Quantum numbers for particles in de Sitter space*

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All subalgebras of the Lie algebra of the de Sitter group O(4,1) are classified with respect to conjugacy under the group itself. The maximal continuous subgroups are shown to be O(4), O(3,1), $D \square E(3)$ (the Euclidean group extended by dilatations), and O(2) ⊗ O(2,1). Representatives of each conjugacy class are shown in the figures, also demonstrating all mutual inclusions. For each subalgebra we either derive all invariants (both polynomial and nonpolynomial ones) or prove that they have none. The mathematical results are used to discuss different possible sets of quantum numbers, characterizing elementary particle states in de Sitter space (or the states of any physical system, described by this de Sitter group).

I. INTRODUCTION

The aim of this paper is to study some properties of the de Sitter group O(4, 1), i.e., the group of motions of a four-dimensional space—time continuum with constant positive curvature. We shall provide a complete analysis of the continuous subgroup structure of O(4, 1), i.e., classify all subgroups into equivalence classes with respect to inner automorphisms of the group itself and construct a lattice of its continuous subgroups. We also consider the Lie algebra of each subgroup and find all its invariants, if they exist, or, as the case may be, prove that none exist. Invariants, in this article, will be defined to include Casimir operators (polynomials in the generators), harmonics (ratios of polynomials), and general nonpolynomial invariants.

The de Sitter group O(4, 1) [as well as the other de Sitter group O(3, 2)] is of considerable interest in relativistic cosmology, elementary particle theory, and also atomic physics. Indeed, the de Sitter spaces with positive or negative curvature[1] are the simplest generalizations of the flat Minkowski space—time of special relativity, capable of providing a model of the expanding universe which we live in. All laws of physics in such a universe would be invariant with respect to one of the de Sitter groups,[2,3] rather than with respect to the Poincaré group. Kinematic conservation laws (energy, linear, and angular momentum, position of the center-of-mass, etc.) will be related to the Lie algebra of the de Sitter group (and its enveloping algebra).

The de Sitter groups are of interest in elementary particle physics for several reasons. First of all, by definition, an elementary physical system in a de Sitter world would be a system described by a wavefunction transforming according to an irreducible unitary representation of the de Sitter group. Complete sets of commuting operators in the enveloping algebra of the de Sitter algebras (i.e., the Casimir operators of the entire group plus, e.g., Casimir operators of a certain chain of subgroups) will then provide the quantum numbers of such particles in definite states. A large amount of literature exists on elementary particle theory in de Sitter space, in particular dealing with problems of localization, the positivity of energy (or lack thereof), generalizations of the Dirac equation and other invariant equations, etc. (see, e.g., Refs. 4—11 and many others). A large body of work also exists on the representation theory of the de Sitter group (see, e.g., the classical papers[12–14]).

Aside from the aspect of considering particle or field theory in curved space and thus incorporating some aspects of gravitational interactions, the de Sitter world may be of interest in that it provides a possible way of avoiding the O'Raifeartaigh theorem.[15] Indeed, while it is not possible to combine the Poincaré group and an internal symmetry group, like SU(3), into a larger group, providing a discrete mass spectrum, such a unification is possible if one of the de Sitter groups is taken as the space—time group.[10,16]

From a different point of view the de Sitter group O(4, 1) is of interest in ordinary elementary particle theory in Minkowski space. Indeed, it has been shown[17] that certain canonical momentum dependent transformations of the ordinary free-particle Dirac equation exist and form an O(4, 1) group. Different subgroups of O(4, 1) then provide different specific transformations of interest, e.g., the Foldy—Wouthuysen transformation[18] is associated with an O(4) subgroup of O(4, 1).

It should also be remembered that the de Sitter groups are among the maximal subgroups of the conformal group of space—time, isomorphic to SO(4, 2). Thus, it may be of interest to consider interactions, breaking down the exact O(4, 2) symmetry, e.g., of relativistic zero-mass equations or of some conformally invariant field theory, to a de Sitter symmetry and further to lower symmetries, corresponding to subgroups of the de Sitter groups. Such reductions of conformal symmetry have been considered in the literature.[19,20]

A further reason why it is of interest to study the de Sitter groups and their subgroups is that within certain restrictions all possible "kinematical groups" can be considered to be contractions[21,22] of the de Sitter groups.[23,24] Indeed, taking the speed of light and/or the radius of curvature to infinity in various ways, we can obtain the Poincaré group, the Galilei group, and several other groups of interest. Again, knowledge of the subgroup structure of the de Sitter groups will make it possible to systematically study contractions with respect to which certain subgroups of physical interest remain invariant.

Finally let us mention that the O(4, 1) group has made its appearance in atomic physics as one of the possible "dynamical noninvariance groups" of the hydrogen atom.[25–35] Indeed, the hydrogen atom is well known to have an O(4) symmetry group, responsible for the accidental degeneracy of its bound state levels, and an O(3, 1) symmetry for the Coulomb scattering states. Both of these can be embedded into a larger group (O(4, 1), O(4, 2), SL(4, R), etc.), the Lie algebras of which contain raising and lowering operators that do not commute with the Hamiltonian. In turn, a study of the subgroups of the corresponding invariance and noninvariance groups will provide a classification of possible symmetry breakings (e. g., by external fields).

At this point it may be appropriate to summarize the reasons why we are interested in the subgroups of the de Sitter group O(4, 1) and more generally in the subgroups of any group of interest in physics (and other applications). Indeed, we have already written four related articles. In the first[36] we found all conjugacy classes of maximal solvable subalgebras of the algebras of the pseudounitary groups SU(p, q) and all subalgebras of LSU(2, 1). In the second[37] we classified all maximal solvable subalgebras of LSO(p, q). In the third[38] and fourth[39] we provided complete lists of all classes of continuous subgroups of the Poincaré group and of the similitude group (the Poincaré group extended by dilations). The general motivation for our program was discussed previously.[36–39] In connection with the O(4, 1) group let us just stress a few points (also having general validity).

1. In a quantum theory in de Sitter space a lattice of subgroups of the de Sitter group will provide us with different complete sets of quantum numbers for elementary physical systems. It should be noted, however, that Casimir operators of continuous subgroups, while providing the simplest types of observables, by no means provide all possible sets of observables. For a discussion of nonsubgroup type observables see, e. g., Refs. 40–43.

2. A knowledge of the subgroup structure is important in group representation theory. Thus different subgroups can be used to induce representations[44] of the group and in particular provide different parametrizations of the group itself. Further, different chains of subgroups provide different bases for representations and lead to different special functions as basis functions.

3. A classification of subgroups provides a classification of different homogeneous manifolds, upon which the group acts transitively.[45] It is often of interest to construct physical wavefunctions as functions on such homogeneous manifolds (and not necessarily simply as functions on the space–time manifolds)[46–48]

4. Since most symmetries in nature are broken ones, it is of considerable interest to discuss symmetry breaking interactions, boundary conditions, etc., reducing the symmetry with respect to a group to that with respect to a subgroup.

It should be noted that we are using Lie algebraic techniques and thus can only provide a classification of continuous subgroups. From the point of view of physical applications in particular those mentioned above, discrete subgroups of Lie groups are also of very considerable interest and we plan to return to this problem. For relevant literature see, e. g., Refs. 49–51.

The subgroup structure of the Lorentz group O(3, 1) has been studied in detail (see, e. g., Refs. 40, 46), in particular in connection with two-variable expansions of relativistic scattering amplitudes. Each subgroup reduction provided a different expansion. Thus, O(3, 1) $\supset$ O(3) was related to partial wave analysis, O(3, 1) $\supset$ O(2, 1) to Regge pole theory, O(3, 1) $\supset$ E(2) to eikonal expansions. For a review of this field see Ref. 52.

In Sec. 2 of this paper we derive a list of all conjugacy classes of subalgebras of the de Sitter algebra LO(4, 1). Results are presented in figures and conjugacy is considered with respect to O(4, 1), SO(4, 1), and $SO_0(4, 1)$ [the continuous component of identity of the O(4, 1) de Sitter group] and in some cases also with respect to the subgroups themselves. In Sec. 3 we find the Casimir operators for all subgroups of O(4, 1) that have them and construct a lattice of these subgroups. We also construct coordinates on an O(4, 1) hyperboloid, allowing the separation of variables in the Laplace operator and corresponding to the individual subgroup chains. We discuss the meaning of the occurring quantum numbers. Our results and future outlook are summarized in Sec. 4.

2. CONTINUOUS SUBGROUPS OF THE DE SITTER GROUP O(4,1)

I. Definitions and general method

We shall make use of two equivalent definitions of the algebra LO(4, 1) of the group O(4, 1). Thus, the usual definition of O(4, 1) as the group of linear homogeneous transformations of a real five-dimensional space x_μ ($\mu = 0, 1, \ldots, 4$), leaving the quadratic form $x^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_0^2$ invariant leads to the Lie algebras of 5×5 real matrices X satisfying

$$X^T I + IX = 0, \tag{1}$$

where

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{2}$$

and T indicates a transposed matrix. The elements g of the group O(4, 1) then satisfy

$$gIg^T = I. \tag{3}$$

This group has four components, similarly as the Lorentz group O(3, 1). Proper de Sitter transformations, constituting the group SO(4, 1), satisfy

$$\det g = 1 \tag{4}$$

in addition to (3), and proper orthochronous transformations $SO_0(4,1)$ satisfy (3), (4), and

$$g_{00} \geqslant 1 \tag{5}$$

[within O(4, 1) we could also have $\det g = -1$ and/or $g_{00} \leqslant -1$].

An alternative realization of LO(4, 1), also useful for our purposes, is obtain by replacing the matrix I of (2) by

$$J = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{6}$$

in Eqs. (1) and (3). In this realization the LO(4, 1) matrices $\tilde{X}$ satisfy

$$\tilde{X}^T J + J\tilde{X} = 0. \tag{7}$$

We choose a basis $M_{\mu\nu}$ ($\mu, \nu = 4, 3, 2, 1, 0$) for LO(4, 1) satisfying

$$[M_{\mu\nu}, M_{\sigma\tau}] = I_{\nu\sigma} M_{\mu\tau} - I_{\nu\tau} M_{\mu\sigma} + I_{\mu\tau} M_{\nu\sigma} - I_{\mu\sigma} M_{\nu\tau} \tag{8}$$

with $I_{ik} = \delta_{ik}$, $i, k = 4, 3, 2, 1$, $I_{00} = -1$, and $I_{0i} = I_{i0} = 0$. In the realization (1) this basis consists of the matrices

$$M_{ik} = Y_{ik} - Y_{ki} \quad \text{and} \quad M_{i0} = M_{0i} = Y_{0i} + Y_{i0}, \quad i, k = 4, 3, 2, 1. \tag{9}$$

The matrices $Y_{\mu\nu}$ have 1 on the intersection of the μth row and νth column and zeros elsewhere.

The matrices $\tilde{X}$ of realization (7) are related to those of realization (1) by the transformation

$$\tilde{X} = ZXZ^{-1}, \tag{10}$$

where

$$Z = \begin{pmatrix} 1/\sqrt{2} & 0 & 0 & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 0 & 0 & 1/\sqrt{2} \end{pmatrix}, \quad Z^{-1} = Z^T. \tag{11}$$

Making use of either of these realizations, we distinguish two types of subalgebras of LO(4, 1), namely those imbedded irreducibly and those imbedded reducibly in LO(4, 1). The irreducibly imbedded ones, by definition, do not leave any nontrivial real subspace in the O(4, 1) space invariant. It can be shown[53] that the LO(4, 1) algebra [contrary to the LO(3, 2) algebra] has no subalgebras of this type. Thus, we only have to classify all reducible subalgebras, and we start out by finding all maximal subalgebras of LO(4, 1). To do this, we simply consider a representative of each conjugacy class of subspaces of the O(4, 1) space [conjugacy is considered under O(4, 1)] and find the subalgebra that leaves this space invariant. We then find all subalgebras of each maximum subalgebra and we can make use of methods and results obtained earlier.[36–39,54]

II. Maximal subgroups of the O(4,1) de Sitter group

A general LO(4, 1) matrix in realization (1) can be written as

$$X = \begin{pmatrix} 0 & a & b & c & d \\ -a & 0 & e & f & g \\ -b & -e & 0 & h & j \\ -c & -f & -h & 0 & k \\ d & g & j & k & 0 \end{pmatrix}. \tag{12}$$

In realization (7) we have

$$\tilde{X} = ZXZ^{-1} = \begin{pmatrix} -d & \frac{a-g}{\sqrt{2}} & \frac{b-j}{\sqrt{2}} & \frac{c-k}{\sqrt{2}} & 0 \\ -\frac{a+g}{\sqrt{2}} & 0 & e & f & -\frac{a-g}{\sqrt{2}} \\ -\frac{b+j}{\sqrt{2}} & -e & 0 & h & -\frac{b-j}{\sqrt{2}} \\ -\frac{c+k}{\sqrt{2}} & -f & -h & 0 & -\frac{c-k}{\sqrt{2}} \\ 0 & \frac{a+g}{\sqrt{2}} & \frac{b+j}{\sqrt{2}} & \frac{c+k}{\sqrt{2}} & d \end{pmatrix}. \tag{13}$$

Let us now consider subspaces of the five-dimensional space of real vectors $(x_4, x_3, x_2, x_1, x_0)$ with metric $x_4^2 + x_3^2 + x_2^2 + x_1^2 - x_0^2 = \text{inv}$. The subspaces will differ by their dimension and signature.

A. One-dimensional subspaces

A1. Timelike subspace [signature (–)]: Consider the space T generated by the column vector (00001) (which we write in row form to save space) and require that the operator X of (12) leaves if invariant: $XT \subseteq T$. This implies $d = g = j = k = 0$ and we obtain the algebra LO(4) of the four-dimensional rotation group, generated by

$$M_{ik} \quad \text{with} \quad i, k = 4, 2, 3, 1. \tag{14}$$

A2. Spacelike subspace [signature (+)]: Consider the space S generated by the vector (10000) (which again should be a column) and require that it be invariant under (12). This implies that $a = b = c = d = 0$ and we obtain the algebra LO(3, 1) of the homogeneous Lorentz group, generated by

$$L_1 = M_{32}, \quad L_2 = -M_{31}, \quad L_3 = M_{21}, \quad K_1 = M_{10},$$
$$K_2 = M_{20}, \quad \text{and} \quad K_3 = M_{30}. \tag{15}$$

A3. Lightlike subspace [signature (0)]: Consider the space L, generated by the vector (1000-1) in the realization (1). Applying operator Z to it, we obtain

$$\tilde{S} = ZS \simeq \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

in realization (7). Requiring that $\tilde{S}$ be invariant under $\tilde{X}$ of (13), we find $g = -a$, $j = -b$, $k = -c$, i.e., we obtain a seven-parameter algebra, generated by

$$D \equiv M_{40}, \quad L_1 \equiv M_{32}, \quad L_2 \equiv -M_{31}, \quad L_3 \equiv M_{21},$$
$$P_1 \equiv M_{41} - M_{10}, \quad P_2 \equiv M_{42} - M_{20}, \quad \text{and} \quad P_3 \equiv M_{43} - M_{30}. \tag{16}$$

These generators satisfy the commutation relations

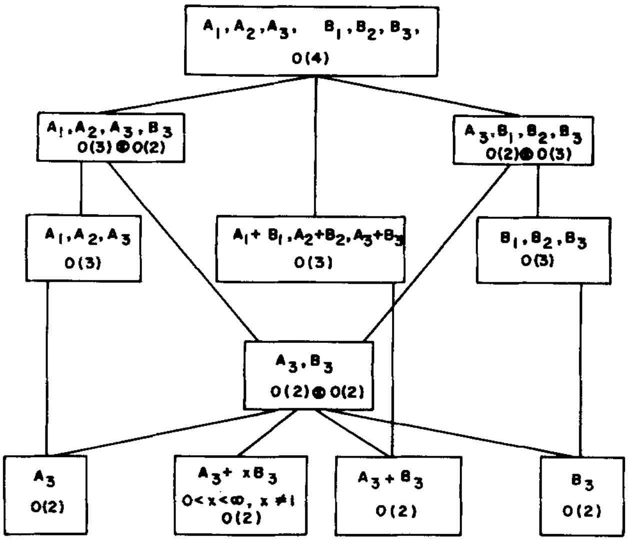

FIG. 1. SO(4) conjugacy classes of subalgebras of LO(4) and the groups they generate.

$$[L_i, L_k] = \epsilon_{ikl} L_l, \quad [L_i, P_k] = \epsilon_{ikl} P_l,$$
$$[D, L_i] = 0, \qquad [D, P_i] = -P_i. \tag{17}$$

Thus, we obtain the algebra D□LE(3), i.e., the algebra of the Euclidean group in three-dimensions, extended by dilations.

B. Two-dimensional subspaces

*B*1. *Signature* (+ +): We use realization (1), consider the space $(SS) = (x_4, x_3, 0, 0, 0)$, and require that $X(SS) \subseteq (SS)$. This implies $b = c = d = e = f = g = 0$ in (12), i.e., we obtain the algebra LO(2) ⊕ LO(2, 1), generated by

$$L_3 \equiv M_{21}, \quad K_1 \equiv M_{10}, \quad K_2 \equiv M_{20}, \quad \text{and} \quad A \equiv M_{43} \tag{18}$$

with

$$[L_3, K_1] = K_2, \quad [K_2, L_3] = K_1, \quad [K_1, K_2] = -L_3,$$
$$[A, K_1] = [A, K_2] = [A, L_3] = 0. \tag{19}$$

*B*2. *Signature* (+ −): Using realization (1), we require that the space $(ST) = (x_4, 0, 0, 0, x_0)$ be invariant. This implies that $a = b = c = g = j = k = 0$ in (12), i.e., we obtain the algebra LO(3) ⊕ LO(1, 1), generated by

$$L_1 = M_{32}, \quad L_2 = -M_3, \quad L_3 = M_{21}, \quad \text{and} \quad D = M_{40}. \tag{20}$$

However, any transformation leaving space *ST* invariant also leaves space *L* invariant and indeed we see that the subalgebra (20) is contained in (17) and is hence not maximal.

*B*3. *Signature* (+ 0): We use realization (1) of LO(4, 1) and require that the space $(x, y, 0, 0, -x)$ remains invariant. This implies $e = f = 0$, $g = -a$, $j = -b$, $k = -c$. This leads us to the subalgebra (D ⊕ LO(2)) □ T_3, generated by

$$L_3 = M_{21}, \quad D = M_{40}, \quad P_1 = M_{41} - M_{10},$$
$$P_2 = M_{42} - M_{20}, \quad P_3 = M_{43} - M_{30}. \tag{21}$$

Again, this subalgebra is contained in (17) and is hence not maximal.

We have thus considered all one- and two-dimensional subspaces of the O(4, 1) space that are not conjugate under O(4, 1). Subalgebras that leave three- or four-dimensional subspaces invariant will automatically also leave their two- or one-dimensional orthogonal complements invariant and will hence coincide with those obtained above, or be contained in them.

To summarize, the algebra LO(4, 1) has exactly four O(4, 1) conjugacy classes of maximal subalgebras, namely LO(4) of (14), LO(3, 1) of (15), D□LE(3) of (16), and LO(2) ⊕ LO(2, 1) of (18). Since an arbitrary one- or two-dimensional subspace of the O(4, 1) space with the corresponding signature can be transformed into one of the spaces *S*, *T*, *L* or (*SS*) by an $SO_0(4, 1)$ transformation the above algebras also represent all $SO_0(4, 1)$ classes of maximal subalgebras.

We now proceed to classify all subalgebras of each maximal subalgebra.

III. Subalgebras of LO(4)

The algebra LO(4) is isomorphic to LO(3) ⊕ LO(3). Its subalgebras can be obtained using the "Goursat twist method" and were originally classified by Goursat.[55] The method, in application to Lie algebras, was discussed in a previous publication,[38] so here we omit all details.

Let us introduce the notation

$$A_1 = \tfrac{1}{2}(M_{32} + M_{41}), \quad A_2 = \tfrac{1}{2}(-M_{31} + M_{42}),$$
$$A_3 = \tfrac{1}{2}(M_{21} + M_{43}),$$
$$B_1 = \tfrac{1}{2}(M_{32} - M_{41}), \quad B_2 = \tfrac{1}{2}(M_{31} + M_{42}),$$
$$B_3 = \tfrac{1}{2}(-M_{21} + M_{43}), \tag{22}$$

so that

$$[A_i, A_k] = \epsilon_{ikl} A_l, \quad [B_i, B_k] = \epsilon_{ikl} B_l, \quad [A_i, B_k] = 0. \tag{23}$$

The algebra LO(4) ~ LO(3) ⊕ LO(3) will have two types of subalgebras. The first type, "nontwisted subalgebras," are obtained by taking direct sums of subalgebras of the one LO(3) with those of the other. The second type, "twisted subalgebras," involve generators that are not conjugate to either A_i nor B_i. Only two such subalgebras exist: the LO(3) algebra generated by $A_1 + B_1$, $A_2 + B_2$, and $A_3 + B_3$, contained reducibly in LO(4) and a one-dimensional subalgebra $A_3 + xB_3$, depending on one parameter x. An SO(4) transformation changing $A_3 + xB_3$ into $A_3 - xB_3$ can easily be constructed; hence we can take $0 < x < \infty$.

A lattice of SO(4) conjugacy classes of subalgebra of LO(4) is given in Fig. 1.

If conjugacy is considered under O(4), rather than SO(4), then A_i and B_i are conjugate

$$gA_i g^{-1} = B_i,$$

$$g = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}. \tag{24}$$

The lattice of Fig. 1 simplifies under O(4), in that all

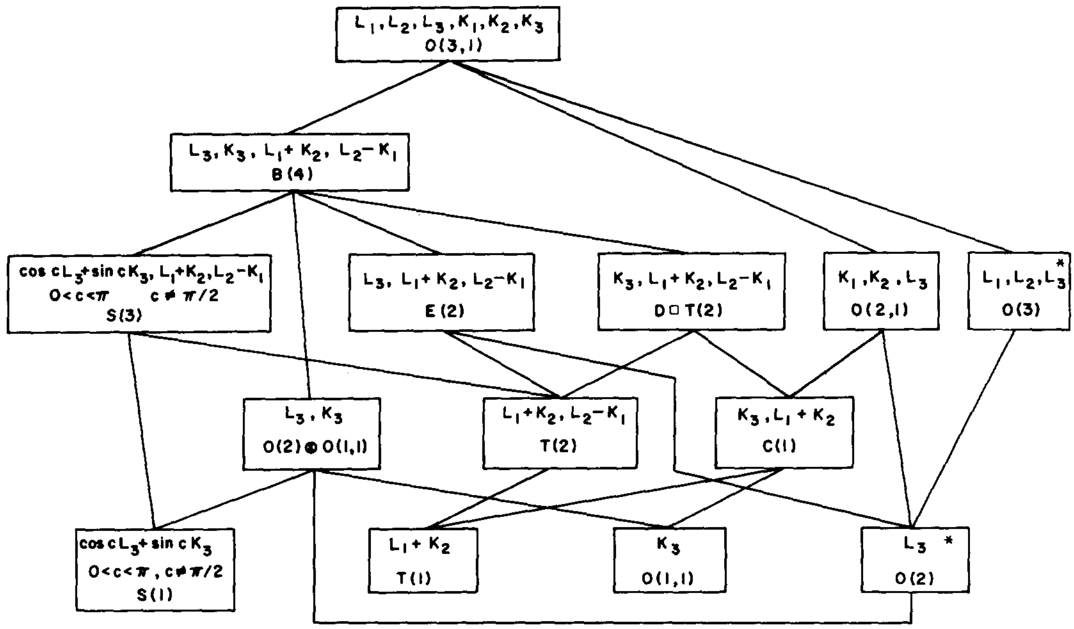

FIG. 2. $SO_0(3,1)$ conjugacy classes of subalgebras of LO(3, 1). The group generated by the algebra is also given. Here B(4) indicates the Borel subgroup, i.e., the maximal solvable subgroup of O(3, 1); the S in S(3) and S(1) stands for "screw" (a combination of a rotation about an axis with a boost along the same axis; T(1) stands for translations along one axis, E(2) for the Euclidean group, and D for dilations. An asterisk indicates subalgebras conjugate to ones contained in Fig. 1.

algebras in the furthest to right column become conjugate to those in the left-hand column and the parameter x can be restricted to $0 < x < 1$.

Imbedding O(4) into O(4, 1), we replace (24) by

$$g=\begin{pmatrix}1&&&&\\&-1&&&\\&&-1&&\\&&&-1&\\&&&&-1\end{pmatrix}. \tag{25}$$

with $\det g = 1$, $g_{00} = -1$. Thus g is contained in SO(4, 1), but not in $SO_0(4,1)$.

IV. Subalgebras of LO(3,1)

The subalgebras of LO(3, 1) are known.[40] For completeness we give a lattice of $SO_0(3,1)$ classes of subalgebras of LO(3, 1) in Fig. 2. If we consider conjugacy under O(3, 1), i.e., include parity, then $0 < c < \pi/2$ in S(3) and S(1). This transformation is contained in $SO_0(4,1)$. Note that the algebras LO(3) and LO(2) in Fig. 2 are already contained in Fig. 1.

V. Subalgebras of D □ LE(3)

The continuous subgroups of the Euclidean group E(3) have been classified earlier.[54] The E(3) conjugacy classes of subalgebras of LE(3) are given in Fig. 3.

If we add parity to E(3) then $a > 0$ in $L_3 + aP_3$ and $\{L_3 + aP_3, P_1, P_2\}$. The corresponding transformation

$$g=\begin{pmatrix}-1&&&&\\&1&&&\\&&1&&\\&&&1&\\&&&&-1\end{pmatrix}. \tag{26}$$

is contained in SO(4, 1) but not in $SO_0(4,1)$.

Let us now add the dilations, generated by D, to E(3). Since $[D, L_3] = 0$ but $[D, P_3] = -P_3$, we can transform the parameter a in $L_3 + aP_3$ into $a = 1$. All subalgebras of Fig. 3 (with $a = 1$) will then also be subalgebras of $D \,\square\, LE(3)$ and none of them will be conjugate to each other. Further subalgebras will involve D and will be of two types. The first type is obtained by simply adding D as a generator to all subalgebras of E(3) that split over their intersection with the translations (i.e., subalgebras not containing the generator $L_3 + P_3$). The second type of algebra can be written as

$$\{D + a_i L_i + x_i P_i ; E_\alpha\}, \tag{27}$$

where E_α is one of the subalgebras of LE(3). We must now run through all subalgebras E_α, spell out the additional generator $\tilde{D} = D + a_i L_i + x_i P_i$, and set all a_i and x_i equal to zero, if the corresponding L_i and P_i are contained in E_α. Then we simplify $\tilde{D}$ further, using trans-

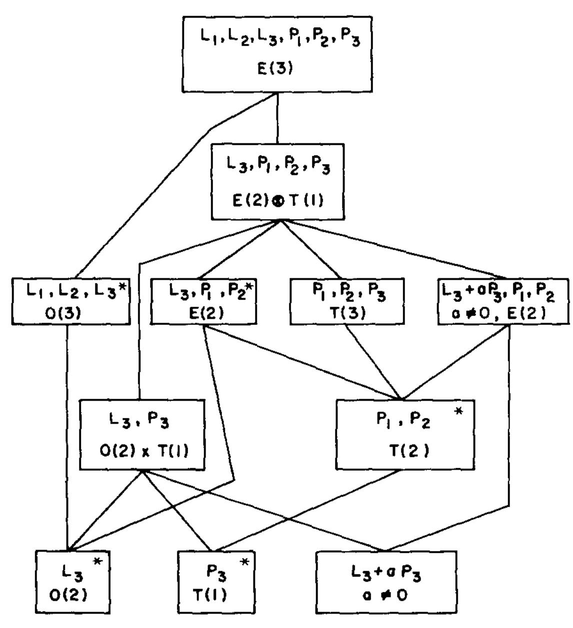

FIG. 3. E(3) conjugacy classes of subalgebras of LE(3) and the groups they generate. An asterisk indicates subalgebras conjugate to ones contained in Figs. 1 or 2.

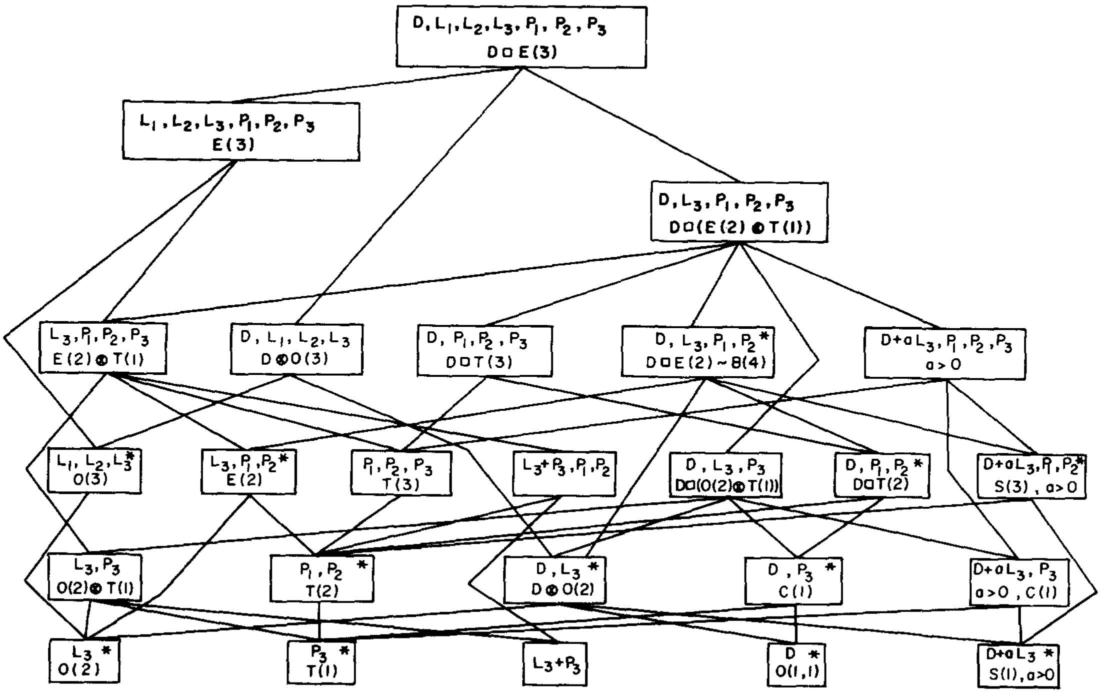

FIG. 4. Subalgebras of $D \square \mathrm{LE}(3)$ and the groups they generate. An asterisk indicates subalgebras conjugate to ones contained in Figs. 1 or 2.

formations leaving E_α invariant and finally we enforce that D and E_α together should form a closed algebra. The last two steps can be performed in an elegant manner, using cohomology theory.[38,39] In the present case the task is so simple that we just proceed in a straightforward manner. Note that new subalgebras are obtained only if at least one of the a_i or x_i is nonzero.

The only subalgebras of LE(3) leading to such nontrivial subalgebras of $D\square \mathrm{LE}(3)$ are those not containing any rotations, i.e., $\{P_1, P_2, P_3\}$, $\{P_1, P_2\}$, $\{P_3\}$, and $\{0\}$. Consider them individually.

a. $\{P_1, P_2, P_3\}$. Write the additional element as $\tilde{\tilde{D}} = D + a_i L_i$. Performing an O(3) rotation, we can transform $\tilde{\tilde{D}}$ into $\tilde{D} = D + aL_3$, $a > 0$. We obtain the algebra

$$D + aL_3, \quad P_1, P_2, P_3, \quad a > 0. \tag{28}$$

b. $\{P_1, P_2\}$. We write $\tilde{D} = D + a_i L_i + xP_3$. Since $[D, P_3] = -P_3$, the group transformation $\exp(yP_3)$, contained in E(3), can be used to transform x into zero. The commutation relations $[\tilde{D}, P_1] = -P_1 - a_2 P_3 + a_3 P_2$ and $[\tilde{D}, P_2] = -P_2 + a_1 P_3 - a_3 P_1$ imply $a_1 = a_2 = 0$ and a rotation through π about axis 1 or 2 can be used to change the sign of a_3. We find the algebra

$$D + aL_3, \quad P_1, P_2, \quad a > 0. \tag{29}$$

c. $\{P_3\}$. We write $\tilde{D} = D + a_i L_i + x_1 P_1 + x_2 P_2$. The transformation $\exp(xP_1)$ and $\exp(yP_2)$ can be used to transform x_1 and x_2 into zero. The relation $[\tilde{D}, P_3] = -P_3 - a_1 P_2 + a_2 P_1$ implies $a_1 = a_2 = 0$. A rotation through π about axis 1 or 2 will change the sign of a_3. We find

$$D + aL_3, \quad P_3, \quad a > 0. \tag{30}$$

d. $\{0\}$. We have $\tilde{D} = D + a_i L_i + x_i P_i$. A rotation can be used to obtain $a_1 = a_2 = 0$. Transformations of the type $\exp(aL_3)$ and $\exp(y_i P_i)$ can be used to obtain $x_1 = x_2 = x_3 = 0$. We obtain the algebra

$$D + aL_3, \quad a > 0. \tag{31}$$

None of the obtained subalgebras of $D\square \mathrm{E}(3)$ can be further simplified by O(4, 1) transformations. The conjugacy classes [under the Euclidean group extended by dilatations and parity and also under O(4, 1)] are summarized in Fig. 4.

All algebras of Fig. 4 leave a one-dimensional lightlike space L invariant. Many of them also leave a one-dimensional timelike or spacelike vector space invariant and are thus contained in LO(4) or LO(3, 1). This can easily be established for each subalgebra separately. We denote by an asterisk in Fig. 4 those subalgebras that are conjugate, under O(4, 1), to algebras in Figs. 1 or 2.

VI. Subalgebras of LO(2) ⊕ LO(2,1)

All subalgebras of this algebra can be obtained either as the direct sums of subalgebras of LO(2) and LO(2, 1) (including the trivial ones) or by applying the Goursat twist method. The results are given in Fig. 5. All of the subalgebras in Fig. 5 also leave a one-dimensional vector space invariant and will thus already have been listed in Figs. 1, 2, or 4. This can easily be established by inspection, and we indicate all previously listed subalgebras by an asterisk in Fig. 5.

VII. All subalgebras of LO(4,1)

Figures 1, 2, 4, 5 can now be used to compile a complete lattice of subalgebras of LO(4, 1) presented in Fig. 6. We consider conjugation under O(4, 1), so as to keep the size of Fig. 6 manageable.

This completes our investigation of the subalgebras of LO(4, 1) and thus also of the continuous subgroups of O(4, 1).

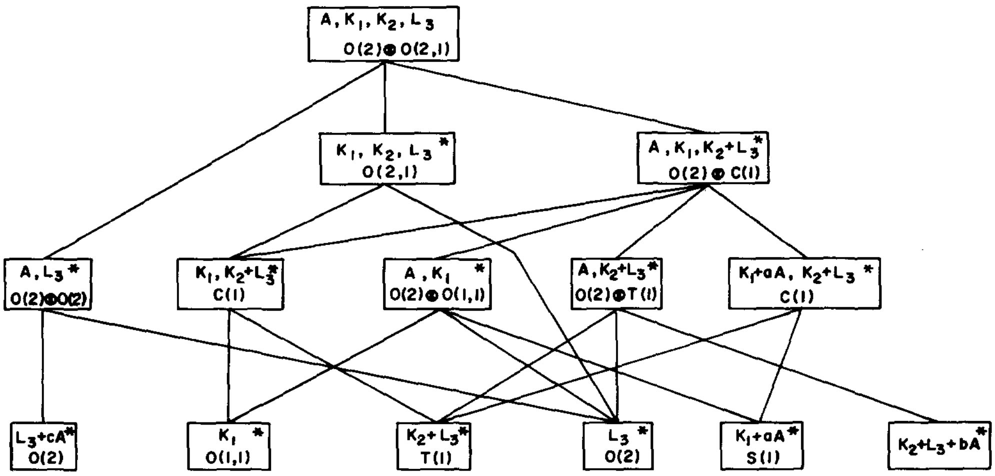

FIG. 5. Subalgebras of $LO(2) \oplus LO(2,1)$. If conjugacy is considered under $S[O(2) \times O_0(2,1)]$ we have $a \neq 0$, $b \neq 0$, $c \neq 0$. Conjugacy under $S[O(2) \times O(2,1)]$ gives $a>0$, $b>0$, $c>0$. Conjugacy under $SO_0(4,1)$ gives $a \neq 0$, $b = \pm 1$, $c \neq 0$. Conjugacy under $SO(4,1)$ gives $a>0$, $b=1$, $c>0$. An asterisk indicates algebras conjugate under $O(4,1)$ to algebras in Figs. 1, 2, or 4.

3. INVARIANTS OF SUBALGEBRAS OF LO(4,1)

I. General method for finding invariants of Lie algebras

Having thus provided a classification of all subalgebras of $LO(4,1)$, we now wish to determine which of the subalgebras have invariants, in particular Casimir operators and to find all of them. These will then provide us with observables and quantum numbers for particles in a de Sitter space [or for any physical system for which $O(4,1)$ is a relevant symmetry group].

Let us briefly discuss our method of searching for invariants. Consider the Lie algebra $\mathcal{L}$ generated by the operators $A_1, \ldots, A_n$, satisfying

$$[A_i, A_k] = \sum_{l=1}^{n} f_{ik}^l A_l. \tag{32}$$

We shall represent the generators A_i as differential operators acting on functions $F(a_1, \ldots, a_n)$ of n variables (n is the dimension of the algebra). Indeed, if we put

$$A_i = \sum_{k,l=1}^{n} f_{ik}^l a_l \frac{\partial}{\partial a_k}, \quad i = 1, \ldots, n, \tag{33}$$

the operators A_i will satisfy (32). We are now interested in finding an operator valued function $P(A_1, \ldots, A_n)$, commuting with all A_i. This is equivalent to finding a numerical function $P(a_1, \ldots, a_n)$, annihilated by all generators (33):

$$A_i P(a_1, \ldots, a_n) = 0, \quad i = 1, \ldots, n, \tag{34}$$

then symmetrizing P with respect to all permutations of a_i and replacing the variables a_i by the operators A_i.

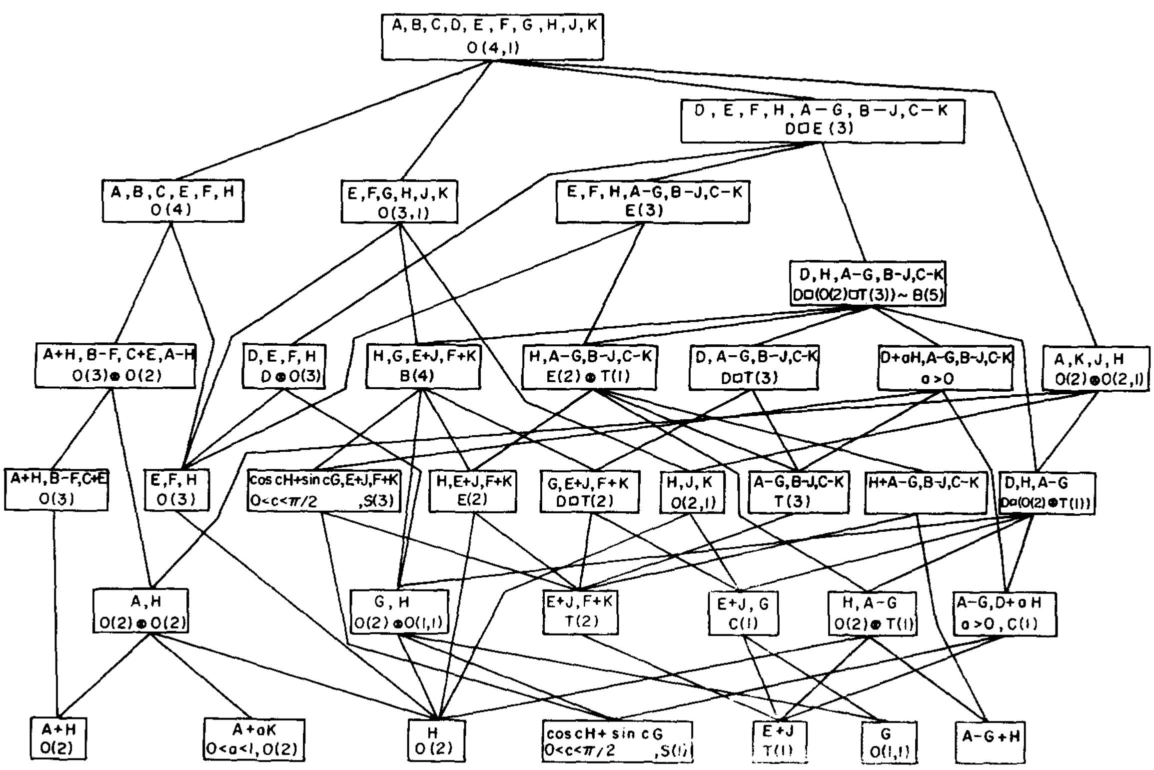

FIG. 6. Subalgebras of $LO(4,1)$ and the groups they generate. We use the notation $A = M_{43}$, $B = M_{42}$, $C = M_{41}$, $D = M_{40}$, $E = M_{32}$, $F = M_{31}$, $G = M_{30}$, $H = M_{21}$, $J = M_{20}$, and $K = M_{10}$. Conjugacy is considered under $O(4,1)$.

We thus reduce the search for Casimir operators to the problem of solving the system of homogeneous linear partial differential equations (34). If the system is contradictory, i.e., does not have a nonzero solution, then the algebra has no Casimir operators. On the other hand, solutions may exist, but not be expressible in terms of polynomials in a_i. We then obtain "generalized Casimir operators," i.e., operators not lying in the enveloping algebra of $\mathcal{L}$ but still commuting with all generators and hence having a fixed numerical value within each irreducible representation of the algebra. If polynomial solutions of (34) exist, they will provide us with Casimir operators. Generally speaking, among such solutions we must find an integrity basis, i.e., a minimal set of operators C_i such that any invariant can be expressed as a polynomial in C_i.

Note that for semisimple algebras the problem is solved—the number of Casimir operators is equal to the rank of the algebra and they are all known.

II. Invariants of the subalgebras of LO(4,1)

Let us go through the algebras of Figs. 1–6 and find their Casimir operators. The Casimir operators of O(4, 1) itself are well known, namely

$$C^{(2)} = M_{\alpha\beta} I_{\beta\gamma} M_{\gamma\delta} I_{\delta\alpha},$$

$$C^{(4)} = M_{\alpha\beta} I_{\beta\gamma} M_{\gamma\delta} I_{\delta\epsilon} M_{\epsilon\zeta} I_{\zeta\eta} M_{\eta\xi} I_{\xi\alpha}.$$

A. LO(4) and its subalgebras

All algebras in Fig. 1 have invariants and they are quite obvious. Thus LO(4) itself has two Casimir operators $\mathbf{A}^2 = A_1^2 + A_2^2 + A_3^2$ and $\mathbf{B}^2 = B_1^2 + B_2^2 + B_3^2$. For a one-dimensional subalgebra the generator itself is an invariant; for an Abelian subalgebra all generators are invariants. The "twisted" LO(3) $(A_1 + B_1,\ A_2 + B_2,\ A_3 + B_3)$ has the Casmimir operator $(\mathbf{A}+\mathbf{B})^2$. The invariants of a direct sum of algebras will be the invariants of each component.

B. LO(3,1) and its subalgebras

The invariants of the subalgebras of LO(3, 1) are known.[40] Thus, using the notations of Fig. 2, we have the following. LO(3, 1) itself has two Casimir operators $\mathbf{L}^2 - \mathbf{K}^2$ and $\mathbf{L}\cdot\mathbf{K}$. The algebras LE(2), LO(2, 1) and LO(3) have the invariants $(L_1 + K_2)^2 + (L_2 - K_1)^2$, $K_1^2 + K_2^2 - L_3^2$, and $L_1^2 + L_2^2 + L_3^2$, respectively. The generators of Abelian or one-dimensional algebras are themselves invariants. The algebras B, S(3), $D \,\square\, (\mathrm{LT}(1) \oplus \mathrm{LT}(1))$, and C(1) have no Casimir operators. However, using the method discussed above, we can show that both S(3) and $D \,\square\, (\mathrm{LT}(1) \oplus \mathrm{LT}(1))$ have a nonpolynomial invariant. Indeed, consider the algebra $D = K_3,\ P = L_1 + K_2,\ Q = -L_2 + K_1$. We have

$$[D, P] = P, \quad [D, Q] = Q, \quad [P, Q] = 0$$

so that

$$D = p\frac{\partial}{\partial p} + q\frac{\partial}{\partial q}, \quad P = -p\frac{\partial}{\partial d}, \quad Q = -q\frac{\partial}{\partial d}. \tag{35}$$

Consider the function $F(p, q, d)$ and require

$$DF = PF = QF = 0.$$

The last two equations imply that F does not depend on d, the first implies that F is an arbitrary function of p/q. Hence, the invariant is the operator

$$X = (L_1 + K_2)/(-L_2 + K_1), \tag{36}$$

which generally speaking is not a well-defined operator. Similarly, consider S(3), generated by

$$R = \cos\phi L_3 + \sin\phi K_3, \quad P = L_1 + K_2, \quad Q = -L_2 + K_1,$$
$$0 < \phi < \pi/2 \quad \text{or} \quad \pi/2 < \phi < \pi.$$

Using the commutation relations for S(3) we write

$$R = (q\cos\phi + p\sin\phi)\frac{\partial}{\partial p} + (q\sin\phi - p\cos\phi)\frac{\partial}{\partial p},$$

$$P = -(q\cos\phi + p\sin\phi)\frac{\partial}{\partial r}, \quad Q = -(q\sin\phi - p\cos\phi)\frac{\partial}{\partial r}. \tag{37}$$

Requiring $RF = PF = QF = 0$, we obtain a nonpolynomial invariant

$$I = (P^2 + Q^2)\left(\frac{P - iQ}{P + iQ}\right)^{i\tan\phi}$$
$$= (P^2 + Q^2)\exp[2\tan\phi \cdot \arctan(Q/P)]. \tag{38}$$

C. $D \,\square\,$ LE(3) and its subalgebras

Consider first the algebra $D \,\square\,$ LE(3) itself. The invariant $F(p_1, p_2, p_3, l_1, l_2, l_3, d)$ could depend on seven variables; however, rotational invariance $L_i F = 0$ implies that it only depends on O(3) scalars $\mathbf{p}^2$, $\mathbf{l}^2$, $\mathbf{p}\cdot\mathbf{l}$, and d. Scale invariance $DF = 0$ implies that F only depends on $\mathbf{l}^2$ and $\mathbf{p}^2/(\mathbf{p}\cdot\mathbf{l})^2$. Finally translational invariance $P_i F = 0$ implies that F depends only on $\mathbf{p}^2/(\mathbf{p}\cdot\mathbf{l})^2$. Thus $D \,\square\,$ LE(3) has no Casimir operators, but has one nonpolynomial invariant, the harmonic

$$I = \mathbf{P}^2/(\mathbf{P}\cdot\mathbf{L})^2. \tag{39}$$

The algebra LE(3) has the two well-known Casimir operators $\mathbf{P}^2$ and $(\mathbf{P}\cdot\mathbf{L})$, i.e., the energy and helicity of a nonrelativistic particle.

The algebra $\{D, L_3, P_1, P_2, P_3\}$ can be shown to have only one invariant, namely $(P_1^2 + P_2^2)/P_3^2$, which again is nonpolynomial.

The algebra $\{L_3, P_1, P_2, P_3\}$ has two Casimir operators: $P_1^2 + P_2^2$ and P_3.

The algebra $\{D, L_1, L_2, L_3\}$ has two Casimir operators: D and $\mathbf{L}^2$.

The algebra $\{D, P_1, P_2, P_3\}$ has no Casimir operators but two nonpolynomial invariants P_1/P_3 and P_2/P_3.

The algebra $\{D + aL_3, P_1, P_2, P_3\}$ is somewhat more complicated. We put

$$R = D + aL_3 = (-p_1 + ap_2)\frac{\partial}{\partial p_1} - (p_2 + ap_1)\frac{\partial}{\partial p_2} - p_3\frac{\partial}{\partial p_3},$$

$$P_1 = (p_1 - ap_2)\frac{\partial}{\partial r}, \quad P_2 = (p_2 + ap_1)\frac{\partial}{\partial r}, \quad P_3 = p_3\frac{\partial}{\partial r}. \tag{40}$$

Requiring $P_i F = 0$ implies that $F = F(p_1, p_2, p_3)$. Requiring $RF = 0$ implies

$$\frac{dp_1}{p_1 - ap_2} = \frac{dp_2}{p_2 + ap_1} = \frac{dp_3}{p_3}. \tag{41}$$

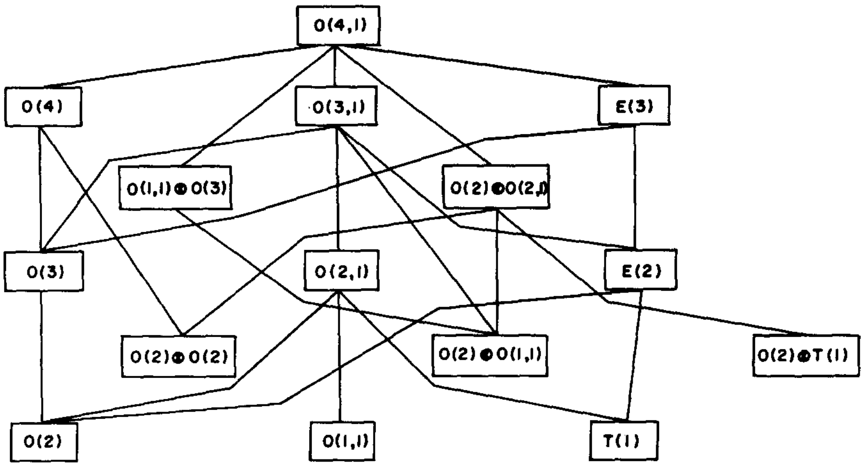

FIG. 7. Subgroups of O(4, 1) that have Casimir operators.

These equations are solved by standard methods,[56] and we obtain two (real) nonpolynomial invariants

$$X_1 = P_3\{(P_1 + iP_2)^{-(1-ia)/(1+a^2)} + (P_1 - iP_2)^{-(1+ia)/(1+a^2)}\},$$
$$X_2 = iP_3\{(P_1 + iP_2)^{-(1-ia)/(1+a^2)} - (P_1 - iP_2)^{-(1+ia)/(1+a^2)}\}. \tag{42}$$

The invariants of all other subalgebras are obvious (or have been obtained above), as are those of subalgebras of $\mathrm{LO}(2) \oplus \mathrm{LO}(2,1)$.

The situation is best summarized by the diagram of Fig. 7, where we give a lattice of subgroups of O(4, 1), listing only those which have Casimir operators. More specifically, we only list a subgroup if it has a new Casimir operator, that is not also a Casimir operator for a larger subgroup, higher in the chain.

III. Quantum numbers

All chains of subgroups providing us with a complete set of observables are shown on Figs. 6 and 7, and we see 16 possible independent sets. In addition to the two Casimir operators of LO(4, 1), characterizing the system as such, we have the following possible choices of operators, characterizing the particle states.

A. Reduction to O(4)

Using the notations (22), we see that the complete set of commuting operators would contain:

$$\mathbf{A}^2, \quad \mathbf{B}^2, \tag{43}$$

and either

$$(\mathbf{A} + \mathbf{B})^2 \quad \text{and} \quad A_3 + B_3 \tag{44a}$$

or

$$A_3 \quad \text{and} \quad B_3. \tag{44b}$$

B. Reduction to O(3,1)

Using the notations (15) we write the observables

$$\mathbf{L}^2 - \mathbf{K}^2 \quad \text{and} \quad \mathbf{L} \cdot \mathbf{K}, \tag{45}$$

supplemented by one of the following pairs:

$$\mathbf{L}^2 \quad \text{and} \quad L_3, \tag{46a}$$

$$K_1^2 + K_2^2 - L_3^2 \quad \text{and} \quad L_3, \tag{46b}$$

$$K_1^2 + K_2^2 - L_3^2 \quad \text{and} \quad K_1, \tag{46c}$$

$$K_1^2 + K_2^2 - L_3^2 \quad \text{and} \quad K_2 + L_3, \tag{46d}$$

$$(L_1 + K_2)^2 + (L_2 - K_1)^2 \quad \text{and} \quad L_3, \tag{46e}$$

$$(L_1 + K_2)^2 + (L_2 - K_1)^2 \quad \text{and} \quad L_1 + K_2, \tag{46f}$$

$$L_3 \quad \text{and} \quad K_3. \tag{46g}$$

C. Reduction to E(3)

Using the notations (16), we write the observables as

$$\mathbf{P}^2 \quad \text{and} \quad \mathbf{P} \cdot \mathbf{L}, \tag{47}$$

supplemented by one of the following pairs:

$$\mathbf{L}^2 \quad \text{and} \quad L_3, \tag{48a}$$

$$P_1^2 + P_2^2 \quad \text{and} \quad L_3, \tag{48b}$$

$$P_1^2 + P_2^2 \quad \text{and} \quad P_3 \quad (\text{or } P_1, P_2 \text{ and } P_3). \tag{48c}$$

D. Reduction to O(3) × O(1,1)

This reduction, using notations (20), provides us with three quantum numbers:

$$\mathbf{L}^2, \quad L_3, \quad \text{and} \quad D. \tag{49}$$

E. Reduction to O(2) ⊕ O(2,1)

Using notations (18), we obtain three quantum numbers given by

$$A, \quad K_1^2 + K_2^2 - L_3^2 \tag{50a}$$

and one of the three operators

$$L_3, \quad K_1, \quad \text{or} \quad K_2 + L_3. \tag{50b}$$

The physical interpretation of each set of observables is open to discussion and depends on the specific physical system considered. We have however clarified the group theoretical significance of each set.

Several comments are in order: (i) While the "canonical" reductions of O(4, 1) to O(4), O(3, 1), and E(3) provide us with complete sets of observables, the reductions to $\mathrm{O}(3) \otimes \mathrm{O}(2)$ and $\mathrm{O}(2) \otimes \mathrm{O}(2,1)$ provide only three quantum numbers and we are faced with a "missing label problem." These have been discussed extensively in the literature in other connections, in particular in relation to the $SU(3) \supset O(3)$ reduction. One way to provide the missing quantum number, completely specifying the states, would be to add a further operator to the set (49) or (50). This would have to lie in the enveloping algebra of $LO(4,1)$, not, however, of the subgroup $O(3)\otimes O(1,1)$ [or $O(2)\otimes O(2,1)$] and be a scalar with respect to the corresponding subgroup. For a discussion of this problem, see Refs. 57, 58 and references therein.

(ii) We have not touched upon "nonsubgroup" type observables, i.e., complete sets of operators that can specify a state, but are not Casimir operators of any Lie subgroup. Examples of such observables in connection with $O(3)$ and other little groups of the Poincaré group have been studied.[40-42] They can be related to discrete subgroups of the corresponding group—a question that is itself of considerable interest.

(iii) A question that to our knowledge has received no attention at all is the significance of invariants of Lie algebras, that do not lie in the enveloping algebra (are not polynomials in the generators) and their possible use in representation theory and physics. We have constructed such invariants for all subalgebras of $LO(4,1)$ for which they exist.

(iv) Diagrams characterizing subgroup reductions of the type shown in Fig. 8 have been used previously[40] for the Lorentz group $O(3,1)$. Similar diagrams have been used to characterize coordinates in $O(n)$ and $O(n,1)$ spaces (the "method of trees")[59,60]

IV. Separable coordinate systems

Let us consider the upper sheet of the two-sheeted hyperboloid

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2 = 1. \tag{51}$$

If we consider a space of scalar functions $\psi(x)$ defined on this hyperboloid and require that a set of such functions transforms irreducibly under the group $O(4,1)$, then $\psi(x)$ must be eigenfunctions of the two Casimir operators of $O(4,1)$. However, the fourth-order Casimir operator is identically zero on such a space and the second order one reduces to the Laplace operator Δ on the hyperboloid (51).

We can now choose a basis by requiring that the functions $\psi(x)$ be eigenfunctions of Δ and of one of the complete sets of commuting operators, corresponding to one of the group reductions discussed above and represented in Fig. 8. It is interesting to note that to each subgroup reduction there corresponds a system of coordinates for which all the equations separate. For future convenience we write out these coordinate systems. It would be quite simple to present the Laplace operator in each system and also the eigenfunctions, but we do not do this here.

A. Reduction $O(4,1) \supset O(4)$

$$x_0 = \cosh a, \quad x_i = \sinh a \tilde{x}_i, \quad 0 \leq a < \infty, \quad i = 1, \ldots, 4,$$
$$\tilde{x}_1^2 + \tilde{x}_2^2 + \tilde{x}_3^2 + \tilde{x}_4^2 = 1. \tag{52}$$

On the sphere $\tilde{x}_i$ we then either introduce spherical coordinates [reduction to $O(3) \supset O(2)$], or cylindrical coordinates [reduction to $O(2)\oplus O(2)$]. For a complete discussion of all subgroup and nonsubgroup type coordinates on an $O(4)$ sphere, see Ref. 61.

B. Reduction $O(4,1) \supset O(3,1)$

$$x_\mu = \sinh a \tilde{x}_\mu, \quad x_4 = \sinh a, \quad -\infty < a < \infty, \tag{53}$$
$$\mu = 0, 1, 2, 3, \quad \tilde{x}_0^2 - \tilde{x}_1^2 - \tilde{x}_2^2 - \tilde{x}_3^2 = 1.$$

On the $O(3,1)$ hyperboloid $\tilde{x}_\mu$ we introduce one of the seven types of subgroup coordinates, discussed earlier.[40,62] These are spherical coordinates for $O(3,1) \supset O(3)$, hyperbolic of three types for $O(3,1) \supset O(2,1)$, horospheric of two types for $O(3,1) \supset E(2)$, and cylindrical for $O(3,1) \supset O(2) \otimes O(1,1)$. Olevskiĭ also lists 27 nonsubgroup type coordinates for the $O(3,1)$ hyperboloid.

C. Reduction $O(4,1) \supset E(3)$

$$x_0 = \cosh\gamma + \tfrac{1}{2}(x^2 + y^2 + z^2)\, e^{-\gamma},$$
$$x_4 = \sinh\gamma + \tfrac{1}{2}(x^2 + y^2 + z^2)\, e^{-\gamma}, \tag{54}$$
$$x_1 = e^{-\gamma}x, \quad x_2 = e^{-\gamma}y, \quad x_3 = e^{-\gamma}z, \quad -\infty < \gamma < \infty.$$

On the Euclidean space x, y, z we can use spherical [$E(3) \supset O(3)$], cylindrical [$E(3) \supset E(2) \supset O(2)$] or Cartesian [$E(3) \supset E(2) \supset T(1)$] coordinates. All 11 subgroup and nonsubgroup type separable coordinates in $E(3)$ space are discussed in the literature.[63,64]

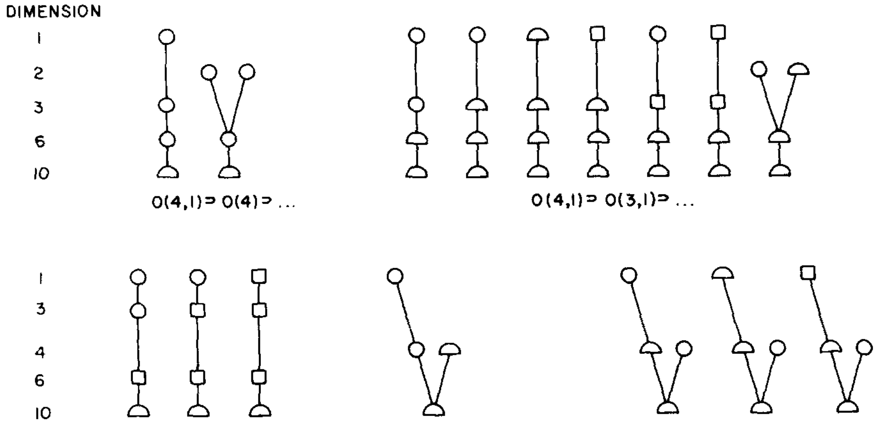

FIG. 8. Chains of subgroups of $O(4,1)$ providing quantum numbers and separable coordinate systems. A semicircle indicates an $O(p,1)$ group, a circle an $O(n)$ group and a square an $E(n)$ group.

D. Reduction $O(4,1) \supset O(3) \oplus O(1,1)$

$$x_0 = \cosh a \cosh b, \quad x_2 = \sinh a \sin\theta \cos\phi,$$
$$x_1 = \cosh a \sinh b, \quad x_3 = \sinh a \sin\theta \sin\phi, \tag{55}$$
$$x_4 = \sinh a \cos\theta,$$
$$0 \le a < \infty, \quad -\infty < b < \infty, \quad 0 \le \theta \le \pi, \quad 0 \le \phi < 2\pi.$$

E. Reduction $O(4,1) \supset O(2,1) \oplus O(2)$

$$x_k = \cosh a \tilde{x}_k, \quad x_3 = \sinh a \cos\phi,$$
$$x_4 = \sinh a \sin\phi, \tag{56}$$
$$k = 0, 1, 2, \ \tilde{x}_0^2 - \tilde{x}_1^2 - \tilde{x}_2^2 = 1, \ 0 \le a < \infty, \quad 0 \le \phi < 2\pi.$$

On the $O(2,1)$ sphere $\tilde{x}_k$ we introduce spherical $[O(2,1) \supset O(2)]$, hyperbolic $[O(2,1) \supset O(1,1)]$ or horospheric $[O(2,1) \supset T(1)]$ coordinates.

The connection between Lie theory and the separation of variables has received a lot of attention in the literature.[40–43, 59–61, 65] The results of this section, aside from listing all subgroup type separable coordinates, suggest a recursive method for introducing separable coordinates in arbitrary $O(p,1)$ and more generally $O(p,q)$ spaces.

4. CONCLUSIONS

The main result of this paper is that we have provided a complete classification of the continuous subgroups of the de Sitter group $O(4,1)$. Thus, we have shown that $O(4,1)$ has four maximal subgroups, namely $O(4)$, $O(3,1)$, $D \square E(3)$ and $O(2) \otimes O(2,1)$. All continuous subgroups of these maximal subgroups were classified into conjugacy classes, where conjugacy was considered under the maximal subgroup, under $SO_0(4,1)$ [the connected component of $O(4,1)$], under $SO(4,1)$ and under $O(4,1)$. The results are summarized in Fig. 1–6. In particular representatives of all $O(4,1)$ conjugacy classes of subalgebras of $LO(4,1)$ are given on Fig. 6 which also shows their mutual inclusions.

In Sec. 3 we have found the invariants of all subalgebras of $LO(4,1)$ (if such exist), both Casimir operators, i.e., polynomial invariants, lying in the center of the enveloping algebra, and also nonpolynomial invariants. These were used to present different possible sets of commuting operators, providing quantum numbers for an elementary physical system, described by an irreducible unitary representation of $O(4,1)$. A lattice of subgroups with Casimir operators is given in Fig. 7. A graphical representation of all different chains of such subgroups is shown in Fig. 8. We have also given a list of all "subgroup type" systems of coordinates, allowing the separation of variables in the Laplace equation on the hyperboloid $x_0^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2 = 1$.

It should be mentioned that the subgroup structure of $O(4,1)$ is relatively simple—much more so than that of the other groups of immediate physical interest, like the Poincaré group,[38] the similitude group,[39] the $O(3,2)$ de Sitter group, or the conformal group of space—time. These last two groups will be the subject of subsequent publications.

We have not gone deeply into any applications, however, the physical context in which the present results should be useful was discussed in the Introduction, and we plan to return to this separately.

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