ON THE ACOUSTIC RADIATION FROM BOUNDARY LAYERS AND JETS

by

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SUMMARY

In the following a general discussion of aerodynamically created sound is given. The study is essentially theoretical in nature, but arrives at a description of the physical phenomena in such a fashion as to yield an immediate access to experiments.

First, the problem of aerodynamic noise is defined and two simple mechanical analogues discussed. Then, the general equations of motion of a viscous, compressible fluid are rearranged in a form suitable for a comparison with Lighthill's approach. However, this approach is not being followed through. Instead, the concept of induced velocities due to displacement effects is put forward and carried through for noise produced by boundary layer flow. The same concept is then extended to describe the sound field created by a jet.

I. INTRODUCTION

Aerodynamically created sound has become a problem of major interest both for practical and academic reasons. The term "aerodynamically created sound" stands for acoustic fields produced by fluid motion without the movement of solid surfaces. The problem includes sound produced by heat release, by turbulence, by vortex motion, like in a Kármán vortex street, and by boundary layers.

The problem has received widespread interest in recent years due to the practical problem of jet noise. In a more or less disguised form the problem has been present and some aspects have been discussed in recent researches on viscous, compressible flow, for example, the studies of Lagerstrom, Cole and Trilling (Ref. 1), and of Cole and Wu (Refs. 2, 3). Lighthill (Ref. 4) has given a theory of noise created from turbulent motion with specific emphasis on the noise created by jets. Lighthill's paper has had a very large influence on the thinking and design of experiments, mainly in England, but also in this country. Lighthill's paper was the first to present the problem with explicit reference to acoustics, and in a form which led to at least some definite results which could be experimentally verified. A considerable number of experimental and theoretical studies have been published in recent years. Reference is made to a survey paper by Richards (Ref. 5).

The present report is written with the aim of giving a physical representation of aerodynamically created noise different from Lighthill's, and presenting a theoretical description in such a form as to lead to definite experiments. The theoretical approach is based largely on the idea of displacement thickness effects. The concept of "displacement thickness" is well known in boundary layer studies. For example, one can describe the effect of a boundary layer along a flat plate upon the external potential field in terms of the displacement thickness δ in the following way: The pressure field in the external flow is the same as the one produced by a downwash distribution of magnitude $U(d\delta/dx)$ in the plane of the plate. Stated in this fashion, the displacement thickness concept is placed on a firm theoretical basis. The same concept can be used for non-stationary phenomena. Indeed, Van Dyke (Ref. 6) has shown that the pressure wave created by a flat plate set into motion impulsively can be described as the pressure pulse due to a piston moving with a velocity equal to the downwash velocity caused by the displacement effect.

On the basis of these considerations a description of the noise field from boundary layers is given in this report. The same concept is then extended to flow in jets and can also be extended to wakes. In this fashion a theory of aerodynamic noise can be formulated which suggests relatively simple and clean experiments. In order to fit this approach within the present theoretical studies, for example, the well-known approach of Lighthill, a short discussion of the methods based on the differential equations of a viscous, compressible fluid is also included. To formulate the ideas use is made of two simple mechanical models.

In the present report the problem of sound radiation from boundary layers and jets is formulated and discussed in general. Work on the problem is continuing, however, both experimentally and theoretically, and detailed computations and studies are left to future reports. The mathematical relation between the present approach and Lighthill's work will also be reported later.

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II. PRESSURE FLUCTUATIONS IN AN INCOMPRESSIBLE FLUID AND ACOUSTIC WAVES. DISCUSSION OF SIMPLE MECHANICAL MODELS

The concepts of pressure and of pressure fluctuations in an incompressible fluid and their relation to acoustic fields requires some discussion. The concept of pressure in an incompressible fluid is not too easy to grasp, and is best illustrated by a mechanical model. Following Sommerfeld (Ref. 7), a simple pendulum is chosen first.

The force exerted by an ideal string of a pendulum does no work. It simply represents a constraint necessary to keep the mass point on a spherical surface. This force enters into the equation of motion in the form of a Lagrangean multiplier, which is used to account for the geometrical relation between the coordinates. The pressure p of an incompressible fluid plays exactly the same role, that is, it represents the constraint necessary to keep the continuity equation $\operatorname{div} \vec{u} = 0$ satisfied. It can be introduced into the equation of motion as a Lagrangean multiplier if the equations are deduced from Hamilton's principle.

One may now elaborate on Sommerfeld's pendulum analogy to formulate a model for the production of aerodynamic noise. For this purpose, imagine the string of the pendulum elastic, that is, extensible, but with a large spring constant. The oscillations of the pendulum mass will exert oscillating forces upon the string and hence the length ℓ of the string will oscillate around a mean value ℓ_0 . If the spring constant is large, the spring oscillations will be small and they will not appreciably affect the transversal motion of the mass of the pendulum.

Hence, to compute the oscillations in the string it is possible to first find the motion of the pendulum on the basis of an inextensible string of length ℓ_0 , next compute the forces acting upon the string due to the motion of the pendulum, and finally obtain the vibrations of the string. It is interesting to note that the ratio of the energy stored in the string to the energy of the pendulum motion is proportional to M^2 , a Mach number based upon the velocity of the pendulum mass and the velocity of sound in the string.

As a second model, similar to the pendulum but somewhat closer to the acoustical problem, consider a small sphere rolling within a hemispherical (or better, paraboloidal) cavity in an elastic solid. In first approximation the solid acts again simply as a constraint and the force normal to the wall of the cavity does no work. In second approximation the fluctuating force gives rise to pressure waves propagating into the elastic solid.

The problem of aerodynamically produced sound is analogous. The fluid motion is first considered without regard to the coupling with the sound field. In second approximation the fluid motion acts as a given force field which gives rise to acoustic radiation. The reaction of the sound field upon the original fluid motion is neglected.

The same idea is actually present in Lighthill's work, and the approach given here later on differs essentially only in the representation and concept of the forcing function.

III. THE EQUATIONS OF MOTION. LIGHTHILL'S APPROACH

The equations of motion of a viscous, compressible fluid can be written in the form:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} = 0 \tag{3-1a}$$

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_j u_k}{\partial x_k} = - \frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ik}}{\partial x_k} + \mathcal{F}_i \tag{3-1b}$$

$$\frac{\partial \rho J}{\partial t} + \frac{\partial \rho u_k J}{\partial x_k} = \frac{\partial p}{\partial t} + \frac{\partial}{\partial x_k} \left(u_i \, \tau_{ik} + q_k \right) + Q \tag{3-1c}$$

where ρ, p, u_i denote density, pressure and velocity. τ_{ik}, q_k are stress tensor and heat flux vector. J denotes the total enthalpy, $\frac{1}{2} |u|^2 + h$, h = enthalpy; and $\mathcal{F}_i$ and Q are external forces and heat sources, respectively.

By forming the divergence of Eq. (3-1b) and the time derivative of Eq. (3-1c) one can combine both into the form of an inhomogeneous wave equation for the pressure p ; a denotes the velocity of sound in the undisturbed fluid.

$$\Box p = \nabla^2 p - \frac{1}{a^2} p_{tt} = - \frac{\partial}{\partial x_i} \left\{ \frac{\partial \rho u_i}{\partial t} + \frac{\partial}{\partial x_k} \left(\rho u_i u_k - \tau_{ik} \right) \right\} + \frac{\partial \mathcal{F}_i}{\partial x_i}$$

$$+ \frac{1}{a^2} \frac{\partial}{\partial t} \left\{ \frac{\partial \rho J}{\partial t} + \frac{\partial}{\partial x_k} \left(\rho u_k J - u_i \, \tau_{ik} - q_k \right) \right\} + \frac{1}{a^2} \frac{\partial Q}{\partial t} \tag{3-2}$$

Eq. (3-2) can be used for the discussion of noise fields. (Lighthill's equation is slightly different from Eq. (3-2), but for the general discussion this difference is immaterial.)

The use of Eq. (3-2) is straightforward if the dominant terms are, for example, an external force field, or external heat sources. In this case one has directly the inhomogeneous wave equation with sound sources related to $\mathcal{F}_i$ and Q, or rather their derivatives. In general, a velocity and stress field will be given which will play the role of the forcing function in Eq. (3-2). In particular, a vortex field, or a field due to a heat conduction problem will in first approximation give no rise to a pressure perturbation, and hence the right-hand side of Eq. (3-2) will vanish. In second approximation, however, a remainder will be left which then acts as the forcing function in Eq. (3-2).

In order to discuss Eq. (3-2) for a specific type of motion, especially for the case of turbulent flow, one has to make an a priori decision concerning the most important term in Eq. (3-2). Lighthill estimates that the most important contribution to the production of sound from low-speed turbulent jets comes from the second term in Eq. (3-2), and hence the equation reduces to

$$\Box p = \frac{\partial^2}{\partial x_i \, \partial x_k} (\rho u_i u_k) \tag{3-3}$$

The fact that the forcing function in Eq. (3-3) is the double divergence of a tensor $T_{ik} \doteq - \rho u_i u_k$ leads then directly to the representation of the sound field in terms of a volume quadrupole distribution.

IV. THE PRESSURE PULSE DUE TO IMPULSIVE HEATING OR IMPULSIVE MOTION OF AN INFINITE FLAT PLATE

As an instructive example we compute in a qualitative way the pressure pulse generated in a gas due to sudden heating or sudden shearing motion of one of its boundaries. This is a problem related to the study of Cole and Wu (Refs. 2, 3) for the heating problem, and treated by Howarth (Ref. 8) and especially Van Dyke (Ref. 6) for the shearing problem. The method used here is qualitative and serves mainly to bring out the important parameters of the problem in a way suitable for later use in discussing boundary layer and jet noise.

Let p_∞, T_∞, ρ_∞ denote pressure, temperature, and density in the gas at rest, k the heat conductivity, $\nu = \mu/\rho$ the kinematic viscosity, and C_p the specific heat at constant pressure. At time $t = 0$ the temperature of the plate is raised by ΔT and kept at this temperature. The layers of gas adjacent to the wall heat up and the density changes by $\Delta\rho$. In first approximation heat conduction is an isobaric process and hence

$$\frac{\Delta\rho}{\rho_\infty} = - \frac{\Delta T}{T_\infty} \tag{4-1}$$

The heat travels into the gas with a velocity c, the velocity of "temperature waves," (Fig. 1), where

$$c \approx \sqrt{\frac{k}{C_p \rho_\infty t}} \tag{4-2}$$

Hence, the rate of change of mass dm/dt per unit cross-sectional area of the plate is

$$\frac{dm}{dt} = (\Delta\rho)\, c \approx - \rho_\infty \frac{\Delta T}{T_\infty} \sqrt{\frac{k}{C_p \rho_\infty t}} \tag{4-3}$$

The pressure pulse produced at large distances* is the same as the

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*That is, for large times t. It is here that the "large Reynolds number" approximation enters.

pulse produced by a piston with the same rate of change of mass. If the piston velocity is w, we have

$$\frac{dm}{dt} = -\rho w \tag{4-4}$$

while the pressure pulse produced by a piston of velocity w is given by

$$\Delta p = \rho_\infty a_\infty w \tag{4-5}$$

where a_∞ denotes the sound velocity in the gas. Comparing Eqs. (4-3), (4-4) and (4-5) we find

$$\frac{\Delta p}{p_\infty} \approx \frac{\rho_\infty}{p_\infty} a_\infty \frac{\Delta T}{T_\infty} \sqrt{\frac{k}{C_p \rho_\infty t}}$$

or, introducing the Prandtl number $Pr = (C_p \mu)/k$ and $a_\infty^2 = \gamma (p_\infty/\rho_\infty)$,

$$\frac{\Delta p}{p_\infty} \approx \gamma \frac{\Delta T}{a_\infty T_\infty} \sqrt{\frac{\nu}{Pr\, t}} \tag{4-6}$$

The pressure pulse due to the impulsive motion of an insulated plate is now obtained from Eq. (4-6) by simply inserting for ΔT the temperature rise due to viscous dissipation. Thus, $\Delta T = T_r - T_\infty$ where T_r is the recovery temperature of the plate corresponding to the Mach number $M = U/a_\infty$. Thus,

$$\frac{\Delta T}{T_\infty} = \frac{\gamma - 1}{2} f(Pr) M^2 \tag{4-7}$$

Inserting Eq. (4-7) into (4-6) we have

$$\frac{\Delta p}{p_\infty} \approx \frac{\gamma(\gamma-1)}{2} \frac{M^2}{a^*} \sqrt{\frac{\nu}{t}} \times g(Pr) = \frac{\gamma(\gamma-1)}{2} M^3 \sqrt{\frac{\nu}{U^2 t}} \times g(Pr) \tag{4-8}$$

where

$$g(Pr) = \frac{f(Pr)}{\sqrt{Pr}}$$

V. THE GENERAL PISTON PROBLEM

To apply the ideas expressed in the Introduction to the radiation from boundary layers and jets, it is necessary to first write down the acoustic field due to a distribution of normal velocities. For the boundary layer the distribution of normal velocities is given in a plane, for the jet along a line. Consequently, we require the solution of this general piston problem for the plane and axial-symmetrical cases. Furthermore, for a boundary layer on a solid surface the velocity of flow at infinity is not zero, and in some cases it is impossible or inconvenient to transform it to zero. Hence, we need the expression for the radiation of sound from a given distribution of normal velocities on a plane into fluid moving with a mean velocity U.

1. Planar Motion

Denote by $\phi(\vec{R},t)$ the velocity potential at the point P and at time t. We deal with an acoustic field and hence with the linearized equations of motion of a non-viscous fluid. ϕ is expressed in terms of the distribution of normal velocities $w(\vec{S},t)$ in the x,y plane by Rayleigh's formula:

$$\phi(\vec{R},t) = -\frac{1}{2\pi} \int_{PLANE} w\left(\vec{S}, t - \frac{r}{a}\right) \frac{d\sigma}{r} \tag{5-1}$$

where $\vec{S}\{x,y\}$; $\vec{r} = \vec{R} - \vec{S}$; a = the velocity of sound (Fig. 2). In linearized theory the perturbation pressure $p(\vec{R},t)$ is related to ϕ and the density in the undisturbed field ρ_∞ by

$$p = -\rho_\infty \frac{\partial\phi}{\partial t} \tag{5-2}$$

Hence, from Eq. (5-1),

$$p = \frac{\rho_\infty}{2\pi} \int w_t\left(\vec{S}, t - \frac{r}{a}\right) \frac{d\sigma}{r} \tag{5-2'}$$

Of main interest in the following will be random distributions in w, and hence it is preferable to deal with the mean square pressure $\overline{p^2}$ from the beginning. Thus,

$$\overline{p^2} = \frac{\rho_\infty^2}{4\pi^2} \iint \overline{w_t\left(\vec{s}_1\,;\, t - \frac{r_1}{a}\right) \quad w_t\left(\vec{s}_2\,;\, t - \frac{r_2}{a}\right)} \, \frac{d\sigma_1 \, d\sigma_2}{r_1 r_2} \tag{5-3}$$

and $\overline{p^2}$ is expressed in terms of the space-time correlation function of the time derivative of the downwash velocity in the x, y plane. If w_t is statistically homogeneous and stationary, then the correlation function ψ will depend only upon the difference in the variables in space and time, that is,

$$\overline{w_t\left(\vec{s}_1\,;\, t - \frac{r_1}{a}\right) \quad w_t\left(\vec{s}_2\,;\, t - \frac{r_2}{a}\right)} = \psi\left(\vec{s}_1 - \vec{s}_2\,;\, \frac{r_1 - r_2}{a}\right) \tag{5-4}$$

Hence Eq. (5-3) can be rewritten:

$$\overline{p^2} = \frac{\rho_\infty^2}{4\pi^2} \iint \psi\left(\vec{s}_1 - \vec{s}_2\,,\, \frac{r_1 - r_2}{a}\right) \frac{d\sigma_1 \, d\sigma_2}{r_1 r_2} \tag{5-5}$$

The sound intensity $\mathcal{J}$, that is, the flux of energy per unit time through unit surface, is related to $\overline{p^2}$ by

$$\mathcal{J} = \frac{\overline{p^2}}{\rho_\infty a} \tag{5-6}$$

Hence if ψ is known, the intensity of sound can be computed from Eq. (5-5). Eq. (5-5) forms the starting point of the discussion. In the following an attempt is made to relate w to the fluctuations in displacement thickness of the boundary layer. Once this is accomplished, Eq. (5-5) yields the sound intensity produced by the boundary layer flow.

2. Axial-Symmetrical Piston Problem (Fig. 3)

For the application to jets and wakes one requires the equivalent of Eq. (5-1) for the case of axial symmetry. Let $A(x,t)$ denote the cross-sectional area of a cylindrical body with its axis along x. At a given axial station the cross-sectional area A may vary in time. The potential in slender-body approximation is then given by

$$\phi(x, r_p, t) = -\frac{1}{4\pi} \int \frac{A_t\left(\xi, t - \frac{\sqrt{(x-\xi)^2 + r_p^2}}{a}\right)}{\sqrt{(x-\xi)^2 + r_p^2}} \, d\xi \tag{5-7}$$

where A_t denotes the derivative of $A(x,t)$ with respect to t. Eq. (5-7) is the fundamental formula for non-stationary flow past slender bodies of revolution (Cole, Ref. 9). Thus, in the axial-symmetrical case the mean square pressure perturbation becomes:

$$\overline{p^2} = \frac{\rho_\infty^2}{16\pi^2} \iint d\xi_1 \, d\xi_2 \, \frac{\overline{\left[A_{tt}\right]_1 \left[A_{tt}\right]_2}}{\sqrt{\left[(x-\xi_1)^2 + r_p^2\right]\left[(x-\xi_2)^2 + r_p^2\right]}} \tag{5-8}$$

where, for example, $\left[A_{tt}\right]_1$ denotes the second time derivative of the cross-sectional area evaluated at ξ_1 and $t - \frac{1}{a}\sqrt{(x-\xi_1)^2 + r_p^2}$. Hence, $\overline{p^2}$ depends upon the correlation function of the second time derivative of the cross-sectional area.

3. Radiation from a Plane into Moving Fluid

For the discussion of radiation from the boundary layer we need the extension of Eq. (5-1) to the case where the fluid moves with a velocity at infinity, relative to the point P. $\phi(\vec{R}, t)$ denotes now a perturbation potential. The formula expressing ϕ in terms of the distribution of normal velocities in the x, y plane can be obtained by transforming Eq. (5-1) to moving coordinates. If M denotes the Mach number of the flow, $M = U/a$, we have for subsonic flow:

$$\phi(\vec{R}, t) = -\frac{1}{2\pi} \iint_{PLANE} \frac{d\sigma}{r} \, w(\vec{S}, t - \tau) \tag{5-9}$$

where now

$$r^2 = (x-\xi)^2 + (1-M^2)\left[(y-\eta)^2 + z^2\right]$$

$$\tau = \frac{r - M(x-\xi)}{a(1-M^2)}$$

The pressure p depends now on $\partial\phi/\partial t$ and $\partial\phi/\partial x$ and is given by

$$p = -\rho\left[\frac{\partial\phi}{\partial t} + U\frac{\partial\phi}{\partial x}\right] \tag{5-10}$$

Combining Eqs. (5-9) and (5-10) we have

$$p = \frac{\rho}{2\pi} \int \frac{d\sigma}{r} \left\{ \frac{r - M(x-\xi)}{r(1-M^2)} \; w_t(\xi, \eta, t-t_1) - \frac{U}{r^2} (x-\xi) \; w(\xi, \eta, t-t_1) \right\} \quad (5\text{-}11)$$

Consider again a velocity distribution w which is statistically homogeneous in the plane and stationary in time. Then, introducing the space-time correlation function ψ (Eq. 5-4), we obtain for the mean square of the pressure at P:

$$\overline{p^2} = \frac{\rho^2}{4\pi^2} \iint \frac{d\sigma_1 d\sigma_2}{r_1 r_2} \left[-f(1) f_2 \psi_{\tau\tau} + g(1) g(2) \psi - \left(f(1) g(2) - f(2) g(1) \right) \psi_\tau \right] (5\text{-}12)$$

where $f(1)$, $g(1)$ etc. stand for

$$f(1) = \frac{r_1 - M(x - \xi_1)}{r_1 (1 - M^2)} \; ; \quad g(1) = \frac{U(x - \xi_1)}{r_1^2} \; ; \text{ etc.}$$

and

$$\psi_\tau = \frac{\partial}{\partial \tau} \psi(\xi, \eta, \tau)$$

with

$$\xi = \xi_1 - \xi_2 \; ; \quad \eta = \eta_1 - \eta_2 \; ; \quad \tau = t_1 - t_2$$

It is evident from Eq. (5-12) how much more involved the expressions become for the case of the moving fluid.

VI. THE CONCEPT OF DISPLACEMENT THICKNESS

Within the frame of boundary layer theory it is possible to express the effects of the dissipative regions upon the potential, outer flow by introducing the concept of "displacement." The concept is well known and often used in applications to boundary layers, but can equally well be formulated for jets and wakes.

Formally, one obtains the induced velocity (normal to the mean flow) by integrating the continuity equation. For the two-dimensional boundary layer one has

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho w)}{\partial z} = 0$$

and thus, denoting the boundary layer thickness by $\delta(x)$,

$$w_\infty \rho_\infty = -\int_0^\delta \frac{\partial(\rho u)}{\partial x}\, dz$$

since $\rho u(\delta) \rightarrow U\rho_\infty$. This can be written

$$w_\infty \rho_\infty = \frac{d}{dx}\int_0^\delta (\rho_\infty U - u\rho)\, dz \tag{6-1}$$

or

$$\frac{w_\infty}{U} = \frac{d}{dx}\int_0^\infty \left(1 - \frac{u\rho}{U\rho_\infty}\right) dz = \frac{d\delta^*}{dx} \tag{6-2}$$

where δ^* denotes the displacement thickness of the boundary layer.

Similarly, for the axial-symmetrical jet one can write:

$$\frac{\partial(\rho u r)}{\partial x} + \frac{\partial(\rho w r)}{\partial r} = 0$$

and, denoting the radius at the "edge" of the jet by δ, again one has

$$w_\delta \rho_\delta \delta = -\int_0^\delta \frac{\partial(u\rho r)}{\partial x}\, dr$$

and since $u(r,x) \rightarrow 0$ for $r = \delta$,

$$w_\delta \rho_\delta \delta = -\frac{d}{dx}\int_0^\delta u\rho r dr \qquad (6\text{-}3)$$

One may define a "displacement cross-section S^*" by

$$\frac{\delta w_\delta}{U} = -\frac{dS^*}{dx} \qquad (6\text{-}4)$$

with

$$S^* = -\int_0^\infty \frac{u\rho}{U\rho_\delta} r dr \qquad (6\text{-}5)$$

U denotes the initial velocity of the jet. In the compressible case one has to distinguish between ρ_δ, that is, the density at the edge of the jet, and ρ_o, the initial density corresponding to U. Since the interest here lies with the incompressible boundary layer and jet, the density drops out and we have

$$\frac{W_\infty}{U} = \frac{d}{dx}\int_0^\infty \left(1-\frac{u}{U}\right) d = \frac{d\delta^*}{dx} \qquad \text{(plane boundary layer)}$$

and

$$\frac{\delta w_\delta}{U} = -\frac{d}{dx}\int_0^\infty \frac{u}{U} r dr = \frac{dS^*}{dx} \qquad \text{(axial-symmetrical jet)}$$

The main idea of the present paper consists of introducing a "displacement fluctuation" expressed as a time-dependent W_∞, δ^* or S^* fluctuation, respectively. This fluctuation in W_∞ is then considered as the piston velocity v in Eq. (4-1), or the time-dependent area A in Eq. (4-6). The resulting sound field is taken to be the noise field produced by the boundary layer or jet.

VII. SOUND RADIATION FROM A BOUNDARY LAYER

We deal first with the acoustic radiation from a boundary layer far removed from the leading edge. That is, the effect of the layer upon the outer flow field can be represented by a homogeneous distribution of normal velocities over a plane. The acoustic intensity at a point P can then be obtained using the simpler form of Rayleigh's solution without the need of transforming into moving coordinates. One can imagine that P moves along with the mean velocity U, or else that the flat surface with the boundary layer slides past P with U. The normal velocity $w(\vec{S},t)$ in the x,y plane is the induced velocity due to the displacement fluctuations of the layer. The space-time correlation function of w is again called ψ, that is,

$$\overline{w(\xi_1,\eta_1,t-t_1)\quad w(\xi_2,\eta_2,t-t_2)} = \psi(\xi_1-\xi_2,\eta_1-\eta_2,t_1-t_2) \tag{7-1}$$

and thus the mean square pressure at P is obtained from Eq. (5-5):

$$\overline{p^2} = \frac{\rho_\infty^2}{4\pi^2}\iint \frac{d\xi_1\,d\xi_2\,d\eta_1\,d\eta_2}{r_1 r_2}\;\psi\left(\xi_1-\xi_2,\eta_1-\eta_2,\frac{r_1-r_2}{a}\right) \tag{7-2}$$

To solve fully the problem of boundary layer radiation, ψ would have to be computed from boundary layer theory. This is not possible at present and hence only some consequences of Eq. (7-2) will be discussed here.

We deal with a very large, "infinite" surface on which a homogeneous distribution of normal velocities is given. It is reasonable to assume first that the details of the correlation function are immaterial, and to write

$$\psi(\xi,\eta,\tau) = \overline{w^2}\,\Lambda^2\,\delta(\xi)\,\delta(\eta)\,\varphi(\tau) \tag{7-3}$$

where $\delta(\xi)$, $\delta(\eta)$ denote the Dirac delta function. Eq. (7-3) expresses the idea that the normal velocities are correlated over areas of order Λ^2, where Λ^2 is small compared to the radiating surface area. Λ represents a scale of turbulence. To set up a correlation function in the form (7-3) is often useful and makes good physical sense (for example, Ref. 10).

With ψ given by Eq. (7-3), Eq. (7-2) can be integrated. To make the arithmetic simpler, the coordinates of P can be conveniently chosen: $P(0,0,z,t)$. Thus, Eq. (7-2) becomes

$$\overline{p^2} = \frac{\Lambda^2 \rho_\infty^2 \overline{w^2}}{4\pi^2} \iiiint \frac{d\xi_1 d\xi_2 \, d\eta_1 d\eta_2}{\sqrt{\xi_1^2+\eta_1^2+z^2} \sqrt{\xi_2^2+\eta_1^2+z^2}} \delta(\xi_1-\xi_2)\,\delta(\eta_1-\eta_2)\,\varphi_{\tau\tau}\left(\frac{r_1-r_2}{a}\right) \quad (7\text{-}4)$$

Integrating over ξ_2 , η_2 yields

$$\overline{p^2} = \frac{\Lambda^2 \rho_\infty^2 \overline{w^2}}{4\pi^2} \iint \frac{d\xi d\eta}{\xi^2+\eta^2+z^2} \varphi_{\tau\tau}(0) \quad (7\text{-}5)$$

or

$$\overline{p^2} = \frac{\Lambda^2 \rho_\infty^2 \overline{w^2}}{2\pi} \varphi_{\tau\tau}(0) \int_0^R \frac{s\,ds}{\sqrt{s^2+z^2}} = \frac{\Lambda^2 \rho_\infty^2 \overline{w^2}}{4\pi} \varphi_{\tau\tau}(0) \log\left(1+\frac{R^2}{z^2}\right) \quad (7\text{-}6)$$

where R denotes the "radius of the flat surface." That $p^2 \to \infty$ as $R \to \infty$ is to be expected since the noise produced by an infinite plate must be infinite.

Eq. (7-6) gives a formal representation of the mean square pressure if ψ can be approximated by Eq. (7-3). Eq. (7-6) can be rewritten in the form

$$\frac{\overline{p^2}}{\rho_\infty a} = \frac{\rho_\infty \overline{w^2} \Lambda^2 \overline{\omega^2}}{4\pi a} \log\left(1+\frac{R^2}{z^2}\right) \quad (7\text{-}6a)$$

where $\varphi_{\tau\tau}(0)$ has been replaced by $\overline{\omega^2}$, the mean square of the frequency of fluctuations. (This follows from using a Fourier transform of φ .)

(a) To derive Eq. (7-6) it was assumed that $\psi(\xi,\eta,\tau)$ can be represented by (7-3). If this is true, Eq. (7-6) or Eq. (7-6a) represents a formal solution. (b) For the application of these equations, $\overline{w^2}$, $\overline{\omega^2}$, Λ^2 have to be related to the characteristics of the boundary layer.

$\psi(x)$ cannot be represented by a δ function if $\int \psi(x)\,dx = 0$. It is here that in the present formalism the difference between elementary sources, dipoles, or quadrupoles enters. It is believed that the form of ψ can be obtained from boundary layer considerations and that here the mathematical equivalence with Lighthill's approach can be demonstrated. This problem will be left to a second report.

By answering (a) some light will be shed on question (b). However, in dealing with turbulent boundary layers all that can be hoped for is a dimensional or similarity argument relating $\overline{W^2}$, Λ^2 and $\overline{\omega^2}$ to the boundary layer characteristics. Indeed, the formal representations of the sound field, like Eqs. (7-6), are useful mainly in that they suggest a fruitful experimental approach and indicate the important quantities to be measured.

VIII. SOUND RADIATION FROM A JET

The model for the radiating jet is a distribution of A_t, that is, of the time rate of change of displacement cross section. The jet is assumed to have a definite length L, later to be defined more precisely. Thus, the problem is to compute the sound intensity at a point P far away from the jet, replaced by distributions of A_t along the x-axis from O to L. The correlation function of A_t in space and time is $\varphi(\xi,\tau)$ say, that is,

$$\overline{A_t\ (x,t)\ \ A_t\,(x+\xi,\ t+\tau)} = \varphi\,(\xi,\tau) \tag{8-1}$$

If it is assumed that the time variations are due to a "frozen" pattern translated with velocity U, one can write

$$\varphi\,(\xi,\tau) = \varphi\,(\xi - U\tau) = \varphi\,(s) \tag{8-2}$$

The sound field is computed from Eq. (5-8): The correlation function $\psi(s)$ of A_{tt} becomes

$$\psi\,(s) = \overline{A_{tt}\ (x+\xi,\ t+\tau)\ A_{tt}\,(x,t)} = -\,U^2\varphi''(s) \tag{8-3}$$

For large distances, that is, where

$$\sqrt{(x-\xi)^2 + r_p^2} \longrightarrow \sqrt{x^2 + r_p^2} \equiv R$$

one obtains from Eq. (5-8)

$$\overline{p^2} = -\,\frac{\rho^2 U^2}{16\,\pi^2 R^2}\iint_0^L \varphi''\left(\xi_1 - \xi_2 - U\,\frac{|r_1 - r_2|}{a}\right)d\xi_1 d\xi_2 \tag{8-4}$$

For large R, r becomes approximately (Fig. 2)

$$r = R - \xi\ \cos\theta$$

and thus,

$$\overline{p^2} = -\,\frac{\rho_\infty^2 U^2}{16\pi^2 R^2}\iint_0^L \varphi''\ \Big[(\xi_1 - \xi_2)\ (1 - M\cos\theta)\Big]\,d\xi_1 d\xi_2 \tag{8-5}$$

Changing variables and reducing the double integral to a single integral, Eq. (8-5) becomes

$$\overline{p^2} = -\frac{\rho_\infty^2 U^2}{16\pi^2 R^2}\frac{2}{(1-M\cos\theta)^2}\int^{L(1-M\cos\theta)}\left[L(1-M\cos\theta)-s\right]\varphi''(s)\,ds \tag{8-6}$$

Integrating,

$$\overline{p^2} = \frac{\rho^2 U^2}{8\pi^2 R^2}\frac{\varphi(0)-\varphi\left[L(1-M\cos\theta)\right]}{(1-M\cos\theta)^2} \tag{8-7}$$

Thus, the sound intensity at P becomes

$$\frac{\overline{p^2}}{\rho_\infty a} = \frac{\rho_\infty U^2 \overline{A_t^2}}{8\pi^2 R^2 (1-M\cos\theta)^2 a}\left[1-\frac{\varphi\left[L(1-M\cos\theta)\right]}{\varphi(0)}\right] \tag{8-8}$$

If the effective jet length is large compared to the "scale of the fluctuations," Eq. (8-8) becomes simply

$$\frac{\overline{p^2}}{\rho_\infty a} = \frac{\rho_\infty U^2 \overline{A_t^2}}{8\pi^2 a R^2 (1-M\cos\theta)^2} \tag{8-9}$$

In the opposite limiting case, that is, if L is small,

$$\frac{\overline{p^2}}{\rho_\infty a} = \frac{\rho_\infty U^2 \overline{A_t^2} L^2}{16\pi^2 R^2 a}\frac{\varphi''(0)}{\varphi(0)} \tag{8-10}$$

Evidently the jet problem is formally simpler than the boundary layer problem. This is due to the finite extent of the jet and the fact that the distribution of normal velocities is given along a line instead of a plane.

A complete solution of the problem involves again a study of the character of possible fluctuations within a jet, and the remarks made at the end of Section VII apply to the jet problem as well.

IX. CONCLUDING REMARKS

In this report it is shown how the problem of acoustic radiation from boundary layers and jets can be formulated in terms of the induced velocities due to displacement thickness fluctuations.

Detailed studies of these acoustical fields, of the mathematical relation of this approach to Lighthill's, etc. are left for future reports. However, it is believed that the theoretical approach outlined here will lead to a more direct comparison between experiments and theory than that of Lighthill.

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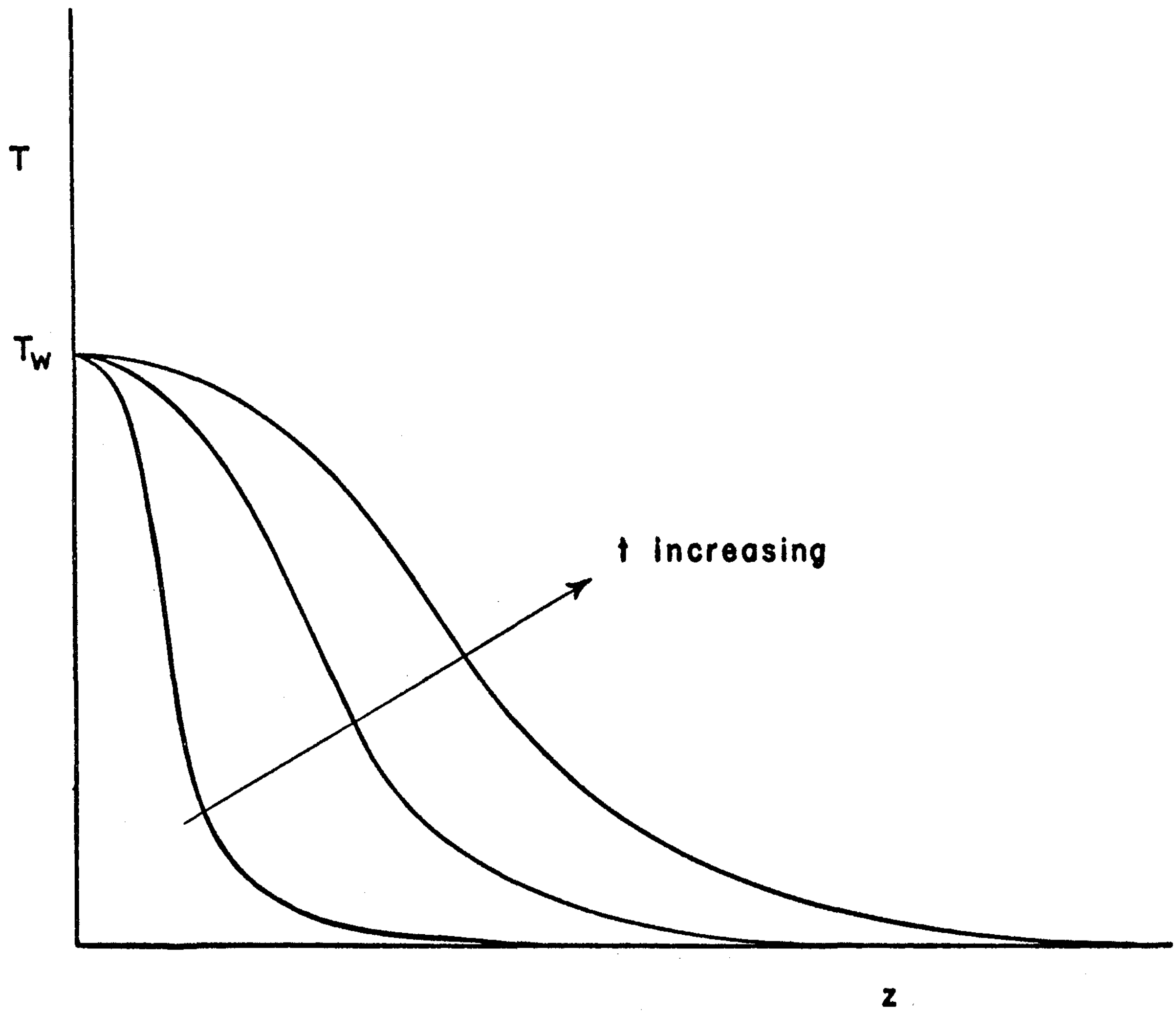

FIG. 1 DEFINITION OF SPREADING VELOCITY c

$$c = \frac{d}{dt}\int_0^\infty \frac{T}{T_W}\, dz$$

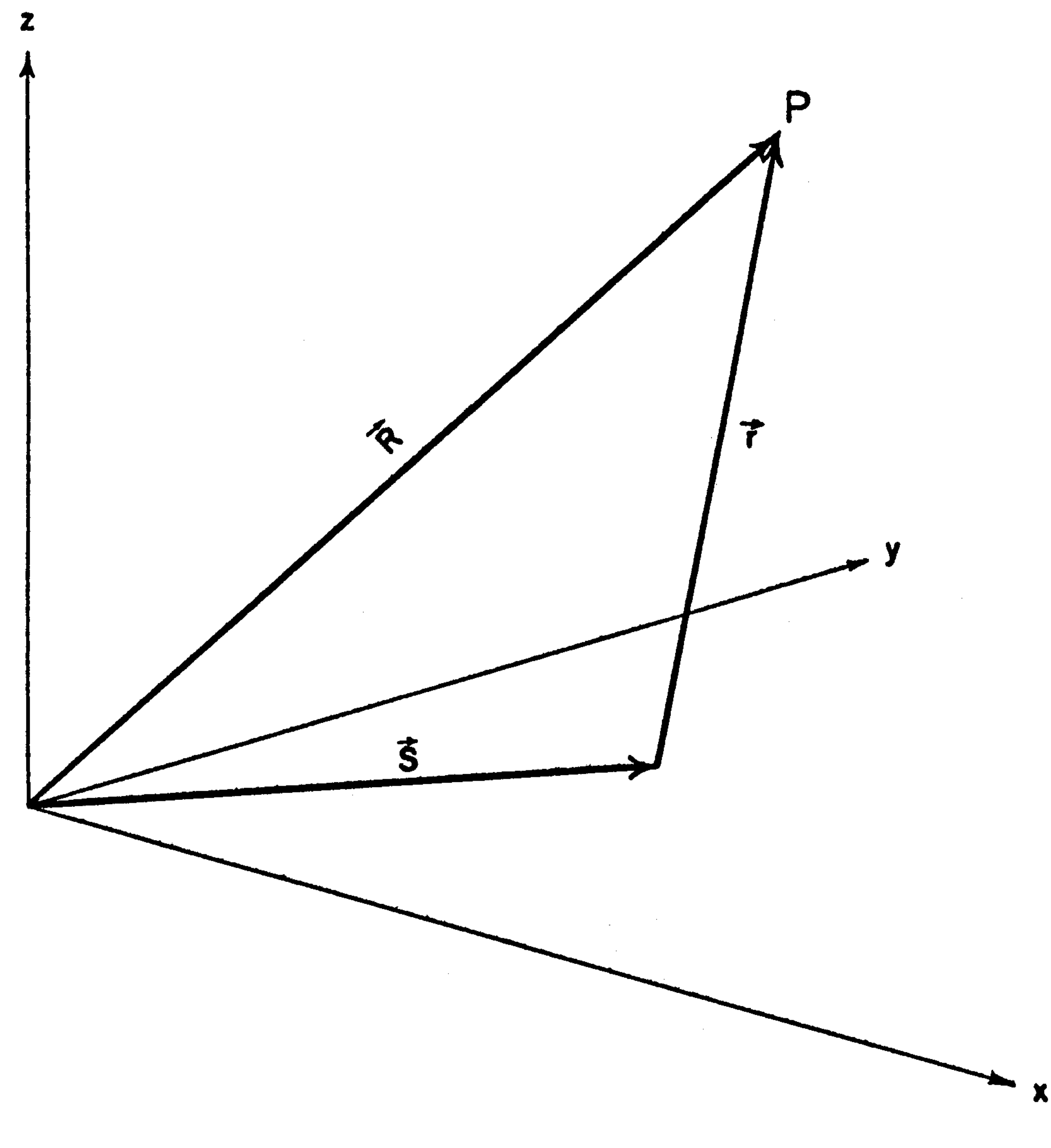

FIG. 2 COORDINATES FOR THE PLANAR PISTON PROBLEM

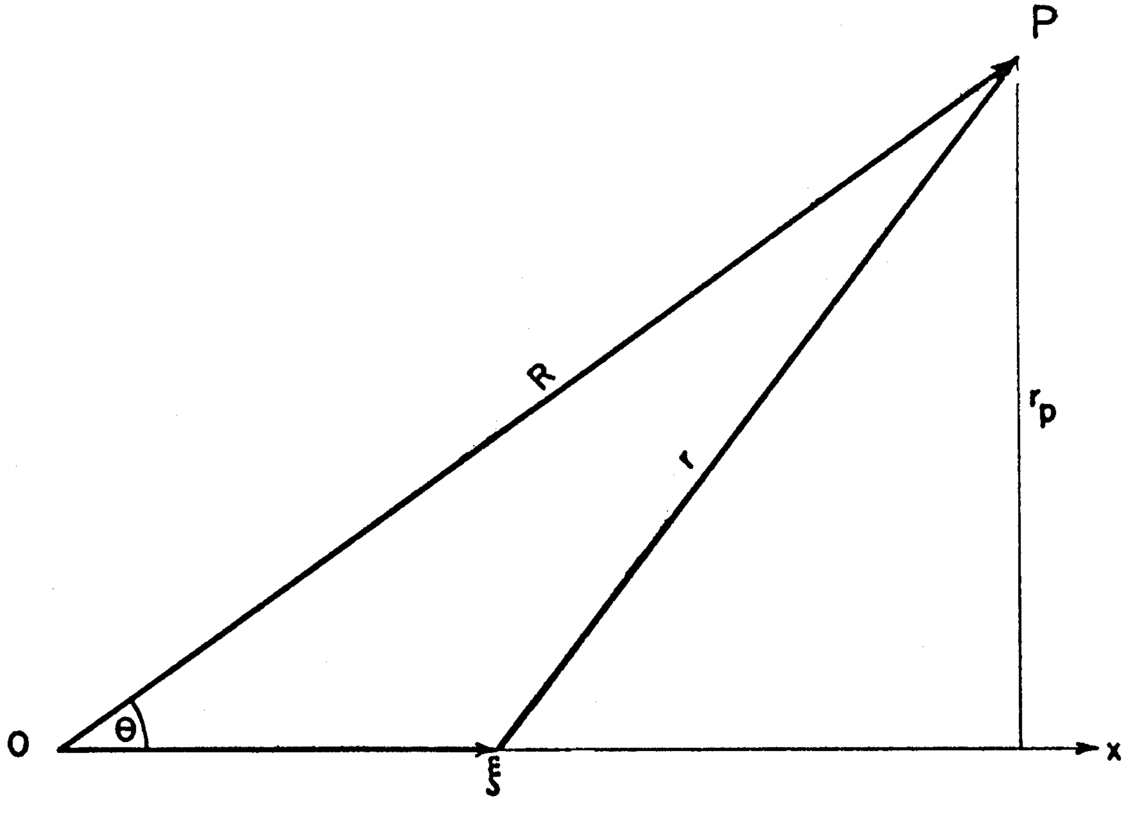

FIG. 3 COORDINATES FOR THE AXISYMMETRICAL PISTON PROBLEM