$ is the distribution of halo masses, $y(\mu) = \mathrm{d}n(\mu)/\mathrm{d}\tau$ and $q(\mu_1, \mu_2;\tau)$ is the merger rate function in these new variables. For the specific case of power-law power spectra (which are, of course, scale-free), there is no explicit time-dependence with this choice of units, and we can write

$$y(\mu) = \int_0^{\mu/2} q(\mu-\mu', \mu')n(\mu-\mu')n(\mu')\mathrm{d}\mu' - \int_0^{\infty} q(\mu, \mu')n(\mu)n(\mu')\mathrm{d}\mu'. \quad (3)$$

Thus, a single solution, valid at all times, can be found for power-law power spectra. For CDM power spectra, which are not scale free, the merger kernel can in principle depend explicitly on time (although the choice of scaled variables will minimize this dependence). In principle, therefore, we could use Smoluchowski's equation at each point in time (τ) to provide constraints on the merger rate function. We retain the variable choices μ and τ as we expect the solutions $q(\mu_1, \mu_2;\, \tau)$ to be only slowly changing with τ in these variables, since the CDM power spectrum is close to a power law over a wide range of masses. In practice, we will use only the $\tau = \tau_0$ (i.e. present day) Smoluchowski equation to constrain the parameters of the merger kernel, although our methods could be easily applied to other redshifts if required. We will, however, utilize this same $z = 0$ merger kernel at higher redshifts when constructing dark matter halo merger trees. This is, of course, slightly inconsistent, but is consistent with the approach of PCH08 and is in the spirit of a first-order approximation to the merger kernel.

It is well known that the PS mass function is not a good description of the mass functions found in *N*-body simulations of CDM structure formation (Jenkins et al. 2001; Sheth et al. 2001). Our methods are applicable to any mass function $n(\mu)$ and so we can replace the PS mass function with a fitting formula such as that from Sheth et al. (2001)

$$n_{\mathrm{SMT}}(\mu;\tau) = \sqrt{\frac{2}{\pi}} A_{\mathrm{SMT}} \left| \frac{\mathrm{d}\ln\sigma(\mu)}{\mathrm{d}\ln\mu} \right| \frac{x'(\mu,\tau)}{\mu^2} \exp[-x'^2(\mu,\tau)/2] \times (1 + 1/x'^{2q_{\mathrm{SMT}}}), \quad (4)$$

where $x'(\mu, \tau) = \sqrt{a_{\mathrm{SMT}}}x(\mu, \tau)$, $x(\mu, \tau) = \exp(-\tau)/\sigma(\mu)$, $a_{\mathrm{SMT}} = 0.707$, $q_{\mathrm{SMT}} = 0.3$ and $A_{\mathrm{SMT}}(\approx 0.3222)$ is chosen such that the mass density in haloes equals the mean density of the Universe.

Smoluchowski's equation involves the time derivative of this function, which is given by

$$y_{\mathrm{SMT}}(\mu,\tau) = n_{\mathrm{SMT}}(\mu,\tau)\left[x'^2(\mu,\tau) + \frac{2q_{\mathrm{SMT}}}{1 + x'^{2q_{\mathrm{SMT}}}(\mu,\tau)} - 1\right]. \quad (5)$$

As shown in Paper I, the standard ePS theory predicts a merger kernel

$$Q_{\mathrm{ePS}}(M_1, M_2;t) = \frac{M_2^2}{\rho_0\sigma_{\mathrm{f}}}\frac{\sigma_2}{M_{\mathrm{f}}}\left|\frac{\dot{\delta}_c}{\delta_c}\right| \left|\frac{\mathrm{d}\ln\sigma_{\mathrm{f}}}{\mathrm{d}\ln M_{\mathrm{f}}}\right| \left|\frac{\mathrm{d}\ln\sigma_2}{\mathrm{d}\ln M_2}\right|^{-1} \times \frac{1}{\left(1-\sigma_{\mathrm{f}}^2/\sigma_1^2\right)^{3/2}} \times \exp\left[-\frac{\delta_c^2}{2}\left(\frac{1}{\sigma_{\mathrm{f}}^2} - \frac{1}{\sigma_2^2} - \frac{1}{\sigma_1^2}\right)\right], \quad (6)$$

where we have adopted the notation $\sigma_2 = \sigma(M_2)$, etc. This is, of course, asymmetric in the two mass arguments. In scaled variables, this becomes

$$q_{\mathrm{ePS}}(\mu_1, \mu_2;\tau) = \frac{\mu_2^2}{\sigma_{\mathrm{f}}}\frac{\sigma_2}{\mu_{\mathrm{f}}} \left|\frac{\mathrm{d}\ln\sigma_{\mathrm{f}}}{\mathrm{d}\ln\mu_{\mathrm{f}}}\right| \left|\frac{\mathrm{d}\ln\sigma_2}{\mathrm{d}\ln\mu_2}\right|^{-1} \times \frac{1}{(1-\sigma_{\mathrm{f}}^2/\sigma_1^2)^{3/2}} \times \exp\left[-\frac{\mathrm{e}^{-2\tau}}{2}\left(\frac{1}{\sigma_{\mathrm{f}}^2} - \frac{1}{\sigma_2^2} - \frac{1}{\sigma_1^2}\right)\right], \quad (7)$$

[3] The equation applies to any additive quantity conserved through the coagulation process.

where, as shown in Paper I, the mean cosmic density, ρ_0, is removed through an appropriate choice of units.

When computing the mass variance, $\sigma(M)$, we use the same power spectrum as used to generate the initial conditions for the MS since we will fit to conditional mass functions measured from that simulation. This power spectrum is integrated under a spherical top-hat real-space window function to obtain $\sigma(M)$.

2.1 Application to merger trees

A primary goal of this study is to provide accurate merger rates for use in the construction of dark matter halo merger trees. 'Accurate' here is defined to mean that the merger trees so constructed should produce the correct distribution of progenitor halo masses at earlier times and, consequently, produce the correct evolution of the halo mass function. Ideally, this evolution should remain precise even when trees are constructed over large fractions of cosmic history. The need for such accurate trees is highlighted by the work of Benson et al. (2006) (see their fig. 9), who were forced to restart their calculations of reionization at periodic intervals in redshift due to accumulating inaccuracies in their merger trees. Constructing merger trees also provides a means of comparing our results for the merger rate with measurements from *N*-body simulations. In this section, therefore, we will describe how we construct merger trees.

2.1.1 Merger-tree construction algorithm

We adopt the algorithm described by Cole et al. (2000) and PCH08 to construct merger trees. This algorithm considers binary splits of the merger tree with a correction for accretion of mass in haloes below the mass resolution of the tree and uses adaptive time-steps to ensure that the probability of non-binary mergers remains small. As Cole et al. (2000) note, their algorithm performs well due to the subtle choice of drawing a progenitor mass from the lower half of the ePS progenitor mass distribution function (i.e. for a parent halo of mass M_0 they use only the $<M_0/2$ region of the progenitor mass function when computing the probability of a binary split). It is interesting to realize that this choice effectively symmetrizes Smoluchowski's merger kernel as

$$q_{\rm ePS,sym}(\mu_1,\mu_2) = \begin{cases} q_{\rm ePS}(\mu_1,\mu_2) & \text{if} \quad \mu_1 \leq (\mu_1+\mu_2)/2 \\ q_{\rm ePS}(\mu_2,\mu_1) & \text{if} \quad \mu_1 > (\mu_1+\mu_2)/2. \end{cases} \quad (8)$$

Other algorithms have been proposed for constructing merger trees (Kauffmann & White 1993; Cole et al. 1994; Somerville & Kolatt 1999) – we select Cole et al. (2000) because it allows only binary mergers and is therefore consistent with a treatment via Smoluchowski's equation. We refer the reader to Cole et al. (2000) and PCH08 (their appendix A) for a full description of the merger-tree construction algorithm, but address the aspects of the implementation unique to the present work below.

Cole et al. (2000) compute two quantities, $F(\mu, \tau)$ and $P(\mu, \tau)$, which give the fraction of mass gained through accretion and the probability of a binary split, respectively, for a halo of mass μ at time τ and during some time period $\delta\tau$. In terms of quantities used in this work, the parameters $F(\mu, \tau)$ and $P(\mu, \tau)$ are given by

$$F(\mu,\tau) = \delta\tau \int_0^{\mu_{\rm min}} \mu' R(\mu';\tau) {\rm d}\mu' \quad (9)$$

and

$$P(\mu,\tau) = \delta\tau \int_{\mu_{\rm min}}^{\mu/2} R(\mu';\tau) {\rm d}\mu', \quad (10)$$

where

$$R(\mu',\tau) = \frac{n(\mu';\tau) n(\mu-\mu';\tau) q(\mu',\mu-\mu';\tau)}{n(\mu;\tau)}, \quad (11)$$

$\mu_{\rm min} = M_{\rm min}/M_*(\tau)$ and $M_{\rm min}$ is the mass resolution of the merger tree. We have expressed this rate in terms of the merger kernel in Smoluchowski's equation. PCH08 give expressions for F and P in terms of the ePS expression for the mean number of progenitors of mass M_1 expected in a small time-step. The two ways of expressing these functions are equivalent – we use the version depending on the merger kernel to make clear the connection to Smoluchowski's equation as used in this work. We adaptively choose time-steps $\delta\tau$ to ensure that $F \ll 1$ and $P \ll 1$ to keep the possibility of multiple fragmentation small and to ensure that the halo mass can change by only a small amount due to accretion in any given time-step.

Using the PS mass function for $n(\mu;\tau)$ and the ePS merger rate $q_{\rm ePS}(\mu_1, \mu_2;\tau)$, equation (7) recovers the expressions given by Cole et al. (2000) for F and P. Equations (9) and (10) are more general however, allowing any mass function $n(\mu;\tau)$ and merger rate function $q(\mu_1, \mu_2;\tau)$ to be used. In particular, PCH08 propose that $R(\mu';\tau)$ be modified by multiplying by a function

$$G(\sigma_1/\sigma_{\rm f}, \delta_{\rm c}(\tau)/\sigma_{\rm f}) = G_0[\sigma_1/\sigma_{\rm f}]^{\gamma_1}[\delta_{\rm c}(\tau)/\sigma_{\rm f}]^{\gamma_2}, \quad (12)$$

where G_0, γ_1 and γ_2 are free parameters. In terms of Smoluchowski's equation, we can similarly multiply the ePS merger kernel by the same function $G(\sigma_1/\sigma_{\rm f}, \delta_{\rm c}(\tau)/\sigma_{\rm f})$. We will call this modified merger kernel $q'_{\rm ePS}$ and label such kernels as (G_0, γ_1, γ_2). However, we must also account for the fact that the PCH08 algorithm implicitly utilizes the PS mass function, while we actually want to reproduce the evolution of the Sheth et al. (2001) mass function and must therefore use that when solving Smoluchowski's equation. Appendix A describes how to correctly implement Smoluchowski's kernel in this case. We note that this form is approximately independent of time τ once expressed in terms of the dimensionless mass μ (it would be exactly independent of time for a scale-free power spectrum). A generalization of equation (12) could include an explicit dependence on τ to capture any required time dependence. Since we expect this dependence to be weak (since it would be non-existent for a scale-free power spectrum), we choose not to include it here. Higher accuracy merger kernel determinations, if required, should consider including such a dependence.

Utilizing the above algorithm, we can, given a halo of mass $\mu_{\rm f}$ at the present day, construct a merger tree back to some earlier time. By constructing a large number of such trees and averaging over their progenitors, we can construct an estimate of the progenitor mass function at that earlier time.

2.1.2 Evolving the halo mass function

Later (see Section 3), we will want to study the evolution of the halo mass function, $n(\mu;\tau)$, as predicted by Smoluchowski's equation given a particular merger kernel. To do this, we begin by drawing a sample of dark matter halo masses at random from that mass function at some initial time τ_1 corresponding to $z = 0$. We use a quasi-random method (specifically a Sobol sequence; Press et al. 2007, section 7.7) to produce a non-uniform sample of masses while minimizing fluctuations in the mass function due to random sampling. We impose a minimum and maximum number of haloes to simulate from each decade of halo mass to ensure good sampling of the entire mass range of interest, and span a large enough range of initial masses to ensure that our results are unaffected by the necessarily limited range of masses probed.

We then apply the methods described above to evolve these haloes backwards in time until a time τ_2 is reached. Summing the progenitor halo masses of all of the τ_1 haloes allows us to construct the halo mass function at τ_2. This can then be compared to the expected mass function $n_{\rm SMT}(\mu;\tau_2)$ which should be matched if our merger kernel is correct.

2.2 Constraining the merger kernel

In this section, we will describe the two methods that we employ to constrain the free parameters of the merger kernel: G_0, γ_1 and γ_2.

2.2.1 *Constraints from progenitor mass functions*

We wish to construct progenitor mass functions to compare to those determined by Cole et al. (2008) from the MS. Cole et al. (2008) estimated progenitor mass functions of $z = 0$ haloes by locating all $z = 0$ haloes with a mass within a factor of $\sqrt{2}$ of some mass $M_{\rm f}$ and then finding all progenitors of those haloes at redshifts 0.5, 1, 2 and 4. To compare to these N-body data,[4] we first produce a sample of halo masses M between $M_{\rm f}/\sqrt{2}$ and $\sqrt{2}M_{\rm f}$ drawn from the Sheth et al. (2001) mass function. We then construct a merger tree for each such halo back to $z = 4$ and cumulate the progenitor halo masses to estimate a mean conditional mass function.

We construct enough trees that our comparison is limited by noise in the N-body results and not by the limited number of trees constructed. We then determine a χ^2 statistic using

$$\chi^2 = \sum \left[\frac{f^{\rm MS}_{\rm cmf}(M_1|M_{\rm f}) - f^{\rm MC}_{\rm cmf}(M_1, M_{\rm f})}{\sigma^{\rm MS}_{\rm cmf}(M_1|M_{\rm f})} \right]^2, \quad (13)$$

where $f_{\rm cmf}(M_1|M_{\rm f})$ is the conditional mass function of haloes of mass M_1 which are progenitors of a halo of mass $M_{\rm f}$ at $z = 0$, and superscripts refer to the MS and Monte Carlo (MC) merger trees constructed using our algorithm. The values of $\sigma^{\rm MS}_{\rm cmf}(M_1|M_{\rm f})$ used are determined from the number of progenitor haloes in the MS assuming Poisson statistics. The sum is taken over all redshifts for which we have progenitor data from the MS (namely $z = 0.5$, 1, 2 and 4) with equal weight applied to each redshift [the actual contribution from the higher redshifts will be automatically down-weighted due to the smaller number of progenitors present at those times and the consequently larger values of $\sigma^{\rm MS}_{\rm cmf}(M_1|M_{\rm f})$].[5] This process is repeated for different values of the parameters G_0, γ_1 and γ_2.

2.2.2 *Constraints from Smoluchowski's equation*

Given a merger kernel, $q'_{\rm ePS}$, we can, for any combination of parameters (G_0, γ_1, γ_2), evaluate the creation and destruction integrals in Smoluchowski's equation and thereby determine $y(\mu)$ for that merger kernel. In general, a merger kernel will not provide a precise solution to Smoluchowski's equation. We can measure the ability of any given merger kernel to solve Smoluchowski's equation by evaluating the mean absolute difference between the resulting $y(\mu)$ and that found by direct differentiation of the mass function with respect to time (which is, of course, the required solution to give the correct evolution of the mass function). Therefore, we compute

$$\langle d \rangle = \frac{1}{N} \sum_{i=1}^{N} \left| \frac{y(\mu_i) - y_{\rm SMT}(\mu_i)}{n_{\rm SMT}(\mu_i)} \right|, \quad (14)$$

where $y_{\rm SMT}(\mu)$ is the rate of change of the Sheth et al. (2001) mass function, and we average the absolute error over $N = 16$ different masses, μ_i, equally spaced in $\log\mu$ between $\mu = 10^{-5}$ and 10^3. It is important to note that equation (14) is evaluated at $z = 0$ only – as noted in Section 2 the merger kernel will be a function of redshift in general when we have a CDM power spectrum, but we have chosen to ignore this dependence for the present work. We have chosen to weight by $1/n_{\rm SMT}(\mu)$ so that we are comparing the rate of change of the mass function per halo. The choice of how to judge any merger kernel's degree of success in solving Smoluchowski's equation is somewhat subjective – the above choice ensures that we require the solution to be most accurate where the time-scale for change in the mass function is most rapid (i.e. for the most massive haloes).

3 RESULTS

We find best-fitting parameters of (G_0, γ_1, γ_2) = (0.605, 0.375, −0.115) and (0.775, 0.275, −0.225) for fits to the MS conditional mass functions and Smoluchowski's equation, respectively. Our best-fitting parameters for matching conditional mass functions differ slightly from those obtained by PCH08 due to our choice to weight our goodness-of-fit measure by the inverse Poisson errors on the N-body conditional mass functions. As expected, both constraints favour $G_0 < 1$ (which reduces the overall merger rate) and $\gamma_1 > 0$ (which boosts the ratio of low-mass to high-mass progenitors) – PCH08 discuss the reasons for these expectations. Fig. 1 shows constraints in the (G_0, γ_1) plane (i.e. on slices of fixed γ_2 in the three-dimensional parameter space), for both fitting to conditional mass functions (blue contours) and Smoluchowski's equation (red contours). In the left- and right-hand panels, we show results corresponding to the best-fitting values of γ_2 for conditional mass functions and Smoluchowski's equation, respectively.

The conditional mass functions provide a good constraint on (G_0, γ_1) with a degree of degeneracy between the parameters. Smoluchowski's equation shows a much larger degeneracy between these two parameters and, in fact, favours regions which lie in a plane in the full 3D parameter space. This plane is, however, thin, thereby strongly ruling out large regions of the parameter space. Remarkably, the degeneracy seen in fitting Smoluchowski's equation is similar to that from fitting conditional mass functions. More importantly, the constraints from both methods intersect, permitting solutions which both match the conditional mass functions and are reasonable solutions to Smoluchowski's equation. This is an important point, as it implies that N-body merger trees are consistent with a binary merger hypothesis.

We could, in principle, combine these two constraints to obtain an overall best-fitting model. Such a procedure is, however, somewhat arbitrary. First, while we have a statistically meaningful (in the sense that we can use it to assign relative probabilities to models) measure of goodness-of-fit for the fits to conditional mass functions, our goodness-of-fit to Smoluchowski's equation is not statistically meaningful. Any combination of the two constraints therefore requires some amount of judgement as to their relative importance. Furthermore, the (G_0, γ_1, γ_2) modification to extended PS merger

[4] Available from http://star-www.dur.ac.uk/∼cole/merger_trees/MS_data/.

[5] We reiterate here that we are adopting a redshift-independent merger kernel (i.e. G_0, γ_1 and γ_2 are assumed to be independent of redshift) even though this is not mathematically guaranteed for a CDM power spectrum. This is consistent with the work of PCH08.

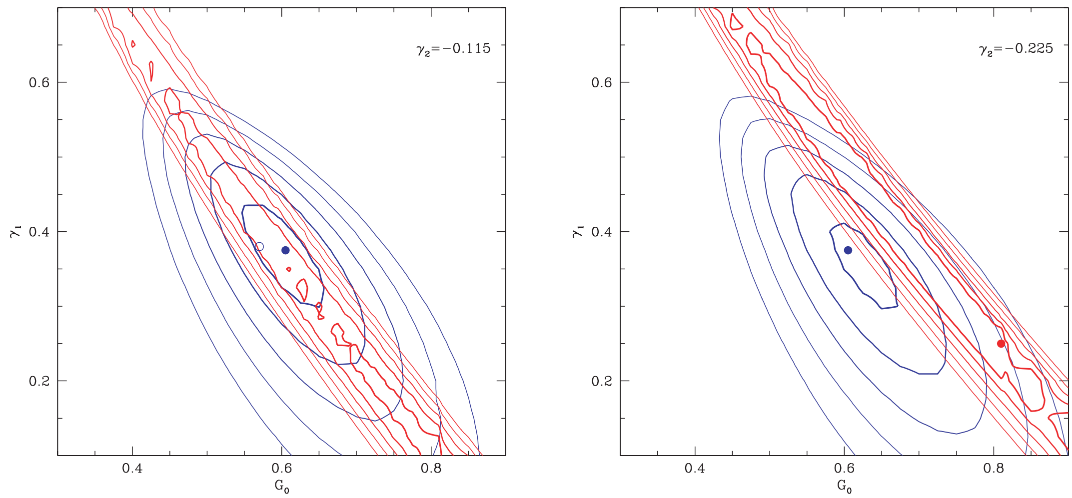

Figure 1. Constraints in the $G_0 - \gamma_1$ parameter plane for two values of $\gamma_2(\gamma_2 = -0.125$ in the left-hand panel, corresponding to the best fit to MS conditional mass functions; $\gamma_2 = -0.225$ in the right-hand panel corresponding to the best solution of Smoluchowski's equation). The solid blue dot shows the best-fitting values obtained by fitting to progenitor mass functions measured from the MS, while blue contours show the constraints obtained from this data set. The red point shows the best-fitting values obtained by fitting to Smoluchowski's equation, while the red contours map locii of constant mean error between $y(\mu)$ and the fitted value as defined in equation (14). The constraints shown correspond to a slice through the 3D parameter space at constant γ_2. The open blue circle shows the constraint obtained by PCH08 using the same MS data set. This differs slightly from our equivalent constraint due to our choice to utilize the measurement errors in the progenitor mass functions in our fit. Nevertheless, the two are quite similar as expected. (Note that PCH08 found a best-fitting value of $\gamma_2 = -0.01$.)

rates does not give a 'good-fit' to conditional mass functions (it is certainly a much better fit than unmodified ePS, but clear systematic differences from the *N*-body data remain) and does not precisely solve Smoluchowski's equation.

It is, however, obvious that any choice of a 'best-fitting' model to both constraints must lie somewhere in the plane defined by Smoluchowski's equation between the red and blue dots in Fig. 1. We will illustrate below the differences in results when these two best-fitting values are obtained – any combined best fit should show intermediate behaviour.

It is instructive to examine how well our various models solve Smoluchowski's equation. Fig. 2 shows the difference between the rate of change of the mass function per halo as a function of halo mass determined from our merger kernels and the value required for a correct solution of Smoluchowski's equation. The standard ePS merger kernel (green line) fails significantly over the entire range of masses plotted (particularly so at the massive end). The blue lines indicate results using merger kernels constrained to match the conditional mass functions from the MS. Over the range where these conditional mass functions provide a good measure of the merger kernel (approximately $0.001 < \mu < 10$), the fitted merger kernels provide a significantly better solution to Smoluchowski's equation than does the unmodified ePS kernel. Note also that our fit (solid blue line) and that of PCH08 (dotted blue line) are very similar over this range. At $\mu \gtrsim 10$, the MS conditional mass functions provide only weak constraints on the merger kernel and, consequently, the kernels fitted to these functions provide worse solutions to Smoluchowski's equation in this regime (although still better than the original ePS kernel). Finally, the red line shows results from the kernel which provides the best solution to Smoluchowski's equation. 'Best' here is in the sense defined by equation (14), which tries to solve Smoluchowski's equation most accurately where the rate of change of the mass function per halo is the largest. Not surprisingly, therefore, this gives the most accurate solution for massive haloes ($\mu \gtrsim 1$), but actually gives a worse solution to Smoluchowski's equation at low

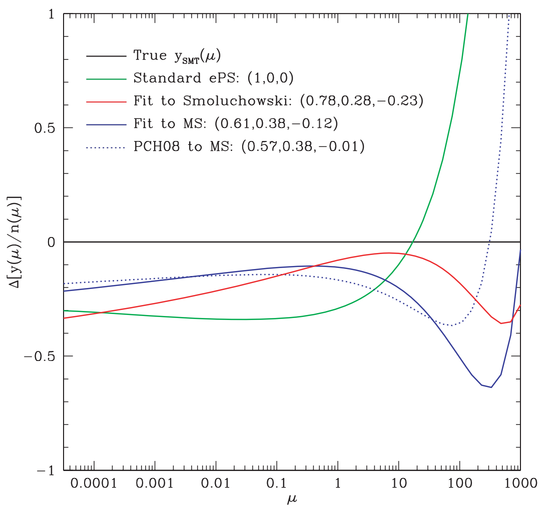

Figure 2. The rate of change of the halo mass function per halo, $y(\mu)/n(\mu)$, relative to that obtained by direct differentiation of the Sheth et al. (2001) mass function (the horizontal black line indicates the true rate of change expressed in this way). Coloured lines show this quantity computed using Smoluchowski's equation using various merger kernels, i.e. using q'_{ePS} with different parameters $(G_0, \gamma_1, \gamma_2)$. The green line shows the result of using the standard ePS kernel, (1, 0, 0), while the red line shows the results for the merger kernel which best matches the true rate of change per halo as judged by equation (14). The solid blue line shows results for the merger kernel that best fits progenitor mass functions in the MS. The dotted blue line shows the same using the fit parameters of PCH08. For all calculations, we use the Sheth et al. (2001) mass function in Smoluchowski's equation.

masses than do kernels fit to conditional mass functions. This simply reflects the relative importance given to different mass ranges by each constraint and the fact that the particular functional form of the merger kernel chosen does not permit a precise solution to

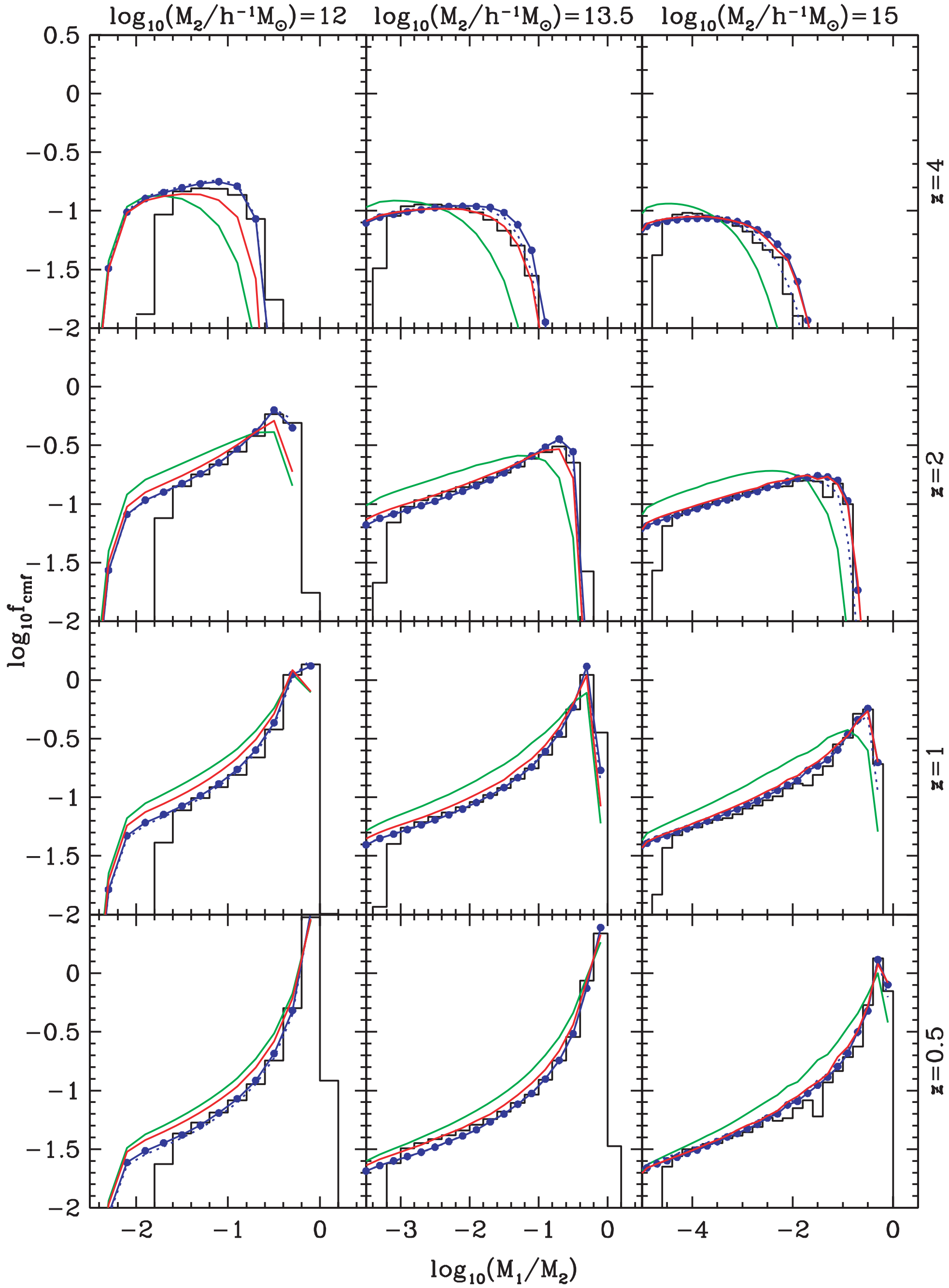

Figure 3. Comparison of progenitor mass functions measured from the MS (black histograms; taken from Cole et al. 2008) and those constructed using merger trees for a variety of merger kernels, (G_0, γ_1, γ_2). The green lines show results for the standard ePS merger kernel, (1, 0, 0), while the red line shows results using the merger kernel which best solves Smoluchowski's equation. The solid blue curve is the best fit to the MS found in this work, while the dotted blue line uses the best fit from PCH08. For all calculations, we use the Sheth et al. (2001) mass function in Smoluchowski's equation.

Smoluchowski's equation, thereby forcing a compromise solution to be adopted.

Fig. 3 (which has the same format as fig. 1 of PCH08) shows conditional mass functions from the MS (black histograms) for three different final halo mass ranges and for four different redshifts. Overplotted are conditional mass functions estimated from merger trees built using different merger kernels. The failure of the unmodified ePS kernel (green lines) is readily apparent, while blue lines – which use kernels constrained by fitting to these same conditional mass functions in this work (solid lines) and PCH08 (dotted lines) – are vastly improved matches as expected. The red line indicates progenitor mass functions obtained using the merger kernel which best solves Smoluchowski's equation. This is clearly intermediate in success between an unmodified ePS kernel and kernels constrained to match the conditional mass functions, as may be expected. (There are cases where it performs significantly better, however, for example, for the $z = 4$ conditional mass function of $10^{13.5}\ h^{-1}\ \mathrm{M}_{\odot}$ haloes.)

Finally, in Fig. 4 we show total (i.e. not conditional) mass functions of haloes from $z = 0$ to 4. The left-hand panel shows this

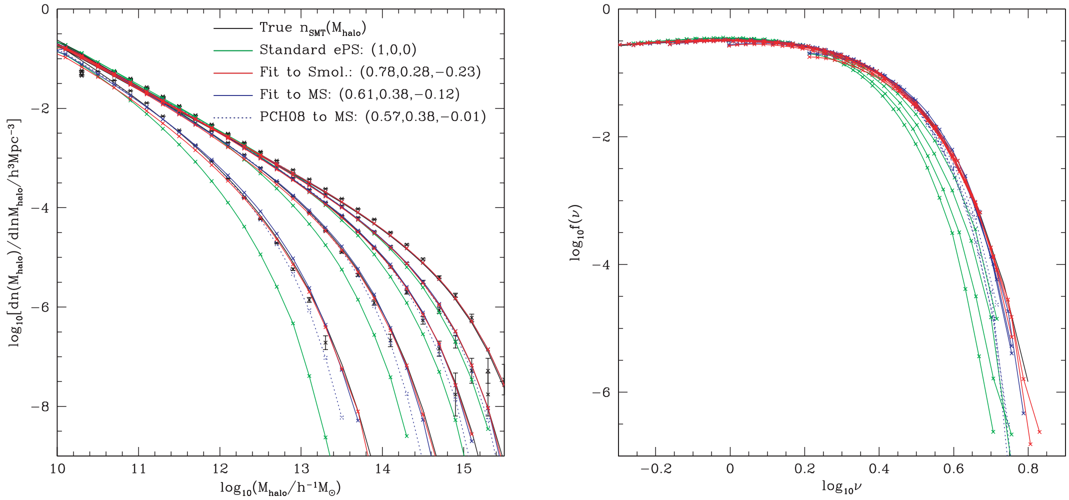

Figure 4. Left-hand panel: the dark matter halo mass function shown at $z = 0$, 0.5, 1, 2 and 4 (from right- to left-hand side). The analytic Sheth et al. (2001) mass function is shown by black lines while black crosses with error bars show the mass function measured directly from the MS (interpolated to the required redshift where necessary). Coloured lines show the result of evolving the mass function from $z = 0$ to higher redshifts using merger trees with different merger kernels, $(G_0, \gamma_1, \gamma_2)$. The green lines show results for the standard ePS merger kernel, (1,0,0), while the red lines show results using the merger kernel which best solves Smoluchowski's equation. The solid blue lines are for the best fit to the MS conditional mass functions found in this work, while the dotted blue lines use the best fit to conditional mass functions from PCH08.

function in physical units, while the right-hand panel shows the fraction of mass in haloes with $\nu = \delta_c(z)/\sigma(M)$ per logarithmic interval of ν (cf. fig. 4 of PCH08). In this latter form, the Sheth et al. (2001) mass function is redshift independent. We probe to very high masses and very low abundances to illustrate the importance of high-accuracy merger kernels when considering rare objects. The unmodified ePS merger kernel performs poorly, quickly resulting in a mass function shifted to low masses. The kernel with parameters identified by PCH08 performs much better, but also begins to underpredict the abundance of the most mass haloes at high redshift. The kernel using parameters obtained in this work by fitting conditional mass functions performs extremely well, remaining remarkably close to the expected Sheth et al. (2001) mass function out to $z = 4$, although statistically significant differences can be detected. The kernel with parameters selected to best solve Smoluchowski's equation (red line) also performs remarkably well. In particular, it is the most successful at matching the evolution of the most massive haloes in the mass function, as would be expected from Fig. 2. It performs somewhat worse than the kernel fit to conditional mass functions at low masses as also expected. We note that the Sheth et al. (2001) mass function is not a perfect fit to the *N*-body data, particularly at high redshifts. Improved fitting formulae (such as that proposed by Reed et al. 2007) could be used with our methods to provide more accurate constraints from Smoluchowski's equation. This, together with a consideration of the need for a redshift-dependent kernel, will be the focus of future work.

4 DISCUSSION

We have described how Smoluchowski's equation, which governs any mass-conserving binary coagulation process, may be used to provide constraints on halo merger rates. These constraints are complementary to those obtained by fitting to conditional mass functions from *N*-body simulations since they span a wide dynamical range of halo masses. Using these methods, and the modified ePS algorithm described by PCH08, we have identified merger kernels which best fit conditional mass functions from the MS and which best solve Smoluchowski's equation. While the functional form of the PCH08 merger rate does not permit an exact solution of Smoluchowski's equation, we are able to find solutions which greatly improve the accuracy of merger trees. Using these best-fitting solutions, we are able to evolve the dark matter halo mass function from $z = 0$ to 4 with remarkably high accuracy, particularly for the most massive objects.

Ideally, we would identify a functional form for the merger kernel which permits a precise solution to Smoluchowski's equation, while simultaneously producing progenitor mass functions consistent with the available *N*-body data. In fact, a perfect solution should agree with *N*-body data when compared using any statistic (e.g. distributions of most massive or second most massive progenitors). In reality, we do not know of such a function and, as mentioned in Section 1, only a handful of analytic solutions to Smoluchowski's equation are known. In that case, the search for the 'best' kernel requires making some decision about what are the most important statistics to fit and accepting that some degree of compromise in matching them is unavoidable. We believe that the most important statistics to match are the evolution of the overall mass function (which is extremely well constrained from *N*-body simulations) and progenitor mass functions. When considering the process of structure and galaxy formation, it is these statistics which control, to first order, the number of galaxies able to form at a given point in cosmic history and how those galaxies were formed.

Practically, these modified merger rates should prove extremely useful in constructing high-accuracy merger trees for use in studies of structure formation, reionization and galaxy formation. If even greater accuracy is required, a different functional form for the merger kernel must be adopted[6] and the techniques described in this work applied to constrain its parameters. As noted in Section 1, only a handful of analytic solutions to Smoluchowski's equation are

[6] The functional form of PCH08's modification of the merger kernel can be considered to be the first-order terms in a Taylor expansion of ln G – adding higher-order terms would be straightforward.

known, none of which provides useful solutions for dark matter halo merger rates in CDM Universes. Nevertheless, a sufficiently general parametrization of the merger kernel should permit arbitrarily accurate solutions to Smoluchowski's equation. If used in a binary split merger-tree algorithm, such a kernel should allow for an arbitrarily accurate evolution of the halo mass function (limited only by the accuracy of the merger-tree construction algorithm).

This work does not provide any further physical insight into dark matter halo merger rates – at least not directly. One might hope that physical insight into the 'micro-physics' (to make an analogy with other areas in which Smoluchowski's equation is applied, e.g. polymer growth) of dark matter halo merging might point towards a functional form for the kernel. Until such insight is gained, the methods described herein provide a practical method for a rapid construction of high-accuracy merger trees.

ACKNOWLEDGMENTS

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APPENDIX A: RELATION BETWEEN MERGER-TREE SPLIT PROBABILITIES AND SMOLUCHOWSKI EQUATION MERGER KERNEL

Smoluchowski's equation is normally employed to take an existing distribution of masses and evolve it forward in time subject to a specified merger kernel. Both creation and destruction of haloes must be considered in this case. However, we can also apply Smoluchowski's equation to the reverse process, taking a distribution of haloes of specified mass and evolving them backwards in time. In this case, we have a series of mass-conserving binary splits reminiscent of merger trees. The split probability is related to the creation term in Smoluchowski's equation.

PCH08 describe a merger-tree binary split algorithm which utilizes a modification of the ePS split probability distribution. Here, we wish to cast that same algorithm in the terms used in Smoluchowski's equation.

Assuming a Sheth et al. (2001) mass function, Smoluchowski's equation is

$$y(\mu) = \frac{1}{2}\int_0^{\mu} n_{\rm SMT}(\mu')n_{\rm SMT}(\mu-\mu')q(\mu',\mu-\mu')\mathrm{d}\mu' - \int_0^{\infty} n_{\rm SMT}(\mu)n_{\rm SMT}(\mu')q(\mu,\mu')\mathrm{d}\mu', \qquad \text{(A1)}$$

where $n_{\rm SMT}(\mu)$ is the Sheth et al. (2001) mass function and $y(\mu)$ is the rate of change of that mass function as predicted by Smoluchowski's equation.

Applying Smoluchowski's equation in reverse, we find that in a binary split algorithm, the split probability distribution for a single halo of mass μ to have a progenitor of mass μ' (and, therefore, a second progenitor of mass $\mu - \mu'$) is given by

$$P_{\rm split}(\mu';\mu) = \frac{n_{\rm SMT}(\mu')n_{\rm SMT}(\mu-\mu')}{n_{\rm SMT}(\mu)} q(\mu', \mu-\mu'). \tag{A2}$$

The PCH08 split probability function is that derived from the PS mass function and ePS techniques:

$$P_{\rm split}(\mu';\mu) = \frac{n_{\rm PS}(\mu')n_{\rm PS}(\mu-\mu')}{n_{\rm PS}(\mu)} q'_{\rm ePS}(\mu', \mu-\mu'), \tag{A3}$$

where $q'_{\rm ePS}(\mu_1, \mu_{\rm f}-\mu_1) = q_{\rm ePS}(\mu_1, \mu_{\rm f}-\mu_1)G(\sigma_1/\sigma_{\rm f}, \delta_{\rm f}/\sigma_{\rm f})$ and $G(\sigma_1/\sigma_{\rm f}, \delta_{\rm f}/\sigma_{\rm f})$ is PCH08's multiplicative modifier of merging rates.

From this, we can deduce that the rate at which systems of mass μ_1 and μ_2 merge together is just

$$\begin{aligned} R(\mu_1,\mu_2) &= n_{\rm SMT}(\mu_1+\mu_2)P_{\rm split}(\mu_1;\mu_1+\mu_2) \\ &= \frac{n_{\rm SMT}(\mu_1+\mu_2)}{n_{\rm PS}(\mu_1+\mu_2)} n_{\rm PS}(\mu_1)n_{\rm PS}(\mu_2)q'_{\rm ePS}(\mu_1,\mu_2). \end{aligned} \tag{A4}$$

Therefore,

$$\begin{aligned} y(\mu) = {} & \frac{n_{\rm SMT}(\mu)}{n_{\rm PS}(\mu)} \frac{1}{2}\int_0^{\mu} n_{\rm PS}(\mu')n_{\rm PS}(\mu-\mu')q'_{\rm ePS}(\mu',\mu-\mu'){\rm d}\mu' \\ & - \int_0^{\infty} \frac{n_{\rm SMT}(\mu+\mu')}{n_{\rm PS}(\mu+\mu')} n_{\rm PS}(\mu)n_{\rm PS}(\mu')q'_{\rm ePS}(\mu,\mu'){\rm d}\mu', \end{aligned} \tag{A5}$$

or, in terms of the rate of change per halo,

$$\begin{aligned} \frac{y(\mu)}{n_{\rm SMT}(\mu)} = {} & \frac{1}{n_{\rm PS}(\mu)} \left[\frac{1}{2}\int_0^{\mu} n_{\rm PS}(\mu')n_{\rm PS}(\mu-\mu')q'_{\rm ePS}(\mu',\mu-\mu'){\rm d}\mu' \right. \\ & \left. - \int_0^{\infty} \frac{n_{\rm PS}(\mu)n_{\rm SMT}(\mu+\mu')}{n_{\rm SMT}(\mu)n_{\rm PS}(\mu+\mu')} n_{\rm PS}(\mu)n_{\rm PS}(\mu')q'_{\rm ePS}(\mu,\mu'){\rm d}\mu' \right]. \end{aligned} \tag{A6}$$

This is the form of Smoluchowski's equation that we utilize throughout this work.

APPENDIX B: ATTEMPTS TO SOLVE SMOLUCHOWSKI'S EQUATION DIRECTLY

In this Appendix, we describe our attempts to directly solve Smoluchowski's equation for the case of CDM power spectra utilizing methods similar to those described in Paper I.

In Paper I, we proceeded by discretizing Smoluchowski's equation and representing the merger rate function $q(\mu_1, \mu_2; \tau)$ by a set of values on a grid which we then interpolated between to find the value at any given (μ_1, μ_2). We chose a grid of $N_\mu \times N_\mu$ points equally spaced in μ between some upper value $\mu_{\rm max}$ and $\mu_{\rm max}/N_\mu$, and interpolated between them using a simple interpolation method such that in the interval (μ_i, μ_j) to (μ_{i+1}, μ_{j+1}) the merger rate was given by

$$q(\mu_1,\mu_2) = q(\mu_i,\mu_j) + \partial q_1\mu_1 + \partial q_2\mu_2 + \partial q_{12}\mu_1\mu_2, \tag{B1}$$

with the coefficients ∂q_1, ∂q_2 and ∂q_12 chosen such that the interpolation matches the values at the four grid-points defining the interval. This particular interpolation choice has the advantage that it allows an exact solution for the special case $n = 0$ for which $q(\mu_1, \mu_2) \propto \mu_1 + \mu_2$.

The disadvantage of the particular discretization used in Paper I is that it permits only a relatively small dynamic range in halo masses to be explored,[7] while we know that the greatest discrepancy between the two ePS predictions occurs for large mass ratios. Furthermore, in constructing halo merger trees we need to know the merger rate function over a large range of halo masses. Consequently, for the present work we chose to represent $q(\mu_1, \mu_2; \tau)$ on a grid of points equally spaced in ln (μ). This has the advantage of permitting a much larger dynamic range of halo masses for relatively small N_μ. The disadvantage of this approach is that it significantly complicates the computations required. In particular, the discretization used in Paper I allows explicit cancellation of the divergent parts of the creation and destruction integrals in Smoluchowski's equation, while the present discretization does not. Care must therefore be taken to ensure that these terms cancel to sufficient precision to permit accurate results to be obtained.

The discretized Smoluchowski equation is written in a simple matrix form

$$\boldsymbol{y} = \mathsf{K}\cdot\boldsymbol{q}. \tag{B2}$$

The vector $\boldsymbol{q}$ has $N_\mu(N_\mu + 1)/2$ components, corresponding to the independent elements of the $N_\mu \times N_\mu$ symmetric array $q(\mu_i, \mu_j)$, while the matrix K has dimensions of $N_\mu \times N_\mu(N_\mu + 1)/2$. The kernel matrix K is found by numerical integration of the creation and destruction terms in Smoluchowski's equation, including the interpolating factors described above (see Paper I for a full discussion of K).

This matrix equation can then be solved by a suitable inversion method. As noted in Paper I, the equation is ill-defined (with more unknowns than constraints) and therefore has no unique solution. We are therefore forced to adopt regularization conditions which impose mathematical and physical constraints on the solution.

In practice, we avoid using the contributions from near the extremes of the tabulated mass range, since the discrete representation of Smoluchowski's equation becomes inaccurate in these regions. We therefore ignore the first and last ΔN_μ terms in equation (B2) when solving for $\boldsymbol{q}$.

B1 Regularization Condition(s)

As shown in Paper I, inversion of Smoluchowski's equation is ill-defined – there are an infinite number of solutions $q(\mu_1, \mu_2;\tau)$ which satisfy equation (2) for any given power spectrum. The vast majority of these will be unphysical, with the merger rate fluctuating wildly as a function of halo mass and containing regions of negative merger rate. We must therefore apply some physically motivated criteria to weed out these unphysical solutions.

The simplest regularization condition is to demand that all elements of $\boldsymbol{q}$ be positive as a negative merger rate does not make physical sense. This constraint, namely $q_{i,j} > 0$ for all (i, j), is non-linear and therefore prevents a solution of equation (B2) being found using linear algebra techniques. While the methods of quadratic programming are in principle well-suited to this problem, we have found that in general a simple matrix-inversion yields a solution $\boldsymbol{q}$ which is everywhere positive.[8] We therefore use linear algebra to find a solution to equation (B2) and simply check that the solution is everywhere positive before proceeding.

[7] The dynamic range is set by N_μ, which is in turn limited by the computational resources available. In Paper I, we were able to achieve $N_\mu = 179$ before exhausting our available computational resources (in terms of both memory and CPU time).

[8] Exceptions to this rule occur when our solution is affected by numerical issues. In such cases, the solution is, of course, not useful anyway.

In Paper I, we adopted a regularization condition which aimed to make the merger rate function $q(\mu_1, \mu_2;\tau)$ smooth in the sense of having the small second derivatives with respect to μ_1 and μ_2. This is clearly a valid condition for the $n = 0$ case (for which these second derivatives are zero everywhere), although it cannot be justified rigorously for other power spectra. This condition imprints the physical constraint that the merger rate should be a smooth function of its arguments (i.e. there is no reason to think that the merger rate should be a rapidly oscillating function of mass).

It is important to note that while this regularization condition does remove all unphysical solutions, it also removes an infinite number of physically reasonable solutions since there are an infinite number of solutions which are smoothly varying as a function of μ_1 and μ_2. This regularization condition will pick out not merely a smooth solution, but also the *smoothest*. For this reason, we have included in the present work an additional regularization condition which makes use of the fact that the ePS merger rate theory is not completely wrong. In fact, ePS merger rates can be used to build merger trees which are statistically similar (although not identical) to merger trees found in *N*-body simulations (Somerville et al. 2000). We therefore seek solutions to Smoluchowski's equation which are both smooth and similar to the ePS expectation for the merger rate function.

Specifically, we include two regularization conditions, the first of which (imposing smoothness) is defined by

$$R_1^2 = \frac{1}{\sigma_{R_1}^2}\left[\int_0^\infty \int_0^\infty \left[\partial^2 q/\partial\mu_1^2\right]^2 + \left[\partial^2 q/\partial\mu_2^2\right]^2 \mathrm{d}\mu_1 \mathrm{d}\mu_2\right], \quad \text{(B3)}$$

where σ_{R_1} is an adjustable parameter and the second of which is defined by

$$R_2^2 = \frac{1}{\sigma_{R_2}^2}\sum_i\sum_j\left[\frac{q_{ij} - q_{\mathrm{ePS,sym}}(\mu_i, \mu_j)}{q_{\mathrm{ePS,sym}}(\mu_i, \mu_j)}\right]^2, \quad \text{(B4)}$$

where σ_{R_2} is a second adjustable parameter.

To find a regularized solution, we want to minimize

$$\sum_i \sigma_i^{-2}\left(y_i - \sum K_{ij}q_j\right)^2 + R_1^2 + R_2^2 \quad \text{(B5)}$$

with respect to all q_i. The σ_i are included to permit different weights to be assigned to the contribution from each y_i. We choose $\sigma_i = |y_i|$. Since the regularization terms can each be written as

$$R^2 = \sum_i\sum_j r_{ij}q_i q_j, \quad \text{(B6)}$$

taking the partial derivative of equation (B5) with respect to each q_i results in a set of linear equations for $\boldsymbol{q}$ which can be solved using simple matrix inversion. The regularization conditions are evaluated using the discretized merger rate function and the interpolation algorithm defined above to find the r_{ij}.

B2 Finding $\mu_{\max}$

Before attempting to solve Smoluchowski's equation numerically, we must determine suitable values of $\mu_{\max}$. The value of $\mu_{\max}$ must meet two requirements:

(i) it should be sufficiently large to encompass all halo masses that we are likely to care about in physical applications;

(ii) it should be sufficiently large that the loss of the contribution to the destruction term from masses between $\mu_{\max}$ and ∞ is negligible.

To quantify the second of these constraints we perform a direct integration of Smoluchowski's equation using the symmetrized ePS merger kernel, $q_{\mathrm{ePS,sym}}$, with an upper limit of $\mu_{\max}$ on the destruction integral. We expect the value of $\mu_{\max}$ to be of greatest importance when considering halo masses close to μ_0, where $y(\mu_0) = 0$, as here the destruction term must precisely cancel the creation term.[9] We thus compute $y(\mu)$ for several values of $\mu_{\max}$ and determine at what value of $\mu_{\max}$ we obtain a converged solution for $y(\mu)$, paying particular attention to the region around μ_0.

A value of $\mu_{\max} = 3000$ was found to be sufficient to provide a converged result and will be adopted from hereon.

B3 Discussion of results

The direct solution algorithm above has been successfully implemented and employed to solve Smoluchowski's equation at $z =$ 0, 0.5 and 1. We have checked that the resulting merger kernels do satisfy Smoluchowski's equation (with the accuracy adopted we found that Smoluchowski's equation was typically solved to within a few parts in 10^5). By construction therefore, these merger kernels correctly evolve the mass function over a wide redshift range with a very high accuracy – that is, they reproduce the expected evolution of $n_{\mathrm{SMT}}(\mu;\tau)$. However, these solutions show a significant failing, namely that when we use them to compute progenitor mass functions of haloes of a given mass there are large and systematic offsets from *N*-body determinations of progenitor mass functions. In fact, they typically perform worse in this respect than the standard ePS theory. The sense of the discrepancy varies with halo mass (both final halo and progenitor halo mass to be specific).

Thus, while we have been able to obtain solutions to Smoluchowski's equation for such power spectra, we find that they *do not* correspond to the merger rates that occur in *N*-body simulations of structure formation, which we consider to provide the 'correct' solution for our purposes. Specifically, while our solutions are correct solutions of Smoluchowski's equation, they do not reproduce the progenitor mass functions of haloes as measured from *N*-body simulations. This again highlights the fact that there is more than one set of progenitor mass functions which give the correct evolution of the overall halo mass function, and our direct solver did not find the correct one.

This is, perhaps, not surprising as our results are controlled by the regularization conditions applied. We required solutions to be smooth, which seems reasonable, but our minimization is forced to pick the *smoothest* solution in order to find a unique answer. It seems, therefore, that the Universe uses a smooth merger rate, but not the smoothest. Trials with alternative regularization conditions have, as yet, failed to find any more successful approach. Physical insight is needed at this point to determine the correct regularization conditions which any solution must obey.

[9] Note that while we would expect $\mu_0 = 1$ for all power spectra assuming a PS mass function, we will in fact find $\mu_0 \neq 1$ since we are using a Sheth et al. (2001) mass function and, in any case, $q_{\mathrm{ePS,sym}}$ is not actually a solution of Smoluchowski's equation.

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