[11] These mobilities when determined from concentration effects are largely independent of partition coefficients. See footnote 3.

[12] Osterhout, W. J. V., *Jour. Gen. Physiol.*, **18,** 992 (1934–1935).

[13] Cf. Osterhout, W. J. V., and Hill, S. E., *Proc. Soc. Exptl. Biol. and Med.*, **32,** 715 (1934–1935).

[14] Unpublished. For changes in mobilities produced by guaiacol in *Valonia*, see Osterhout, W. J. V., *Jour. Gen Physiol.*, **20,** 13, 685 (1936–1937).

RAYLEIGH WAVES

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1. *The Interaction of the Air and the Ground.*—The transmission of plane longitudinal waves of unlimited extent from the ground to the air was investigated by C. G. Knott[1] many years ago. He found that the resulting air waves, which are propagated in almost a vertical direction, generally have only a small amount of energy. Earthquake sounds have been studied by many writers. C. Davison[2] put forward the theory that they originate from the margin of the region disturbed by the earthquake and travel some distance through the earth before being transmitted to the air. A summary of results relating to earthquake noises has been given by Landsberg.[3] The type of air motion considered here is not the simple progressive wave in an unlimited atmosphere but is a type of free vibration of the air and ground having the characteristics of a Rayleigh wave except that its velocity of propagation is less than the velocity of sound in air instead of being slightly less than the velocity of a shear wave in the ground. The mathematical analysis is very similar to that used by Stoneley[4] in his study of Rayleigh waves in a plane homogeneous elastic earth below a compressible sheet of water of unlimited extent. It is assumed here, however, that the vertical velocity of the air is negligible at a height H above the ground while in Stoneley's work the boundary condition at the free surface of the water is one of constant pressure. His remarks on nodal planes indicate that his analysis may be applicable in our case but it has been thought worth while to give the analysis again in a form in which the velocity of the wind is taken into consideration and some of Stoneley's approximations are omitted. It is thought that the analysis may be of some interest in connection with the interpretation of the ground roll observed in geophysical field work. For information relating to the ground roll I am indebted to Dr. Gutenberg, Mr. Martin Gould and other members of the group connected with the Pasadena Seismological Laboratory. It

has generally been assumed, of course, that the influence of the air on the propagation of seismic waves is slight but such an assumption ought to be justified by numerical work in the case of waves produced by an artificial explosion for there are some features of the phenomena that are not fully elucidated. The problem resembles that of the loud speaker with infinite baffle, the disturbed area of the earth corresponding to the membrane that is set in vibration. Now in the theory of the loud speaker the short circuiting of energy is a familiar phenomenon, there is not simply a radiation of energy outwards. If, then, there is a similar short circuiting of energy in the air after an explosion, an interaction of air and ground is to be expected. If the air and ground are treated as a coupled system, an explosion may be expected to give rise to a subsequent motion that is composed of free vibrations of the system and the particular type of motion to be studied is, indeed, a free vibration. The whole problem is, then, one of the partition of energy among a number of free vibrations including in particular the ordinary Rayleigh wave and the new type of Rayleigh wave. This second type of Rayleigh wave is called "new" merely to distinguish it from the old type but it has been known for a long time that there is more than one type of Rayleigh wave for a stratified medium. With regard to the likelihood of the existence of a marked interaction between the earth and the air it should be mentioned that many years ago the late Lord Rayleigh[5] concluded that in the vicinity of a vibrating body of linear dimensions small in comparison with the wave-length, the air acts as if it were almost incompressible while the great mass of air at some distance from the body is slightly compressed periodically. A similar conclusion has been reached more recently by Lennard Jones[6] after some elaborate calculations. In trying to apply this result to our problem we are led to surmise that when the ground rises initially after an explosion the air immediately above it will either move away laterally and produce a reaction on the ground somewhere else or will try to lift or compress the great body of air above it.

2. *The Problem of a Flat Earth and a Homogeneous Atmosphere.*—Taking the axis of x along the ground in the direction of propagation and the axis of z vertically downward, the component displacements, u, w, in the earth, which for simplicity is supposed to be homogeneous, can, as in the theory of the late Lord Rayleigh,[7] be expressed in the form.

$$u = u_1 + u_2, \qquad w = w_1 + w_2,$$

where

$$u_1 = ifPe^{-rz - ifX}, \quad w_1 = rPe^{-rz - ifX}$$
$$u_2 = isQe^{-sz - ifX}, \quad w_2 = fQe^{-sz - ifX}$$

$$X = x - vt, \quad v = \text{velocity of propagation,}$$
$$r^2 = f^2(1 - v^2/a^2) = f^2C^2(v), \text{ say,}$$
$$s^2 = f(1 - v^2/b^2) = f^2S^2(v), \text{ say.}$$

The symbol a is used here to denote the velocity of propagation of longitudinal waves and the symbol b to denote the velocity of propagation of waves of shear.

Supposing for simplicity that the undisturbed air has a velocity V (independent of z, x and t) parallel to the axis of X, a constant density ρ and a constant velocity of sound c, the velocity potential, ϕ, of a small irrotational perturbation satisfies the partial differential equation

$$c^2\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2}\right) = \left(\frac{\partial}{\partial t} + V\frac{\partial}{\partial x}\right)^2\phi.$$

This equation has a solution of the form

$$\phi = Ae^{-mz - ifX} + Be^{mz - ifX}$$

in which A, B, m are constants, if m is given by the equation

$$m^2 = f^2[1 - V^2/c^2] = f^2M^2(v), \text{ say.}$$

When $(v - V)^2 < c^2$, m is real and the air wave may be regarded as of Rayleigh's type. When $(v - V)^2 > c^2$, the air wave consists of a wave of sound and its reflection at some reflecting layer.

The component velocities of the air are U, W, where

$$U = \frac{\partial\phi}{\partial x} = -if(Ae^{-mz} + Be^{mz})e^{-ifX}$$

$$W = \frac{\partial\phi}{\partial z} = m(Be^{mz} - Ae^{-mz})e^{-ifX}.$$

The condition $W = 0$ when $z = -H$ gives $A = Ke^{-mH}$, $B = Ke^{m}$, where K is a quantity which may be found from the condition that

$$W = \frac{\partial w}{\partial t} \text{ when } z = 0.$$

Thus
$$2mK\, sh(mH) = ifv(rp + fQ).$$

When the viscous drag of the air on the ground is neglected the absence of a shearing traction on the ground gives the equation

$$2frP + (s^2 + f^2)Q = 0.$$

Since the normal excess pressure of the air on the ground due to the perturbed air motion is wholly responsible for the normal traction on the ground, we find that

$$\sigma a^2\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) - 2\sigma b^2\frac{\partial u}{\partial x} = \rho\left(\frac{\partial\phi}{\partial t} + V\frac{\partial\phi}{\partial x}\right)$$

where σ is the density of the ground. Hence

$$\sigma a^2(f^2 - r^2)P - 2\sigma b^2 f(fP + sQ) = 2Kif\,\rho(v - V)\text{ ch }(mH).$$

Substituting the values already found for the ratios of P, Q, K, we obtain the equation

$$\sigma b^4 M(v)\tanh\,[fHM(v)].R(v) = \rho v^3(v - V)\,C\,(v) \qquad \text{(A)}$$

where
$$R(v) = 4S(v)C(v) - [1 + S^2(v)]^2.$$

This equation may be discussed in much the same way as Stoneley's equation. For our present purpose we remark first that when $v < c + V$ and both quantities are less than the velocity v_1 of the ordinary Rayleigh wave which makes $R(v_1) = 0$, then $R(v) > 0$ and if also $v > V$ both sides of equation (A) are positive. If, moreover, v is chosen so that

$$\sigma b^4 M(v)R(v) > \rho v^3(v - V)C(v)$$

there is a value of tanh $[fHM(v)]$ equal to a positive proper fraction for which equation (A) is satisfied and so H can be found uniquely when v is given. The critical value of v is that given by the equation

$$\sigma b^4 M(v)R(v) = \rho v^3(v - V)C(v).$$

With ordinary values for the elastic constants of the ground this critical value v_c is only slightly less than $c + V$. When $v < v_c$ there is a Rayleigh wave in the ground as well as in the air.

Since r and s decrease as v increases from 0 to b, the amplitudes of the constituents of a Rayleigh wave in the ground drop more rapidly with increasing depth when v is close to c than when v is close to v_1. The drop in the amplitude of the Rayleigh wave in the air is not quite exponential for

$$U = -2ifK\text{ ch }(mz + mH)\,e^{-ifX},\; W = 2mK\text{ sh }(mz + mH)e^{-ifX}.$$

When $c + V < v < v_1$ and $N^2(v) = (v - V)^2/c^2 - 1$, the equation for H may be written in the form

$$\sigma b^4 N(v)\tan\,[fHN(v)]R(v) = -\rho v^3(v - V)C(v). \qquad \text{(B)}$$

When v is assigned tan $[fHN(v)]$ has a definite sign when the foregoing equation is satisfied and so the height H of the reflecting layer is confined to certain ranges that are equally spaced. This may mean that a value of v for which $c + V < v < v_1$ can only exist under certain atmospheric conditions. To find these it will be necessary to assume that the temperature and wind velocity vary with altitude. The equation for ϕ is then much harder to solve.

The existence of a large rock or mountain from which air waves can be reflected may also be influential in determining the possible values of v but then the problem of propagation must be studied in three dimensions.

3. *Remarks on the Theory of the Ground Roll.*—Though some seismolo-

gists think that the ground roll has some of the characteristics of a Rayleigh wave yet another explanation has been sought on account of the low velocity of the ground roll as compared with that of the ordinary Rayleigh wave. The existence of a second type of Rayleigh wave which arises from a marked interaction between the air and the earth puts a different complexion on the matter. The part which this type of free vibration plays can only be judged by a study of the waves produced by a shock of short duration. An extension of the work of Lamb,[8] Banerji,[9] Coulomb[10] and the Japanese seismologists[11] is necessary. The work of Jeans,[12] in which seismic waves are regarded as free vibrations (of high order) of a spherical earth must also be extended. Such extensions may be useful for a study of Stoneley's problem and for a study of the effect of a swiftly moving river on the propagation of Rayleigh waves.

The results of Lamb's calculations are only partly supported by the observations.[13] Andreotti,[14] who has used them in his study of the microseisms produced by waves of the Adriatic dashing against the coast, thinks that there is good agreement. His numerical calculations are based, however, on the assumption that the whole coast is affected by a wave in one phase all along its length. When, moreover, waves dash against a cliff, diffraction effects must be taken into consideration; the problem is not simply one of a solid with plane face. It is possible, however, that edge effects can be taken into consideration in the simple problem by using surface singularities represented by wave-potentials of type

$$(x + iz)^{-1/2}F(t - \omega/a),$$

where $\omega^2 = x^2 + z^2$. In this connection it is interesting to note that a relativity transformation gives a corresponding solution for a similar type of source moving with velocity v and an integration over a set of such moving sources, with an appropriate choice of the individual functions F, leads to the wave-function

$$\int_{-\infty}^{\infty} [z + 1s - i(x - vt)(1 - v^2/a^2)]^{-1/2}e^{-iks}ds,$$

which, when evaluated, leads to just the type of wave-function used in the representation of Rayleigh waves in two dimensions.

The study of elastic waves in two dimensions is complicated by the lack of simplicity in the solution for a line source. Various forms of the principle of Huygens have been found by Volterra,[15] Picht[16] and others, but have not been used very often. Picht[17] has, however, tried to use the principle of Huygens to interpret theoretically the boundary wave which in a stratified earth, travels with the velocity of the upper medium. It is assumed that the strongly damped wave issuing from the focus sets the boundary between the two media into vibration. Such an assumption is very similar to our assumption that an explosion in the earth sets the boundary between air and earth into vibration producing a wave which travels with a velocity close to that of the upper medium (air). Dr. Gutenberg has furnished me with the information that the velocity of the ground roll is sometimes greater than the velocity of sound in air but is usually between 100 meters a second and 500 meters a second. The amplitude of the ground roll decreases when the depth of the source increases, being small when the depth is 10 meters and negligible when the depth is 20 meters or more. Banerji's calculations indicate a marked decrease in amplitude of the ordinary Rayleigh wave when the depth of the source increases and we might expect by analogy that there would be a similar decrease of amplitude for the Rayleigh wave of low speed. An attempt must therefore be made to test this surmise. Banerji's calculations were for the three dimensional case to which we must now turn. Here there is the advantage that there is a simple wave function for a point source but much analytical ground work is needed before a complete solution of a problem can be obtained. Coulomb[10] has made some progress by endeavoring to extend Boussinesq's solution of statical elastic problems to wave problems. Much of the analysis depends on the properties of a certain transcendental function which he has studied in some detail. Some further properties of this function are developed in an accompanying paper.

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[5] Lord Rayleigh, *Theory of Sound,* Vol. 2, p. 158; Sc. Papers, Vol. 4, No. 230.

[6] J. E. Lennard Jones, *Proc. London Math. Soc.* (2) **20,** 347–364 (1920).

[7] Lord Rayleigh, *Ibid.* (1) **17,** 4–11 (1885); Sc. Papers, Vol. 2, p. 441.

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[10] J. Coulomb, *Am. de Toulouse* (3) **23,** 91–137 (1931).

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