BRANCHING RATIOS INTO $K_{\mu 2}$, K_{e3}, AND K_{e4} IN THE PARTIALLY CONSERVED AXIAL-VECTOR CURRENT*

Mahiko Suzuki

California Institute of Technology, Pasadena, California

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The partially conserved axial-vector current hypothesis (PCAC)[1] has been successfully applied to the parity-nonconserving amplitudes of the nonleptonic hyperon decays[2] and to the nonleptonic K decays.[3] The $\Delta I=\frac{1}{2}$ rules are proved for all the processes except for the Σ decays in the approximation of putting the pion four-momenta equal to zero.[4] In addition, the branching ratio of $K_{3\pi}/K_{2\pi}$ is calculated and found to be in good agreement with experiment.

In the present paper we further apply the same method to the leptonic K-meson decays. PCAC for the strangeness-changing currents or SU(3) is not assumed. We assume only charge independence for the strong interactions. We calculate here the branching ratios of the K_{l2}, the K_{l3}, and the K_{l4} decays. The ratio of K_{e4}/K_{e3} turns out to be in good agreement with experiment if we make corrections due to the momentum-transfer dependence of the form factors and the final-state interactions of the di-pion. The ratio $K_{e3}/K_{\mu2}$ is dependent on the ratio of the two form factors in K_{e3}. We predict the value of the parameter $\xi=f_-/f_+$ by fitting the ratio to experiment.

We first write down the relations among these three kinds of leptonic decays in the framework of PCAC. The hadronic part of the K_{l3}^+ decay matrix element is given by

$$\langle K^+|S_\mu(0)|\pi^0\rangle = i\int e^{iqx}(\Box-\mu^2)\langle K^+|T\{S_\mu(0),\varphi^3(x)\}|0\rangle d^4x, \quad (1)$$

where S_μ is the strangeness-changing weak current with $I=\frac{1}{2}$ having the transformation property of $\bar{\psi}\gamma_\mu(1+\gamma_5)\psi$ in the Lorentz space, and state vectors are normalized as $\langle\alpha|\beta\rangle = 2\omega_\alpha\delta_{\alpha\beta}\delta(\vec{p}_\alpha-\vec{p}_\beta)$. By use of PCAC for the pion,

$$\partial_\mu \mathfrak{F}_{5\mu}^{\ i} = (i/c)\varphi^i, \quad (2)$$

we replace the pion field φ^i by $\partial_\mu \mathfrak{F}_{5\mu}^{\ i}$ and carry out the partial integration to get

$$\langle K^+|S_\mu(0)|\pi^0\rangle = c\int e^{iqx}(\Box-\mu^2)\langle K^+|[S_\mu(0),\mathfrak{F}_{50}^{\ 3}(x)]_-|0\rangle \times\delta(x_0)d^4x + icq_0\int e^{iqx}(\Box-\mu^2)\langle K^+|T\{S_\mu(0),\mathfrak{F}_{50}^{\ 3}(x)\}|0\rangle d^4x. \quad (3)$$

Let q go to zero and eliminate $\Box$ by partial integration. Since the second term apart from q_0 does not have a singularity like $1/q$, we have

$$\lim_{q\to 0}\langle K^+|S_\mu(0)|\pi^0\rangle = c\mu^2\langle K^+|[S_\mu(0),F_5^{\ 3}(0)]_-|0\rangle, \quad (4)$$

with the definition

$$F_5^{\ i}(0)=\int d^3x\,\mathfrak{F}_{50}^{\ i}(\vec{x},0). \quad (5)$$

The commutator $[S_\mu,F_5]_-$ gives $\frac{1}{2}S_\mu$, since F_5 is a generator of the chiral U(2)⊗U(2)[3] and S_μ has $I=\frac{1}{2}$. We can thus obtain the relation between the K_{l2} and the K_{l3} amplitudes.

The relation between the K_{l3} and the K_{l4} amplitudes can be obtained in the same manner; contract either of the two pions in K_{l4}, use PCAC to convert the T product into the equal-time commutator, and let the four-momentum of the contracted pion go to zero, keeping the other on the physical mass shell. The only difference from the case of K_{l3} is that we must be careful about the Bose statistics of the final di-pion in the K_{l4} decays. Since we are not interested in the detailed shape of the spectrum, we suppose that the two pions are in an s wave.[5,6] We symmetrize the K_{l4} matrix element with respect to the isospin indices of the pions to recover the symmetry property required by the Bose statistics. Otherwise the Bose statistics would be badly violated. Then we get, in the limit of one of the pion four-momenta equal to zero,[7]

$$\lim_{q\to 0}\langle K^+|S_\mu(0)|\pi^i\pi^j\rangle = (c\mu^2/2)\{\langle K^+|[S_\mu(0),F_5^i(0)]_-|\pi^j\rangle + \langle K^+|[S_\mu(0),F_5^j(0)]_-|\pi^i\rangle\}. \quad (6)$$

The right-hand side essentially reduces to the K_{l3} decay amplitude since the commutators give back $S_\mu(0)$ with a different charge state, in general. In terms of $K^+\to\pi^0+l+\nu$ and $K^+\to\pi^++\pi^-+l+\nu$, for example, the above equation is rewritten as

$$\lim_{q\to 0}\langle K^+|S_\mu(0)|\pi^+\pi^-\rangle = -i(c\mu^2/2)\langle K^+|S_\mu(0)|\pi^0\rangle, \quad (7)$$

where we have again used the fact that $S_\mu(0)$ is the $\Delta I=\frac{1}{2}$ current.

The hadronic parts of the leptonic K decays are parametrized as

$$\langle K^+|S_\mu|0\rangle=(1/\sqrt{2})p_\mu f, \quad (8a)$$

$$\langle K^+|S_\mu|\pi^0\rangle = (1/\sqrt{2})(p+q)_\mu f_+ + (1/\sqrt{2})(p-q)_\mu f_-, \quad (8b)$$

$$\langle K^+|S_\mu|\pi^+\pi^-\rangle = (1/\sqrt{2})(q_++q_-)_\mu g_1 + (1/\sqrt{2})p_\mu g_2, \quad (8c)$$

where p and q denote the four-momenta of the kaon and the pion, respectively, and terms responsible for the p- and d-wave di-pions have been neglected in Eq. (8c) according to our assumption. As far as the rate is concerned, this is a sufficiently good approximation. Now let either q_+ or q_- go to zero, keeping the other on the physical mass shell. Since Eq. (8c) is related to Eq. (8b) through Eq. (7) in this limit, we get

$$g_1^0=-(c\mu^2/2)(f_+-f_-), \quad (9a)$$

$$g_2^0=-(c\mu^2/2)(f_++f_-), \quad (9b)$$

where the superscript 0 means the value of $g_{1,2}$ in which one of the pion four-momenta goes to zero. In the same way,

$$f_+^0+f_-^0=-(c\mu^2/2)f$$

follows. The superscript 0 again means the value of $f_\pm$ in which the pion four-momentum goes to zero. We should notice that in K_{e3} and K_{e4} ($m_e\approx 0$), Eqs. (8b) and (8c) can be rewritten by use of Dirac equations into

$$\langle K^+|S_\mu|\pi^0\rangle=\sqrt{2}p_\mu f_+, \quad (10a)$$

$$\langle K^+|S_\mu|\pi^+\pi^-\rangle=(1/\sqrt{2})p_\mu(g_1+g_2). \quad (10b)$$

We can, therefore, calculate the K_{e3}/K_{e4} ratio, if the energy and momentum-transfer dependences of the form factors $g_{1,2}$ and $f_\pm$ are known. We shall continue the functions from the vanishing pion four-momentum to the physical value.

The dependence on energy and momentum-transfer is supposed to come from the final-state interactions of the s-wave di-pion and the p-wave $K\pi$ interaction. The former is a consequence of a large scattering length of the s-wave di-pion, which is often replaced by the so-called σ meson. The latter is due to the K^* resonance (890 MeV) and the lower continuum state of $K\pi$. Having this in mind, we make the following corrections to get the physical form factors:

$$f_\pm=\{K_{K\pi}((p-q)^2)/K_{K\pi}(m^2)\}f_\pm^0, \quad (11a)$$

$$g_{1,2}=\{K_{\pi\pi}((q_++q_-)^2)/K_{\pi\pi}(\mu^2)\}g_{1,2}^0, \quad (11b)$$

where $g_{1,2}^0$ still should contain the momentum-transfer dependence due to $K_{K\pi}$. They depend on energy and momentum like

$$g_{1,2}\sim \text{const}\times K_{\pi\pi}((q_++q_-)^2)K_{K\pi}((p-q)^2) \quad (11c)$$

in total. According to conventions, we adopt a linear function of $(p-q)^2$ for $K_{K\pi}$,

$$K((p-q)^2)=1+(p-q)^2/m^{*2}, \quad (12a)$$

while we adopt the one-pole approximation to $K_{\pi\pi}$,

$$K((q_++q_-)^2)=[1-(q_++q_-)^2/(m_\sigma+\tfrac{1}{2}i\gamma)^2]^{-1}. \quad (12b)$$

Since the proportional constant c connecting f, $f_\pm$, and $g_{1,2}$ is known from PCAC,

$$c=g_\gamma K_{\pi N}(0)/M_N\mu^2 g_A, \quad (13)$$

we can straightforwardly calculate the decay

Table I. Estimated values of K_{e4}/K_{e3} for various values of the parameters m_σ and m^*. The width γ has been chosen to be 100 MeV. The unit is 10^{-3}.

m_σ \ m^*	4μ	5μ	6μ	∞	Experiment
3μ	1.5	1.4	1.3	1.1	
4μ	0.99	0.88	0.84	0.70	
5μ	0.73	0.65	0.62	0.52	
∞	0.54	0.48	0.46	0.38	
Experiment					0.86 ± 0.19[a]

[a]A. H. Rosenfeld, A. Barbaro-Gartieri, W. H. Barkas, P. L. Bastien, J. King, and M. Roos, Rev. Mod. Phys. 37, 633 (1965).

rates. We have carried out numerical calculations[8] for a few values of the characteristic masses, m^* and m_σ. The ratio K_{e4}/K_{e3} (Table I) is uniquely determined when m^*, m_σ, and γ are fixed. In contrast, the ratio $K_{e3}/K_{\mu 2}$ (Table II) is obtained as a function of $\xi = f_-/f_+$ for each each value of m^*.

We find in Table I that the ratio K_{e4}/K_{e3} is correctly reproduced if we choose $m^* \approx 5\mu$ and $m_\sigma \approx 4\mu$ with the width $\gamma = 100$ MeV. If we completely neglect both the s-wave final-state interaction and the momentum-transfer dependence of the form factors, we would have a value approximately half as large as the experimental value. The values of m^*, m_σ, and γ which make our theory fit experiment are of reasonable magnitudes. On the other hand, the ratio $K_{e3}/K_{\mu 3}$ is not unique for a fixed value of m^*. Recent results of experiments suggest that m^* is around 700 MeV for K_{e3}^+. If we take this value seriously, ξ is predicted to be ~-0.1. The present experiment is, however, not so accurate as to give a conclusive value of ξ. Since the value of m^* also has a large ambiguity at present, our result on $K_{e3}/K_{\mu 2}$ should be tested by future experiment.

Table II. Estimated values of $K_{e3}/K_{\mu 2}$ for various values of the parameter m^*. The ratio $\xi = f_-/f_+$ has been calculated in each case.

m^*	$(K_{e3}/K_{\mu 2}) \times (1+\xi)^2$	$K_{e3}/K_{\mu 2}$	ξ
4μ	0.050		-0.23
5μ	0.066		-0.09
6μ	0.078		0.01
∞	0.130		0.23
Experiment		0.076 ± 0.004[a]	

[a]A. H. Rosenfeld, A. Barbaro-Gartieri, W. H. Barkas, P. L. Bastien, J. Kirz. and M. Roos, Rev. Mod. Phys. 37, 633 (1965).

In conclusion, our theory incorporating PCAC of the pion can predict correct orders of magnitudes for the ratios among $K_{\mu 2}$, K_{e3}, and K_{e4}. For reasonable values of the parameters involved in the form factors, the experimental values of the rates can be correctly reproduced. The present discussions can be applied to the electromagnetic and the strong decays with minor modifications. Further applications will be given elsewhere.

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[3]M. Suzuki, to be published.

[4]The $\Delta I = \frac{1}{2}$ rules can be proved only within the charge independence of the strong interactions. See Ref. 3.

[5]It can be shown that the p-wave configuration does not change the rate of K_{e4} very much. See L. B. Okun' and E. P. Shablin, Zh. Eksperim. i Teor. Fiz. 37, 1775 (1959) [translation: Soviet Phys.—JETP 10, 1256 (1960)].

[6]If we take the asymmetric limit without caring about Bose statistics, we would have

$$\lim_{p_+ \to 0} \langle K^+ | S_\mu | \pi^+ \pi^- \rangle = 0,$$

$$\lim_{p_- \to 0} \langle K^+ | S_\mu | \pi^+ \pi^- \rangle \neq 0,$$

where p_+ and p_- denote the four-momenta of π^+ and π^-, respectively. We could obtain some restrictions from the first relation, but the matrix element in this limit seems to be a worse approximation to the physical one as compared with the matrix element given in the text, unless the continuation is made. The limit taken in the text does not contradict with s-wave configuration of the di-pion in contrast to the limit given above. In our case, $K_{e4}(\pi^+\pi^-)/K_{e4}(\pi^0\pi^0)=2$ is implicitly predicted. In the case of $K_{e4}(\pi^0\pi^0)$ decay, there does not occur such a problem, and so our estimate of $K_{e4}(\pi^+\pi^-)/K_{e3}$ should be translated into $K_{e4}(\pi^0\pi^0)/K_{e3}$, if one does not adopt explicit symmetrization.

[7]If we use a symmetrized spatial wave function normalized to unity for the di-pion, the factor $c\mu^2/2$ should be replaced by $c\mu^2/\sqrt{2}$, but accordingly the phase-volume integral over final states must be restricted to a hemisphere.

[8]The decay rates with constant form factors are given in Ref. 4 and L. B. Okun', Ann. Rev. Nucl. Sci. 9, 61 (1959).

CURRENT ALGEBRAS AND THE SUPPRESSION OF LEPTONIC MESON DECAYS WITH $\Delta S=1$*

Reinhard Oehme

The Enrico Fermi Institute for Nuclear Studies and the Department of Physics, The University of Chicago, Chicago, Illinois

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Some time ago, we obtained a set of relations between the semileptonic amplitudes of mesons, which gave an indication that the suppression of $\Delta S=1$ transitions relative to $\Delta S=0$ decays cannot be due to strong-interaction effects.[1] These relations were obtained essentially on the basis of the chiral $U(3)\otimes U(3)$ algebra of hadron currents,[2] partially conserved axial-vector currents (PCAC),[3] and the assumption that the divergence of the axial-vector current acts approximately like a creation or destruction operator for the corresponding pseudoscalar mesons.

In this note we show that the same relations can be derived without the latter assumption, provided we allow one of the meson mass variables to be zero. Irrespective of this mass extrapolation, we find that our formulas give a very direct indication for the absence of specific strangeness-dependent renormalization effects in matrix elements of axial-vector (and vector) currents. Especially, we find that those form factors which are relevant for the determination of the Cabibbo angle[4] are essentially uneffected by the large $K\pi$ mass splitting.

Let us write the matrix elements for semileptonic K decays in the form

$$\langle 0|A_{4\alpha}-iA_{5\alpha}|K^+\rangle = iK_\alpha B_K,$$

$$\langle\pi^0|V_{4\alpha}-iV_{5\alpha}|K^+\rangle = (1/\sqrt{2})F_{\pi K}\{(K+\pi)_\alpha+\xi_{\pi K}(K-\pi)_\alpha\}, \quad (1)$$

with similar expressions for the corresponding π decays. The invariant functions $F_{\pi K}$ and $\xi_{\pi K}$ have the arguments $(-\pi^2, -K^2; -(K-\pi)^2)$. We can write the relations obtained in Ref. 1 in the form

$$F_{\pi K}(1+\xi_{\pi K})=[F_{\pi K}(1-\xi_{\pi K})]^{-1}=B_K/B_\pi,$$

$$F_{\pi\pi}(0)=F_{KK}(0)=1, \quad (2)$$

with $-\pi^2=m_\pi^2$, $-K^2=m_K^2$. As far as the momentum-transfer variable $q^2=(K-\pi)^2$ is concerned, we have assumed that $F_{\pi K}$ and $\xi_{\pi K}$ are slowly varying functions for $(m_K-m_\pi)^2 \leq -q^2 \leq (m_K+m_\pi)^2$.

In order to give a more direct derivation of Eqs. (2), we use the reduction formula and PCAC. Then we have an equation like

$$\langle\bar{K}^0|V_{40}(0)-iV_{50}(0)|\pi^+\rangle = \frac{i}{m_K^2B_K}\int d^4x\, e^{-iK\cdot x}(\Box-m_K^2)\langle 0|\theta(-x_0)[\partial_\alpha\{A_{6\alpha}(x)+iA_{7\alpha}(x)\}, V_{40}(0)-iV_{50}(0)]|\pi^+\rangle, \quad (3)$$

and a corresponding one for the matrix element $\langle\pi^0|V_{40}-iV_{50}|K^+\rangle$, where the dependence upon π_α is exhibited. Taking the limit $K_\alpha\to 0$ (or $\pi_\alpha\to 0$, respectively), and using partial integration with vanish-