Reeh–Schlieder-type density results in one- and n-body Schrödinger theory and the "unique continuation problem"

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A couple of Reeh–Schlieder-type density results are proved to hold in one- and n-body Schrödinger theory, that is, it is proved that states localized at time zero in an arbitrarily small open set of $\mathbb{R}^n$ are already total after an arbitrarily small time (which implies much more than the well-known acausal behavior of nonrelativistic theories). It is shown that there exists a close connection to the so-called "unique continuation property" of elliptic partial differential operators. Furthermore, a certain machinery of analytic continuation is developed and the notion of generalized propagation kernels is introduced, which also might be of use elsewhere (e.g., in scattering theory).

I. INTRODUCTION

Analytic continuation of physically interesting quantities, e.g., n-point functions or S-matrix elements, has proved to be of great importance in quantum field theory, in particular in the so-called "Wightman theory." The proofs of many of the central results in this field rely, sometimes almost exclusively, on the technique of extending objects, originally only defined over $\mathbb{R}$ or $\mathbb{R}^{(\nu+1)\cdot n}$ (where ν is the space dimension and $\nu+1$ is the dimension of space-time) into certain domains of $\mathbb{C}$ or $\mathbb{C}^{(\nu+1)\cdot n}$, $n \in N$.

This is evidently not so in Schrödinger theory, and it is probably a widespread belief that the applicability of powerful methods like these is typically restricted to the relativistic regime, where one has the so-called "spectrum condition," "locality," etc. We have shown previously, however, that several of the results carry over to the nonrelativistic regime.[1] In this paper, we will pursue two different strategies to prove a couple of results, which we think would be difficult to prove, without using the techniques developed below, in Schrödinger theory proper. One relies on an interplay of relatively deep results of functional analysis and a simple analytic continuation argument; the other employs exclusively spectral properties of energy momentum to develop a certain machinery of analytic continuation. Even in the simplest case, free motion of one particle, the results seem not to be easily accessible without using the methods described here.

We will proceed as follows. In the Sec. II we will exhibit the close connection between what we will call a Reeh–Schlieder property of an arbitrary domain $U \subset \mathbb{R}^\nu$ or $\mathbb{R}^{\nu\cdot n}$ (for the origin of the notion in Wightman theory see Ref. 2) and two seemingly different groups of concepts and ideas from the realm of classical functional analysis, one running under the catchword "unique continuation property," the other comprising the various notions of "generalized eigenfunction expansions" of Schrödinger operators. By "Reeh–Schlieder property" we mean roughly that the wave functions being localized in an arbitrarily small domain $U \subset \mathbb{R}^{\nu\cdot n}$ at time 0 are already total in the full Hilbert space $L^2(\mathbb{R}^{\nu\cdot n})$ after an arbitrarily short time interval. (Note that this implies much more than the feature, well-known at least for the free time evolution, that a wave function is more or less "everywhere" after an arbitrarily short time. However, without this property the stronger result could, of course, not hold.) These notions will be made more precise in Sec. II.

While in the Sec. II analyticity plays only a minor role, we have to rely heavily on it in the rest of the paper where we develop another sequence of ideas, pursuing more or less the goal of analytically extending both space and time translations into certain domains of $\mathbb{C}^{(\nu+1)\cdot n}$. That is, the first part carries a more functional analytic flavor while the latter draws more on procedures known to be successful in relativistic quantum field theory. Furthermore, we think the concepts we develop in the latter part, such as, e.g., "generalized propagation kernels," also will be of use elsewhere (for example, in a paper on a new general approach to scattering theory in energy-momentum space, which is in preparation).

To indicate some of the technical steps, we will start by rewriting n-body Schrödinger theory in the form of a certain time-dependent bilinear functional $W(X,Y;t)$ lying in $\mathscr{S}'(\mathbb{R}^{2\nu\cdot n})$, and acting on the wave functions at time 0 by using the nuclear theorem. These functionals contain the full physical information of the theory. We show that these "Wightman functions" of Schrödinger theory can be naturally viewed as restrictions of more complex functions lying in a bigger space, i.e.,

$$W(x_1,\dots,x_n,y_1,\dots,y_n;t) \to W(x_1t_1,\dots;y_1t'_1,\dots)\,. \tag{1.1}$$

The distributional Fourier transform of this extended W has nice support properties in the energy-momentum variables $\{(\omega_i,k_i)\}$ corresponding to $\{(x_i,t_i)\}$, which allows us to make an analytic extension of the original $W(X,Y;t)$ [$X \simeq (x_1,\dots,x_n)$, etc.] into a certain domain of $\mathbb{C}^{(\nu+1)\cdot n}$. The "values" $W(X,Y;t)$ then turn out to be the boundary values of an analytic function over $\mathbb{R}^{(\nu+1)\cdot n} + i\Gamma^n \subset \mathbb{C}^{(\nu+1)\cdot n}$ when we approach the real boundary $\mathbb{R}^{\nu\cdot n+1}$ (resp. $\mathbb{R}^{(\nu+1)\cdot n}$).

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II. THE CONNECTION BETWEEN THE REEH-SCHLIEDER (RS) AND THE UNIQUE CONTINUATION PROPERTY

To begin with, we have to define exactly what the RS property is to mean. So let $\mathscr{H} = L^2(\mathbb{R}^m)$, $m = \nu \cdot n$, $H = -\Delta + V$, U be an open set $\subset \mathbb{R}^m$, and I be an arbitrary time interval about $t = 0$, then

$$S(U, I) := \{\exp(-itH)\Phi; \operatorname{supp} \Phi \subset U, t \in I\}. \quad (2.1)$$

Definition 1: U has the RS property if $S(U, I)$ is total in $\mathscr{H}$ for some I.

We want to show that, in fact, for a large class of potentials and practically every arbitrarily small U and I, $S(U, I)$ has this property. To prove this we draw on two relatively advanced topics of elliptic partial differential operators, described by the catchwords (i) unique continuation property and (ii) generalized eigenfunction expansion. Both are topics of currently active research, and to prove them we need relatively advanced machinery. We do not want to go into detail in this paper concerning these two problems, but prefer to restrict ourselves to giving some definitions and references that show that both features are actually fulfilled for a sufficiently large class of potentials.

We need the unique continuation property in the following form.

Definition 2: Let Φ lie in the local Sobolev space $H^2_{\text{loc}}(R^m)$, i.e., $u \cdot \Phi \in D(-\Delta)$ for all $u \in C_0^\infty(\mathbb{R}^m)$. Furthermore, let Φ satisfy the following differential inequality for some $\lambda \in \mathbb{R}$ and V the potential:

$$|\Delta\Phi(x)| \leqslant |(\lambda - V(x)) \cdot \Phi(x)|. \quad (2.2)$$

We say a *unique continuation* property holds if (2.2) implies the following: If we assume that Φ vanishes around a certain point x, then it vanishes everywhere (in the sense of L^2_{loc}).

Remark: The phenomenon mentioned above has a long history (see, e.g., the notes in Ref. 3 to the appendix to Chap. XIII.13 or Ref. 4). In recent years, the conditions imposed on the potential V have been more and more relaxed (see, e.g., Refs. 5–8 and further references given there).

The next property we shall need is the existence of an eigenfunction expansion of the Hamiltonian with the generalized eigenfunctions being "sufficiently nice." Also, here we do not aim at optimal results, but content ourselves with showing that something like this actually does exist for a sufficiently large class of potentials. In order not to struggle with perhaps nasty measure theoretic problems, we restrict ourselves to the class of so-called Agmon potentials (a treatment of long-range potentials can, e.g., be found in the book of Saito[9]). An approach more in the original spirit of Ikebe and Povzner can also be found in Ref. 10, see also Ref. 11, Sec. C 5. So, with V being an Agmon potential (cf., e.g., Ref. 12 for the necessary details) we have the following theorem.

Theorem 1: With V of short-range type in the sense of Agmon, we have a complete set of generalized continuum eigenfunctions $\phi(\cdot,k)$, labeled by $k \in \mathbb{R}^m$, lying in H^2_{loc} such that the following holds: With $\Phi, \Psi \in L^2(\mathbb{R}^m)$, and g a bounded continuous function,

$$(\Psi | g(H)\Phi) = \sum_{n=1}^{N} g(\lambda_n)(\Psi|\phi_n)(\phi_n|\Phi) + \int d^m k\, g(k^2)\langle \Psi|\phi(\cdot,k)\rangle \times \langle \phi(\cdot,k)|\Phi\rangle, \quad (2.3)$$

where N may be infinite, the ϕ_n are the usual L^2-eigenfunctions with eigenvalue λ_n, and $\langle \phi(\cdot,k)|\Phi\rangle$ is an abbreviation for $\int \overline{\phi(x,k)} \cdot \Phi(x) \cdot d^m x$, defined in an appropriate sense, where the k^2 are the continuum eigenvalues.

Proof: For the proof see Ref. 12. Note in particular that there is no singular continuous spectrum in this case.

Now we are ready to prove the main result of this section. We assume throughout the paper that H is bounded below (which is of course fulfilled for, e.g., Agmon potentials). With U an arbitrary open set in $\mathbb{R}^m$ and I a fixed but arbitrarily small time interval about $t = 0$, we would like to show that $S(U, I)$ is total in $\mathscr{H} = L^2(\mathbb{R}^m)$. We assume the contrary, i.e., there exists a nonzero $\Psi \in L^2(\mathbb{R}^m)$ such that

$$(\Psi, e^{-itH}\Phi) \equiv 0, \quad \text{for all } \Phi \in S(U), \ t \in I \quad (2.4)$$

[where $S(U) \cong S(U, t = 0)$, i.e., the functions exactly localized in U]. Evidently,

$$(\Psi, e^{-itH}\Phi) = \int e^{-i\lambda t} d(\Psi, E_\lambda \Phi)$$

has an analytic continuation into the lower half plane, i.e.,

$$F(t - i\tau) := \int e^{-i\lambda(t - i\tau)} d(\Psi, E_\lambda \Phi) \quad (2.5)$$

is analytic for $\tau > 0$. Defining

$$G(t + i\tau) := \overline{F\ \overline{(t + i\tau)}}, \quad (2.6)$$

G is analytic in the upper half plane. We have

$$G(I) \equiv F(I) \equiv 0 \quad (2.7)$$

by assumption, that is, on an open set of the common real boundary of G, F. Hence we see that G is the analytic continuation of F through the real open set I, where the analytic function $F \cup G$ is zero. This implies $F \equiv 0$ in the lower half plane, and by continuity F vanishes also on the real boundary, i.e., we have

$$(\Psi, e^{itH}\Phi) \equiv 0, \quad \text{for all } t. \quad (2.8)$$

(This simple reasoning, in fact quite common in Wightman theory, was also exploited in Ref. 13 in order to study the localization properties in quantum theory.)

By uniqueness of the Fourier transform the measure $d(\Psi, E_\lambda \Phi)$ is zero. Assuming now that H has a generalized eigenfunction expansion according to Theorem 1 we can conclude

$$\int_{k^2 = \lambda} dS_{m-1} \langle \Psi, \phi(\cdot,k)\rangle \cdot \langle \phi(\cdot,k), \Phi\rangle + \sum_{\lambda_n = \lambda} (\Psi, \phi_n) \cdot (\phi_n, \Phi) = 0, \quad (2.9)$$

for a.e. $\lambda \in \mathbb{R}$ (with respect to the measure $[\theta(\lambda) \cdot d\lambda + \Sigma_n \delta(\lambda - \lambda_n) d\lambda]$!).

Remark: The above restricted integration over the sphere $k^2 = \lambda$ is well defined, cf. Ref. 12, Theorem 5.1.

Abbreviating now $\langle \Psi, \phi(\cdot,k)\rangle$ [resp. (Ψ, ϕ_n)] by $c(k)$ (resp. c_n) we can write this as

$$\left\langle \int_{k^2=\lambda} dS_{m-1}\, c(k)\, \phi(\cdot,k) + \sum_{\lambda_n=\lambda} c_n\, \phi_n, \Phi \right\rangle = 0, \quad \text{a.e.}, \tag{2.10}$$

$$u_\lambda := \left(\int_{k^2=\lambda} dS_{m-1}\, c(k)\, \phi(\cdot,k) + \sum_{\lambda_n=\lambda} c_n\, \phi_n \right) \tag{2.11}$$

is again a solution of $(H-\lambda)u = 0$, $u \in H^2_{\text{loc}}$.

It is our aim to show that (2.9) implies, in fact, that $c_n = 0$, $c(k) = 0$ for all n (resp. almost all k). This then would yield that $\Psi = 0$ in L^2, i.e., that $S(U,I)$ is total. To show this we need the unique continuation property of Definition 2; u_λ of (2.11) fulfills the hypotheses of Definition 2, furthermore,

$$\langle u_\lambda, \Phi\rangle = 0, \quad \text{for all} \quad \Phi \in S(u) \Rightarrow u_\lambda \equiv 0 \quad \text{on} \quad U. \tag{2.12}$$

By the unique continuation property this entails that $u_\lambda \equiv 0$ a.e. (in L^2_{loc}). But the expansion with respect to $\{\phi(\cdot,k), \phi_n\}$ is "orthogonal," that is, $u_\lambda \equiv 0$ for almost all λ implies that $\{c(k), c_n\} \equiv 0$ a.e., hence $\Psi = 0$ in L^2. This proves the first part of the following theorem.

Theorem 2: Assuming that H has the unique continuation property and a generalized eigenfunction expansion in the sense of Theorem 1, the following statements hold.

(i) Given an arbitrary open set $U \subset \mathbb{R}^m$ and an arbitrarily small time interval I around $t = 0$, then $S(U,I)$ is already total in the full $L^2(\mathbb{R}^m)$.

(ii) Conversely, if a RS property holds for every open $U \subset \mathbb{R}^m$, then the generalized eigenfunctions have the unique continuation property in the sense of Definition 2.

Proof: The proof of (ii) is easy. Assuming that u is a nontrivial generalized eigenfunction vanishing, e.g., on a certain open $U \subset \mathbb{R}^m$, we have for all $\Phi \in S(U)$ and all $t \in \mathbb{R}$

$$0 = e^{-i\lambda t}\langle u_\lambda, \Phi\rangle = \langle e^{iHt}u_\lambda, \Phi\rangle = \langle u_\lambda, e^{-itH}\Phi\rangle. \tag{2.13}$$

But, by assumption, $\{e^{itH}\phi\}$ is total in L^2, in particular in every L^2_Ω, Ω compact in $\mathbb{R}^m$. Hence u_λ (which lies in L^2_{loc}) vanishes in L^2_{loc}.

We would like to mention that the above result is much stronger than that implied by the well-known feature of nonrelativistic quantum theory, namely, that wave functions have the tendency to spread out to infinity almost instantaneously. The latter says only that the wave function cannot be orthogonal to certain functions that have their support concentrated in possibly very small neighborhoods of points $x \in \mathbb{R}^m$. Theorem 2 says that even arbitrary extended and oscillating functions cannot be orthogonal to $S(U,I)$. The physics behind the result is perhaps even more striking. Theorem 2 tells us that the physical content of the theory is already contained in an arbitrarily small space-time neighborhood of an arbitrary point.

III. ANOTHER APPROACH AND THE NOTION OF GENERALIZED PROPAGATION KERNELS

In the rest of the paper we will develop a different approach to the problem with slightly different results and completely different methods, the whole approach being more in the spirit of the original relativistic context. Stated somewhat sloppily, it consists of extending time and space translations together into certain domains of $\mathbb{C}^{(\nu+1)\cdot n}$. This analytical continuation is, however, a little bit subtle since momentum is not bounded below. It would be tempting to use a scale of auxiliary spaces (which are possibly no longer Hilbert spaces) to give meaning to expressions like $\{\exp i(a+ib)P\}\,\phi$, $\phi \in L^2(\mathbb{R}^m)$, where P is the momentum operator. We postpone this approach to the future, however, and choose another strategy in this paper.

In a first step we want to present n-body Schrödinger theory in a way slightly different from the conventional approach, i.e., by means of so-called propagation kernels. It will turn out that these distributions contain all the physics of the theory and are amenable to a certain embedding of ordinary Schrödinger theory into a larger theory, in which certain spectral and analyticity properties of the objects of interest can be visualized more easily.

We start from the expression

$$(\phi, e^{-itH}\psi), \quad \phi, \psi \in L^2(\mathbb{R}^{\nu\cdot n}). \tag{3.1}$$

Evidently this defines a sesquilinear functional over $L^2(\mathbb{R}^{\nu\cdot n}) \times L^2(\mathbb{R}^{\nu\cdot n})$ which depends on t. We will restrict this functional to the dense subset $\mathscr{S}(\mathbb{R}^{\nu\cdot n})$. In contrast to, e.g., L^2, $\mathscr{S}$ is a *nuclear space*, which has a far-reaching consequence. In this case the kernel or *nuclear theorem* holds, which allows us to prove the following theorem.

Theorem 3: With $\phi, \psi \in \mathscr{S}(\mathbb{R}^{\nu\cdot n})$ there exists a time-dependent tempered distribution $W \in \mathscr{S}'(\mathbb{R}^{\nu n} \times \mathbb{R}^{\nu n})$ such that

(i) $$(\phi, e^{itH}\psi) = \int \bar{\phi}(X_n)\cdot \psi(Y_n) \cdot W(X_n, Y_n; t)\, dX_n\, dY_n,$$

(ii) $$\int W(X_n, Y_n; t)\cdot \psi(Y_n)\, dY_n = \psi_t = e^{-itH}\psi \quad \text{in } L^2\text{-sense, and}$$

(iii) $W(X_n, Y_n; 0) = \delta(X_n - Y_n)$ and $W(X_n, Y_n; t)$ "solves" the Schrödinger equation with respect to $X_n = (x_1, \ldots, x_n)$.

Proof: Continuity in $\mathscr{S}$ implies continuity in L^2 such that (ϕ, ψ_t) is separately continuous in $\mathscr{S} \times \mathscr{S}$ for every fixed t. The simple proof goes as follows:

$$f_n \to f \text{ in } \mathscr{S} \Rightarrow \sup_x\, (1+|X|)^k \cdot |f - f_n|^2 \to 0, \tag{3.2}$$

with $|X| := \Sigma_{i=1}^n |x_i|$ and every $k \in \mathbb{N}$. Thus we have

$$\int |f - f_n|^2\, dX = \int |f - f_n|^2 \cdot (1+|X|)^k (1+|X|)^{-k}\, dX \leqslant \sup_x \{(1+|X|)^k |f - f_n|^2\} \cdot \int (1+|X|)^{-k}\, dX \to 0, \tag{3.3}$$

for k large enough and $n \to \infty$. So the kernel theorem can be applied. By varying ϕ over a dense set in $L^2 \cap \mathscr{S}$, we see that (ii) must hold, and (iii) is evident.

Remarks: (i) That the kernel theorem yields nontrivial information can be seen by the following observation. The scalar product (ϕ,ψ) is also a sesquilinear, even jointly continuous, functional over $L^2 \times L^2$. If a kernel theorem would hold in this situation, the W under discussion would be an element from $L^2(\mathbb{R}^{\nu n} \times \mathbb{R}^{\nu n})$! But we already know what W looks like, namely for $\phi, \psi \in \mathscr{S}$, $W = \delta(X_n - Y_n)$, which is not in L^2 but in $\mathscr{S}'$.

(ii) The above distribution W has special additional continuity properties such that we can hope to be able to restrict the general structure of W further. The structure of distributions lying in $\mathscr{S}'$ is well known (see, e.g., Refs. 14 and 15):

$$W(X_n,Y_n) = D^{\alpha}(X_n,Y_n)m(X_n,Y_n), \tag{3.4}$$

with D a certain differential operator of degree α, $m(X_n,Y_n)$ a measure or a function on $\mathbb{R}^{\nu n} \times \mathbb{R}^{\nu n}$. In the simplest case, e.g., H_0, the free Hamiltonian on $L^2(\mathbb{R}^{\nu})$, the free propagator has the well-known form

$$P_0(x,y;t) = (4\pi i t)^{-\nu/2} \exp(i|x-y|^2/4t). \tag{3.5}$$

Other kernel representations are known for, e.g., $(H-E)^{-1}$ in the context of eigenfunction expansions and the Lippmann–Schwinger equation. In the restricted case of potential scattering we have, for example,

$$((H-E)^{-1}\cdot\psi)(x) = \int G(x,y;E)\,\psi(y)\,dy, \tag{3.6}$$

$G \in L^1 \cap L^2$ for a.e. fixed x with respect to y and certain classes of potentials (see, e.g., Ref. 10, Chap. XI.6). We must, however, emphasize that the situation in (3.6) is considerably simpler since one makes heavy use of the fact that certain operators are Hilbert–Schmidt (which is typical for the case of resolvents).

(iii) Note that the distribution W of Theorem 3 occurs also as a path integral in the Feynman–Kac theory.

The observation above motivates the name propagation kernel or Wightman function of n-particle Schrödinger theory. The W defined in Theorem 3 is of the form

$$W(x_1t,\ldots,x_nt;y_1,\ldots,y_n), \tag{3.7}$$

i.e., all time coordinates are equal. In the next step we want to make a natural extension to a more general distribution, depending on $t_1,\ldots,t_n$. To this end we will assume (whereas this is not strictly necessary) that the potential occurring in the n-particle Hamiltonian is a sum of pair potentials, i.e.,

$$H = H_0 + V = \frac{1}{2}\sum_{i=1}^{n} -\Delta_i + \sum_{i<j} V_{ij}(x_i - x_j) \tag{3.8}$$

(for simplicity all masses are normalized to 1). This makes V translation invariant, more precisely, invariant under overall translations. Thinking now of the coordinates as ordered n-tuples $(x_n,\ldots,x_1)$ and correspondingly $L^2(\mathbb{R}^{\nu\cdot n}) = L^2(\mathbb{R}^{\nu}_n) \otimes \cdots \otimes L^2(\mathbb{R}^{\nu}_i)$, we can define individual time evolutions in each subspace

$$\mathscr{H}_1 := L^2(\mathbb{R}^{\nu}_1),\quad \mathscr{H}_2 := L^2(\mathbb{R}^{\nu}_2) \otimes L^2(\mathbb{R}^{\nu}_1),\ldots$$

and

$$\begin{aligned} H_1 &:= -\tfrac{1}{2}\Delta_1, \\ H_2 &:= -\frac{1}{2}\sum_{i=1}^{2}\Delta_i + V_{12}(x_1 - x_2), \\ &\vdots \\ H_n &:= -\frac{1}{2}\sum_{i=1}^{n}\Delta_i + \sum_{i<j} V_{ij}(x_i - x_j). \end{aligned} \tag{3.9}$$

For notational simplicity we assume the overall wave function at time zero, $\phi(x_n,\ldots,x_1)$, to be given as a product (the general case is analogous):

$$\phi(x_n,\ldots,x_1) := \phi_n(x_n)\cdot\cdots\cdot\phi_1(x_1). \tag{3.10}$$

We can then extend $\phi(x_n,\ldots,x_1;t)$ in the following way: We let $\exp -i(t_1 - t_2)\cdot H_1$ operate on $\phi(x_1)$, $\exp -i(t_2 - t_3)\cdot H_2$ operate on the product $\{\phi_2(x_2)\cdot\phi_1(x_1;t_1 - t_2)\}$, etc., that is we get the following definition.

Definition 3: Let $\phi(x_n,\ldots,x_1) = \phi_n(x_n)\cdot\cdots\cdot\phi_1(x_1)$ with $\phi_i \in L^2(\mathbb{R}^{\nu})$. We define $\phi(x_nt_n,\ldots,x_1t_1)$ by

$$\begin{aligned} \phi(x_nt_n,\ldots,x_1t_1) &:= e^{-it_nH_n}[\phi_n(x_n)\cdot\exp(-i(t_{n-1} - t_n)H_{n-1})] \\ &\quad \times [\phi_{n-1}(x_{n-1})\exp(-i(t_{n-2} - t_{n-1})H_{n-2})] \\ &\quad \cdots[\cdots[\phi_2(x_2)\cdot e^{-i(t_1 - t_2)H_1}\phi_1(x_1)]\cdots]. \end{aligned} \tag{3.11}$$

We have the simple following corollary, which shows that this actually defines an embedding.

Corollary 1: $\phi(x_n,\ldots,x_1;t)$ is recovered by setting

$$t_n = t_{n-1} = \cdots = t_1 = t.$$

This procedure can be extended in an evident manner to every function of $L^2(\mathbb{R}^{\nu\cdot n})$ since by the Fubini–Tonelli theorem, every function of $L^2(\mathbb{R}^{\nu\cdot n})$ is an L^2-function in, e.g., $\{x_k,\ldots,x_1\}$ for almost all coordinates $\{x_n,\ldots,x_{k+1}\}$ being held fixed.

In a completely analogous way we can extend space translations.

Definition 4: With ϕ of Definition 3, we define

$$\begin{aligned} \phi(x_n + a_n;t_n,x_{n-1} + a_{n-1};t_{n-1},\ldots,x_1 + a_1;t_1) &:= e^{-it_nH_n}\cdot e^{ia_nP_n} \\ &\quad \cdot[\phi_n(x^n)\cdot\exp(-i(t_{n-1} - t_n)H_{n-1}) \\ &\quad \cdot\exp(i(a_{n-1} - a_n)P_{n-1})[\phi_{n-1}(x_{n-1})\cdots]\cdots], \end{aligned} \tag{3.12}$$

with

$$P_k := \sum_{j\le k} p_j,\quad p_j = -i\cdot\partial_{x_j},\quad a_j \in \mathbb{R}^{\nu}.$$

Remark: Again we see that we get an overall translation by setting $a_n = a_{n-1} = \cdots = a_1 = a$. Note furthermore that H_k and P_k commute on $\mathscr{H}_k$ and that, in fact, the above-defined extended space translations shift the individual coordinates $\{x_k\}$ by vectors $\{a_k\}$.

Employing the above definitions we can make corresponding extensions of the propagation kernels.

Proposition 1: By

$$\int W(x_n + a_n; t_n, \ldots, x_1 + a_1, t_1 | y_n, \ldots, y_1)$$
$$\times \phi(y_n, \ldots, y_1) dY_n$$
$$:= \int W(x_n, \ldots, x_1 | y_n, \ldots, y_1)$$
$$\times \phi(y_n + a_n, t_n; \ldots, y_1 + a_1, t_1) dY_n, \qquad (3.13)$$

a *generalized propagation kernel* is defined, which is again a distribution in $\mathscr{S}'$ depending on the parameters $\{t_n, a_n; \ldots; t_1, a_1\}$.

Proof: The continuity properties with respect to X_n and Y_n in the $\mathscr{S}$-topology can be shown as in (3.2).

The next natural step is to make a Fourier transformation with respect to the variables $\{t_n a_n, \ldots, t_1 a_1\}$ and employ certain support properties of the Fourier transform $\hat{W}$ in the variables $\{\omega_n k_n, \ldots, \omega_1 k_1\}$. While we do not intend to talk about scattering theory in this paper we would nevertheless like to make a short aside about the possible use of the above extension method in this field. The above approach offers the possibility of dealing with scattering phenomena on a very broad scale, that is, in quantum field theory, Schrödinger theory, temperature states, ground states, etc. The particular advantage is that one need not worry (at least openly) about reference dynamics, range of interactions, spectral gaps around the mass shells, etc. This is replaced by a study of the spectral properties of the $\hat{W}$'s along certain submanifolds in the support of $\hat{W}$ on which, in the limit $t \to \pm \infty$, the "scattering states" will live. For temperature states this has been done in a recent paper.[16] By means of certain geometric arguments we can show when and why scattering states exist and exactly which properties, in the neighborhood of the mass shells of ingoing and outgoing clusters, particles in the spectral support of $\hat{W}$ yield a nontrivial S-matrix. The whole approach is more in the spirit of Buchholz's treatment of scattering of massless particles and will be developed in a forthcoming paper.

IV. SUPPORT PROPERTIES OF THE FOURIER TRANSFORMS OF $\phi(t_n a_n, \ldots, t_1 a_1)$ AND $W(t_n a_n, \ldots, t_1 a_1)$

We will now investigate the support properties of the Fourier transforms of $\phi(t_n a_n, \ldots, t_1 a_1)$ [resp. $W(t_n a_n, \ldots, t_1 a_1)$] (where we dropped the remaining coordinates X_n, Y_n) with respect to $(t_n, a_n, \ldots, t_1 a_1)$. This is the same as investigating the support of the joint energy-momentum spectrum of (H_m, P_m) in each subspace $\mathscr{H}_m$ [cf. (3.9)].

Proposition 2: Let all V_{ij} be infinitesimally H_0-bounded, either in operator sense or form sense, or, slightly weaker, each $V^{(m)}$ in $\mathscr{H}_m$ relatively bounded with all relative bounds $\{a_m\}$ smaller than 1. Then the joint (H_m, P_m)-spectrum, a set in $\mathbb{R}^4$, can be bounded from below by the hypersurface $(a_m < 1)$

$$\omega = c_m \cdot (1 - a_m) \cdot k^2 - b_m, \qquad (4.1)$$

with c_m, b_m certain constants and (ω, k) the energy-momentum variables corresponding to (H_m, P_m) in the subspace $\mathscr{H}_m$ of the particles $(1), \ldots, (m)$.

Proof: An analogous proof for the more general case of general nonrelativistic quantum field theory can be found in Ref. 1, so we give only a sketch of the proof. In the first step we show that the joint spectrum of $H_0^{(m)} := -\frac{1}{2} \Sigma_{i=1}^m \Delta_i$ and $P_m := \Sigma_{i=1}^m p_i$ can be bounded below by a parabolic hypersurface (bounding, e.g., Σp_i^2 by $c \cdot |\Sigma p_i|$). In the second step we exploit the relative boundness to show that the interaction results only in a *finite* overall shift of this paraboloid.

Corollary 2: (i) The joint spectrum of (H_m, P_m) can be embedded in a domain $K_m \cup \Gamma_c \subset \mathbb{R}^4$, where K_m is a ball around $(0,0) \in \mathbb{R}^4$ with sufficiently large radius and Γ_c the so-called forward cone $\{(\omega, k); \omega > c \cdot |k|, c > 0\}$.

(ii) By choosing the radius of K_m large enough, c also can be chosen arbitrarily large.

Proof: Each paraboloid of the form (4.1) intersects every cone Γ_c for sufficiently large $|k|$. So there exists always a finite ω_0, depending on a_m, b_m, and c such that all (ω, k) lying above the surface (4.1) are ultimately contained in Γ_c for $\omega \geqslant \omega_0$.

Now we take a $\phi(x_n + a_n; t_n, \ldots, x_1 + a_1; t_1)$ defined in Definition 4 and observe that each $\exp(-i(t_{k-1} - t_k) H_{k-1}) \cdot \exp(i(a_{k-1} - a_k) \cdot P_{k-1})$ standing between the functions ϕ_k and $[\phi_{k-1} \ldots]$ acts on a Hilbert space of particle number $k-1$. We know from the above discussion that the joint (H_{k-1}, P_{k-1})-spectrum is contained in some $K_{k-1} \cup \Gamma_c \subset \mathbb{R}^4$. Inserting the spectral resolution for each of the above operators, we get

$$\phi(a_n; t_n, \ldots, a_1; t_1) = (2\pi)^{-2n} \int \exp[-i(t_n \omega - a_n k_n)] \exp[-i(t_{n-1} - t_n) \omega_{n-1} - (a_{n-1} - a_n) k_{n-1}]$$
$$\cdot \cdots \cdot \{E^{(n)}(d\omega_n\, dk_n) \cdot \phi_n E^{(n-1)}(d\omega_{n-1}\, dk_{n-1}) \phi_{n-1} \cdot \cdots \cdot E^{(1)}(d\omega_1\, dk_1)\, \phi_1\} \qquad (4.2)$$
$$= (2\pi)^{-2n} \int \exp[-i(t_n \omega_n - a_n\, k_n)] \cdot \cdots \cdot \hat{W}(\omega_n k_n, \ldots, \omega_1 k_1 | X_n, Y_n) \phi(y_n \cdots y_1) dY_n. \qquad (4.3)$$

It should be noted that, whereas the various $E\,(d\omega\, dk)$ occurring in (4.2) are spectral measures, the curly bracket is a measure only in each of the coordinates $(\omega_i k_i)$ with the remaining variables $(\omega_j k_j)$ being integrated over (with appropriate test functions). Taken as a whole, the curly bracket in (4.2) is a vector valued distribution in the variables $(\omega_n k_n, \ldots, \omega_1 k_1)$ with $\exp[-i(t_n \omega_n - a_n\, k_n) \cdot \cdots]$ lying in the domain of definition. By the same token $\hat{W}$ is a distribution with respect to $\{(\omega_i k_i)\}$. The support properties of the joint spectrum of $\{(H_k\, P_k)\}$ now entail that the curly bracket on the right-hand side (rhs) of (4.2) (resp. $\hat{W}$) are distributions with support:

$$\text{supp}\,\hat{W} \quad [\text{resp. supp}\,\hat{\phi} \subset \{(\omega_n k_n,...,\omega_1 k_1) \text{ such that each } (\omega_i\, k_i) \in K_i \cup \Gamma_c\}], \tag{4.4}$$

that is, we have proved the following theorem.

Theorem 4: $\phi(t_n a_n,...,t_1 a_1)$ is the Fourier transform [with respect to $(t_n a_n,...,t_1 a_1)$] of a vector valued distribution $\hat{\phi}$, given as the curly bracket in (4.2), with (ω_i,k_i)-support contained in $K_i \cup \Gamma_c$. The same support properties hold for $\hat{W}$ the Fourier transform of the generalized propagation kernel.

V. THE ANALYTIC CONTINUATION OF $\phi(t_n a_n,...,t_1 a_1)$ AND $W(t_n a_n,...,t_1 a_1)$

We will now exploit the special support properties of ϕ, $\hat{W}$ to show that ϕ, W can be analytically continued into a domain of $\mathbb{C}^{(\nu+1)\cdot n}$. We have seen that the support of $\hat{\phi}$, $\hat{W}$ is restricted by $(\omega_i\, k_i) \in K_i \cup \Gamma_c$. We observe now that the rhs of, e.g., (4.2) still exists if the exponents

$$(t_n\omega_n - a_n k_n),$$
$$((t_{n-1} - t_n)\cdot\omega_{n-1} - (a_{n-1} - a_n)k_{n-1}),...$$

are replaced by

$$(z_n\cdot\omega_n - \zeta_n\cdot k_n),$$
$$((z_{n-1} - z_n)\cdot\omega_{n-1} - (\zeta_{n-1} - \zeta_n)\cdot k_{n-1}),..., \tag{5.1}$$

with

$$z_n := t_n - i\tau_n, \quad \zeta_n := a_n - ib_n,..., \tag{5.2}$$

where $(\tau_n,b_n),...$, fulfill the support condition

$$(\tau_n\omega_n - b_n\, k_n) > 0,$$
$$((\tau_{n-1} - \tau_n)\omega_{n-1} - (b_{n-1} - b_n)k_{n-1}) > 0,..., \tag{5.3}$$

for $(\omega_n,k_n)\in\Gamma_c$, $(\omega_{n-1},k_{n-1})\in\Gamma_c$,... .

This can be seen as follows. We can split the support of $\hat{W}$ with respect to each (ω_i,k_i) into the sets $K_i \cup \Gamma_c \setminus \Gamma_c$ and Γ_c. Since K_i has a finite diameter the analytic continuation of

$$\int_{K_i \cup \Gamma_c \setminus \Gamma_c} (\cdots) d\omega_i\, dk_i \quad \text{with respect to } (z_i, \zeta_i) \tag{5.4}$$

always exists. To continue the integral over Γ_c, $\int_{\Gamma_c} (\cdots)\, d\omega_i \times dk_i$, we need the special support properties of the (τ_i, ζ_i) mentioned in (5.3). These properties guarantee that

$$\exp[-(\tau_n\omega_n - b_n\, k_n)],$$
$$\exp[-((\tau_{n-1} - \tau_n)\omega_{n-1} - (b_{n-1} - b_n)k_{n-1})],... \tag{5.5}$$

are globally bounded on the domain, Γ_c, of integration.

This observation allows us to prove the following theorem.

Theorem 5: $\phi(t_n a_n,...,t_1 a_1)$, $W(t_n a_n,...,t_1 a_1)$, defined in Sec. III can be analytically continued into the domain $T^n \subset \mathbb{C}^{(\nu+1)\cdot n}$, given by

$$(z_n, \zeta_n), \quad ((z_{n-1} - z_n),(\zeta_{n-1} - \zeta_n)),...$$
$$\in \mathbb{R}^{\nu+1} + i\{(\tau,b);\ b\in\mathbb{R}^\nu,\ \tau > 0\} := T, \tag{5.6}$$

with $z_n = t_n - i\tau_n$, $\zeta_n = a_n - ib_n$,... .

Proof: The set of pairs (τ_n,b_n),... given by (5.3) span the interior of the so-called dual cone $\tilde{\Gamma}_c$ of Γ_c. We showed in Sec. IV that the (ω_i,k_i)-support of $\hat{\phi}$, $\hat{W}$ is bounded below by a paraboloid that intersects eventually every cone Γ_c for arbitrarily large c. This implies that ϕ, W can be analytically continued into $\mathbb{R}^{(\nu+1)} + i\,\tilde{\Gamma}_c$ in each of the variables given in (5.2) for $c\to\infty$. For $c\to\infty$, $\tilde{\Gamma}_c$ becomes the whole half space $(b\in\mathbb{R}^\nu, \tau > 0)$.

It is perhaps instructive to apply the above machinery to the simplest example we can think of, the free time evolution in $\mathbb{R}^3$. In this case we have $(H_0 := -\Delta)$

$$(e^{-itH_0}\phi)(a) = (2\pi)^{-3/2}\cdot\int e^{-itp^2}\cdot e^{ipa}\hat{\phi}(p)d^3p. \tag{5.7}$$

The energy spectrum is supported on the hypersurface $\omega = p^2$, that is,

$$\hat{W}(\omega,k) = \delta(\omega - p^2), \tag{5.8}$$

and we see that

$$(2\pi)^{-3/2}\int e^{-i(t-i\tau)p^2}\cdot e^{i(a-ib)p}\hat{\phi}(p)d^3p \tag{5.9}$$

exists provided that $\tau > 0$ and is analytic in the domain

$$\{(z,\zeta);\ z = t - i\tau, \quad \zeta = a - ib, \quad \tau > 0\} \tag{5.10}$$

(since p^2 wins out against $|p|$ for $|p|\to\infty$). The generalized propagation kernel is

$$W(a - ib, y; t - i\tau) := (4\pi i(t - i\tau))^{-3/2} \cdot\exp(i\cdot(a - y - ib)^2/4(t - i\tau)), \tag{5.11}$$

and we see again that

$$\int W(a - ib, y, t - i\tau)\phi(y)dy$$

is complex differentiable with respect to $\{(a - ib), (t - i\tau)\}$ in L^2 as long as $\tau > 0$ since this provides us with a term $\sim\exp(-|a - y^2|/4\tau)$.

VI. ANOTHER VERSION OF THE REEH–SCHLIEDER THEOREM

We now prove another version of the RS theorem, which is more in the spirit of the original version proved in Wightman theory.

Theorem 6: With a Hamiltonian H and generalized states $\phi(t_n\, a_n,...,t_1 a_1)$ as given in Sec. III, the following holds: The set

$$S'(U_n, I) := \{\phi(x_n t_n,...,x_1 t_1),$$
$$t_i\in I, \quad \text{supp}\,\phi(x_n,...,x_1) \subset U_n \subset \mathbb{R}^{\nu\cdot n}\}$$

is already total in $\mathscr{H}_n = L^2(\mathbb{R}^{\nu\cdot n})$, where U_n is an arbitrarily small open set in $\mathbb{R}^{\nu\cdot n}$ and I is an arbitrarily small time interval around $t = 0$.

Remark: It is already sufficient to choose the wave functions of the form

$$\phi(x_n,...,x_1) = \phi_n(x_n)\cdot\cdots\cdot\phi_1(x_1).$$

Proof: We assume the contrary, i.e., there exists a wave function $\psi(x_n,...,x_1)$ such that

$$(\psi,\phi) = 0, \quad \text{for all } \phi\in S'(U_n,I). \tag{6.1}$$

We proved in Sec. V that the function

$$F(t_n a_n,...,t_1 a_1) := (\psi, \phi(t_n a_n,...,t_1 a_1)) \tag{6.2}$$

can be analytically continued, with respect to $\{(t_i a_i)\}$, into an open domain T^n of $\mathbb{C}^{(\nu+1)\cdot n}$ and that $F(t_n a_n,...,t_1 a_1)$ [resp. $F(a_n,...,a_1)$] are the boundary values for $\{(\mathrm{Im}\, z_i, \mathrm{Im}\, \zeta_i)\} \to 0$.

Choosing the ϕ's to have their supports in a subset $U_n' \subset U_n$ such that for $\{a_i\}$ sufficiently small their space translates, $\phi(a_n,...,a_1)$ have their support still contained in U_n, we see that we can arrange matters such that

$$F(a_n t_n,...,a_1 t_1) \equiv 0,$$

for an open set $\mathscr{O}$ of

$$\{(t_i a_i)\} \subset \mathbb{R}^{(\nu+1)\cdot n}. \tag{6.3}$$

This set $\mathscr{O}$ is part of the boundary of the analytic continuation of F into T^n. Proceeding as in Sec. II, we define

$$G(z_n \zeta_n,...,z_1 \zeta_1) := \overline{F(\bar{z}_n \bar{\zeta}_n,...,\bar{z}_1 \bar{\zeta}_1)}, \tag{6.4}$$

where G is now analytic in $\overline{T^n}$ and $\overline{T^n}$, T^n having a common real boundary set $\mathscr{O}$, where

$$F \equiv G \equiv 0 \tag{6.5}$$

holds.

Again we conclude that $F \equiv 0$ in T^n (by using the "edge of the wedge" theorem, see Ref. 2) which, by continuity, holds also for the real boundary, i.e.,

$$F(t_n a_n,...,t_1 a_1) \equiv 0 \tag{6.6}$$

on $\mathbb{R}^{(\nu+1)\cdot n}$. Now we can set all time coordinates $\{t_i\}$ equal to zero and vary the a_i's independently in $\mathbb{R}$ yielding

$$\int \bar{\psi}(x_n,...,x_1) \prod_{i=1}^{n} \phi_i(x_i + a_i) dX_n \equiv 0, \tag{6.7}$$

for all $\{a_1\}$. The ϕ_i's can be chosen to be arbitrary functions as long as $\Pi_{i=1}^n \phi_i$ has its support contained in $U_n' \subset U_n$. This, together with (6.7), implies that $\psi = 0$ in $L^2(\mathbb{R}^{\nu\cdot n})$, which proves the theorem.

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