Generating coherent phonon waves in narrow-band materials: a twisted bilayer graphene phaser - Supplementary Material

**PHONON INSTABILITY FROM THE EQUATION OF MOTION FORMALISM**

In this section, we derive the equations of motion for the electronic and bosonic operators and demonstrate an instability of the resonant phononic mode. For simplicity, here and throughout these notes we work in the units in which \( \hbar = 1 \). We consider the electronic Hamiltonian

\[
\mathcal{H}_D = \sum_{k,\alpha=\pm} \varepsilon_\alpha(k) \hat{c}^\dagger_{k\alpha} \hat{c}_{k\alpha},
\]

where \( \varepsilon_\alpha(k) = \alpha n|k| \). Similarly the Hamiltonian for the acoustic phonons reads

\[
\mathcal{H}_p = \sum_q \omega_0(q) \hat{b}^\dagger_q \hat{b}_q,
\]

where \( \omega_0(q) = c_p|q| \). The electron-phonon coupling is given by

\[
\mathcal{H}_{ep} = \sum_{k,q,n} g_n \{ \hat{c}^\dagger_{k+Q_n,\alpha} \hat{b}_q + \hat{c}^\dagger_{k-Q_n,\alpha} \hat{b}^\dagger_q \},
\]

where \( Q_n = q + nk \) and \( n = \{-1, 0, 1\} \). The full Hamiltonian of the system is given by the sum, \( \mathcal{H} = \mathcal{H}_D + \mathcal{H}_p + \mathcal{H}_{ep} \). The equations of motion for the operators in the Heisenberg picture are given by \( -i\partial_t \hat{c}_{k\alpha} = [\mathcal{H}, \hat{c}_{k\alpha}] \) and \( -i\partial_t \hat{b}_q = [\mathcal{H}, \hat{b}_q] \). Explicit calculation of the commutator yields

\[
-i\partial_t \hat{c}_{k\alpha} = -\varepsilon_\alpha(k) \hat{c}_{k\alpha} - \sum_{q,n} g_n \{ \hat{c}^\dagger_{k+Q_n,\alpha} \hat{b}_q + \hat{c}^\dagger_{k-Q_n,\alpha} \hat{b}^\dagger_q \}.
\]

The equation for \( \hat{c}^\dagger_{k\alpha} \) is obtained from the complex conjugate. Next, we compute \( -i\partial_t \langle \hat{c}^\dagger_{k\alpha} \hat{c}^{k',\alpha'} \rangle = -i\langle [\partial_t \hat{c}^\dagger_{k\alpha} \hat{c}^{k',\alpha'}] \rangle \), which reads

\[
-i\partial_t \langle \hat{c}^\dagger_{k\alpha} \hat{c}^{k',\alpha'} \rangle = [\varepsilon_\alpha(k) - \varepsilon_{\alpha'}(k')] \langle \hat{c}^\dagger_{k\alpha} \hat{c}^{k',\alpha'} \rangle + g \sum_{q,n,\beta} \{ \langle \hat{c}^\dagger_{k+Q_n,\beta} \hat{b}_q \rangle + \langle \hat{c}^\dagger_{k-Q_n,\beta} \hat{b}^\dagger_q \rangle \} -
\]

\[
- g \sum_{q,n,\beta} \{ \langle \hat{c}^\dagger_{k\alpha} \hat{c}^{k+Q_n,\beta} \rangle \langle \hat{b}^\dagger q \rangle + \langle \hat{c}^\dagger_{k\alpha} \hat{c}^{k-Q_n,\beta} \rangle \langle \hat{b}_q \rangle \}. \tag{5}
\]

Similarly,

\[
-i\partial_t \langle \hat{b}_q \rangle = -\omega_0(q) \langle \hat{b}_q \rangle - \sum_{n,k} g_n \langle \hat{c}^\dagger_{k+Q_n,\alpha} \hat{c}_{k\alpha} \rangle \tag{6a}
\]

\[
-i\partial_t \langle \hat{b}^\dagger_q \rangle = \omega_0(q) \langle \hat{b}^\dagger_q \rangle + \sum_{n,k} g_n \langle \hat{c}^\dagger_{k+Q_n,\alpha} \hat{c}^{k,\alpha} \rangle. \tag{6b}
\]

Next, we perform the transformation \( -i\partial_t \to \omega \) and solve for the eigenmodes. To simplify this problem, we assume a single coherent phonon mode with momentum \( q \) corresponding to \( Q_n \), and set \( k' = k + Q_n \) in Eq. (5), yielding

\[
\omega \langle \hat{c}^\dagger_{k\alpha} \hat{c}^{k+Q_n,\alpha'} \rangle = [\varepsilon_\alpha(k) - \varepsilon_{\alpha'}(k+Q_n)] \langle \hat{c}^\dagger_{k\alpha} \hat{c}^{k+Q_n,\alpha'} \rangle + g_n [f_k+Q_n,\alpha' - f_{k,\alpha}] \langle \hat{b}_q \rangle.
\]

Here, we defined \( f_{k\alpha} = \langle \hat{c}^\dagger_{k\alpha} \hat{c}_{k\alpha} \rangle \) and neglected expectation values with momentum \( 2Q_n \). Combining Eq. (7) and Eq. (6a), we obtain a linear equation. Imaginary eigenvalues correspond to an exponentially growing mode of phonons hybridized with electron-hole excitations. An approximate solution can be obtained if we assume \( \varepsilon_{\alpha'}(k+Q_n) - \varepsilon_\alpha(k) = \Delta \varepsilon \), and \( \alpha = \alpha' = + \). Then one can sum both sides of Eq. (7) over \( k \), leading to and eigenvalue equation

\[
(\omega + \Delta \varepsilon)(\omega + \omega_0(q)) = -g_n^2 \sum_k \Delta f_k,\tag{8}
\]

where \( \Delta f_k = f_k + Q_n, + - f_{k, +} \). The solution reads

\[
\omega = \frac{\Delta \varepsilon + \omega_0(q)}{2} \pm \frac{1}{2} \sqrt{(\Delta \varepsilon - \omega_0(q))^2 - 4g_n^2 \sum_k \Delta f_k}. \tag{9}
\]

Therefore, near the resonance the eigenvalues obtain an imaginary component, when \( (\Delta \varepsilon - \omega_0(q))^2 < 4g_n^2 \sum_k \Delta f_k \).

**BOLTZMANN EQUATION FROM THE EQUATION OF MOTION FORMALISM**

Here, we derive the Boltzmann equation [Eq. (6) in the main text] from the equations of motion, Eq. (4) and Eq. (6). First, we transform the operators to the “interaction picture”, \( \hat{c}_{k\alpha} = e^{-i\varepsilon_{\alpha}(k)t} \hat{c}_{k\alpha}^0 \) and \( \hat{b}_q = e^{-i\omega_0(q)t} \hat{b}_q^0 \). Next, we integrate both sides of Eq. (4) over \( t \), leading to

\[
\hat{c}_{k\alpha}(t) = e^{i\varepsilon_{\alpha}(k)t} \hat{c}_{k\alpha}^0 -
\]

\[
-i \int_0^t dt' \sum_{q,n} g_n e^{(\Delta \varepsilon - \omega_0(q))t'} \hat{c}^\dagger_{k+Q_n,\alpha'} \hat{b}^\dagger_q - \]

\[
-i \int_0^t dt' \sum_{q,n} g_n e^{(\Delta \varepsilon - \omega_0(q))t'} \hat{c}^\dagger_{k-Q_n,\alpha} \hat{b}_q, \tag{10}
\]

where \( \Delta \varepsilon \equiv \varepsilon_{k+Q} - \varepsilon_k \) and \( \hat{c}_{k\alpha}^0 = \hat{c}_{k\alpha}(0) \). To the linear order in \( g_n \), we can approximate the operators \( \hat{c}^\dagger \) and \( \hat{b}^\dagger \)
on the r.h.s. by $\delta^0$ and $\hat{\delta}^0$, defined for $g_n = 0$. Therefore, to this order in $g_n$,  
\begin{equation}
\hat{c}^0_{\alpha\beta}(t) = i\hat{c}^0_{\alpha\beta} + \sum_{q,n} g_n F(\Delta \varepsilon_{Q_n} - \omega_0(q), t) \hat{c}^0_{\alpha Q_n,\alpha'} \hat{\delta}^0_q + \\
+ \sum_{q,n} g_n F(\Delta \varepsilon_{Q_n} - \omega_0(q), t) \hat{c}^0_{\alpha Q_n,\alpha'} (\hat{\delta}^0_q)^\dagger,
\end{equation}

where $F(\Delta \varepsilon, t) = -i \int_0^t dt' e^{i \Delta \varepsilon t'} = \frac{1}{1 - e^{-i \Delta \varepsilon \tau}}$. Next, we compute $\langle (\hat{c}^0_{\alpha\beta})^\dagger \hat{c}^0_{\alpha\beta} \rangle$, yielding  
\begin{equation}
\hat{f}_{\alpha\beta} = f_{\alpha\beta}^0 + \sum_{q,n} |F(\Delta \varepsilon_{Q_n} - \omega_0(q), t)|^2 f_{\alpha Q_n,\alpha n q}^0 + \\
+ \sum_{q,n} |F(\Delta \varepsilon_{Q_n} - \omega_0(q), t)|^2 f_{\alpha Q_n,\alpha n q}^0.
\end{equation}

Here, we analytically estimate the gain in the toy model [given in Eq. (1) in the main text], deriving Eq. (8) in the main text. Our goal is to evaluate the period-averaged power, $P^0(\omega)$, of the phonon mode with frequency $\omega$, given by Eq. (6) in the main text in the small-gain limit, $\gamma_\omega \rightarrow 0$. For simplicity, we assume $\mathcal{M}_{k,k'} = (2\pi^2 q^2)^{-1} |u_0|^2 g^2(\Omega)^2 \sum_{n=\pm} \delta(k - k' - Q_n)$, and we defined $Q_n = (q \pm k_\alpha)$. The gain can be written as $P_\alpha(\omega) = \sum_{\alpha \alpha'} \mathcal{P}_{\alpha \alpha'}(\omega - \Omega_{\alpha \alpha'}(\omega))$, where  
\begin{equation}
\mathcal{P}_{\alpha \alpha'} = 2\pi P_0 \int \frac{d^2k}{(2\pi)^2} f_{\alpha k} \delta(\varepsilon_\alpha(k) - \varepsilon_\alpha(k - Q_n) + \omega),
\end{equation}

$P_0 = \omega q^2 |u_0|^2 g^2(\Omega)^2$, and $\Omega = \pm$. We also assume $f_{\alpha k} = 0$ for $0 < \varepsilon_\alpha(k) < V$, and $f_{\alpha k} = 1$, otherwise. The expression breaks up into overlaps of two cones in the energy-momentum space, $\varepsilon_\alpha(k)$ and $\varepsilon_\alpha(k - Q_n) \pm \omega$, where $\varepsilon_\alpha(k) = \alpha \varepsilon_0 |k|$, for $\alpha = \pm$ [see below Eq. (1) in the main text].

First, we evaluate $\mathcal{P}_{\alpha \alpha'}$ corresponding to the gain in the system. It accounts for the intersection of $\varepsilon_\alpha(k)$ and $\varepsilon_\alpha(k - Q_n) + \omega$ in the range $\max(V, \omega) < \varepsilon_\alpha(k) < V + \omega$. For $\omega > \omega_{R \alpha}$ [see Eq. (3) in the main text for the definition of $\omega_{R \alpha}$], the two cones do not intersect, giving rise to zero contribution to the gain. In the opposite case, when $\omega < \omega_{R \alpha}$, the two cones intersect along a line. The maximal overlap is expected when $\omega \lesssim \omega_{R \alpha}$, where the two cones are nearly tangential. Focusing on this case, we define momentum in a spherical system of coordinates, such that $k = (k \cos(\theta), k \sin(\theta))$, where $k = |k|$ and $\theta$ is the angle of $k$ relative to the $z$ axis. The intersection of the cones occurs near $\theta_1 = 0$ when $q + n k_n > 0$ or near $\theta_2 = \pi$ when $q + n k_n < 0$. We expand $\varepsilon_\alpha(k - Q_n)$, to the leading (quadratic) order in $\delta\theta = \theta - \theta_{1,2}$, yielding $\varepsilon_\alpha(k - Q_n) = \varepsilon_\alpha[k - Q_n] + \frac{\varepsilon_\alpha k Q_n}{2\varepsilon_\alpha Q_n} \delta\theta^2 + O(\delta\theta^4)$, where $Q_n = |Q_n|$. At this order in $\delta\theta$, the intersection $\varepsilon_\alpha(k) = \varepsilon_\alpha(k - Q_n) + \omega$, occurs at $\delta\theta_0^2 = \pm \sqrt{(v_0 Q_n - \omega)^2 - \varepsilon^2}$, where we considered $k > Q_n$, corresponding to the dominant contribution. Note, that $v_0 Q_n = \omega_{R \alpha}$ identically.

Next, we split the integral in Eq. (1) to an integral over the energy $\varepsilon = v_0 k$ and the angle $\delta\theta$, as  
\begin{equation}
\int d^2k e^{i \omega t} = \int d\varepsilon N_D(\varepsilon) e^{i \omega t / \varepsilon}.
\end{equation}

The density of states for the dispersion $\varepsilon(k)$ is given by $N_D(\varepsilon) = \frac{\pi}{2 v_0^2}$. The energy conserving $\delta$-function can be simplified in the angular coordinates as $\delta(\varepsilon_\alpha + \varepsilon_\alpha(k - Q_n) - \omega) = \frac{\pi}{2 \varepsilon_{\omega R} (\delta\theta_0^2)^{1/2}} \sum_{\alpha = \pm} \delta(\delta\theta - \delta\theta_0)$.

The angular integral therefore can be trivially performed due to the $\delta$-function, leaving the energy integral  
\begin{equation}
\mathcal{P}_{\alpha \alpha'}(\omega) = P_0 \int_{\max(0, \omega, V)}^{\omega + V} \frac{2(\varepsilon - \omega_{R \alpha})}{\varepsilon} N_D(\varepsilon). 
\end{equation}

We implicitly assume that $\mathcal{P}_{\alpha \alpha'}(\omega) = 0$ for $\omega > \omega_{R \alpha}$ or $\varepsilon < \omega_{R \alpha}$. We therefore obtain  
\begin{equation}
\mathcal{P}_{\alpha \alpha'}(\omega) = \frac{P_0 T_{\alpha \alpha'}(\omega, V) \Theta(\omega_{R \alpha} - \omega)}{\pi v_0^2 \sqrt{2 \omega_{R \alpha} (\omega_{R \alpha} - \omega)}},
\end{equation}

where $T_{\alpha \alpha'}(\omega, V) = \int_{\max(0, \omega, V)}^{\omega + V} d\varepsilon \sqrt{\varepsilon (\omega_{R \alpha} - \omega)}$, which has an exact analytic expression. In the limit $V \gg \omega_{R \alpha}$, $T_{\alpha \alpha'}(\omega, V) = V \omega_{R \alpha}$.

Next, we evaluate $\mathcal{P}_{- \alpha \alpha}$, accounting for an intersection of $\varepsilon_\alpha(k) - \varepsilon_\alpha(k + Q_n)$ in the range $-\omega < \varepsilon_\alpha(k) < \min(0, V - \omega)$ contributing to a negative gain (i.e., absorption of the phonons by the electrons). The intersection occurs at $\delta\theta_0^2 = \pm \sqrt{(\omega - v_0 Q_n)^2 - \varepsilon^2}$, for $k < Q_n$. Integration over the energy and angle in the range $\varepsilon \in [\max(0, \omega, V), \omega]$, using $\delta(\varepsilon_\alpha(k) - \varepsilon_\alpha(k + Q_n) + \omega) = \frac{\omega_{R \alpha} - \omega}{\varepsilon_{\omega R} (\delta\theta_0^2)^{1/2}} \sum_{\alpha = \pm} \delta(\delta\theta - \delta\theta_0)$, yields  
\begin{equation}
\mathcal{P}_{- \alpha \alpha'}(\omega) = \frac{P_0 T_{\alpha \alpha'}(\omega, V) \Theta(\omega_{R \alpha} - \omega)}{\pi v_0^2 \sqrt{2 \omega_{R \alpha} (\omega_{R \alpha} - \omega)}},
\end{equation}

where $T_{\alpha \alpha'}(\omega, V) = \int_{\min(\omega_{R \alpha}, \max(0, \omega, V))}^{\omega + V} d\varepsilon \sqrt{\varepsilon (\omega_{R \alpha} - \omega)}$. The total contribution is shown in Fig. 2b in the main text. For $\omega = \omega_{R \alpha} - \delta\omega_n$, where $\delta\omega_n \ll \omega_{R \alpha}$ and in the limit $V \gg \omega_{R \alpha}$, we approximate $\mathcal{P}_{\alpha}(\omega) \approx P_0 N_D(V) \sqrt{\frac{2 \omega_{R \alpha}}{\delta\omega_n}}$. Therefore, the gain, as
follows from Eq. (7) in the main text reads \( \gamma_\omega = \frac{\omega_0^2 (\omega)^2}{c_{ph}^2 M} N_D(V) \sqrt{\frac{2e}{\hbar a}} \).

**THE BISTRIZER-MACDONALD MODEL OF THE TBG**

In this section, we present the continuum model that describes the low-energy physics of the TBG near the charge neutrality. Our goal is to outline the model that we used in the numerical simulations and to estimate the energy scale of the spatially modulated electron-phonon coupling due to the uniaxial strain and an array of screening gates. Therefore, we begin with a continuum model with a generic weak strain and screened electron-phonon interaction.

The TBG consists of two graphene monolayers twisted by a relative angle \( \theta \). Each untwisted graphene monolayer, denoted by \( l = 1, 2 \), comprises a honeycomb lattice of carbon atoms with a lattice constant of \( a = 0.246 \) nm and reciprocal vectors \( \mathbf{G}_{l,2} = \frac{2\pi}{a} (1, \pm 1) \). The graphene band structure exhibits two valley points at \( K_\pm = \mp (\mathbf{G}_1 + \mathbf{G}_2)/3 \), where the electrons have nearly Dirac dispersion. Our model is linearized around the valley points giving rise to the valley number, \( \xi = \pm \). Under a uniaxial strain and a twist, the low energy physics of each monolayer, at valley \( \xi \), is given by [1, 2]

\[
h^{(i)}_\xi = -v_F [(1 + \xi \mathbf{\mathcal{E}}^T) (\mathbf{k} - \mathbf{D}^{(i)}_\xi) + \xi \mathbf{\mathcal{A}^{(i)}_{ph}}] \cdot (\xi \sigma^\tau, \sigma^y) + \Phi^{(i)}_{ph}. \tag{18}
\]

Here, \( v_F/a = 2.14 \) eV, \( \sigma^\tau \) and \( \sigma^y \) are Pauli matrices acting in the sublattice basis \( \{ \mathcal{A}, \mathcal{B} \} \) of each monolayer \( \{ \mathcal{A} = \mathcal{B} \} \). \( \mathcal{E}_l \) is the strain and rotation tensor, given for small deformations by \( E_i = \begin{pmatrix} \epsilon_{xx} & \epsilon_{xy} - \theta_{1,2} \\ \epsilon_{yx} + \theta_{1,2} & \epsilon_{yy} \end{pmatrix} \), where \( \theta_{1,2} = \pm \theta/2 \) is the rotation angle of each monolayer. The vector \( \mathbf{D}^{(i)}_\xi = (1 - \xi \mathbf{\mathcal{E}}^T) \mathbf{K}_\xi - \xi \mathbf{\mathcal{A}} \) includes the modified position of the valley points due to the deformation and an effective gauge connection imposed by the strain, reading \( \mathbf{\mathcal{A}} = \frac{\sqrt{2}}{2a} \beta (\epsilon_{xx} - \epsilon_{yy}, \epsilon_{xy}) \), with \( \beta \approx 3.14 \). Eq. (18) also includes coupling to acoustic phonons, represented by the effective gauge connection \( \mathbf{\mathcal{A}^{(i)}_{ph}} = \frac{\sqrt{2}}{2a} \beta (\xi_{ux}, \xi_{uy}, \xi_{ux}) \) and by the diagonal term \( \Phi^{(i)}_{ph} = D (u_{ux} + u_{uy}) \). Here, \( u_{ij} = (\partial_i u_{ij} + \partial_j u_{ij})/2 \), and \( \xi_{ux}, \xi_{uy} \) is the displacement operator in the direction \( i = \{ x, y \} \) and layer \( l \). Each monolayer has an additional spin degree of freedom which is degenerate in the model.

For \( \epsilon_{ij} = 0 \) and \( \beta = D = 0 \), the twisted structure exhibits an emergent moiré lattice with a honeycomb structure, described by the reciprocal vectors \( \mathbf{G}_{m,2} = \frac{2\pi}{a_{cm}} (1, \pm 1) \), where \( a_{cm} = \frac{2\pi}{a} \) and \( a_{cm} = a/(2 \sin(\theta/2)) \).

The Hamiltonian describing the low-energy physics of the TBG is obtained by combining Eq. (18) describing monolayers and interlayer hopping terms \( T_\xi \), yielding [1, 3, 4] for the valley \( \xi \),

\[
\mathcal{H}_\xi = \left( \frac{h^{(1)}_\xi}{T_\xi \xi} \frac{T^\dagger_\xi}{h^{(2)}_\xi} \right). \tag{19}
\]

Here, the interlayer hopping is approximated by

\[
T_\xi(r) = \begin{pmatrix} u & u' \\ u' & u \end{pmatrix} e^{i G p^1 r} + \begin{pmatrix} u & u' \omega^{-1} \\ u' & u \end{pmatrix} e^{i G p^2 r}
\]

in the sublattice basis, where \( w = e^{i 2\pi \xi/3} \), and we consider \( u = 0.0797 \) eV, \( u' = 0.0975 \) eV. Fig. 3a in the main text shows the spectrum of a single valley of the TBG, described by Eq. (19), for \( \epsilon_{ij} = \beta = D = 0 \).

Next, we discuss the realizations of the nanoundulator. We begin by discussing a spatially modulated uniaxial strain along the \( x \) direction. We parametrize such a strain by \( \epsilon_{xx} = \epsilon_0 \cos(k u x) \), where \( \epsilon_0 \) is the amplitude of the strain, assumed to be small \( \epsilon_0 \ll 1 \), and \( \epsilon_{xy} = \epsilon_{yx} = 0 \). The effective electron-phonon coupling in the presence of the strain is given by the linear term \( \epsilon_0 \xi^{(l)}_{ux} \) in Eq. (18). Expanding to this order we arrive at

\[
h^{e-p}_l = -v_F \frac{\sqrt{3}}{2a} \beta \epsilon_0 \cos(k u x)(\xi^{(l)}_{ux} - \xi^{(l)}_{uy}) \sigma^y. \tag{20}
\]

By comparing Eq. (20) and Eq. (2) in the main text, we find \( g_1 = \frac{\sqrt{3}}{2a} v_F \beta \epsilon_0 \). For \( \epsilon_0 \approx 5\% \) strain, we estimate \( g_1 \approx 0.15 \) eV, corresponding to \( \gamma_0 \approx 0.02 \mu m^{-1} \).

Another realization is based on a periodic array of gates placed at the distance \( d \) from the TBG. The gates modify the screening efficiency of the interaction between the electronic charge density and the lattice ions[5–11]. For a phonon of momentum \( q \), the renormalized coupling term near the gate can be estimated by \( D = D_{TF} / [q + q_{TF}(1 - e^{-2q d})] \), where we use the estimate for the Thomas Fermi wavevector[12] \( q_{TF} \approx 1 \) nm. For a periodic structure of gates along \( \mathbf{x} \) with periodicity \( \lambda_u \), and in the limit \( q_{TF} \gg q, 1/d \), the coupling approximately oscillates between \( D_{min} \approx \frac{D_0}{q_{TF}} \) and \( D_{max} \approx \frac{D_0}{1 + 2D_0 q_{TF}} \). Comparing with Eq. (2) in the main text, we estimate \( g_1 \approx \frac{1}{2} \frac{D_0}{1 + 2D_0 q_{TF}} \) and \( \mathcal{O} \sim 1 \). For \( q_{TF} d \approx 3 \) and \( D_0 = 50 \) eV, we estimate \( g_1 \approx 1.8 \) eV. For this value of \( g_1 \), we obtain \( \gamma_0 \approx 3 \mu m^{-1} \).

**THE EFFECT OF THE PERIODIC PERTURBATION ON THE ELECTRONIC BANDSTRUCTURE**

Here, we study electronic dispersion in the presence of periodic distortion in the nano-undulator. The array of gates can only change the screening of the electronic interactions but does not produce a field. The periodic uniaxial strain, on the other hand, could have a small effect as described below. We can model this periodic
distortion by an extra term in the Hamiltonian $\hat{H}_d = \hat{V}_u \cos(\lambda_u x)$. This term creates a new superlattice for the electrons. The Bloch wavefunctions for the model with the perturbation can be written as

$$\psi_{k,a}^n(r,t) = e^{-i\varepsilon^{n}_{k,a} t / \hbar} \sum_{k,m} c_{k,m}^{n} |\varphi_{k-m,a}(r)\rangle,$$

where $|\varphi_{k,a}(r)\rangle$ is the Bloch function of the unperturbed $a$-band of the TBG and crystal momentum $k$ defined in the mini-Brillouin zone, and $c_{k,a}^{n}$ are coefficients determined by the solution to the full Schrödinger equation with $\hat{H}_d$, and $\varepsilon^{n}_{k,a}$ denotes the bands defined in the new superlattice. The predominant effect of the periodic uniaxial homostrain, arises from the term $\nu_F \epsilon_0 \cos(k_u x) k_u \sigma z$, which corresponds to a rescaling of the moiré unit cell. This term is a small perturbation, with a characteristic energy scale for $\epsilon_0 = 5\%$ less than 5\% of the bandwidth of the narrow band [13].

Fig. 1 shows the bandstructure of the model, $\varepsilon^{n}_{k,a}$ for $\xi = 1$, where we take $\hat{V}_u = \nu_F k_u \epsilon_0 \sigma z$, for $k_u = 1$ and $\epsilon_0 = 5\%$. The intensity indicates the overlaps of the undulated states with the unperturbed bandstructure of the TBG, $V_{k,a}^\alpha = |\langle \psi_{k,a}^\alpha |\varphi_{k,a}\rangle|^2$. Most of the spectral weight near the $K$ point remains in the original bands. The perturbed bandstructure is modified only near the level crossings between the shifted TBG bands, where it opens small energy gaps of the order of $V_u = |\langle \psi_{k} |\hat{V}_u |\varphi_{k-k_u}\rangle|^2$.

![Electronic bandstructure](image1.png)

**FIG. 1. Electronic bandstructure in the undulator.** The bandstructure of the electrons perturbed by the modulated uniaxial strain, along the cut in the mini-Brillouin zone shown in the inset. The intensity indicates the overlap between the perturbed bands and the original bands, $V_{k,a}^\alpha$. Here, we take $\lambda_u \approx 60 \text{ nm}$ and $V_u = \epsilon_0 \nu_F k_u \sigma z$, for $\epsilon_0 = 5\%$.

**COUPLING OF THE PHONONS TO PLASMON MODES**

In this section, we discuss the coupling of phonons to plasmon modes, giving rise to an evanescent THz-oscillating electromagnetic field. To describe this effect, we assume a uniform density of electrons $\rho_0^e$ modulated by small fluctuations represented by the displacement operator $\hat{u}_e(r,t)$, resulting in the density operator, $\hat{\rho}_e(r,t) = \rho_0 [1 - \nabla \cdot \hat{u}_e(r,t)]$. Similarly, the density operator of the ions modulated by the phonon displacement field $\hat{u}_i(r,t)$ [see the main text], is given by $\hat{\rho}_L(r,t) = \rho_L^0 [1 - \nabla \cdot \hat{u}_i(r,t)]$, where $\rho_L^0$ is the density of ions. For uniform densities (when $\hat{u}_e = \hat{u}_i = 0$), the sample is neutral, corresponding to $\rho_0 = Z_L \rho_L^0$, where $Z_L$ is the ions’ charge. Plasma modes in the electronic density and phonon modes in the ions, give rise to local electric charge density given by $\delta \hat{\rho}(r,t) = \rho_0 \nabla \cdot [\hat{u}_e(r,t) - \hat{u}_i(r,t)]$. This charge density, in turn, creates an oscillating electric field.

The dynamics of the electronic density fluctuations is described by the Hamiltonian

$$\hat{H}_e = \int \frac{d^2r}{a_0^2} \left[ \frac{1}{2m_e} |\hat{\pi}_e(r)|^2 + \frac{\kappa_e}{2} |\partial_r \hat{u}_e(r)|^2 \right].$$

(22)

Here, $\hat{\pi}_e(r)$ is the conjugate momentum of $\hat{u}_e(\varepsilon)$, $m_e$ is an effective electron mass [14] and $\kappa_e$ is related to the electronic compressibility, which near the Dirac cone [see Eq. (1) in the main text], can be approximated as $\kappa_e = \nu_e / a_m$, where $a_m = a / (2 \sin(\theta/2))$. Similarly, the lattice displacement is described by the Hamiltonian

$$\hat{H}_L = \int \frac{d^2r}{a^2} \left[ \frac{1}{2M} |\hat{\pi}_L(r)|^2 + \frac{M c_{ph}^2}{2} |\partial_r \hat{u}(r)|^2 \right].$$

(23)

The electronic and the ionic densities are coupled by the Coulomb repulsion between the charge densities $\hat{\rho}_e$ at different positions, which is described by the Hamiltonian

$$\hat{H}_C = \int d^2r d^2r' V_C(r-r') \delta \hat{\rho}(r) \delta \hat{\rho}(r'),$$

(24)

where $V_C(r-r') = e^2 / |r-r'|$. We also consider the electron-phonon coupling [see Eq. (2) in the main text] written in the form

$$\hat{H}_{ep} = - \int d^2r g(r) \rho_0^e (\nabla \cdot \hat{u}_e)(\nabla \cdot \hat{u}).$$

(25)

The equation of motion for the displacement operators in the Heisenberg picture driven by the Hamiltonian $\hat{H} = \hat{H}_e + \hat{H}_L + \hat{H}_C + \hat{H}_{ep}$, reads $\partial_t \hat{u}_e = \hat{\pi}_e / m_e$ and $\partial_t \hat{u} = \hat{\pi} / M$. In turn, the equation of motion for the conjugate momenta reads

$$\partial_t \hat{\pi}_e = \kappa_e \partial_r^2 \hat{\pi}_e - 2 \alpha_m^2 \partial_r \delta \hat{\rho}(r) - \alpha_m^2 g(r) \rho_0 \partial_r^2 \hat{u},$$

(26)
where $\dot{\phi}(r) = \int d^2r' V_C(r-r')\rho_0 \delta \rho(r')$. Similarly, for the phonon conjugate momentum, we obtain

$$\partial_t \pi = M c_p^2 \partial_r^2 \hat{u}_e + 2a^2 \partial_{rr} \phi(r) - a^2 g(r) \rho_0 \partial_r^2 \hat{u}_e. \quad (27)$$

Differentiating over time the equations of motion of the displacement fields and combining with equations of motion for the conjugate momenta, we arrive at

$$M \partial_t^2 \hat{u} = M c_p^2 \partial_r^2 \hat{u} + 2a^2 \partial_{rr} \phi(r) - a^2 g(r) \rho_0 \partial_r^2 \hat{u}, \quad (28a)$$

$$m_e \partial_t^2 \hat{u}_e = -\kappa_e \partial_r^2 \phi(r) - a^2 g(r) \rho_0 \partial_r^2 \hat{u}, \quad (28b)$$

To find the eigenmodes of the coupled differential equation, we substitute $\hat{u}(r,t) = \frac{1}{\sqrt{M}} \hat{u}_e(q,\omega) e^{i(q \cdot r - \omega t)}$ and $\hat{u}_e(r,t) = \frac{1}{\sqrt{M_e}} \hat{u}_e(q,\omega) e^{i(q \cdot r - \omega t)}$, leading to

$$\omega^2 \hat{u} = \frac{c_p^2 M}{M} \hat{u} + \frac{a q v_C^2}{M} (\hat{u} - \hat{u}_e) - \frac{a^2 q v_D^2}{M} \hat{u}_e, \quad (29a)$$

$$\omega^2 \hat{u}_e = \frac{\kappa_e q^2}{m_e} \hat{u}_e - \frac{a^2 q v_C^2}{M} \hat{u}_e - \frac{a^2 q v_D^2}{m_e} \hat{u}_e, \quad (29b)$$

where we used the surface Fourier transform of $F \{ V_C \} = 2\pi e^2/q$, take only the constant in space component of $g(r)$, and defined $v_C = 4\pi e^2/m_0 e$, $v_D = g_0 \rho_0$. We can rewrite the latter equation as an eigenvalue problem

$$\omega^2 \hat{U} = \mathcal{K} \hat{U}, \quad (30)$$

where $\mathcal{K} = \left( \begin{array}{cc} \frac{c_p^2 q^2}{M} + \frac{a q v_C^2}{M} - \frac{a^2 q v_D^2}{M} & -\frac{a^2 q v_C^2}{M} \\
-\frac{a^2 q v_D^2}{m_e} & \frac{\kappa_e q^2}{m_e} \end{array} \right) \frac{1}{\omega^2 - \frac{q^2}{1 + \frac{a^2 q^2}{m_e} M}}$, $\hat{U} = (\hat{u}/\sqrt{m_e} \hat{u}_e/\sqrt{M}) M$, and $M = \sqrt{M m_e}$. The eigenvalues to the leading order in $q$ read

$$\omega^2_+ = \frac{a^2 q v_C^2}{m_e} \left( 1 + \frac{a^2 m_e}{c_p^2 M} \right) + \mathcal{O}(q^4), \quad (31a)$$

$$\omega^2_- = \frac{q^2}{1 + \frac{a^2 q^2}{m_e} M} \left( \frac{c_p^2 q^2}{M} + \frac{\kappa_e q^2}{m_e} \right) - 2a^2 q v_D + \mathcal{O}(q^4), \quad (31b)$$

respectively corresponding to the plasmon and phonon modes. The plasmon eigenmode, corresponding to $\omega_+$, reads $\hat{u}_e/\hat{u} = -\frac{a m_e}{c_p^2 M} + \mathcal{O}(q)$. Similarly, the phonon eigenmode, corresponding to $\omega_-$, reads $\hat{u}_e/\hat{u} = 1 + \lambda q + \mathcal{O}(q^2)$, where

$$\lambda = \frac{a^2 m_e M - a^2 m_e}{(a^2 m_e + a^2 m_e) \omega^2_{-}} \frac{2 c_p^2 m_e M - M \kappa_e}{(a^2 m_e + a^2 m_e) \omega^2_{-}}. \quad (32)$$

Therefore, for $q \to 0$ phonons, $\hat{u}_e \to \hat{u}_0$, corresponding to $\delta \rho \to 0$. For a finite momentum excitation, we find

$$\delta \rho(r,t) = \rho_0 \lambda q^2 \hat{u}(r,t). \quad (33)$$

For $q = q z$ and $\omega = \omega_0$, the charge density wave in the $z$ direction gives rise to an oscillating current density in the same direction, as follows from the continuity equation $J_z(r,t) = -\frac{\partial}{\partial t} (\delta \rho(r,t))$. The current density generates an oscillating magnetic field which near the sample plane ($z = 0$) is oriented mostly in the $\hat{y}$ direction and can be estimated by Ampère’s law as $B_y(r,t) = \frac{2\pi}{c_t} J_z(r,t)$, where $r = (x, y, z = 0)$ and $c_t$ is the speed of light. For $z > 0$, this oscillating magnetic field propagates according to the electro-magnetic wave equation. Assuming $B_y(r,t) = B_y(z = 0,t)e^{i k_z z}$, we find $k_z^2 = \omega^2/c_t^2 - q^2$. Since, $c_t \gg c_p, k_z$ obtains imaginary values, corresponding to an evanescent electromagnetic wave. The effective electric field is given by $E = -i \frac{2}{\omega} \hat{E} \times \hat{B}$, yielding $\hat{E}(r,t) = -\frac{e}{\omega} (\hat{x} k_z - \hat{z} q) B_y(r,t)$. In terms of the charge densities, the electric field reads

$$E_x = 2\pi e^2 (\delta \rho e^{-|k_z| z}(k_z/q) \quad (34a)$$

$$E_z = -2\pi e^2 (\delta \rho) e^{-|k_z| z}, \quad (34b)$$

corresponding to the amplitude

$$|\hat{E}| = 2\pi e \rho_0 \lambda q^2 |(\hat{u}(r,t))| e^{-|k_z| z} \sqrt{1 + |k_z| q^2}. \quad (35)$$

We estimate Eq. (32) by $\lambda \approx a v_D/v_C$. Taking $\rho_0 = 1/a_{m}^2$, and $g_0 \approx 50$ eV, we find $\lambda \approx 2a$. Assuming at the laser saturation the displacement vector is $|\hat{u}(r)| \approx 0.1a$, taking $qa \approx 10^{-2}$, and focusing on the near field $z \ll |k_z|^{-1}$, we estimate $|\hat{E}| \approx 30$ kV/m.


[13] A qualitatively similar conclusion was obtained in [15] for a uniform homostrain, yet with a more significant effect near the charge neutrality point. As the phonon emission processes in the phaser are expected to occur around the gate potential level, $V$, such small mass-gap renormalization would weakly affect the operation.
