Inequalities of Schwarz and Hölder type for random operators

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Let A and B be random operators on a Hilbert space, and let $\langle\ \rangle$ denote averages (expectations). We prove the inequality $\|\langle A^*B\rangle\| \leqslant \|\langle A^*A\rangle\|^{1/2}\|\langle B^*B\rangle\|^{1/2}$. A generalized Hölder inequality involving traces is also proved.

I. SCHWARZ INEQUALITY

In this paper we prove two inequalities, one of which was announced and extensively used before.[1] Despite the simple proof, the inequalities seem not to have been published elsewhere; only a special case of the Schwarz-type inequality for commuting operators has appeared.[2]

A random operator A is an operator-valued function $A(\cdot)$ on some space Ω, with a given probability measure μ on Ω. Averages or expectations are denoted interchangeably by $\langle\ \rangle$ or E. Thus

$$\langle A\rangle \equiv EA := \int_\Omega A(\omega)d\mu(\omega).$$

We have to be a little bit more precise. Let $\mathscr{H}$ be a separable Hilbert space and $B(\mathscr{H})$ the set of all bounded operators on $\mathscr{H}$. Let Σ be the σ algebra of sets in Ω on which μ is defined. We assume that the complex-valued function $(\varphi, A(\cdot)\psi)$ is Σ measurable for all $\varphi, \psi \in \mathscr{H}$. Thus, a random operator is a weakly measurable $B(\mathscr{H})$-valued function. A typical example is a random matrix.

It may happen that the expectation or average $E(\varphi, A(\cdot)\psi)$ exists for all $\varphi, \psi \in \mathscr{H}$; if in addition there is a bounded operator, denoted by EA or $\langle A\rangle$, such that

$$(\varphi, EA\psi) = E(\varphi, A(\cdot)\psi),$$

we say that the expectation of A exists (in the Pettis sense) and is given by EA, or $\langle A\rangle$. Clearly, $\langle A\rangle$ exists if $\langle\|A\|\rangle$ does, and then $\|\langle A\rangle\| \leqslant \langle\|A\|\rangle$.

Theorem 1: Let A and B be random operators on a Hilbert space $\mathscr{H}$. Then

$$\|\langle A^*B\rangle\| \leqslant \|\langle A^*A\rangle\|^{1/2}\|\langle B^*B\rangle\|^{1/2}, \tag{1.1}$$

where the existence of the right-hand side (rhs) implies the existence of the left-hand side (lhs) and where $\|\cdot\|$ is the usual operator norm on $B(\mathscr{H})$.

The consequences are analogous to those of Schwarz's inequality and are proved in a similar way.

Corollary 1: For a random operator A we define

$$\|A\|_\mu := \|\langle A^*A\rangle\|^{1/2},$$

if the rhs exists. Then

$$\|A+B\|_\mu \leqslant \|A\|_\mu + \|B\|_\mu \quad \text{(triangle inequality)},$$

$$|\|A\|_\mu - \|B\|_\mu| \leqslant \|A-B\|_\mu,$$

$$\|\langle A^*A - B^*B\rangle\| \leqslant \{\|A\|_\mu + \|B\|_\mu\}\|A-B\|_\mu.$$

If $\|A\|_\mu = 0$, then $A=0$ with probability 1. Thus $\|\cdot\|_\mu$ is a norm on (equivalence classes of) random variables.

Proof of Theorem 1: We use the ordinary Schwarz inequality, first for the $d\mu$ intergral (i.e., for expectations) and then for the scalar product in $\mathscr{H}$. Let the rhs of Eq. (1.1) exist. Then

$$\langle\|A\varphi\|^2\rangle = E(\varphi, A^*A\varphi) = \|\langle A^*A\rangle^{1/2}\varphi\|^2 \tag{1.2}$$

and similarly for B. The two Schwarz inequalities now give

$$\langle\|A\varphi\|^2\rangle^{1/2}\langle\|B\psi\|^2\rangle^{1/2} \geqslant \langle\|A\varphi\|\|B\psi\|\rangle \geqslant |E(\varphi, A^*A\psi)|, \tag{1.3}$$

with existence implied. We now take the sup over φ and ψ with $\|\varphi\| = \|\psi\| = 1$. This shows that the rhs of Eq. (1.3) defines a bounded operator $\langle A^*B\rangle$. Since $\|\langle A^*A\rangle^{1/2}\|^2 = \|\langle A^*A\rangle\|$, its norm is bounded by the rhs of Eq. (1.1). Q.E.D.

Using the existence statement of Theorem 1, the following analog of an inequality of Lieb and Ruskai[3] is an easy consequence.

Corollary 2: Let $\langle A^*A\rangle$ and $\langle B^*B\rangle$ exist. Then for any $\epsilon > 0$

$$\langle A^*B\rangle\{\langle B^*B\rangle + \epsilon\}^{-1}\langle B^*A\rangle \leqslant \langle A^*A\rangle. \tag{1.4}$$

As a consequence

$$\langle A^*B\rangle\langle B^*A\rangle \leqslant \|\langle B^*B\rangle\|\langle A^*A\rangle. \tag{1.5}$$

Proof: Let $Q := \{\langle B^*B\rangle + \epsilon]^{-1}\langle B^*A\rangle$, which is a nonrandom operator. Then one has

$$0 \leqslant (A - BQ)^*(A - BQ) + \epsilon Q^*Q.$$

Expanding and taking expectation gives Eq. (1.4). From

$$\|\langle B^*B\rangle + \epsilon\|^{-1}{}_1 \leqslant \{\langle B^*B\rangle + \epsilon\}^{-1},$$

one then obtains Eq. (1.5). Both also follow directly from Eq. (1.3). Q.E.D.

An alternative proof of Theorem 1 was proposed to me, which is based on the observation that

$$\begin{pmatrix} A^*A & A^*B \\ B^*A & B^*B \end{pmatrix} \geqslant 0.$$

Sandwiching this with $\phi \equiv \begin{pmatrix} \lambda\varphi \\ \psi/\lambda \end{pmatrix}$ from both sides and then taking expectations and the sup over φ, ψ yields Eq. (1.1). In a similar way Corollary 2 can be derived directly, with B^*B replaced by $B^*B + \epsilon$.

We remark in passing that for $\dim \mathscr{H} < \infty$ the normed space $\{A; \|\langle A^*A\rangle\|^{1/2} \equiv \|A\|_\mu < \infty\}$ is complete, and the norms $\|A\|_\mu$ and $\|A^*\|_\mu$ are equivalent. For $\dim \mathscr{H} = \infty$ this is in general not true, and $\|\langle A^*A\rangle\| < \infty$ does not imply $\|\langle AA^*\rangle\| < \infty$.

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II. HÖLDER INEQUALITY

Defining $|A|^p := (A^*A)^{p/2}$, Eq. (1.1) can be written as

$$\|\langle A^*B\rangle\| \leqslant \|\langle |A|^2\rangle\|^{1/2} \|\langle |B|^2\rangle\|^{1/2}.$$

The corresponding Hölder-type inequality for $p \neq 2$ does not hold in general if $\dim \mathscr{H} \geqslant 2$. This can be shown by counterexamples. For trace norms, however, one has the following.

Theorem 2: Let A and B be random operators on a Hilbert space $\mathscr{H}$. Then, for $r \geqslant 1$ and $1/p + 1/q = 1/r, p, q > 0$,

$$\{\mathrm{Tr}|\langle A^*B\rangle|^r\}^{1/r} \leqslant \mathrm{Tr}\{\langle |A^*B|\rangle^r\}^{1/r}$$

$$\leqslant \{\mathrm{Tr}\langle |A|^p\rangle\}^{1/p} \{\mathrm{Tr}\langle |B|^q\rangle\}^{1/q}, \quad (2.1)$$

where existence of the rhs implies existence of the rest. Here A^* may be replaced by A in the middle and the lhs.

Similarly as before, we use Hölder's inequality for integrals and then for trace norms. But first we note a simple fact.

Lemma 1: On positive random operators, trace and expectations commute,

$$E\,\mathrm{Tr}|A| = \mathrm{Tr}\,E\,|A|, \quad (2.2)$$

and existence of either side implies that of the other. In this case A is trace class almost surely, EA exists and is trace class, and

$$\mathrm{Tr}\,EA = E\,\mathrm{Tr}\,A. \quad (2.3)$$

Proof: Let $\{\varphi_n\}$ be an orthonormal basis in $\mathscr{H}$. Then

$$E\,\|A\| \leqslant E\,\mathrm{Tr}|A| = \sum_n E(\varphi_n, |A|\varphi_n),$$

by positivity. Hence, if the rhs is finite then $E|A|$ and EA exist as bounded operators and the rhs equals $\mathrm{Tr}\,E\,|A|$. Equation (2.3) then follows from Lebesgue's bounded convergence. Q.E.D.

Proof of Theorem: By Hölder's inequality,[4] first for integrals and then for trace norms, we have

$$\{E\,\mathrm{Tr}|A|^p\}^{1/p}\{E\,\mathrm{Tr}|B|^q\}^{1/q}$$

$$\geqslant \{E\,[(\mathrm{Tr}|A|^p)^{1/p}(\mathrm{Tr}|B|^q)^{1/q}]^r\}^{1/r}$$

$$\geqslant \{E\,\mathrm{Tr}|A^*B|^r\}^{1/r}. \quad (2.4)$$

By Lemma 1, this proves the second part of Eq. (2.1), together with existence. The remainder follows from Lemma 2.

Lemma 2: Let A be a random operator and let $p \geqslant 1$. Then

$$\mathrm{Tr}|EA|^p \leqslant \mathrm{Tr}\,E\,|A|^p, \quad (2.5)$$

where existence of the rhs implies that of the lhs.

Proof: Let the rhs exist. By Lemma 1, $|A|^p$ is trace class almost surely. Since $\|A\|^p \leqslant \mathrm{Tr}|A|^p$ one has

$$E\,\|A\| \leqslant \{E\,\|A\|^p\}^{1/p} < \infty.$$

Hence EA exists as a bounded operator.

Now let X be any nonrandom operator with $|X|^q$ trace class,[5] $1/p + 1/q = 1$. Then, by the second half of Eq. (2.1), $E\,|XA|$ exists and is trace class. Thus, by Lemma 1, $EXA = XEA$ is also trace class. By duality one now has[5]

$$\{\mathrm{Tr}|EA|^p\}^{1/p} = \sup_{\mathrm{Tr}|X|^q = 1} |\mathrm{Tr}\,XEA|$$

$$\leqslant E \sup_{\mathrm{Tr}\,|X|^q = 1} |\mathrm{Tr}\,XA|$$

$$= E\,\{\mathrm{Tr}|A|^p\}^{1/p}$$

$$\leqslant \{E\,\mathrm{Tr}|A|^p\}^{1/p}. \quad (2.6)$$

Q.E.D.

It was pointed out to me that the argument in Eq. (2.4) can be replaced by an equivalent linear version of Hölder's inequality, i.e.,

$$r^{-1}\,\mathrm{Tr}|A^*B|^r \leqslant p^{-1}\lambda^p\,\mathrm{Tr}|A|^p + q^{-1}\lambda^{-q}\,\mathrm{Tr}|B|^q,$$

for all $\lambda > 0$. Taking expectation and using Lemma 1 also gives the second part of Eq. (2.1).

Remark: Finiteness of the measure μ has only entered in the proof of Lemma 2. For nonfinite μ, Theorem 1, Theorem 2 with $r = 1$, and Lemma 2 with $p = 1$ still hold, as does the second inequality of Theorem 2 for any $r \geqslant 1$.

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[5]If $p = 1$, we take X bounded and replace $\{\mathrm{Tr}|X|^q\}^{1/q}$ by $\|X\|$.