COULOMB'S FUNCTION

By H. Bateman

Norman Bridge Laboratory of Physics, California Institute of Technology

Communicated June 18, 1938

1. In his work on Rayleigh waves Coulomb[1] has studied the function

$$\psi_n(hw,\ hw \operatorname{sh} a) = i^n \int_a^\infty e^{-ihw \operatorname{ch} u} \operatorname{ch}(n\,u)du. \tag{1.1}$$

$$\operatorname{ch} a = \cosh a,\ \operatorname{sh} a = \sinh a$$

The function ψ_0 with a complex value of h occurs in the work of Buchholz[2] on the propagation of alternating currents in the earth between two electrodes connected above ground by a rectangular loop of wire whose vertical ends support the horizontal piece. Use will be made here of the notation

$$C_n(a,\ x) = \int_a^\infty e^{-x \operatorname{ch} u} \operatorname{ch}(n\,u)du,\ S_n\ (a,\ x) = \int_a^\infty e^{-x \operatorname{sh} u} \operatorname{ch}(nu)du \tag{1.2}$$

wherein $R(x) > 0$ and $a > 0$. When $n = 0$ expansions of these functions are readily obtained by putting $x \operatorname{ch} u = v$, $x \operatorname{ch} a = c$ in the first integral and $x \operatorname{sh} u = w$, $x \operatorname{sh} a = s$ in the second. With the notation $(m/,\ n)$ for the binomial coefficient $\mathcal{C}_{m,n}$ and the notation

$$Q(z,\ k) = \int_z^\infty e^{-t} t^{k-1} dt \tag{1.3}$$

for the incomplete Gamma function of the second kind, the expansions obtained by using the binominal theorem are

$$C_o(a,\ x) = \sum_{n=0}^{\infty} (-)^n(-{}^1/_2/,\ n)x^{2n}\ Q(c,\ -2n) \tag{1.4}$$

$$S_o(a,\ x) = \sum_{n=0}^{\infty} (-{}^1/_2/,\ n)x^{2n}\ Q(s,\ -2n).$$

The first of these expansions is given by Coulomb and Buchholz. The convergence of the series may be established by using the formula

$$Q(c,\ -m) = e^{-c}c^{-m}/m - e^{-c}c^{-m+1}/m(m-1) + \ldots (-)^m\, Q(c, o)/m!$$

The second series converges absolutely when $\operatorname{sh} a > 1$. When each term is transformed by using the formula

$$\Gamma(2n+1)e^s\, Q(s,\ -2n) = \int_0^\infty e^{-st}\, t^{2n}\, dt/(1 + t)$$

as in Buchholz's transformation of the series for $C_o(a,\ x)$, we find that

$$S_o(a, x) = \int_0^\infty \exp\,[-(1+t)x \operatorname{sh} a]\, J_o(xt)\, dt/(1+t), \quad (1.5)$$

while the corresponding formula of Buchholz is

$$C_o(a, x) = \int_0^\infty \exp\,[-(1+t)x \operatorname{ch} a]\, I_o(xt)\, dt/(1+t). \quad (1.6)$$

It should be noticed that by expanding $1/(1 + t)$ in powers of t and integrating term by term we obtain the same asymptotic series for $S_o(a, x)$ as is obtained from (1.2) by repeated integration by parts.

A relation between $C_n(a, x)$ and $S_n(a, x)$ may be found by putting $s = \operatorname{sh} u$ in the integral

$$\int_0^\infty e^{-sz} J_o[(z^2 + 2xz)^{1/2}]\, dz = (1 + s^2)^{-1/2} \exp[-x\{(1 + s^2)^{1/2} - s\}], \quad (1.7)$$

multiplying by $\operatorname{ch} nu\, e^{-x \operatorname{sh} u}\, du$ and integrating from a to ∞. This gives

$$C_n(a, x) = -(d/dx) \int_a^\infty e^{-x \operatorname{ch} u} \operatorname{ch}(nu) du/\operatorname{ch} u \quad (1.8)$$

$$= -(d/dx) \int_0^\infty S_n(a, x \operatorname{sh} a \operatorname{ch} v) J_o(x \operatorname{sh} v) x \operatorname{sh} v\, dv.$$

If, on the other hand, we multiply (1.7) by $e^{-x \operatorname{sh} u} \operatorname{ch} u\, du$ and integrate from a to ∞ we find that

$$C_o(a, x) = \int_0^\infty e^{-x \operatorname{ch} v \operatorname{sh} a} J_o(x \operatorname{sh} v) \tanh v\, dv. \quad (1.9)$$

2. Another expansion for $C_o(a, x)$ may be found by using the function

$$V_n(x) = \int_0^\infty e^{-xt} P_n\left(\frac{t-1}{t+1}\right) dt/(t+1) = e^x \int_x^\infty e^{-u} P_n(1 - 2x/u) du/u \quad (2.1)$$

which has been studied in a former paper.[3] If $V_n(x) = e^x W_n(x)$ there is an expansion

$$W_n(x) = \sum_{m=0}^{n} (-n/, m)(n + m/, m) x^m Q(x, -m) \quad (2.2)$$

a differential equation

$$x^2 W_n''' + (x^2 + 3x) W_n'' + (2x + 1) W_n' - n(n + 1) W_n = 0 \quad (2.3)$$

and recurrence relations

$$x(W_n' + W_{n-1}') = n(W_n - W_{n-1}).$$

$$(n + 1) W_{n+1}' + n W_{n-1}' - (2n + 1) W_n' = 2(2n + 1) W_n.$$

$$(4n + 2)x(W_n' + W_n) = (n + 1)^2 (W_{n+1} - W_n) + n^2(W_n - W_{n-1}),$$

$$x(W_{n+1}'' - W_{n-1}'') = 2(2n + 1)W_n + W_{n-1}' - W_{n+1}',$$

$$n^2W_{n-1} = 2x^2W_n'' + [2x^2 - 2(n - 1)x]W_n' - n(2x - n)W_n$$

$$(n + 1)^2W_{n+1} = 2x^2 W_n'' + [2x^2 + 2(n + 2)x]W_n' + (n + 1)(2x + n + 1)W_n$$

$$(2n - 1)(n + 1)^2 (W_{n+1} - W_n) - 2n(2n^2 - 1)(W_n - W_{n-1}) + (n - 1)^2(2n + 1)(W_{n-1} - W_{n-2}) = 2x(4n^2 - 1)(W_n + W_{n-1}). \quad (2.4)$$

The differential equation for $W_n(x)$ is adjoint to the differential equation

$$x^2Z_n''' + (3x - x^2)Z_n'' + (1 - 2x)Z_n' + n(n + 1)Z_n = 0, \quad (2.5)$$

which is satisfied by the function $Z_n(x) = F(-n, n + 1; 1,1; x)$ which was studied at the same time[3] as $V_n(x)$. This function $Z_n(x)$ may be used to obtain the representation

$$V_n(z) = \lim_{x \to 1} Z_n(-d/dx) V_0(zx). \quad (2.6)$$

The generating function of $W_n(x)$ suggests the expansion

$$\int_a^\infty e^{-z \operatorname{ch} u} du = (1 - e^{-a}) \sum_{n=0}^{\infty} e^{-na} W_n[z(\operatorname{ch} a - 1)] \quad (2.7)$$

which is certainly convergent when $z > 0$, $a > 0$ but may be valid under more general conditions.

3. In the physical investigations the wave potential connected with $C_o(a, x)$ $C(a, x)$ is

$$W = \int_z^\infty e^{-kR} ds/R, \quad (3.1)$$

where k is a complex constant and $R^2 = s^2 + w^2 = s^2 + x^2 + y^2$, x, y and z being rectangular coördinates. With $s = w \operatorname{sh} u$, $z = w \operatorname{sh} a$ the integral is $C_o(a, kw)$ and if $r^2 = z^2 + w^2 = x^2 + y^2 + z^2 = w^2 \operatorname{ch}^2 a$ the expansion of the integral W is

$$W = e^{-kw}(1 - e^{-a}) \sum_{n=0}^{\infty} e^{-na} W_n(kr - kw)$$

$$= e^{-kw} \sum_{n=0}^{\infty} [(r - z)/w]^n [W_n(kr - kw) - W_{n-1}(kr - kw)] \quad z>0 \quad (3.2)$$

where it is understood that $W_{-1}(x) \equiv 0$. It is thought that this expansion will converge rapidly. This surmise can be checked as soon as the tables of the function $W_n(x)$ have been completed.

If $v = w\,(\text{ch}\, a - 1) = r - w$, there is also an expansion

$$\int^{\infty} e^{-kw\,\text{ch}\,u} du = \sum_{n=0}^{\infty} (-{}^1/_2/, n)(2w)^{2n}\, Q(kv, -n). \qquad (3.3)$$

4. There is an integral relation

$$\int_0^{\infty} t^{s-1}\, W_n(t) dt = \Gamma(s) G_n(s) \quad R(s) > 0 \qquad (4.1)$$

in which

$$s(1 + s)(2 + s) \ldots (n + s) G_n(s) = (1 - s)(2 - s) \ldots (n - s). \qquad (4.2)$$

This is readily derived from (2.2) and suggests the new definition

$$W_n(t) = ({}^1/_{2\pi i}) \int_{c - i\infty}^{c + i\infty} t^{-s} \Gamma(s) G_n(s) ds \qquad (4.3)$$

which, when $c > 0$, may be found directly by an attempt to solve the differential equation by means of a definite integral. The function $G_n(s)$ occurs as a coefficient in the expansion

$$(\text{ch}\, a - 1)^s \int_a^{\infty} (\text{ch}\, u - 1)^{-s} du = (1 - e^{-a}) \sum_{n=0}^{\infty} e^{-na} G_n(s). \quad 0 < s < 1 \qquad (4.4)$$

5. Another representation of $W_n(x)$ which may be useful in finding new properties of the function is

$$W_n(x) = (\pi/x)^{1/2} \int_0^{\infty} J_0(u) I_{n+1/2}(u^2/8x) \exp(-u^2/8x) du. \qquad (5.1)$$

This is valid so long as $R(x) > 0$. The formula may be checked by means of the recurrence formulae for $W_n(x)$ and the finite series for $I_{n+1/2}(z)$.

6. An asymptotic expansion for $V_n(x)$ for large values of z such that $R(z) > 0$ may be obtained from the series

$$V_n(z) = \sum_{m=0}^{n} (-n/, m)(n + m/, m) \int_0^{\infty} e^{-zt}(t + 1)^{-m-1} dt \qquad (6.1)$$

by using the asymptotic expansion of each of the integrals, it is

$$V_n(z) \sim \sum_{r=0}^{\infty} \sum_{m=0}^{n} (-n/, m)(n + m/, m)(-m - 1/, r) z^{-r-1}\, r! = \sum_{r=0}^{\infty} F_n(2r + 1) z^{-r-1} (-)^n \qquad (6.2)$$

where $F_n(x)$ is the polynomial studied in a former paper.[4]

7. It follows at once from the formula

$$V_n(x) = \int_0^\infty e^{-t} Z_n(t)dt/(x+t) \tag{7.1}$$

that

$$V_n(x) = V_o(x)Z_n(-x) + p_{n-1}(x) + p_{n-2}(x) + \ldots + p_o(x), \tag{7.2}$$

where $p_{n-1}(x)$ is a polynomial of degree $n - 1$ in x. Hence as $x \to 0$ $V_n(x) - V_{n-1}(x) \to p_{n-1}(0)$. To verify that this is equal to $-2/n$ we may use the last recurrence formula (2.4), this value being readily found when $n = 1$ and $n = 2$. This leads to the formula

$$\lim_{x \to 0} [V_o(x) - V_n(x)] = 2\left(1 + \frac{1}{2} + \ldots + \frac{1}{n}\right). \tag{7.3}$$

On account of this relation it is often convenient to transform a series of type $\sum c_n V_n(x)$ into one of type

$$\sum_{n=0}^{\infty} b_n [V_n(x) - V_{n-1}(x)]$$

on the understanding that $V_{-1}(x) = 0$. The convergence of the resulting series as $n \to 0$ may then be readily tested. In particular it is found that the series (3.2) fails to converge when $z = 0$ as is to be expected.

[1] J. Coulomb, *Annales de Toulouse* (3), **23**, 91–137 (1931).

[2] H. Buchholz, *Arkiv für Elektrotechnik*, **30**, 1–33 (1936).

[3] H. Bateman, *Duke Math. Jour.*, **2**, 569–577 (1936).

[4] H. Bateman, *Tôhoku Math. Jour.*, **37**, 23–38 (1933).

[5] S. O. Rice, *Bell System Technical Jour.*, **16**, 101–109 (1937).

[6] H. Bateman and S. O. Rice, *Amer. Jour. Math.*, **60**, 297–308 (1938).

[7] The function $C_o(a, x)$ occurs also in a paper by S. O. Rice[5] in which a transformation is given of van der Pol's expression for the value on the ground of the wave-function of a vertical dipole placed at the surface of a plane earth. Expansion for $C_o(a, x)$, $C_n(a, x)$ may also be found by using the integrals (14) and (15) in a recent paper by the author and S. O. Rice.[6]