LIMIT GROUPS FOR RELATIVELY HYPERBOLIC GROUPS, II: MAKANIN-RAZBOROV DIAGRAMS

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ABSTRACT. Let Γ be a torsion-free group which is hyperbolic relative to a collection of free abeian subgroups. We construct Makanin-Razborov diagrams for Γ. We also prove that every system of equations over Γ is equivalent to a finite subsystem, and a number of structural results about Γ-limit groups.

1. INTRODUCTION

This paper is a continuation of [17]. Throughout this paper, Γ will denote a torsion-free group which is hyperbolic relative to a collection of free abelian subgroups. For an arbitrary finitely generated group G, we wish to understand the set $\mathrm{Hom}(G, \Gamma)$ of all homomorphisms from G to Γ.

In [17] we considered a sequence of homomorphisms $\{h_i : G \to \Gamma\}$ and extracted a limiting G-action on a suitable asymptotic cone, and then extracted an $\mathbb{R}$-tree with a nontrivial G-action. This $\mathbb{R}$-tree allows much information to be obtained. In particular, in case $G = \Gamma$, we studied $\mathrm{Aut}(\Gamma)$ and also proved that Γ is Hopfian. In this paper, we continue this study, in case G is an arbitrary finitely generated group. In particular, we construct a *Makanin-Razborov diagram* for G, which gives a parametrisation of $\mathrm{Hom}(G, \Gamma)$ (see Section 6 below). We build on our work from [17], which in turn builds on our previous work of [15] and [16]. The strategy is to follow [27, §1], though there are extra technical difficulties to deal with.

To a system of equations Σ over Γ in finitely many variables there is naturally associated a finitely generated group G_Σ, with generators the variables in Σ, and relations the equations. The solutions to Σ in Γ are in bijection with the elements of $\mathrm{Hom}(G_\Sigma, \Gamma)$. Thus, Makanin-Razborov diagrams give a description of the set of solutions to a given system of equations over Γ. For free groups, building on the work of Makanin and Razborov, Makanin-Razborov diagrams were

Date: 2 March, 2005.
2000 *Mathematics Subject Classification.* 20F65, (20F67.20E08,57M07).

constructed by Kharlampovich and Miasnikov [20], and also by Sela [26]. For torsion-free hyperbolic groups, Makanin-Razborov diagrams were constructed by Sela [27], and it is Sela's approach that we follow here. Alibegović [2] constructed Makanin-Razborov diagrams for limit groups.

Limit groups are hyperbolic relative to their maximal non-cyclic abelian subgroups (see [8] and [1]). Limits groups are also torsion-free. Therefore, the main result of this paper (the construction of Makanin-Razborov diagrams) generalises the main result of [2]. Alibegović has another approach to the construction of Makanin-Razborov diagrams for these relatively hyperbolic groups (see [2, Remark 3.7]).

The main results of this paper are the following:

Theorem 5.9 *Suppose that Γ is a torsion-free relatively hyperbolic group with abelian parabolics, and that G is a finitely generated group. Then G is a Γ-limit group if and only if G is fully residually Γ.*

Proposition 5.10 *Suppose that Γ is a torsion-free relatively hyperbolic group with abelian parabolics. Then there are only countably many Γ-limit groups.*

Definition 5.14 *A group G is a called* equationally Noetherian *if every system of equations over G in finitely many variables is equivalent to a finite subsystem.*

Theorem 5.15 *Let Γ be a torsion-free relatively hyperbolic group with abelian parabolics. Then Γ is equationally Noetherian.*

Theorem 6.4 *Let G be a finitely generated group and Γ a torsion-free relatively hyperbolic group with abelian parabolics. Associated to G is a Makanin-Razborov diagram, with vertices G and Γ-limit quotients of G, and edges a canonical quotient map. Any homomorphism $h \in Hom(G, \Gamma)$ can be given by compositions of modular automorphisms of the Γ-limit groups in the diagram with the canonical maps from Γ-limit groups into their maximal proper shortening quotients, and finally with either embeddings of a Γ-limit group in the diagram into Γ, or general homomorphisms of the terminal free groups that appear in the diagram into Γ.*

See Sections 3, 4, 5 and 6 for definitions and discussion of the terminology in Theorem 6.4 above. The output of Theorem 6.4 is a parametrisation of $\mathrm{Hom}(G, \Gamma)$ for an aribtrary finitely generated group G, in terms of successive proper quotients of G (with a fixed canonical quotient), modular automorphisms, embeddings into Γ, and homomorphisms from a free group to Γ. Note that for a fixed finitely generated free group F of rank d, the set $\mathrm{Hom}(F, \Gamma)$ can be naturally parametrised by F^d, by the universal property of free groups.

Sela [28, I.8] asked whether Theorems 5.15 and 6.4 hold in the context of CAT(0) groups with isolated flats. We believe that relatively hyperbolic groups with abelian parabolics are a natural setting for these questions.

An outline of this paper is as follows. In Section 2 we recall the definition of relatively hyperbolic groups, and recall the construction of limiting $\mathbb{R}$-trees from [15] and [17], as well as other useful results. In Section 3 we improve upon our version of Sela's shortening argument from [16] and [17] to deal with arbitrary sequences of homomorphisms $\{h_n : G \to \Gamma\}$ where G is an arbitrary finitely generated group. In Section 4 we recall Sela's construction of *shortening quotients* from [26], and adapt this construction to our setting. In Section 5 we prove Theorem 5.2, one of the main technical results of this paper. We also prove Theorems 5.10, 5.15, and a number of structural results about Γ-limit groups. Finally in Section 6 we construct Makanin-Razborov diagrams over Γ.

In the future work [18], we will continue to study the elementary theory of torsion-free relatively hyperbolic groups with free abelian parabolic subgroups.

Acknowledgement. *I would like to thank Zlil Sela for providing me with the proof of* [27, Proposition 1.21]*, which is repeated in the proof of Proposition 5.13 in this paper.*

2. Preliminaries

2.1. Relatively hyperbolic groups.

Relatively hyperbolic groups were first defined by Gromov in his seminal paper on hyperbolic groups [14]. Another definition was given by Farb [13], and further definitions given by Bowditch [6]. These definitions are all equivalent (see [7] and [29]). Recently there has been a large amount of interest in these groups (see [1, 7, 9, 10, 11, 21, 30], among others). The definition we give here is due to Bowditch [6].

Definition 2.1. *A group Γ with a family $\mathcal{G}$ of finitely generated subgroups is called* hyperbolic relative to $\mathcal{G}$ *if Γ acts on a δ-hyperbolic graph $\mathcal{K}$ with finite quotient and finite edge stabilisers, where the stabilisers of infinite valence vertices are the elements of $\mathcal{G}$, so that $\mathcal{K}$ has only finitely many orbits of simple loops of length n for each positive integer n.*

The groups in $\mathcal{G}$ are called parabolic subgroups of Γ*.*

In this paper we will be exclusively interested in relatively hyperbolic groups which are torsion-free and have abelian parabolic subgroups.

2.2. **The limiting $\mathbb{R}$-tree.** In this subsection we recall a construction from [17] (see also [15] for more details). Suppose that Γ is a torsion-free relatively hyperbolic group with abelian parabolic subgroups. In [17], we constructed a space X on which Γ acts properly and cocompactly by isometries. For each parabolic subgroup P (of rank n, say) there is in X an isometrically embedded copy of $\mathbb{R}^n$, with the Euclidean metric, so that the action of P leaves this Euclidean space invariant and this P-action is proper and cocompact with quotient the n-torus.

Suppose now that G is a finitely generated group, and that $\{h_n : G \to \Gamma\}$ is a sequence of homomorphisms, and suppose that the h_n do not differ only by post-composition with an inner automorphism of Γ. By considering the induced actions of G on X, and passing to a limit, we extract an isometric action of G on the asymptotic cone X_ω of X such that this action has no global fixed point. There is a separable G-invariant subset $\mathcal{C}_\infty \subseteq X_\omega$, and by passing to a subsequence $\{f_i\}$ of $\{h_i\}$ we may assume that the (appropriately scaled) actions of G on X converge in the Gromov-Hausdorff topology to the G-action on $\mathcal{C}_\infty$.

The space $\mathcal{C}_\infty$ is a *tree-graded* metric space, in the terminology of Druţu and Sapir [10]. Informally, this means that there is a collection of 'pieces' (in this case finite dimensional Euclidean spaces), and otherwise the space is 'tree-like' (see [10] for the precise definition and many properties of tree-graded metric spaces). By carefully choosing lines in the 'pieces', and projecting, an $\mathbb{R}$-tree T is extracted from $\mathcal{C}_\infty$. This tree T comes equipped with an isometric G-action with no global fixed points and the kernel of the G-action on T is the same as the kernel of the G-action on $\mathcal{C}_\infty$. For more details on this entire construction, see [17] and [15].

Definition 2.2. *Suppose that $\{h_n : G \to \Gamma\}$ is a sequence of homomorphisms. The* stable kernel *of $\{h_n\}$, denoted $\underrightarrow{Ker}(h_n)$, is the set of all $g \in G$ so that $g \in ker(h_n)$ for all but finitely many n.*

The following theorem recalls some of the properties of the G-action on the $\mathbb{R}$-tree T.

Theorem 2.3 (cf. Theorem 4.4, [15] and Theorem 6.4, [17])**.** *Suppose that Γ is a torsion-free group that is hyperbolic relative to a collection of free abelian subgroups and that G is a finitely generated group. Let $\{h_n : G \to \Gamma\}$ be a sequence of pairwise non-conjugate homomorphisms. There is a subsequence $\{f_i\}$ of $\{h_i\}$ and an action of G on an $\mathbb{R}$-tree T so that if K is the kernel of the action of G on T and $L := G/K$ then*

(1) The stabiliser in L of any nondegenerate segment in T is free abelian;
(2) If T is isometric to a real line then L is free abelian, and for all but finitely n the group $h_n(G)$ is free abelian;
(3) If $g \in G$ stabilises a tripod in T then $g \in \underrightarrow{\mathrm{Ker}}\,(f_i) \subseteq K$;
(4) Let $[y_1, y_2] \subset [y_3, y_4]$ be non-degenerate segments in T, and assume that $Stab_L([y_3, y_4])$ is nontrivial. Then

$$Stab_L([y_1, y_2]) = Stab_L([y_3, y_4]).$$

In particular, the action of L on T is stable; and
(5) L is torsion-free.

Thus T is isometric to a line if and only if L is abelian. If L is not abelian then $K = \underrightarrow{\mathrm{Ker}}\,(f_i)$.

We now recall the definition of Γ-*limit groups*. There are many ways of defining Γ-limit groups. We choose a geometric definition using the above construction.

Definition 2.4 (cf. Definition 1.11, [27], Definition 1.2, [17])**.** *A strict Γ-limit group is a quotient G/K where G is a finitely generated group, and K is the kernel of the action of G on T, where T is the $\mathbb{R}$-tree arising from a sequence of non-conjugate homomorphisms $\{h_n : G \to \Gamma\}$ as described above.*

A Γ-limit group is a group which is either a strict Γ-limit group or a finitely generated subgroup of Γ.

Remark 2.5. *There are finitely generated subgroups of torsion-free hyperbolic groups which are not finitely presented (see, for example,* [22]*). Therefore, when Γ is a torsion-free relatively hyperbolic group with free abelian parabolic subgroups, a Γ-limit group need not be finitely presented. This presents substantial complications (many of which are already dealt with by Sela in* [27]*), some of which are solved by the application of Theorem 5.7 below.*

2.3. **Acylindrical accessibility and JSJ decompositions.** In [24], Sela studied *acylindrical* graph of groups decomposition, and proved an accessibility theorem for k-acylindrical splittings. Unlike other accessibility results such as [12] and [3], Sela's result holds for finitely generated groups, rather than just for finitely presented groups.

We can apply acylindrical accessibility to our limiting construction because (i) tripod stabilisers are trivial; and (ii) maximal abelian subgroups of Γ-limit groups are malnormal. See [16] for a more detailed discussion of this and of JSJ decompositions.

The construction of the limiting $\mathbb{R}$-tree immediately implies that the abelian JSJ of a non-abelian, freely indecomposable strict Γ-limit group is nontrivial.

We can also apply the arguments of [26, Theorem 3.2] and [25, Theorem 3.2], as adapted in [16] (this adaptation also applies to the results in [17]) to prove that the cyclic JSJ decomposition of such a group is nontrivial – see Theorem 5.1 below. But first, we recall the shortening argument.

3. The shortening argument

In [16] and [17] we described a version of Sela's shortening argument which worked for sequences of *surjective* homomorphisms to Γ, and described in [16] why this notion is insufficient for all sequences of homomorphisms.

In this section we present another version of the shortening argument, which works for all sequences of homomorphisms $\{h_n : G \to \Gamma\}$, for *any* finite generated group G. This version was stated but not proved in [17], and we give the proof here.

There are two equivalent approaches to this version of the shortening argument. The first is to find a group $\widehat{G}$ which contains G and shorten using elements of $\text{Mod}(\widehat{G})$, rather than just elements of $\text{Mod}(G)$ (this approach was used in the proof of [16, Theorem 7.9]) . The second approach is to use the 'bending' moves of Alibegović [2]. We use the second approach, because it yields a simpler parametrisation of $\text{Hom}(G, \Gamma)$ when we construct Makanin-Razborov diagrams in Section 6.

Recall the definition of $\text{Mod}(G)$ for a finitely generated group G.

Definition 3.1. *Let G be a finitely generated group. A* Dehn twist *is an automorphism of one of the following two types:*

1. *Suppose that $G = A *_C B$ and that c is contained in the centre of C. Then define $\phi \in Aut(G)$ by $\phi(a) = a$ for $a \in A$ and $\phi(b) = cbc^{-1}$ for $b \in B$;*
2. *Suppose that $G = A*_C$, that c is in the centre of C, and that t is the stable letter of this HNN extension. Then define $\phi \in Aut(G)$ by $\phi(a) = a$ for $a \in A$ and $\phi(t) = tc$.*

Definition 3.2 (Generalised Dehn twists)**.** *Suppose G has a graph of groups decomposition with abelian edge groups, and A is an abelian vertex group in this decomposition. Let $A_1 \leq A$ be the subgroup generated by all edge groups connecting A to other vertex groups in the decomposition. Any automorphism of A that fixes A_1 elementwise can be* *naturally extended to an automorphism of the ambient group G. Such an automorphism is called a* generalised Dehn twist *of G.*

Definition 3.3. *Let G be a finitely generated group. We define* $\mathrm{Mod}(G)$ *to be the subgroup of* $\mathrm{Aut}(G)$ *generated by:*

(1) Inner automorphisms;
(2) Dehn twists arising from splittings of G with abelian edge groups; and
(3) Generalised Dehn twists arising from graph of groups decompositions of G with abelian edge groups.

Similar definitions are made in [26, §5] and [4, §1].

We will try to shorten homomorphisms by precomposing by elements of $\mathrm{Mod}(G)$. However, as seen in [16, §3], this is not sufficient to get the most general result. Thus, we also define a further kind of move (very similar to Alibegović's *bending* move, [2, §2].

Definition 3.4. *Suppose that Γ is a torsion-free group which is hyperbolic relative to free abelian subgroups, that G is a finitely generated group and that $h : G \to \Gamma$ is a homomorphism. We define two kinds of 'bending' moves as follows:*

(B1) Let Λ be a graph of groups decomposition of G, and let A be an abelian vertex group of G. Suppose that $h(A)$ is contained in a parabolic subgroup $P \le G$. A move of type (B1) replaces h by a homomorphism h' which is such that (i) $h'(A) \le P$; and (ii) h' agrees with h on all edge groups adjacent to A, and all vertex groups other than A.

(B2) Let Λ be a graph of groups decomposition of G, and let A be an abelian edge group associated to an edge e. Suppose that $h(A)$ is contained in a parabolic subgroup $P \le \Gamma$. A move of type (B2) replaces h by a map which either (i) conjugates a component of $\pi_1(\Lambda \smallsetminus e)$ by an element of P, in case e is separating; or (ii) multiplies the stable letter associated to e by an element of P, in case e is non-separating.

Definition 3.5 (cf. Definition 4.2, [4]; Definition 2.11, [2])**.** *We define the relation '$\sim$' on the set of homomorphisms $h : G \to \Gamma$ to be the equivalence relation generated by setting $h_1 \sim h_2$ if h_2 is obtained from h_1 by:*

(1) precomposing with an element of Mod(G);
(2) postcomposing with an inner automorphism of Γ; or
(3) a bending move of type (B1) or (B2).

Definition 3.6. *Let $\mathcal{A}$ be an arbitrary finite generating set for G, and let X be the space upon which Γ acts properly, cocompactly and isometrically, with basepoint x. For a homomorphism $h : G \to \Gamma$ define $\|h\|$ by*

$$\|h\| := \max_{g \in \mathcal{A}} d_X(x, h(g).x).$$

A homomorphism $h : G \to \Gamma$ is short *if for any h' such that $h \sim h'$ we have $\|h\| \leq \|h'\|$.*

The following is the main result of this section.

Theorem 3.7. *Suppose that Γ is a non-abelian, freely indecomposable and torsion-free group which is hyperbolic relative to a collection of free abelian subgroups. Let G be a freely indecomposable finitely generated group and $\{h_n : G \to \Gamma\}$ be a sequence of non-conjugate homomorphisms which converges to a faithful action of G on $\mathcal{C}_\infty$ as above. Then, for all but finitely many n, the homomorphism h_n is not short.*

Proof. Suppose that the sequence $\{h_n : G \to \Gamma\}$ converges into a faithful G-action on $\mathcal{C}_\infty$. From this we extract a faithful G-action on an $\mathbb{R}$-tree T.

The group G is freely indecomposable and the stabiliser in G of any tripod in T is trivial, so we can apply the decomposition theorem of Sela – [24, Theorem 3.1] – and decompose T into subtrees of three types: axial, IET, and discrete (note that because G is freely indecomposable and tripod stabilisers are trivial, there are no thin components in T). This decomposition of T induces a graph of groups decomposition of G, which will allow us to shorten h_n for sufficiently large n. See [24] or [16, §4] for more information.

Note that there are two sources for segments in T. There are segments in $\mathcal{C}_\infty$, and there are flats in $\mathcal{C}_\infty$ which are projected to lines in T. We treat these as two separate cases. However, we can make the following simplifications (let $\mathbb{P}$ be the collection of lines in T which are projections of flats in $\mathcal{C}_\infty$):

(1) Suppose that Y is an IET subtree of T and that $p_E \in \mathbb{P}$ is a line in T. Then the intersection $Y \cap p_E$ contains at most a point ([16, Proposition 4.3]);
(2) Suppose that a line $l \subset T$ is an axial subtree of T and and let the line p_E be in $\mathbb{P}$. If $l \cap p_E$ contains more than a point then $l = p_E$ ([16, Proposition 4.5]);
(3) If an edge e in the discrete part of T has an intersection of positive length with $p_E \in \mathbb{P}$ then $e \subset p_E$ ([16, Lemma 4.7]).

Fix a finite generating set $\mathcal{A}$ for G. Let y be the basepoint in T, and consider the paths $[y, u.y]$ for $u \in \mathcal{A}$. If there is any IET component of T which intersects any segment $[y, u.y]$ nontrivially then we can apply [23, Theorem 5.1] and [16, Corollary 4.4] to shorten these intersections whilst leaving the remaining segments unchanged (to see that we can have G finitely generated rather than finitely presented, see [16, Remark 4.8]).

Suppose that some segment $[y, u.y]$ has an intersection of positive length with some axial component $l \subset T$ so that l is not contained in any $p_E \in \mathbb{P}$. Then [16, Theorem 5.1] can be used to shorten those segments $[y, u_i.y]$ intersecting the orbit of p_E nontrivially, and leaving other segments unchanged.

Suppose that $[y, u.y]$ intersects some line p_E nontrivially, and that p_E is an axial component of T. The only place where the proof of [16, Theorem 5.2] breaks down is that the images $h_i(G)$ may not intersect parabolics in its image in denser and denser subsets (when measured with the scaled metric). However, this is exactly what the bending move (B1) is designed to deal with.

We have the following analogue of [16, Proposition 5.4]: Let E denote the flat in $\mathcal{C}_\infty$ which projects to p_E. The subgroup $\mathrm{Stab}_G(E)$ is an abelian subgroup of G. There is a sequence of flats $E_i \subset X_i$ so that $E_i \to E$ in the Gromov topology. The subgroups $h_i(\mathrm{Stab}_G(E))$ are abelian, and fix the flat E_i, for sufficiently large i. Thus $h_i(\mathrm{Stab}_g(E))$ is contained in a unique maximal abelian subgroup A_{E_i} of Γ. If we fix a finite subset W of $\mathrm{Stab}_G(E)$ and $\epsilon > 0$, then for sufficiently large i, there is an automorphism $\sigma_i : A_{E_i} \to A_{E_i}$ so that

(1) For every $w \in W$, and every $r_i \in E_i$,
$$d_{X_i}(r_i, h_i(\sigma_i(w)).r_i) < \epsilon,$$
(2) For any $k \in \mathrm{Stab}_G(E)$ which acts trivially on E we have $\sigma_i(h_i(k)) = h_i(k)$.

The proof of the existence of such a σ_i is the same as the proof of [16, Proposition 5.4]. Such a σ_i induces a move of Type (B1) in a straightforward manner, since the adjacent edge groups to the vertex group $\mathrm{Stab}_G(E)$ contain elements which act trivially on E, therefore we replace h_i by the homomorphism which agrees with h_i on all edge groups and on all vertex groups which are not $\mathrm{Stab}_G(E)$, and replaces $h_i|_{\mathrm{Stab}_G(E)}$ by $\sigma_i \circ h_i|_{\mathrm{Stab}_G(E)}$.

We now construct shortening elements for all but finitely many of the intervals $[y_n, h_n(u).y_n]$ by following the proof of [23, Theorem 5.1] (see [16, §5] for more details).

Finally, we are left with the case where $[y, u.y]$ is contained entirely in the discrete part of T. We follow the proof of [16, Theorem 6.1], which in turn followed Section 6 of [23]. This argument naturally splits into a number of cases.

Case 1: y is contained in the interior of an edge e.

Case 1a: e is not contained entirely in a line $p_E \in \mathbb{P}$ and $\bar{e} \in T/G$ is a splitting edge.

This case follows directly as in [16, §6].

Case 1b: e is not completely contained in a line $p_E \in \mathbb{P}$ and $\bar{e}$ is not a splitting edge.

This case also follows directly as in [16, §6].

Case 1c: e is contained in a line $p_E \in \mathbb{P}$ and $\bar{e}$ is a splitting edge.

In this case, we have a graph of groups decomposition $G = H_1 *_{A_E} H_2$, where $A_E = \mathrm{Stab}_G(E)$ (and E is the flat in $\mathcal{C}_\infty$ which projects to p_E). The Dehn twist which is found in [16, §6] is naturally replaced by a bending move of type (B2).

Case 1d: e is contained in a line $p_E \in \mathbb{P}$ and $\bar{e}$ is not a splitting edge.

Once again the Dehn twist is replaced by a bending move of Type (B2).

Case 2: y is a vertex of T.

Once again here there are four cases, depending on whether on edge adjacent to y is or is not a splitting edge and is or is not contained in a line $p_E \in \mathbb{P}$. In case an edge e is not contained in a line $p_E \in \mathbb{P}$, we proceed exactly as in [16], following [23] directly. In case $e \subset p_E$, we replace the shortening Dehn twists by bending moves of type (B2) as in Case 1 above.

Therefore, in any case, we can find moves which shorten all but finitely many of the h_i, as required. □

4. Shortening quotients

We now recall the concept of *shortening quotients* from [26] and [27].

Let G be a finitely generated group, Γ a torsion-free relatively hyperbolic group with abelian parabolics and $\{h_n : G \to \Gamma\}$ a stable sequence of homomorphisms, with associated Γ-limit group L_∞, and suppose that L_∞ is d-generated. The shortening procedure constructs a sequence of homomorphisms $\{\nu_i : F_d \to \Gamma\}$ which has a subsequence converging to a Γ-limit group Q_∞, equipped with a canonical epimorphism $\eta : L_\infty \to Q_\infty$. We follow the construction from [26, §3] and [25] (see [16] for more details in this context).

Given the situation described in the previous paragraph, we now describe the construction of $\{\nu_i\}$, Q_∞ and η. Let Λ_{L_∞} be the canonical abelian JSJ decomposition for L_∞, with vertex groups $V_\infty^1, \ldots, V_\infty^m$ and edge groups $E_\infty^1, \ldots, E_\infty^s$.

As in [26] and [16] we do not yet know that the edge groups are finitely generated (though this will eventually turn out to be the case; see Proposition 5.11 below).

We can 'approximate' the finitely generated group L_∞ by finitely presented groups U_n, each equipped with a graph of groups decomposition Λ_n which is a 'lift' of Λ_∞. See [26, §3] and [16] for more details of this. The output of this is a commutative diagram:

$$\begin{array}{ccccccc}
W_1 & \xrightarrow{\kappa_1} & W_2 \cdots & \xrightarrow{\kappa_{n-1}} & W_{n_1} & \xrightarrow{\kappa_n} & W_n \\
{\scriptstyle \iota_1}\downarrow & & {\scriptstyle \iota_2}\downarrow & & {\scriptstyle \iota_{n-1}}\downarrow & & {\scriptstyle \iota_n}\downarrow \\
U_1 & \xrightarrow{\kappa_1} & U_2 \cdots & \xrightarrow{\kappa_{n-1}} & U_{n_1} & \xrightarrow{\kappa_n} & U_n \\
{\scriptstyle \lambda_0}\downarrow & & {\scriptstyle \lambda_1}\downarrow & & {\scriptstyle \lambda_{n-1}}\downarrow & & {\scriptstyle \lambda_n}\downarrow \\
\Gamma & & \Gamma & & \Gamma & & \Gamma
\end{array}$$

The Γ-limit group L_∞ is the direct limit of the sequence $\{(W_i, \kappa)\}$.

The group U_n comes with a generating set, and a graph of groups decomposition. The equivalence relation used to define short homomorphisms in the previous section is naturally defined also for the set of homomorphisms $\mathrm{Hom}(U_n, \Gamma)$. When defining the equivalence relation for 'short', we restrict to those elements of $\mathrm{Mod}(U_n)$ which come from the graph of groups decomposition Λ_n.

The group W_i is a subgroup of U_n (and the map ι_n is inclusion), and comes with a generating set $\{x_1^1, \ldots, x_{l_1}^1, \ldots, x_1^m, \ldots, x_{l_m}^m, y_1, \ldots, y_b\}$, which corresponds to a generating set for L_∞ coming from the graph of groups decomposition Λ_{L_∞}. For a homomorphism $h : W_i \to \Gamma$ define the following stretching constant:

$$\begin{aligned}
\mu_i(h) &= \max_{1 \leq j \leq l_i} d_X(x, h(x_j^i).x), \text{ and} \\
\chi_r(h) &= d_X(x, h(y_r).x).
\end{aligned}$$

Also define the corresponding $(m+b)$-tuple

$$\mathrm{tup}(h) = (\mu_1(h), \ldots, \mu_m(h), \chi_1(h), \ldots, \chi_r(h)).$$

Now for each $n \geq 1$ choose a homomorphism $\widehat{\lambda_n} : W_n \to \Gamma$ so that $\widehat{\lambda_n} \sim \lambda_n \circ \iota_n$ and so that $\mathrm{tup}(\widehat{\lambda_n})$ is minimal amongst all homomorphisms equivalent to $\lambda_n \circ \iota_n$ (the set of $(m+b)$-tuples is given the lexicographic order).

Passing to a subsequence of $\{\widehat{\lambda}_n\}$, we obtain an associated Γ-limit group Q_∞. There are two cases to consider here: (i) Q_∞ is a strict Γ-limit group; and (ii) Q_∞ is not a strict Γ-limit group. In case (ii), let $\pi_n : F_d \to \Gamma$ be given by $\pi_n = \widehat{\lambda}_n \circ \psi_n$, where F_d is the free group of rank d and $\psi_n : F_d \to U_n$ is the canonical quotient map. Since Q_∞ is not a strict Γ-limit group, we may assume that the (convergent) subsequence of $\{\pi_n\}$ is constant, and Q_∞ is isomorphic to a subgroup of Γ. In this case, since each image $\widehat{\lambda}_n(U_n)$ is isomorphic to $\lambda(U_n)$ via the natural map between generating sets, it is not hard to see that L_∞ and Q_∞ are isomorphic via the natural map between generating sets.

In case Q_∞ is a strict Γ-limit group, each vertex group V^i_∞ is embedded canonically in Q_∞ (see [26, §3] or [16, §7] for more details). We also claim that there is a canonical epimorphism $\pi : L_\infty \to Q_\infty$. To see this, it remains to see that each of the relations corresponding to stable letters in Λ_{L_∞} are preserved when the canonical generating set for L_∞ is mapped to the canonical generating set for Q_∞. For each such relation w, there is some n so that for all $m > n$ the group U_m includes $w = 1$ as a defining relation. This defining relation is preserved by the shortening moves, and so holds in the Γ-limit group Q_∞.

The above group Q_∞ is called the *shortening quotient* of L_∞ associated to $\{h_n : G \to \Gamma\}$.

Although we speak of *the* shortening quotient, it depends on the choices of shortest homomorphism in the equivalence class of h_n, and also on the convergent subsequence of $\{\widehat{h_n}\}$ chosen. Of course, it also depends on the choice of finite generating set for G, but we assume this is fixed.

Proposition 4.1 (cf. Proposition 1.15, [27])**.** *Let G be a finitely generated group, and Γ a torsion free relatively hyperbolic group with abelian parabolics, and let $\{u_s : G \to \Gamma\}$ be a sequence of homomorphisms that converges into an action of a non-abelian, freely indecomposable strict Γ-limit group L on an $\mathbb{R}$-tree T. If for every index s, the group $u_s(G)$ is not isomorphic to L by the natural map that sends the images of the generators of G in $u_s(G)$ to the images of these generators in L, then every shortening quotient of L which is obtained from the sequence $\{u_s\}$ is a proper quotient of L.*

Proof (of Proposition 4.1). Suppose that Q is the shortening quotient of L associated to $\{\widehat{u_s}\}$ as described above, and let $\{\pi_n : F_d \to \Gamma\}$ be the homomorphism arising from $\widehat{\lambda}_n \circ \psi_n$, where $\psi_n : F_d \to U_n$ is the canonical quotient map. Let $\{\pi_{s_n}\}$ denote the convergent subsequence.

The hypothesis of the proposition implies that Q is a strict Γ-limit group, and that $\{\pi_{s_n}\}$ do not belong to finitely many conjugacy classes. Since each of the π_{s_n} is short, the shortening argument implies that the limiting action cannot be faithful, which is to say that Q is a proper quotient of L. □

5. Γ-LIMIT GROUPS

In this section we follow [27] in order to understand Γ-limit groups, and $\mathrm{Hom}(G, \Gamma)$, where Γ is an aribitrary finitely generated group. The main technical results of this section are Theorem 5.1, Theorem 5.2 and Theorem 5.7.

These results are then applied to prove various applications of these results: Theorem 5.9, Proposition 5.10, Theorem 5.15, and in the next section the construction of Makanin-Razborov diagrams.

Theorem 5.1 (cf. Theorem 3.2, p. 14, [26]). *Let Γ be a torsion-free group which is hyperbolic relative to a collection of free abelian subgroups, and let L be a strict Γ-limit group which is nonabelian and freely indecomposable. Then L admits a principal cyclic splitting.*

Proof. The strategy is to follow the proof of [26, Theorem 3.2], which is very similar to the proof of [25, Theorem 3.2], the difference being in [26] that there might be abelian subgroups of the limit group which are not locally cyclic.

In the proof of the Hopf property for Γ, we have already adapted [25, Theorem 3.2] to the current context (see [16] and [17]), and we already there had to deal with abelian subgroups of the Γ-limit group which are not locally cyclic. Therefore, the proof of this theorem proceeds almost exactly as in [16] (with the translations made in [17]).

The only substantial difference is that we need to use the improved version of the shortening argument presented in Section 3 of this paper. One of the key points of the proof is a construction very similar to that of shortening quotients as in the previous section, although for this proof this construction is used in the process of deducing an elaborate contradiction. □

Let G be a fixed finitely generated group. Define an order on the set of Γ-limit groups that are quotients of G as follows: suppose R_1 and R_2 are both Γ-limit groups that are quotients of G, and that $\eta_i : G \to R_i$ are the (fixed) canonical quotient maps. We say $R_1 > R_2$ if there exists an epimorphism with non-trivial kernel $\tau : R_1 \to R_2$ so that $\eta_2 = \tau \circ \eta_1$. We say that R_1 and R_2 are *equivalent* if there is an isomorphism $\tau : R_1 \to R_2$ so that $\eta_2 = \tau \circ \eta_1$.

The following is one of the main technical results of this paper.

Theorem 5.2 (cf. Theorem 1.12, [27])**.** *Let Γ be a torsion-free group which is hyperbolic relative to free abelian subgroups, and let G be a finitely generated group. Every decreasing sequence of Γ-limit groups that are quotients of G:*

$$R_1 > R_2 > R_3 > \dots,$$

terminates after finitely many steps.

For limit groups, the analogous result has a short proof using algebraic geometry (see [4]). However (as observed by M. Kapovich; see, for example, [5, §1.4]), not all hyperbolic groups are linear and the same is therefore true for relatively hyperbolic groups.

Before we prove Theorem 5.2, we prove the following lemma (implicit in [27, p. 7]):

Lemma 5.3. *Let Ξ be a finitely generated group, let L be a Ξ-limit group and suppose that L is d-generated. Then L can be obtained as a limit of homomorphisms $\{h_n : F_d \to \Xi\}$, where F_d is the free group of rank d.*

Proof. If L can be embedded in Ξ, then we can take the constant sequence $\{h : F_d \to \Xi\}$, where $h = \iota \circ \pi$, for a fixed surjection $\pi : F_d \to L$ and fixed embedding $\iota : L \to \Xi$.

Otherwise, suppose that L is obtained from a sequence $\{h_n : G \to \Xi\}$. Suppose that $\{x_1, \dots, x_d\}$ is a generating set for L. For each n, define $G_n \leq G$ to be generated by elements $y_{1,n}, \dots, y_{d,n}$, where $h_n(y_{i,n}) = x_i$. Now define $\pi_n : F_d \to G$ by $f_i = y_{i,n}$, where $\{f_1, \dots, f_d\}$ is a basis of F_d. It is not difficult to see that L is realised as the limit of the sequence of homomorphisms $\{h_n \circ \pi_n : F_d \to \Xi\}$. □

Proof (of Theorem 5.2). We follow the proof of [27, Theorem 1.12]. In order to obtain a contradiction, we suppose that there exists a finitely generated group G, for which there exists an infinite descending sequence of Γ-limit groups:

$$R_1 > R_2 > R_2 > \dots.$$

Without loss of generality, we may assume that $G = F_d$, the free group of rank d. Let $\{f_1, \dots, f_d\}$ be a basis for F_d, and let C be the Cayley graph of F_d with respect to this generating set. We construct a particular descreasing sequence of Γ-limit groups as follows. Let R_1 be a Γ-limit group with the following properties:

(1) R_1 is a proper quotient of F_d;

(2) R_1 can be extended to an infinite decreasing sequence of Γ-limit groups: $R_1 > L_2 > L_3 > \ldots$;
(3) The map $\eta_1 : F_d \to R_1$ maps to the identity the maximal number of elements in the ball of radius 1 about the identity in C among all possible maps from F_d to a Γ-limit group L that satisfies the first two conditions.

Continue to define the sequence inductively. Suppose that the finite sequence $R_1 > R_2 > \ldots > R_{n-1}$ has been constructed, and choose R_n to satisfy:

(1) R_n is a proper quotient of R_{n-1};
(2) The finite decreasing sequence of Γ-limit groups $R_1 > R_2 > \ldots > R_n$ can be extended to an infinite decreasing sequence; and
(3) The map $\eta_n : F_d \to R_n$ maps to the identity the maximal number of elements in the ball of radius n about the identity in C among all possible maps from F_d to a Γ-limit group L_n satisfying the first two conditions.

It is worth noting that we do not insist that the Γ-limit groups R_n be strict Γ-limit groups. This will be important later, because to study a single homomorphism, we consider a constant sequence, which leads to a Γ-limit group which need not be strict.

Since each of the Γ-limit groups R_n is a quotient of F_d, each R_n is d-generated. Let $\{r_{1,n}, \ldots, r_{d,n}\}$ be a generating set for R_n. By Lemma 5.3, R_n can be obtained as a limit of a sequence of homomorphisms $\{v_i^n : F_d \to \Gamma\}$, with the quotient map $\eta_n : F_d \to R_n$ sending f_i to $r_{i,n}$.

For each n, choose a homomorphism $v_{i_n}^n : F_d \to \Gamma$ for which:

(1) Every element in the ball of radius n about the identity in C that is mapped to the identity by $\eta_n : F_d \to R_n$ is mapped to the identity by $v_{i_n}^n$. Every such element that is mapped to a nontrivial element by η_n is mapped to a nontrivial element by $v_{i_n}^n$; and
(2) There exists an element $f \in F_d$ that is mapped to the identity by $\eta_{n+1} : F_d \to R_{n+1}$ for which $v_{i_n}^n(f) \neq 1$.

Denote the homomorphism $v_{i_n}^n$ by h_n. By construction, the set of homomorphisms $\{h_n : F_d \to \Gamma\}$ does not belong to a finite set of conjugacy classes. Therefore, from the sequence $\{h_n\}$ we can extract a subsequence that converges into a (strict) Γ-limit group, denoted R_∞. By construction, the Γ-limit group R_∞ is the direct limit of the sequence of (proper) epimorphisms:

$$F_d \to R_1 \to R_2 \to \cdots .$$

Let $\eta_\infty : F_d \to R_\infty$ be the canonical quotient map.

Lemma 5.4 (cf. Lemma 1.13, [27])**.**

(1) the set of homomorphisms $\{h_n : F_d \to \Gamma\}$ do not belong to finitely many conjugacy classes, hence R_∞ is a strict Γ-limit group;
(2) R_∞ is not finitely presented;
(3) R_∞ is not the free product of a finitely presented group and freely indecomposable Γ-limit groups that do not admit a cyclic splitting;
(4) Let $R_\infty = U_1 * \cdots * U_t * F$ be the most refined (Grushko) free decomposition of R_∞, where F is a finitely generated group. Then there exists an index j, with $1 \leq j \leq t$, for which:
  (a) U_j is not finitely presented;
  (b) If B is a finitely generated subgroup of F_d such that $\eta_\infty(B) = U_j$ then the restrictions $h_n|_B$ do not belong to finitely many conjugacy classes. Furthermore, if $b_1, \ldots, b_p$ is a generating set for B, then for every index n, the group $h_n(B)$ is not isomorphic to $\eta_\infty(B)$ by an isomorphism that sends $h_n(b_i)$ to $\eta_\infty(b_i)$.

Proof (of Lemma 5.4). Given Theorem 5.1, the proof of this is identical to that of [27, Lemma 1.13]. □

The Γ-limit group R_∞ is a proper quotient of each of the Γ-limit groups R_n. For each index n, the group R_n was chosen to maximise the number of elements in the ball of radius n about the identity in C that are mapped to the identity in Γ among all Γ-limit groups that are proper quotients of R_{n-1} and that admit an infinite descending chain of Γ-limit groups. Therefore, it is not difficult to see that R_∞ does not admit an infinite descending chain of Γ-limit groups.

We now obtain a *finite resolution* of R_∞:

Proposition 5.5 (cf. Proposition 1.16, [27])**.** *Let $\{h_n : F_d \to \Gamma\}$ be the sequence of homomorphisms constructed above. Then there exists a finite sequence of Γ-limit groups:*

$$R_\infty \to L_1 \to \cdots \to L_s,$$

so that

(1) The epimorphisms along the sequence are proper epimorphisms;
(2) Let $L_s = H_1 * \cdots * H_r * F$ be the (possibly trivial) Grushko free decomposition of L_s. There exists a subsequence $\{h_{n_t}\}$ of $\{h_n : F_d \to \Gamma\}$ so that each of the homomorphisms h_{n_t} can be

written in the form;

$$h_{n_t} = \nu_t \circ \eta_{s-1} \circ \phi^t_{s_1} \circ \eta_{s-2} \cdots \circ \phi^t_1 \circ \eta_0,$$

where each $\phi^t_i \in Mod(L_i)$*, and* $\nu_t : L_s \to \Gamma$ *is a homomorphism that embeds each of the factors* H_j *of* L_s *into* Γ*;*

(3) *The sequence of homomorphism* $\{h_{n_t}\}$ *converges into a faithful action of* R_∞ *on ann* $\mathbb{R}$*-tree* T*. Furthermore, the entire sequence of homomorphisms* h_{n_t} *factors through the epimorphism* $\eta_\infty : F_d \to R_\infty$*.*

Proof (of Proposition 5.5). Given Lemma 5.4 and Proposition 4.1 the proof is identical to that of [27, Proposition 1.16].

We note that the groups $L_1, \dots, L_s$ are found by taking successive shortening quotients, so that L_1 is a shortening quotient of R_∞. □

The homomorphisms $\{h_n : F_d \to \Gamma\}$ were chosen so that for every index n there exists some element $f \in F_d$ for which $\eta_{n+1}(f) = 1$ and $h_n(f) \neq 1$. Since R_∞ is a proper quotient of all of the Γ-limit groups R_n, for every index n and every element $f \in F_d$, if $\eta_{n+1}(f) = 1$ then $\eta_\infty(f) = 1$. By Part 3 of Proposition 5.5 it is possible to extract a subsequence $\{h_{n_t} : F_d \to \Gamma\}$ that factors through the group R_∞, which is to say that there is a homomorphism $\pi_t : R_\infty \to \Gamma$ so that $h_{n_t} = \pi_t \circ \eta_\infty$. Hence, for every index t, and every element $f \in F_d$, if $\eta_{n_t+1}(f) = 1$ then $\eta_\infty(f) = 1$, which implies that $h_{n_t}(f) = 1$, in contradiction to the way that the homomorphisms h_n were chosen. This finally ends the proof of Theorem 5.2. □

Corollary 5.6. *Let* Γ *be a torsion-free relatively hyperbolic group with abelian parabolics, and let* L *be a* Γ*-limit group. Then* L *is Hopfian.*

Corollary 5.6 generalises one of the main results of [17], and implies that the relation defined on Γ-limit groups which are quotients of a fixed group G is a partial order. The proof of the following theorem is identical to that of [27, Theorem 1.17].

Theorem 5.7 (cf. Theorem 1.17, [27])**.** *Let* G *be a finitely generated group and* Γ *a torsion-free relatively hyperbolic group with abelian parabolics, and let* $\{h_n : G \to \Gamma\}$ *be a sequence of homomorphisms. Then there exists a finite sequence of* Γ*-limit groups:*

$$G \to L_1 \to L_2 \to \cdots \to L_s,$$

for which

(1) $\eta_0 : G \to L_1$ *is an epimorphism and* $\eta_i : L_i \to L_{i+1}$ *is a proper epimorphism for each* $1 \leq i \leq s-1$*;*

(2) *Let $L_s = H_1 * \cdots * H_r * F$ be the Grushko free decomposition of L_s. Then there exists a subsequence $\{h_{n_t}\}$ os $\{h_n\}$ so that each of the homomorphisms h_{n_t} can be decomposed as:*

$$h_{n_t} = \nu_t \circ \eta_{s-1} \circ \phi^t_{s-1} \circ \eta_{s-2} \circ \cdots \circ \phi^t_1 \circ \eta_0,$$

where $\phi^t_i \in Mod(L_i)$ and $\nu_t : L_s \to \Gamma$ embeds each of the freely indecomposable factors H_j of L_s into Γ;

(3) *The sequence of homomorphisms $h_{n_t} : G \to \Gamma$ is either constant or converges into a faithful action of L_1 on a tree-graded space $\mathcal{C}_\infty$. Furthermore, the entire sequence of homomorphisms h_{n_t} factor through the epimorphism $\eta_0 : G \to L_1$.*

Given Theorem 5.7, if a sequence $\{h_n : G \to \Gamma\}$ converges into a Γ-limit group L, then by passing to a subsequence we may assume that each h_n factors through the canonical quotient map $\eta : G \to L$, so we may replace $\{h_n\}$ by a sequence of homomorphisms from L to Γ. This simplifies the definition of shortening quotients considerably, since we may start with a sequence $\{h_n : L \to \Gamma\}$. Proposition 4.1 still holds in this context (in fact the proof is easier).

Definition 5.8. *Let Γ and G be finitely generated groups. We say that G is* fully residually Γ *is for every finite $\mathcal{F} \subset G$ there is a homomorphism $h : G \to \Gamma$ which is injective on $\mathcal{F}$.*

Theorem 5.9. *Let Γ be a nonabelian torsion-free relatively hyperbolic group with abelian parabolics. A finitely generated group G is a Γ-limit group if and only if it is fully residually Γ.*

Proof. If G is a fully residually Γ then it is certainly a Γ-limit group. Suppose that G is a Γ-limit group, obtained from a sequence $\{h_n : U \to \Gamma\}$. By Theorem 5.7, by passing to a subsequence we may assume that each of the h_n factor through the Γ-limit group G. Now it is clear that G is fully residually Γ. □

Proposition 5.10. *Let Γ be a torsion-free relatively hyperbolic group with abelian parabolics. Then there are only countably many Γ-limit groups.*

Proof. If Γ is abelian, then all Γ-limit groups are finitely generated abelian groups, of which there are only countably many.

Suppose then that Γ is nonabelian. We apply the proof of Theorem 5.7, along with the construction of shortening quotients. By Theorem 5.7, for any Γ-limit group L there is a finitely generated free group G and a sequence

$$G \to L \to L_2 \to \ldots \to L_s,$$

where each of the non-free factors in the Grushko decomposition of L_s admits an embedding into Γ, and each L_i is a shortening quotient of the previous term in the sequence. We consider the limit groups which arise with sequences of increasing lengths, and note that there are only countably many for each length.

If $L = L_s$, then L is the free product of finitely many finitely generated subgroups of Γ and (possibly) a finitely generated free group. Since there are countably many finitely generated subgroups of Γ, there are countably many such Γ-limit groups.

We now consider hot to obtain L_{i-1} from L_i. First, L_i admits a Grushko free decomposition into freely indecomposable nonabelian Γ-limit groups, along with possibly a free group and a free product of finitely generated free abelian groups. Each freely indecomposable nonabelian factor $H_{i-1,j}$ of L_{i-1} admits a graph of groups decomposition $\Lambda_{H_{i-1,j}}$, whose edge groups are free abelian groups. By induction on i, and the fact that abelian subgroups of Γ are finitely generated, we may assume that these edge groups are finitely generated. The vertex groups of $\Lambda_{H_{i-1,j}}$ are either abelian groups, surface groups, or embed in L_i (by the construction of shortening quotients; see Section 4). Therefore, each such free factor of L_{i-1} can be formed by taking finitely many HNN extensions and amalgamated free products of finitely generated subgroups of L_i over finitely generated abelian subgroups. There are only countably many such constructions. This completes the proof. □

The following result follows from the proof of Proposition 5.10.

Corollary 5.11. *Any abelian subgroup of a Γ-limit group is finitely generated free abelian.*

Proposition 5.12 (cf. Proposition 1.20, [27])**.** *Let G be a finitely generated group and Γ a torsion-free relatively hyperbolic group with abelian parabolics. Let $R_1, R_2 \dots$ be a sequence of Γ-limit groups that are all quotients of G so that*

$$R_1 < R_2 < \cdots .$$

Then there exists a Γ-limit group R, a quotient of G, so that $R > R_m$ for all m.

Proof. For each m, choose a homomorphism $h_m : G \to \Gamma$ that factors through the quotient map $\eta_m : G \to R_m$ as $h_m = h'_m \circ \eta_m$ and so that h'_m is injective on the ball of radius m in R_m. This is possible by Theorem 5.7.

A subsequence of $\{h_m\}$ converges to a Γ-limit group R, which is a quotient of G. By Theorem 5.7, we may assume that each element of this subsequence factors through the canonical quotient map $\eta : G \to R$.

We prove that $R > R_m$ for each m. We have quotient maps $\eta : G \to R$, and $\eta_i : G \to R_i$. Since $R_i < R_{i+1}$ there exists $\tau_i : R_{i+1} \to R_i$ so that $\eta_i = \tau_i \circ \eta_{i+1}$. In particular, $\ker(\eta_{i+1}) \subseteq \ker(\eta_i)$.

Let $\mathcal{A}$ be the fixed finite generating set for G. We attempt to define a homomorphism $\kappa_i : R \to R_i$ as follows: for $a \in \mathcal{A}$, define $\kappa_i(\eta(a)) = \eta_i(a)$. This is well-defined if and only if $\ker(\eta) \subseteq \ker(\eta_i)$. Therefore, suppose that $g \in \ker(\eta)$. Since each h_j factors through η, we have $h_j(g) = 1$ for all j. Suppose that g lies in the ball of radius n about the identity in the Cayley graph of G. Then for all j, the element $\eta_j(g)$ lies in the ball of radius n about the identity in R_j. Since $h_j(g) = 1$, for all j, and by the defining property of the h_j, if $k \geq n$ then $\eta_k(g) = 1$. Thus since for all j we have $\ker(\eta_{j+1}) \subseteq \ker(\eta_j)$ we have $\eta_i(g) = 1$, as required. We have constructed a homomorphism $\kappa_i : R \to R_i$ so that $\eta_i = \kappa_i \circ \eta$, which is to say that $R > R_i$. This finishes the proof. □

Propositions 5.10 and 5.12 imply that there are maximal elements for the set of Γ-limit groups which are quotients of a fixed finitely generated group G, under the order described before Theorem 5.2.

Recall that we say that two Γ-limit groups which are quotients of G, $\eta_1 : G \to R_1$ and $\eta_2 : G \to R_2$ are equivalent if there is an isomorphism $\tau : R_1 \to R_2$ so that $\eta_2 = \eta_1 \circ \tau$.

Proposition 5.13 (cf. Proposition 1.21, [27])**.** *Let G be a finitely generated group and Γ a torsion-free group hyperbolic relative to free abelian subgroups. Then there are only finitely many equivalence classes of maximal elements in the set of Γ-limit groups that are quotients of G.*

Proof. The following proof was explained to me by Zlil Sela in the context of torsion-free hyperbolic groups. The same proof works in the current context.

Suppose on the contrary that there are infinitely many non-equivalent maximal Γ-limit groups $R_1, R_2, \ldots$, each a quotient of G. Let $\eta_i : G \to R_i$ be the canonical quotient map. Fixing a finite generating set $\mathcal{A}$ for G, we fix a finite generating set for each of the R_i, and hence obtain maps $\nu_i : F_d \to R_i$, where $d = |\mathcal{A}|$. There is a fixed quotient map $\pi : F_d \to G$ so that for each i we have $\nu_i = \eta_i \circ \pi$.

For each i, consider the set of words of length 1 in F_d that are mapped to the identity by ν_i. This set is finite for each i, and there is a bound on its size, so there is a subsequence of the R_i so that this set is the same for all i. Starting with this subsequence, consider those words of length 2 in F_d which are mapped to the identity by ν_i, and again there is a subsequence for which this (bounded) collection is the same for all i. Continue with this process for all lengths of words in F_d, passing to finer and finer subsequences, and consider the diagonal subsequence. We continue to denote this subsequence by $R_1, R_2, \ldots$.

Now, for each i, choose a homomorphism $h_i : F_d \to \Gamma$ so that for words w of length at most i in F_d, we have $h_i(w) = 1$ if and only if $\nu_i(w) = 1$, and so that h_i factors through the quotient map $\pi : F_d \to G$. This is possible because each R_i is a Γ-limit group which is a quotient of G.

A subsequence of $\{h_i : F_d \to \Gamma\}$ converges into a Γ-limit group M, which is a quotient of G since all h_i factor through π. Let $\psi : F_d \to M$ be the canonical quotient, and $\phi : G \to M$ the quotient for which $\psi = \phi \circ \pi$. Note that a word w of length at most i in F_d maps to the identity under ψ if and only if $\nu_i(w) = 1$.

Now, $R_1, R_2, \ldots$ are non-equivalent maximal Γ-limit quotients, so (possibly discarding one R_i which is equivalent to M) are all non-equivalent to M. Therefore, for each i there does not exist a homomorphism $\mu : M \to R_i$ so that $\nu_i = \mu \circ \psi$. That is to say that for each i there exists $u_i \in F_d$ so that $\psi(u_i) = 1$ but $\nu_i(u_i) \neq 1$.

We now construct a new sequence of homomorphisms $\tau_i : F_d \to \Gamma$ that all factor through $\pi : F_d \to G$ so that

- a word $w \in F_d$ of length at most i satisfies $\tau_i(w) = 1$ if and only if $\nu_i(w) = 1$; and
- $\tau_i(u_i) \neq 1$.

By Theorem 5.5 there is a subsequence $\{\tau_{n_i}\}$ of $\{\tau_i\}$ which converges into a Γ-limit group (which must be M) so that each τ_{n_i} factors through $\psi : F_d \to M$. Therefore, there is $r_i : M \to \Gamma$ so that $\tau_{n_i} = r_i \circ \psi$. However, we have that $\psi(u_{n_i}) = 1$, but $1 \neq \tau_{n_i}(u_{n_i}) = r_i(\psi(u_{n_i})) = r_i(1) = 1$, a contradiction. This contradicts the existence of $R_1, R_2, \ldots$, and finishes the proof. □

Definition 5.14. *A group G is a called* equationally Noetherian *if every system of equations over G in finitely many variables is equivalent to a finite subsystem.*

The following theorem answers a question (essentially) asked by Sela [28, I.8(ii)]. We believe that relatively hyperbolic groups with abelian parabolics form a more natural context for this question than CAT(0) groups with isolated flats. In this context, [28, I.8(i)] was answered by the author in [17] and [28, I.8(iii)] is answered in Theorem 6.4 below.

Theorem 5.15 (cf. Theorem 1.22, [27])**.** *Suppose that Γ is a torsion-free relatively hyperbolic group with abelian parabolics.*

Proof. We follow the proof of [27, Theorem 1.22].

Let Σ be a system of equations in finitely many variables over Γ. We iteratively construct a directed locally finite tree as follows. Start with the first equation σ_1 in Σ, and associate with it a one relator group G_1, generated by the variables of Σ with relator corresponding to σ_1. By Proposition 5.13, to G_1 is associated finitely many maximal Γ-limit groups $R_1, \ldots, R_m$ which are quotients of G_1. Place G_1 at the root node of a tree, and a directed edge from G_1 to each R_i.

Now let σ_2 be the second equation in Σ, and consider each R_i in turn. If σ_2 represents the trivial element of R_i, leave it unchanged. If σ_2 is nontrivial in R_i, define $\widehat{R_i} = R_i / \langle \sigma_2 \rangle^{R_i}$. With $\widehat{R_i}$, we associate its finite collection of maximal Γ-limit quotients, and extend the locally finite tree by adding new vertices for these quotients of R_i, and directed edges joining R_i to each of its quotients.

Continue this procedure iteratively. By Theorem 5.2, each branch of this locally finite tree is finite, and therefore by Konig's Lemma the entire tree is finite. This implies that the construction of this tree terminates after finitely many steps, which implies that Σ is equivalent to a finite subsystem. □

Guba [19] proved the analogous theorem for free groups, whilst Sela [27] proved it for torsion-free hyperbolic groups.

6. Makanin-Razborov diagrams

In this final section, we describe the construction of Makanin-Razborov diagrams for Γ, which give a description of the set $\mathrm{Hom}(G, \Gamma)$, where Γ is an arbitrary finitely generated group. This is analogous to the constructions in [26, §5] and [27, §1].

Let R be a freely indecomposable Γ-limit group, and let $r_1, \ldots, r_m \in R$ be a fixed generating set for R. We assume that we always use the generating set $\{r_1, \ldots, r_m\}$ to define the length of homomorphisms, and hence to find short homomorphisms.

We need to understand those shortening quotients of R obtained from sequences of homomorphisms $\{h_n : R \to \Gamma\}$ so that each $h_n(R)$ is a proper quotient of R. By Proposition 4.1, each such shortening quotient is a proper quotient of R.

Following [26, 27] we say that two proper shortening quotients S_1, S_2 of R are *equivalent* if there is an isomorphism $\tau : S_1 \to S_2$ so that the canonical quotient maps $\eta_i : R \to S_i$, for $i = 1, 2$ satisfy $\eta_2 = \tau \circ \eta_1$.

This defines an equivalence relation on the set of shortening quotients of R, paired with the canonical quotient maps: $\{(S_i, \eta_i : R \to S_i)\}$.

Let $SQ(R, r_1, \ldots, r_m)$ be the set of (proper) shortening quotients of R. On the set $SQ(R, r_1, \ldots, r_m)$ we define a partial order as follows: given two proper shortening quotients S_1, S_2 of R, along with canonical quotients $\eta_i : R \to S_i$, we say that $S_1 > S_2$ if there exists a proper epimorphism $\nu : S_1 \to S_2$ so that $\eta_2 = \nu \circ \eta_1$.

Lemma 6.1 (cf. Lemma 1.23, [27]). *Let R be a freely-indecomposable Γ-limit group. Let $S_1 < S_2 < S_3 < \cdots$ (where $S_j \in SQ(R, r_1, \ldots, r_m)$) be a properly increasing sequence of (proper) shortening quotients of R. Then there exists a shortening quotient $S \in SQ(R, r_1, \ldots, r_m)$ so that for each j we have $S > S_j$.*

Proof. Restricting to short homomorphisms throughout, the proof is identical to that of Proposition 5.12 above. □

Lemma 6.2 (cf. Lemma 1.24, [27]). *Let R be a freely-indecomposable Γ-limit group. The set, $SQ(R, r_1, \ldots, r_m)$, of (proper) shortening quotients of R contains only finitely many equivalence classes of maximal elements with respect to the partial order.*

Proof. Once again, restricting throughout to short homomorphisms, the proof is identical to the of Proposition 5.13 above. □

We can now use shortening quotients to 'encode and simplify' all homomorphisms from a freely-indecomposable Γ-limit group into Γ.

Proposition 6.3 (cf. Proposition 1.25, [27]). *Suppose that R is a freely-indecomposable Γ-limit group. Let $r_1, \ldots, r_m \in R$ be a generating set for R, and let $M_1, \ldots, M_k$ be a set of representatives of the (finite) set of equivalence classes of maximal (proper) shortening quotients in $SQ(R, r_1, \ldots, r_m)$, equipped with the canonical quotient maps $\eta_i : R \to M_i$, for $i = 1, \ldots, k$.*

Let $h : R \to \Gamma$ be a homomorphism which is not an embedding. Then there exist a (not necessarily unique) index $1 \le i \le k$, a modular automorphism $\phi_h \in Mod(R)$, and a homomorphism $h_{M_i} : M_i \to \Gamma$ so that $h \circ \phi_h = h_{M_i} \circ \eta_i$/

Proof. Choose $\widehat{h} \sim h$ so that $\widehat{h}$ is short. The constant sequence $\widehat{h}, \widehat{h}, \ldots$ converges into a proper shortening quotient S of R. Now, $S \cong \widehat{h}(R)$, and the canonical quotient map is just $\widehat{h}$. There exists some M_i so that $M_i > S$ or M_i is equivalent to S. In either case, we get the required conclusion. □

Finally, we now construct Makanin-Razborov diagrams over Γ. Let G be an aribtrary finitely generated group. By Proposition 5.13, G has finitely many (equivalence classes of) maximal Γ-limit quotients, $R_1, \dots, R_s$, say. We now continue with each of the R_i in parallel. Let R be one such maximal Γ-limit quotient of G. The there is a free factorisation, $R = H_1 * \dots * H_l * F_g$, where each H_i is a freely-indecomposable non-cyclic subgroup of R and F_g is a finitely generated free group. Let $r_1^1, \dots, r_{m_1}^1 \in R$ generate H_1, and $r_1^2, \dots, r_{m_2}^2 \in R$ generate H_2, etc.

By Lemma 6.1 the set of (proper) shortening quotients of H_i contain maximal elements, and by Lemma 6.2 there are only finite many equivalence classes of maximal Γ-limit quotients of each H_i. For $i = 1, \dots, l$, let $M_1^i, \dots, M_{k_i}^I$ be a collection of representatives of the equivalence classes of maximal proper shortening quotients in $SQ(H_i, r_1^i, \dots, r_{m_i}^i)$, and let $\eta_j^i : H_i \to M_j^i$ be the canonical quotient map.

We now define the *Makanin-Razborov diagram* of G iteratively. Start by mapping G to its finite collection of maximal Γ-limit quotients, and continue with each of the maximal Γ-limit quotients in parallel. Denote such a maximal Γ-limit quotient of G by R. Factor R into $H_1 * \cdots * H_l * F_g$ as above. To each of the factors H_i associate k_i directed edges starting at H_i and terminating at M_j^i, a maximal shortening Γ-limit quotient of H_i. Do not proceed from F_g.

Now for each M_j^i we find a free product factorisation and to each (nonfree) factor associate the finitely many maximal shortening quotients. This procedure terminates after finitely many steps by Theorem 5.2 (and Konig's Lemma). We have constructed the Makanin-Razborov diagram associated to G.

In summary, we have

Theorem 6.4 (cf. Theorem 1.26, [27])**.** *Let G be a finitely generated group and Γ a torsion-free relatively hyperbolic group with abelian parabolics. Associated to G is a Makanin-Razborov diagram, with vertices Γ-limit quotients, and edges the canonical quotient map. Any homomorphism $h \in Hom(G, \Gamma)$ can be given by compositions of modular automorphisms of the Γ-limit groups in the diagram with the canonical maps from Γ-limit groups into their maximal proper shortening quotients, and finally with either embeddings of a Γ-limit group in the diagram into Γ, or general homomorphisms of the terminal free groups that appear in the diagram into Γ.*

Theorem 6.4 answers a question (essentially) asked by Sela [28, Problem I.8(iii)]. See the discussion above Theorem 5.15 above.

In the future work [18] we will continue the study of the elementary theory of Γ.

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