to r_π. Scattering $\pi^\pm$ from both ^{3}He and ^{3}H would be similar in principle to the $\pi^\pm$-α experiment, but much more complex to analyze, since ^{3}He and ^{3}H have different charge distributions. Such experiments might be of more interest in the context of probing nuclear structure, e.g., in determining the amount of S' or mixed-symmetry state.[20,21]

[20] K. Ananthanarayanan, Phys. Letters **19**, 43 (1965).
[21] G. Ramachandran and K. Ananthanarayanan, Nucl. Phys. **64**, 652 (1965).

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A Sum Rule Based on Unitarity*

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Unitarity, analyticity, Regge asymptotic behavior, and a resonance approximation are combined to derive a new sum rule. The sum rule is very convergent; the contribution of high-mass resonances is suppressed by a decreasing weight function. The spin-flip and non-spin-flip residues of the ρ meson in the $\pi\pi \to N\bar{N}$ amplitude are evaluated at the mass of the ρ, and in conjunction with the first-moment finite-energy sum rule, a calculation of the ρ-meson mass is performed. The results are in good agreement with experiment. A calculation of the ρ and f_0 resonance parameters in the $\pi\pi \to \pi\pi$ amplitude is also discussed.

I. INTRODUCTION

THE recent calculations of strong-interaction parameters from the finite-energy sum rules have been quite successful. The results have been in agreement with experiment to within the limits of error imposed by the model. Moreover, they have provided the bootstrap problem with a new approach which has already enjoyed some successes.[1,2]

The finite-energy sum rules relate all the moments of the discontinuity of the amplitude over a finite region in energy to the Regge parameters. However, in practice only the first few positive-moment sum rules have been used. The higher-moment sum rules emphasize higher-energy behavior so that in the context of most models they become redundant. As the negative-moment sum rules each contain the value of the amplitude or one of its derivatives at some point, they supply no additional constraints without prior knowledge of these unknown constants. It would be useful to have sum rules in which the weight function decreases, since even the low-positive-moment sum rules already put an uncomfortable emphasis on the higher-energy behavior of the discontinuity of the amplitude.

In Sec. II of this paper we derive a sum rule with a decreasing weight function by using two-body unitarity in the complex-J plane in addition to analyticity and Regge behavior. The weight function that multiplies the imaginary part of the amplitude is $Q_J(z)$. The derivation involves a small-width resonance approximation (*not* the usual narrow-width approximation), which we discuss in detail. By a small-width approximation we mean the width of the resonances we consider are small enough that the Breit-Wigner formula is reasonably accurate, but we do not take the limit $\mathrm{Im}\alpha \to 0$ in the discussion of this paper.

Finite-energy sum rules in general contain a parameter N, the upper limit of the integral of the imaginary part of the amplitude multiplied by some weight function. In order for these sum rules to be useful in bootstrap-type calculations, N must correspond to the "intermediate energies," so that the integrand may be parametrized by a sum of resonances. We find that for the sum rule presented here, the value of N depends on the magnitude of $\mathrm{Im}\alpha$. If N is to correspond to intermediate energies, $\mathrm{Im}\alpha$ can not be very small. Thus, for the

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[1] R. Dolen, D. Horn, and C. Schmid, Phys. Rev. **166**, 1768 (1968); C. Schmid, Phys. Rev. Letters **20**, 628 (1968).
[2] S. Mandelstam, Phys. Rev. **166**, 1539 (1968); D. Gross, Phys. Rev. Letters **19**, 1303 (1967).

conclusions of this paper it is crucial that experimentally near the mass of many prominent resonances, $\mathrm{Im}\alpha\approx 0.1$ and not much smaller. Note that 0.1 is small enough to allow neglecting terms of order $\mathrm{Im}\alpha$ compared to one, without causing a very large error. Of course, since we do not take the limit $\mathrm{Im}\alpha\to 0$, we must be careful that the coefficient of $\mathrm{Im}\alpha$ in such terms is not much larger than one. For example, when $\mathrm{Im}\,\alpha$ is multiplied by $2\ln(2N)$, as is the case of this sum rule, nonlinear terms in $\mathrm{Im}\alpha$ should also be included. Our small-width approximation does introduce an intrinsic error of about 15% into our calculations.

In Sec. III we apply the sum rule to a calculation of the non-spin-flip and spin-flip residues of the ρ trajectory at $t=m_\rho^2$ in the $\pi\pi\to N\bar{N}$ scattering amplitudes. The nucleon plus all the established πN resonances below 2-GeV center-of-mass (c.m.) energy constitute the input for the sum rule. The dominant contribution comes from the nucleon. Since the identical model can be applied to the finite-energy sum rules, it is interesting to ask whether our sum rule has dynamical content beyond that contained in the positive-moment sum rules. Our numerical calculations indicate that they have very different content and, therefore, our sum rule may be used in addition to the finite-energy sum rules to restrict further the resonance parameters of the model. In this case, we have four equations for three unknowns, so we have a check on the internal consistency of the model. The calculated mass of the ρ is about 900 MeV.

In Sec. IV we discuss a simple model of $\pi\pi\to\pi\pi$ scattering in which the amplitude is dominated by the ρ and f_0 resonances. Again, we obtain reasonable constraints on the resonance parameters. Together with the finite-energy sum rules, these equations provide a nearly complete bootstrap system. The slopes of the Regge trajectories [which are arbitrarily set at 1 $(\mathrm{GeV})^{-2}$] and a scale for the resonance widths can not be determined from the equations. (Our sum rule is not nonlinear enough in $\mathrm{Im}\alpha$ to obtain a scale for the resonance widths.) For such a simple model, the determined values of the masses and ratio of the widths are quite reasonable.

II. DERIVATION AND DISCUSSION OF THE SUM RULE

Let $A(t,z)$ be the amplitude for the elastic scattering of two spinless particles for which t is the square of the c.m. energy and z is the cosine of the scattering angle in the t channel. The asymptotic behavior of $A(t,z)$ as $z\to\infty$ is assumed dominated by the leading t-channel Regge pole which we denote by $R(t,z)$. Although the functional form of the Regge term is somewhat ambiguous, we require that $R(t,z)$ reduce to the correct resonance formula when $\alpha(t)$ is near an even (or odd) integer. Other modifications of $R(t,z)$ do not affect our derivation, and we choose the explicit form

$$R(t,z)=-\pi[2\alpha+1]\beta(t)\frac{P_{\alpha(t)}(-z)\pm P_{\alpha(t)}(z)}{2\sin\pi\alpha(t)},\qquad(1)$$

where $\pm$ refers to signature.

If t is near M^2, the mass of a resonance of spin L, then Eq. (1) reduces to the Breit-Wigner expression with $\mathrm{Im}\alpha=\alpha' M\Gamma$, where Γ is the decay width of the resonance.

The resonance pole also corresponds to a pole in the complex-J plane. The unitarity relation can be continued to complex J:

$$a^{\pm}(J,t)-a^{\pm *}(J^*,t)=2i\rho a^{\pm}(J,t)a^{\pm *}(J^*,t),\qquad(2)$$

where $\pm$ again refers to signature and t is in its physical region. If $a(J,t)$ has a pole at $J=\alpha$ and if $\mathrm{Im}\alpha$ is small compared to the distance to other singularities, then $a^*(J^*,t)$ may also be represented by a single pole plus a background of order $\mathrm{Im}\alpha$. Equation (2) then implies the relation[3]

$$\rho\beta=\mathrm{Im}\alpha+O((\mathrm{Im}\alpha)^2).\qquad(3)$$

Since $\mathrm{Im}\alpha$ is real, the phase of β is of order $\mathrm{Im}\alpha$,

$$\mathrm{Im}\beta=O((\mathrm{Im}\alpha)^2).\qquad(4)$$

Let $\alpha(t)$ be the position of the leading Regge pole. $a^{\pm *}(J^*,t)$ has a pole at $J=\alpha^*(t)$. With the Froissart-Gribov definition of $a(J,t)$, Eq. (2) implies

$$-\frac{1}{2i\rho(t)}=\lim_{J\to\alpha^*}\frac{1}{\pi}\int_{z_0}^{\infty}dz\,A_z^{\pm}(t,z)Q_J(z),\qquad(5)$$

where

$$A_z^{\pm}(t,z)=(1/2i)[A(t,z+i\epsilon)-A(t,z-i\epsilon)]\pm(1/2i)[A(t,-z-i\epsilon)-A(t,z+i\epsilon)].\qquad(6)$$

Since the integral in Eq. (5) is divergent for $\mathrm{Re}\alpha\geqslant\mathrm{Re}J$, it should be evaluated for $\mathrm{Re}\alpha<\mathrm{Re}J$ and then continued to $J=\alpha^*$.

If $A^{\pm}(t,z)$ is approximated to any preassigned accuracy by its leading Regge trajectory for $z\geqslant N$, we may rewrite Eq. (5) as

$$-\frac{1}{2i\rho}=\frac{1}{\pi}\int_{z_0}^{N}dz\,A_z^{\pm}(t,z)Q_{\alpha^*}(z)+\lim_{J\to\alpha^*}\frac{1}{\pi}\int_{N}^{\infty}dz\,R_z^{\pm}(t,z)Q_J(z)+\frac{1}{\pi}\int_{N}^{\infty}dz\,\bar{R}_z(t,z)Q_{\alpha^*}(z),\qquad(7)$$

where $\bar{R}$ is the contribution of the other J-plane singularities.

We propose that the real part of Eq. (7) may be applied to the calculation of some high-energy param-

[3] H. Cheng and D. Sharp, Ann. Phys. (N. Y.) **22**, 481 (1963); S. C. Frautchi, P. E. Kaus, and F. Zachariasen, Phys. Rev. **133**, B1607 (1964).

eters. In order to do so we will make the following two approximations. We will argue that for $\mathrm{Im}\alpha \approx 0.1$ and N reasonably large, the last integral in Eq. (7) is small compared to the first two, and that in the second integral, terms containing $\mathrm{Im}\beta$ are negligible.

We consider the integrals

$$\lim_{J\to\alpha^*}\int_N^\infty dz\, P_\alpha(z)Q_J(z) = \frac{(-\sin y + i\cos y)}{2\,\mathrm{Im}\alpha\,(2\,\mathrm{Re}\alpha+1)}[1+O(\mathrm{Im}\alpha)] \quad (8)$$

and

$$\lim_{J\to\alpha^*}\int_1^N dz\, P_\alpha(z)Q_J(z) = \frac{[\sin y + i(1-\cos y)]}{2\,\mathrm{Im}\alpha\,(2\,\mathrm{Re}\alpha+1)}[1+O(\mathrm{Im}\alpha)], \quad (9)$$

where $y = 2\,\mathrm{Im}\alpha\ln(2N)$, and we include only the leading terms in N. (For $N \geqslant 3$, the error is only a couple of percent.) The integral in Eq. (8) is proportional to the Regge integral in Eq. (7). Since experimentally the Regge contribution approximates the average magnitude of the imaginary part of the amplitude for small values of z (or s),[1] the integral in Eq. (9) can give a good estimate of the size of the first integral in Eq. (7).

From Eq. (1), the explicit expression for $R_z^\pm(t,z)$ is

$$R_z^\pm(t,z) = \pi(2\alpha+1)\beta(t)P_\alpha(z)\,, \text{ for right signature}$$
$$= 0\,, \qquad \text{for wrong signature.} \quad (10)$$

Substituting this in Eq. (7) and with the aid of Eq. (8), we obtain

$$\frac{\mathrm{Re}[\beta(M^2)]}{\mathrm{Im}\alpha}\sin y\left[1-\frac{\mathrm{Im}[\beta(M^2)]}{\mathrm{Re}[\beta(M^2)]}\cot y\right] = \mathrm{Re}\left[\frac{2}{\pi}\int_{z_0}^N dz\, A_z^\pm(t,z)Q_{\alpha^*}(z)\right] + \mathrm{Re}\left[\frac{2}{\pi}\int_N^\infty dz\, \bar{R}_z(t,z)Q_{\alpha^*}(z)\right]. \quad (11)$$

From Eq. (3) $\mathrm{Re}\beta/\mathrm{Im}\alpha$ is of order unity (for this discussion ρ may be set equal to one). Equation (4) implies that the quantity $\mathrm{Im}\beta/\mathrm{Re}\beta$ is of order $\mathrm{Im}\alpha$. Therefore, if $\sin y$ is not very small, or equivalently, if $\cot y$ is not much greater than one, we can neglect the term $(\mathrm{Im}\beta/\mathrm{Re}\beta)\cot y$. The left-hand side will then be equal to $(\mathrm{Re}\beta/\mathrm{Im}\alpha)\sin y$, a quantity which is of order one. The magnitude of the second integral on the right of Eq. (11) can be estimated by the contribution of the next leading singularity in the J plane. If α_2 and β_2 are the position and the residue of the singularity, then this contribution is of order $\beta_2/N^{(\alpha-\alpha_2)}$. As for the finite-energy sum rules, for large enough N this term is much smaller than the left-hand side. However, in order to justify neglecting this term we must show that the first integral on the right is also of order one. As mentioned before, the magnitude of the latter quantity can be estimated by considering the integral of Eq. (9). As N is increased from its minimum value, this integral starts from zero and grows in magnitude. For some value of N its magnitude actually becomes comparable to that of the integral in Eq. (8). This condition is realized when $\cot y \lesssim 1$. Thus $\cot y$ is a measure of the accuracy of both approximations, and the condition $\cot y \approx 1$ gives the desired relation between N and $\mathrm{Im}\alpha$. If $\mathrm{Im}\alpha$ is very small, $\cot y \approx 1$ demands an extremely large N. However, for the experimental value of $\mathrm{Im}\alpha \approx 0.1$, this condition is satisfied when $N \approx 10$, which corresponds to intermediate energies when t is in the resonance region. The next oscillation of $\cot y$ occurs for a very large value of N and does not concern us here.

Including the above approximations in Eq. (11), we write our sum rule in the final form of Eq. (12), correct to leading order in $1/N$ and $\mathrm{Im}\alpha$:

$$\sin(2\,\mathrm{Im}\alpha\ln 2N) = 2\rho(M^2)\,\mathrm{Re}\left\{\frac{1}{\pi}\int_{z_0}^N dz\, A_z^\pm(M^2,z)Q_{\alpha^*}(z)\right\}. \quad (12)$$

The successful calculations with the finite-energy sum rules suggest that, for $z<N$, $A_z^\pm(t,z)$ can be approximated by s- and u-channel resonances, even though t is outside the ellipse of convergence of the partial-wave expansions in the s and u channels. We have parametrized $A_z^\pm(M^2,z)$ by a sum of resonances and applied Eq. (12) to the calculation of some parameters in the $\pi\pi \to N\bar{N}$ and $\pi\pi \to \pi\pi$ processes. We have used δ functions for the s- and u-channel resonances for the sake of simplicity. When compared to the Breit-Wigner formula, this approximation causes only a 1% error for $\mathrm{Im}\alpha < 0.2$. However, the problem of a correct parametrization of $A_z^\pm(t,z)$ in the region $t>0$ needs a separate and thorough investigation.

III. CALCULATION OF ρ RESIDUES IN $\pi\pi \to N\bar{N}$

The derivation of sum rules like Eq. (12) for inelastic amplitudes with spin contains no essential complications. The procedure, which is similar to that for the spinless elastic case, is: (a) Substitute the fixed-t dispersion relation into the partial-wave formula (i.e., derive the Froissart-Gribov formula); (b) continue to $J=\alpha^*$ after introducing the leading Regge pole; (c) use unitarity to evaluate the amplitude at $J=\alpha^*$; (d) analyze the resulting equation as in Sec. II. There is only one modification of the basic formula, Eq. (7). If inelastic two-body intermediate states are included in the unitarity relation, the left-hand side of Eq. (7) will be $-1/2i\rho$ times a ratio of Regge residues. However, in our resonance approximation the real part of this ratio is proportional to the phase of the residues, which

TABLE I. Contribution of each resonance to our sum rules, Eqs. (21) and (22), and to the finite-energy sum rule, Eq. (24). The numerical values of the masses, widths, and inelasticities are found in Ref. 5. We used a value $g^2=14$ for the πN coupling.

Resonance	$L_{2I,2J}$ identification	Eqs. (21) and (22) r_+	r_-	Finite-energy sum rule r_+	r_-
Nucleon		2.70	8.533	0.05	0.62
$N'(1470)$	P_{11}	0.09	0.30	0.09	0.43
$N(1518)$	D_{13}	0.21	0.72	0.28	1.27
$N(1550)$	S_{11}	0.01	0.003	0.01	0.01
$N(1680)$	D_{15}	0.10	-0.19	0.29	-0.55
$N(1688)$	F_{15}	0.44	1.54	1.31	4.59
$N'(1710)$	S_{11}	0.02	0.01	0.08	0.04
$\Delta(1236)$	P_{33}	-0.99	1.42	-0.25	0.76
$\Delta(1640)$	S_{31}	-0.01	-0.003	-0.02	-0.01
$\Delta(1920)$	F_{37}	-0.12	0.25	-0.97	1.50
Total		2.45	12.6	0.87	8.66
$N(2190)$	G_{17}	0.11	0.36	2.02	3.79
$\Delta(2420)$	$H_{3,11}$	-0.05	0.11	-1.66	1.79

is of order $\mathrm{Im}\alpha$. Consequently, it can be neglected compared to terms of order 1. We apply this procedure to the spin-flip and non-spin-flip amplitudes in the $\pi\pi\to N\bar{N}$ channel and calculate the ρ residues.

The t channel is $\pi\pi\to N\bar{N}$, the usual invariant amplitudes are A and B, and some useful kinematical factors are

$$\begin{aligned} 2p&=(t-4M^2)^{1/2},\\ 2q&=(t-4\mu^2)^{1/2},\\ \Delta&=\tfrac{1}{2}(t-2\mu^2-2M^2),\\ z&=(s+\Delta)/2pq,\\ x&=|z|,\end{aligned}$$

and

$$\nu=2M(s+\Delta)/(4M^2-t). \tag{13}$$

The nucleon and pion masses are denoted by M and μ, respectively. The value $t=m_\rho^2$ is below the $N\bar{N}$ threshold, so that p and z are both pure imaginary ($z=-ix$).

The amplitudes $A'=A+\nu B$ and B are proportional to the helicity amplitudes in the t channel:

$$\begin{aligned} T_+(t,z)&=A(t,z)+\nu B(t,z),\\ T_-(t,z)&=\sin\theta_t\, B(t,z), \end{aligned}\tag{14}$$

where θ_t is the c.m. scattering angle, and $+$ and $-$ refer to the $++$ and $+-$ helicity amplitudes. The isotopic-spin projections are

$$A'^{(-)}=\tfrac{1}{3}[A'(I=\tfrac{1}{2})-A'(I=\tfrac{3}{2})], \tag{15}$$

and a similar relation for B. The $(-)$ amplitudes are pure $I=1$ in the t channel. The partial-wave unitarity relations for $T_\pm(t,z)$ with only the two-pion intermediate states are

$$T_\pm(J,t)-T_\pm^*(J^*,t)=2i\rho T_\pm(J,t)T_\pi^*(J^*,t), \tag{16}$$

where $T_\pi(J,t)$ is the amplitude for $\pi\pi\to\pi\pi$.

We assume the ρ trajectory gives the asymptotic behavior of $A'^{(-)}$ and $B^{(-)}$ as $z\to\infty$. The Regge terms are[4]

$$A_R'(t,z)=-\pi(2\alpha+1)\frac{P_\alpha(-z)-P_\alpha(z)}{2\sin\pi\alpha}\times\frac{16\pi M}{4M^2-t}\left(\frac{pq}{M^2}\right)^\alpha r_+(t) \tag{17}$$

and

$$B_R(t,z)=\pi\frac{P_\alpha'(-z)+P_\alpha'(z)}{2\sin\pi\alpha}\frac{8\pi}{M^2}\left(\frac{pq}{M^2}\right)^{\alpha-1}r_-(t), \tag{18}$$

where P_α' is the derivative of the Legendre polynomial. The normalization of the residues is identical to the normalization of the $r_+(t)$ and $r_-(t)$ defined by Desai in Ref. 4. We continue the Froissart-Gribov formula to $t=m_\rho^2$ and $J=\alpha^*$ and use unitarity to evaluate the amplitude at this point. To leading order in N, we find

$$-\frac{1}{2i\rho}\frac{r_+^*(m_\rho^2)}{\beta_\pi^*(m_\rho^2)}=\frac{4M^2-m_\rho^2}{8\pi^2M}\left(\frac{M^2}{i\bar{p}q}\right)^\alpha\times\int_{x_0}^{\bar{N}}(-i)dx\,A_z'(m_\rho^2,-ix)Q_{\alpha^*}(-ix)-\frac{r_+(m_\rho^2)}{2i\,\mathrm{Im}\alpha}(2i\bar{N})^{2i\mathrm{Im}\alpha} \tag{19}$$

and

$$-\frac{1}{2i\rho}\frac{r_-^*(m_\rho^2)}{\beta_\pi^*(m_\rho^2)}=\frac{M^2}{4\pi^2}\left(\frac{M^2}{i\bar{p}q}\right)^{\alpha-1}\int_{x_0}^{\bar{N}}(-i)dx\,B_z(m_\rho^2,-ix)\times[Q_{\alpha^*-1}(-ix)-Q_{\alpha^*+1}(-ix)]-\frac{r_-(m_\rho^2)}{2i\,\mathrm{Im}\alpha}(2i\bar{N})^{2i\mathrm{Im}\alpha}. \tag{20}$$

We have already dropped the contribution of the other J-plane singularities [the $\bar{R}$ term of Eq. (7)], and in evaluating the Regge integral, some trivial factors of order $\mathrm{Im}\alpha$ have also been dropped. The bars denote absolute values of quantities that become complex for $t<4M^2$: $N=-i\bar{N}$, $p=i\bar{p}$, and $z=-ix$.

When the real parts of Eqs. (19) and (20) are evaluated, we find the terms on the left are of order $\mathrm{Im}\alpha$. To leading order in $\mathrm{Im}\alpha$, the sum rules for $r_\pm(m_\rho^2)$ are

$$r_+(m_\rho^2)=-\lambda^{-1}\frac{(4M^2-m_\rho^2)}{8\pi^2\bar{p}q}\times\int_{x_0}^{\bar{N}}dx\,A_z'(m_\rho^2,-ix)Q_1(-ix) \tag{21}$$

and

$$r_-(m_\rho^2)=\lambda^{-1}\frac{M^2}{4\pi^2}\times\int_{x_0}^{\bar{N}}dx\,B_z(m_\rho^2,-ix)[-iQ_0(-ix)+iQ_2(-ix)], \tag{22}$$

[4] B. Desai, Phys. Rev. 142, 1255 (1966).

where

$$\lambda = e^{\pi I m\alpha}[\sin(2\,\mathrm{Im}\alpha\,\ln 2N)]/(2\,\mathrm{Im}\alpha)$$

and, for example,

$$-iQ_0(-ix) = \arctan(1/x).$$

In order to test these sum rules, we saturated the discontinuity of the amplitudes with the s- and u-channel πN resonances. We parametrized these resonances as δ functions and included the nucleon and all the established πN resonances up to 2 GeV (N corresponds to 2 GeV). The masses, widths, and inelasticities were obtained from the most recently available experimental data.[5]

The resonances and their contributions to the sum rule are listed in Table I. We have also listed the contribution of several prominent resonances above 2 GeV to show the size of these terms in the sum rule. (Of course, they are not included in the final sum.) We set $m_\rho = 775$ MeV and $\Gamma_\rho = 140$ MeV, and obtain

$$r_-(m_\rho^2) = 12.6$$

and

$$r_+(m_\rho^2) = 2.45. \tag{23}$$

These results are increased by only 10% if we set $\Gamma_\rho = 90$ MeV; the sensitivity to changes of m_ρ is much greater. As in other calculations of the ρ residues, the ratio of r_- to r_+ is large, $r_-/r_+ = 5.1$. As discussed in Sec. II, $\cot y$ should not be too much greater than one. However, we are forced by the experimental data to choose N corresponding to 2 GeV. Then $\cot y = 1.7$, so that the intrinsic error of the sum rule is at least 15%. Our determination of $r_\pm(m_\rho^2)$ is about 3 times larger than Desai's form-factor calculation.[4] [Desai quotes the numbers $r_+(m_\rho^2) = 0.87$ and $r_-(m_\rho^2) = 3.98$.] The models involved in all of these calculations are crude enough that this probably is not a serious discrepancy.

Comparison with the finite-energy sum rules leads to some interesting conclusions. We may use the same model in both sum rules. If this model is a "good" model, we should be able to use sum rules of different "physical content" to derive restrictions on the free parameters of the model. Our sum rule makes some use of direct-channel unitarity, and therefore it has a chance of differing with finite-energy sum rules. It is almost obvious from Table I that our sum rule is not equivalent to the positive-moment sum rules. Our rule emphasizes the nucleon and low-mass resonances, but the principal contribution to the lowest-moment finite-energy sum rule is the $N(1688)$. The higher-moment sum rules emphasize the higher-mass resonances even more.

There is yet a better check on the nonequivalence of the two sum rules. If the sum rules had the *same* content, then a plot of $r_\pm(m_\rho^2)$ and m_ρ from our sum rule should be coincident with the curve from the

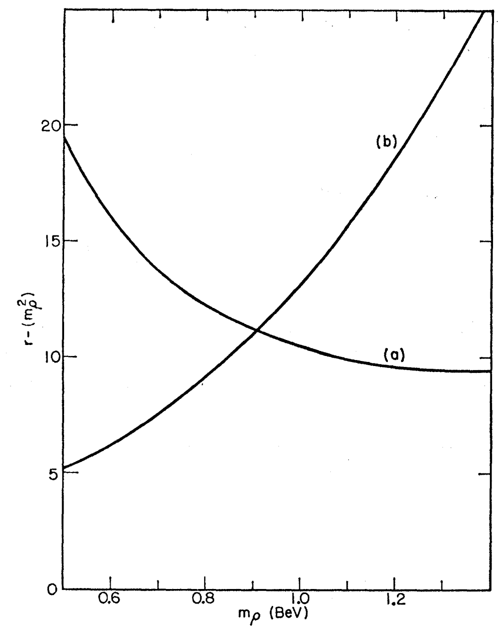

FIG. 1. The ρ residue $r_-(m_\rho^2)$ is plotted as a function of m_ρ. Curve (a) results from our sum rule, Eq. (22). The finite-energy sum rule, Eq. (24), produces curve (b). In both calculations we have used the resonances listed in Table I; the numerical values of the parameters are listed in Ref. 5.

finite-energy sum rule. (If the resonance model is not perfect, then the curves would only approximately duplicate each other.) If the sum rules have different content, the requirement of consistency within the model places bootstraplike restrictions on other model parameters. In this case, the only free parameter is the ρ mass. (Our sum rule is not dependent enough on $\mathrm{Im}\alpha$ to calculate it; the finite-energy sum rule does not depend on $\mathrm{Im}\alpha$ at all.) Since there are four sum rules and only three parameters [$r_\pm(m_\rho^2)$ and m_ρ], we also have one internal check on the model.

The ρ residues are calculated from the finite-energy sum rules using equations like

$$r_-(m_\rho^2) = \frac{M^2}{2\pi^2 \bar{N}^2} \int_{x_0}^{\bar{N}} dx\, x B_z(m_\rho^2, -ix), \tag{24}$$

and a similar relation for $r_+(m_\rho^2)$. We plot $r_-(m_\rho^2)$ in Fig. 1 from Eqs. (21) and (24). From this plot m_ρ is seen to be about 900 MeV. Similar results are obtained for $r_+(m_\rho^2)$ with $m_\rho = 1040$ MeV. This is consistent with the results of the $r_-(m_\rho^2)$ sum rules, because the $r_+(m_\rho^2)$ finite-energy sum rule is a rapidly varying function of m_ρ at $m_\rho \approx 900$ MeV and is very sensitive to the exact parameters (or existence) of the more massive resonances. Thus, there is enough internal consistency in the model so that we conclude that our sum rule has different content from the finite-energy sum rule, and may even be used with the finite-energy sum rules to obtain restrictions on the model parameters.

[5] A. H. Rosenfeld *et al.*, January 1968 Wallet Sheets, Lawrence Radiation Laboratory, Berkeley (unpublished).

Our sum rule together with the finite-energy sum rule and the resonance model of A_z' and B_z yield $r_-(m_\rho^2)=11$, $r_+(m_\rho^2)=2.5$, and $m_\rho=900$ MeV. It is difficult to attach errors to these numbers, but in view of the intrinsic error of 15% in our sum rule and a possibly incomplete model for the finite-energy sum rule, an error of at least 40% is reasonable.

IV. A MODEL OF THE ρ AND f_0 MESONS

We consider a model of the $\pi\pi\rightarrow\pi\pi$ amplitude in which only the ρ and f_0 resonances are included. The poles at $t=m_f^2$ and at $t=m_\rho^2$ lie on the leading Regge trajectories for isospin zero and one, respectively. We may then write the sum rule, Eq. (12), at each of these values of t. If we also saturate the discontinuity of the cross-channel amplitudes with the ρ and f_0, we obtain a set of two equations,

$$\sin(\alpha_i' M_i\Gamma_i \ln 2N_i)=\sum_j 2X^{ij}\frac{m_j^2\Gamma_j}{m_i q_i q_j} \times(2L_j+1)P_{L_j}(z_{ij})Q_{L_i}(z_{ij}), \quad (25)$$

where i and j both correspond to the ρ or f_0. Other symbols are

$$4q_i^2=m_i^2-4\mu^2,$$

$$z_{ij}=1+m_j^2/(2q_i^2),$$

and

$$N_i=1+s_1^2/(2q_i^2). \quad (26)$$

Also, m_i, Γ_i, and L_i denote the mass, width, and spin, respectively, of the ρ or f_0; X^{ij} is a 2×2 submatrix of the isospin crossing matrix,

$$X=\begin{pmatrix} \frac{1}{3} & 1 \\ \frac{1}{3} & \frac{1}{2} \end{pmatrix}, \quad (27)$$

and s_1 is a c.m. energy between 1250 and 1600 MeV. We set $\alpha_\rho'=\alpha_f'=1$ $(\text{GeV})^{-2}$ in the following calculations.

Equation (25) is then two relations among the masses and widths of the ρ and f_0 mesons. The solutions of these equations can be studied numerically. The results are very encouraging, considering that only the grossest features of π-π scattering have been included in this model. (For example, we have assumed that the imaginary part of the $I=0$ amplitude is zero up to $t=m_f^2$.)

We search for solutions of Eq. (25) by scanning over values of the parameters 400 MeV$\leqslant m_\rho\leqslant$800 MeV, 800 MeV$\leqslant m_f\leqslant$1400 MeV, and $0.3\leqslant(\Gamma_f/\Gamma_\rho)\leqslant3$. Because of the factor $\sin y$, the equation we are using is somewhat nonlinear. Although this nonlinearity serves to exclude some of the solutions, it cannot give an absolute scale for the widths. In one set of solutions we find that m_ρ increases from 500 to 700 MeV as (Γ_ρ/Γ_f) increases from 0.3 to 1.5 and as m_f decreases from 1200 to 900 MeV. To complete the bootstrap, we use the finite-energy sum rules which pick out the solution $m_\rho=540$ MeV; $m_f=1150$ MeV; and $(\Gamma_f/\Gamma_\rho)=3$. Although the dependence on N is large in the finite-energy sum rules, it is sufficiently small in Eq. (25) to limit the uncertainty in these numbers due to variations in N to ±20 MeV.

We should explain why m_ρ is too small in this model and too large in Sec. III. The bootstrap of the ρ mass in the $\pi\pi\rightarrow N\bar{N}$ system depends on the accuracy of the finite-energy sum rule, which we discussed in Sec. III. In this bootstrap we have not taken proper account of the low-energy π-π scattering in Eq. (25). Thus, the two-resonance model is probably untrustworthy in our sum rule.

Another disjoint set of solutions contain the physical masses of the ρ and the f meson but give a width of about 350 MeV for the f_0. We find that we can bring the solutions of Eq. (25) into agreement with experiment by either of two mechanisms. We can fit the experimental parameters by inserting a scalar meson. The scalar meson is broad with a mass of about 500 meV. Also, simple models for the $\pi\pi$ threshold bring the results into good accord with experiment.

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