ANALYSIS OF APERTURE ANTENNAS

IN INHOMOGENEOUS MEDIA

by

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ABSTRACT

The object of this report is to calculate the admittance and the radiation pattern of aperture antennas fed by waveguides of arbitrary cross-section and radiating into dielectric slabs, whose constitutive parameters may be functions of position along the direction normal to the slab faces.

For a given aperture field distribution the antenna aperture admittance and the radiation field are expressed here, for the first time, in terms of two auxiliary quantities directly related to the plane wave reflection and transmission coefficients of the dielectric slab. These quantities are the input admittance of the dielectric slab and the ratio of the total electric field amplitude transmitted at one end of the slab to the transverse field at the other, both calculated for plane waves as a function of incident propagation direction. This approach introduces a great simplification in the solution of the problem, particularly in the case of an antenna radiating into an inhomogeneous dielectric slab.

A simple and powerful method has been devised for the computation of the input admittance of an inhomogeneous dielectric slab as well as for the electric field ratio. In this case the impracticability of obtaining analytical results has necessitated the use of numerical techniques.

Examples of the application of the theory to typical dielectric slabs are given and the results are discussed.

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1. INTRODUCTION

The study of antennas covered by dielectrics has been of particular interest to scientists and engineers in the last decade. Interest in this problem is generated because of two situations of practical importance that scientists have been confronted with. The first arises in connection with maintaining radio communication with space vehicles at reentry. High speed space vehicles reentering the earth's atmosphere are surrounded by an ionized gas layer or a plasma sheath which strongly affects the propagation of electromagnetic waves. The presence of the plasma sheath causes a mismatch between transmitter or receiver and the antenna, and a distortion of the antenna radiation pattern, thus creating problems in the maintenance of radio communication between the vehicle and ground stations. As a result, the study of the admittance and the radiation pattern of an antenna covered by plasma sheaths of varying properties is of primary importance. The second situation arises in connection with plasma diagnostics. Electromagnetic waves in plasma make it possible to extract information about such plasma characteristics as electron densities or relaxation times. Often the first step in the measurement of these properties is the determination of the admittance of an antenna in the plasma.

This report presents a new and simple method for the calculation of the admittance and radiation pattern of antennas consisting of waveguide-fed apertures in infinitely conducting ground planes, and radiating into dielectric slabs of varying properties. The usefulness and simplicity of the method stem from the fact that the usual boundary-value problem approach, involving lengthy calculations for each special situation, is bypassed.

As is well known, for each of the two independent polarization directions of an incident plane wave and as a function of its propagation direction, a dielectric slab has a reflection coefficient and a transmission coefficient. Given the aperture field distribution, the expression for the aperture admittance of the antenna is shown to involve these reflection coefficients or equivalently the input admittances of the slab. Similarly the slab transmission coefficients enter into the expression for the radiation field. The problem is formulated in such a way that it can readily be applied for any aperture shape and corresponding waveguide cross-section. In most instances we are interested in infinite slots and circular, annular or rectangular apertures in flat ground planes. Hence, we will be concerned particularly with these configurations.

Furthermore the expressions derived are valid for any kind of dielectric medium of practical interest making up the slab - homogeneous, inhomogeneous or turbulent with constitutive parameters varying only along the direction normal to the slab faces. The present method is particularly useful for slabs which are not homogeneous isotropic media, since the difficulty involved in these cases is reduced to the calculation of the appropriate reflection and transmission coefficients which then are used in the evaluation of antenna admittance and radiation field.

Antennas in lossy, dielectric media have been the subject of many studies for almost half a century. However since this report deals essentially with the admittance and radiation pattern of radiating apertures, only those investigations which have some bearing on the present problem will be mentioned here.

Levine and Schwinger (1,2), starting with integral equation formulations, have arrived for the first time at variational expressions for the far field from an aperture in an infinite plane screen on which a plane scalar or electromagnetic wave is incident. Levine and Papas (3) in a similar way have found variational expressions for the admittance of an annular aperture in an infinite plane conducting screen fed by a coaxial waveguide and radiating into free space. Using similar techniques Lewin (4) has calculated the admittance of a rectangular waveguide-fed aperture radiating into free space.

Variational techniques have consistently been used in the calculation of the admittance of dielectric covered waveguide-fed slots as well. In almost all cases the fields everywhere are expressed in terms of their Fourier transforms and the boundary-value problem is solved assuming the aperture field to consist of the dominant waveguide mode. The admittance of a rectangular waveguide-fed slot covered by a homogeneous plasma layer has been calculated by Galejs (5). However his formulation is quite involved and does not apply to plasma layers whose thickness is small compared to a wavelength. Galejs (6,7) has also computed the admittance of plasma-covered annular and rectangular slot antennas assuming they radiate into a wide waveguide instead of an unbounded half-space. This approximation has allowed him to represent the fields by a discrete sum of modes. Furthermore he has shown (8) that the same summation is obtained if a constant mesh size approximation is used for the numerical computation of the integral representing the admittance. Villeneuve (9) too has calculated the admittance of a rectangular waveguide radiating into a homogeneous plasma layer, through an application of the reaction concept of Rumsey. Both Galejs and Villeneuve, however, have considered only homogeneous plasmas whose electron densities are below the critical density, and hence the relative permittivity varies between zero and unity. In precisely this range the plasma does not support surface waves and hence they have not had to consider surface wave contributions to the aperture admittance. Compton (10) has presented the most straightforward method of calculating the admittance of aperture antennas fed by parallel-plate and rectangular waveguides and radiating into a lossy dielectric. His original formulation has been modified by Croswell, Rudduck and Hatcher (11) to account for the surface-wave pole contributions to the admittance for low-loss dielectric slabs with permittivity greater than one. Fante (12) has described a simple technique for the admittance and the radiation pattern calculations of thin plasma slabs based on the impedance sheet notion. Finally, Bailey and Swift (13) have calculated the admittance of a circular waveguide aperture covered by a homogeneous dielectric slab with permittivity greater than unity. Recently Croswell, Taylor, Swift and Cockrell (14) suggested a method for the calculation of the admittance of a rectangular waveguide-fed aperture covered by an inhomogeneous plasma slab. However their method, based on the evaluation of the fields in the plasma region, is too involved even for numerical computations. It calls for the numerical solution of the Helmholtz equation in inhomogeneous media, which is very complicated, but quite unnecessary for the solution of the problem in question.

Two points need to be mentioned in connection with the above-mentioned analyses of the plasma-covered flush-mounted antennas. First, all investigations rely on boundary-value problem techniques. Second, the important case of antennas covered by inhomogeneous plasma slabs has not been studied at all in a useful way.

The radiation pattern of aperture antennas covered by homogeneous plasma slabs have been analyzed by various authors. Tamir and Oliner (15) have calculated the radiation field of an infinite slot covered by a plasma layer and have also considered the effect of surface wave poles on the radiation field. Knop and Cohn (16) have found the radiation field from apertures in ground planes covered by dielectrics. The radiation from infinite slots and apertures covered by anisotropic plasma sheaths have been studied by Hodara and Cohn (17) and Hodara (18). Before the present report, the radiation pattern of aperture antennas covered by inhomogeneous plasma slabs had not been analyzed.

In the second chapter of this report a stationary expression for the aperture admittance is obtained, using as the aperture distribution, the dominant mode of the waveguide as well as a combination of the dominant and higher-order modes.

The third chapter is the main body of the report. Here the aperture distribution is thought as resulting from a superposition of plane waves. Using this idea, the aperture admittance and the radiation pattern are calculated in terms of the plane wave reflection and transmission coefficients. Specific calculations are made for a number of common geometries.

The treatment of aperture antennas radiating into inhomogeneous dielectric slabs requires a discussion of the properties of such media. This is done in the fourth chapter. Besides the differential equations for the reflection and transmission coefficients, other equations are derived which yield directly the input admittance and the ratio of the total electric field amplitude at one end of the slab to the transverse field at the other.

The fifth and final chapter is devoted to a dicussion of the results obtained from some specific examples.

2. VARIATIONAL TREATMENT OF WAVEGUIDE-FED APERTURE ADMITTANCE

A stationary expression is derived in this chapter for the admittance of a waveguide-fed aperture in an infinite conducting ground plane. The application of the variational principle to problems of this nature is well known, however it will be wise to start our analysis from this point for the sake of presenting a complete treatment of the subject.

We consider an aperture in an infinitely conducting ground plane at $z = 0$, fed by a cylindrical waveguide of arbitrary cross-section located in the region $z < 0$, and radiating into a region $z > 0$ whose parameters may vary along the z-direction (Fig. 2.1).

Assuming the waveguide to be operating in its dominant mode, only this mode will be incident on the aperture. However higher order modes will be generated at the discontinuity and reflected back from the aperture, along with the reflected dominant mode. The transverse electric and magnetic fields at the aperture can be expressed as (19),

$$\underline{E}_o(x,y) = V_o\, \underline{e}_o(x,y) + \sum_n V_n\, \underline{e}_n(x,y) \tag{2.1}$$

$$\underline{H}_o(x,y) = I_o\, \underline{h}_o(x,y) + \sum_n I_n\, \underline{h}_n(x,y) \tag{2.2}$$

Here V_o and I_o represent the total (incident plus reflected) amplitude of the dominant mode, V_n and I_n represent the amplitudes of the reflected high order TE and TM modes. The transverse mode functions

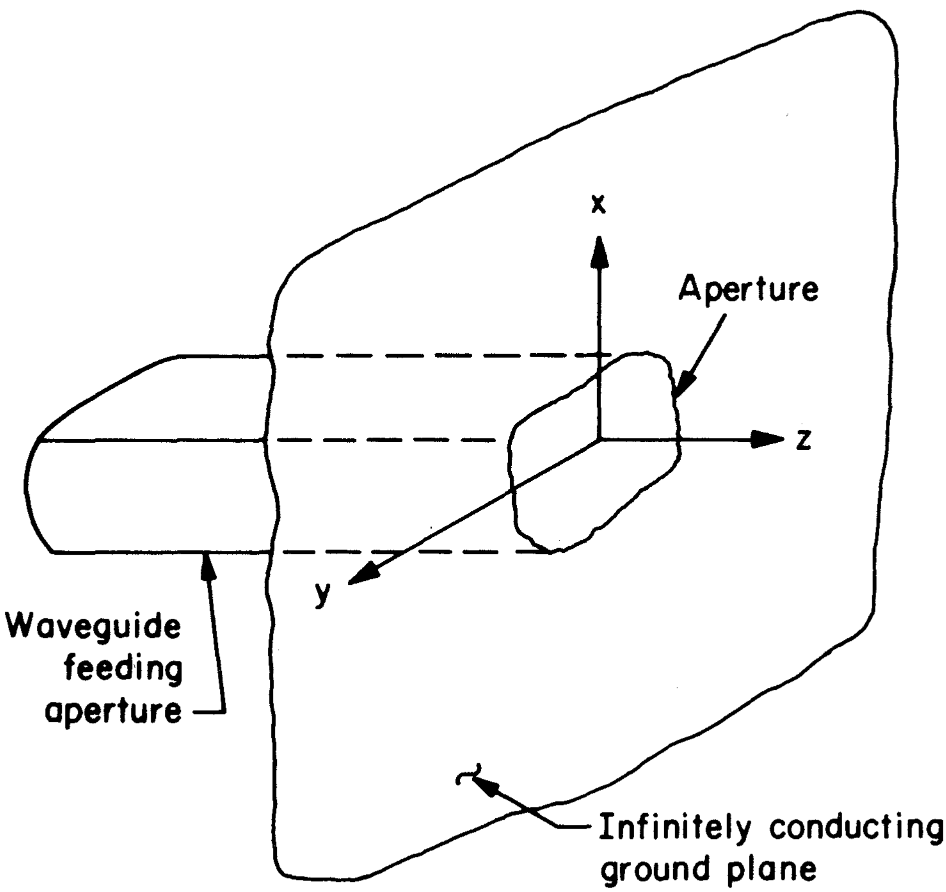


Fig. 2.1 A waveguide-fed aperture antenna

$\underline{e}_o(x,y)$, $\underline{h}_o(x,y)$ for the dominant mode, and $\underline{e}_n(x,y)$, $\underline{h}_n(x,y)$ for the higher-order modes, depend on the particular waveguide cross-section and satisfy the relations (19)

$$\underline{h}_i = \underline{u} \times \underline{e}_i \tag{2.3}$$

and

$$\iint_A \underline{e}_i(x,y) \cdot \underline{e}_j(x,y)\, dA = \delta_{ij} \tag{2.4}$$

where $\underline{u}$ is the unit vector in the z direction, and A denotes integration over the aperture area.

For the reflected higher-order modes,

$$I_n = - Y_n V_n \tag{2.5}$$

where Y_n are the characteristic mode admittances. Now making use of relations (2.3), (2.4), (2.5) and (2.1), equation (2.2) can be written as

$$\begin{aligned} \underline{H}_o(x,y) &= I_o\, \underline{u} \times \underline{e}_o(x,y) \\ &\quad - \sum_n Y_n \underline{u} \times \underline{e}_n(x,y) \int_A \underline{E}_o(x,y) \cdot \underline{e}_n(x,y)\, dA \end{aligned} \tag{2.6}$$

As a result of the well-known existence theorem for the electromagnetic fields, the magnetic field within the region $z > 0$ is uniquely determined by the tangential component of the electric field on the plane $z = 0$. Hence through the use of a dyadic Green's function

$\underline{\underline{G}}$, the magnetic field in the region $z > 0$ can be expressed as (2)

$$\underline{H}(x,y,z) = \int_A \underline{\underline{G}}\,(x,x',y,y',z,0)\cdot\underline{u} \times \underline{E}_o(x',y')\;dA' \tag{2.7}$$

The Green's function depends on the parameters of the region $z > 0$ and satisfies an appropriate differential equation with boundary conditions. The integration in (2.7) is over the aperture area only, since the tangential electric field vanishes over the conducting ground plane.

The continuity of the tangential magnetic field at the plane $z = 0$ results in the relation

$$\underline{u} \times \underline{H}_o(x,y) = \underline{u} \times \underline{H}\,(x,y,0) \tag{2.8}$$

Substituting (2.6) and (2.7) in (2.8) we obtain an integral equation for the aperture electric field:

$$\begin{aligned} I_o\underline{e}_o(x,y) = \sum_n Y_n\underline{e}_n(x,y) \int_A \underline{E}_o(x,y)\cdot\underline{e}_n(x,y)\;dA \\ - \int_{A'} \underline{u} \times \underline{\underline{G}}(x,x',y,y',0,0) \times \underline{u} \cdot \underline{E}_o(x',y')\;dA' \end{aligned} \tag{2.9}$$

The admittance of the waveguide-fed aperture is defined as the ratio of the total dominant mode magnetic to electric field amplitudes.

$$Y = I_o/V_o \tag{2.10}$$

A variational expression for Y can be obtained if equation (2.9) is scalarly multiplied by $\underline{E}_o(x,y)$, integrated over the aperture area A and then divided through by $V_o^2 = [\int_A \underline{E}_o(x,y) \cdot \underline{e}_o(x,y)\, dA]^2$. The result is

$$Y = \frac{1}{\left[\int_A \underline{E}_o(x,y) \cdot \underline{e}_o(x,y)\, dA\right]^2} \left\{ \sum_n Y_n \left[\int_A \underline{E}_o(x,y) \cdot \underline{e}_n(x,y)\, dA\right]^2 \right.$$

$$\left. - \iint_{AA'} \underline{E}_o(x,y) \cdot (\underline{u} \times \underline{\underline{G}}(x,x',y,y',0,0) \times \underline{u}) \cdot \underline{E}_o(x',y')\, dA dA' \right\} \tag{2.11}$$

If the exact aperture electric field were known, the admittance Y could readily be found from (2.11). The exact field could only be found by solving the integral equation (2.9), which, in general, is an almost impossible task. Hence approximate methods must be devised, and one such method consists in writing Y as a variational expression. The expression (2.11) is stationary with respect to small variations δE_o of the aperture electric field E_o about its exact value determined by the integral equation (2.9). This means that substitution of an approximate aperture electric field into (2.11) will still yield a good estimate for Y. Furthermore, expression (2.11) is seen to be homogeneous in E_o.

It is easy to verify that the variation of Y due to small variations δE_o of E_o disappears. Making use of the symmetry property of the Green's function (2),

$$G_{ij}(x,x',y,y',0,0) = G_{ji}(x',x,y',y,0,0) \quad , \tag{2.12}$$

we obtain for the variation of Y,

$$\delta Y\left[\int_A \underline{E}_o \cdot \underline{e}_o \, dA\right]^2 = -2Y\left[\int_A \underline{E}_o \cdot \underline{e}_o \, dA\right]\left[\int_A \delta\underline{E}_o \cdot \underline{e}_o \, dA\right]$$

$$+ 2\sum_n Y_n\left[\int_A \underline{E}_o \cdot \underline{e}_n \, dA\right]\left[\int_A \delta\underline{E}_o \cdot \underline{e}_n \, dA\right] - 2\iint_{AA'} (\underline{u} \times \underline{\underline{G}} \times \underline{u}) \cdot \underline{E}_o' \cdot \delta\underline{E}_p \, dA dA' \tag{2.13}$$

The right-hand side of (2.13) may be rewritten as

$$2\int_A \left\{- I_o \underline{e}_o + \sum_n Y_n \underline{e}_n \int_A \underline{E}_o \cdot \underline{e}_n \, dA - \int_{A'} (\underline{u} \times \underline{\underline{G}} \times \underline{u}) \cdot \underline{E}_o' \, \delta A'\right\} \cdot \delta\underline{E}_o \, dA$$

The expression in brackets under the integral sign vanishes because of the integral equation (2.9). Hence

$$\delta Y = 0 \tag{2.14}$$

The variational expression (2.11) can be put in the more convenient form

$$Y = \frac{\int_A \underline{E}_o(x,y) \times \underline{H}_o(x,y) \cdot \underline{u} \, dA}{\left[\int_A \underline{E}_o(x,y) \cdot \underline{e}_o(x,y) \, dA\right]^2} + \sum_n Y_n \frac{\left[\int_A \underline{E}_o(x,y) \cdot \underline{e}_n(x,y) \, dA\right]^2}{\left[\int_A \underline{E}_o(x,y) \cdot \underline{e}_o(x,y) \, dA\right]^2} \tag{2.15}$$

provided it is understood that the function to be varied is $\underline{E}_o(x,y)$, and that $\underline{H}_o(x,y) = H(x,y,0)$ is related to $\underline{E}_o(x,y)$ through equation (2.7). This requires finding an appropriate Green's function, which, in practice, may be quite difficult. Instead it is easier to find a relation between the Fourier components of the electric and magnetic fields, as will be shown in the next chapter.

The most simple and logical assumption is that the aperture electric field has the form of the dominant mode. Then

$$\underline{E}_o(x,y) = V_o \, \underline{e}_o \, (x,y) \tag{2.16}$$

and the second term in (2.15) involving the summation vanishes because of (2.4). Thus for dominant mode aperture electric field approximation the admittance becomes

$$Y = \frac{1}{V_o^2} \int_A \underline{E}_o(x,y) \times \underline{H}_o(x,y) \cdot \underline{u} \, dA \tag{2.17}$$

where it is understood that $\underline{H}_o$ is linearly related to $\underline{E}_o$ through a relation like (2.7).

Comparisons of theoretical and experimental results have shown that the dominant mode aperture electric field approximation is adequate in most instances (11, 14, 20). However, for a more exact treatment, the aperture electric field could be assumed as a superposition of the dominant mode and some higher order modes. For simplicity let us consider only one higher order mode, with mode function $\underline{e}_i$, which would be the next most highly excited mode, as indicated by the geometry of the problem. The generalization to more than one, but a finite number of higher order modes will be evident. Accordingly, the aperture electric field could be written as

$$\underline{E}_o(x,y) = V_o\left[\underline{e}_o(x,y) + a_i\ \underline{e}_i(x,y)\right] \tag{2.18}$$

$$= \underline{E}_{oo}(x,y) + a_i\ \underline{E}_{oi}(x,y)$$

The resulting aperture magnetic field would be

$$\underline{H}_o(x,y) = \underline{H}_{oo}(x,y) + a_i\ \underline{H}_{oi}(x,y) \tag{2.19}$$

with $$\underline{H}_{oo,i}(x,y) = \int_A \underline{\underline{G}}\ (x,x',y,y') \cdot \underline{u} \times \underline{E}_{oo,i}(x',y')\ dx'dy'$$

Making use of the symmetry of the Green's function, the aperture admittance becomes in this case,

$$Y = \frac{1}{V_o^{\,2}}\left\{\int_A \underline{E}_{oo} \times \underline{H}_{oo} \cdot \underline{u}\ dA + 2a_i \int_A \underline{E}_{oo} \times \underline{H}_{oi} \cdot \underline{u}\ dA\right.$$

$$\left. + a_i^{\,2} \int_A \underline{E}_{oi} \times \underline{H}_{oi} \cdot \underline{u}\ dA\right\} + a_i^{\,2}\ Y_i$$

$$= y_{oo} + 2a_i\ y_{oi} + a_i^{\,2}\ (y_{ii} + Y_i) \tag{2.20}$$

where

$$y_{mn} = \frac{1}{V_o^2} \int_A E_{om} \times H_{on} \cdot \underline{u}\, dA \tag{2.21}$$

Since Y is stationary, a_i is determined from the condition that $dY/da_i = 0$. Hence,

$$a_i = - \frac{y_{oi}}{y_{ii} + Y_i} \tag{2.22}$$

Substituting (2.22) into (2.20) we finally obtain

$$Y = y_{oo} - \frac{y_{oi}^2}{y_{ii} + Y_i} \tag{2.23}$$

The first term in (2.23) is the result of assuming an aperture electric field of the form of the dominant mode, and is the same as (2.17). It is also evident from equation (2.23) that when higher order modes are included a mutual coupling exists between the modes. Furthermore, the terms in equation (2.23), y_{oo}, y_{oi}, y_{ii}, are of the form of (2.17), which suggests that once a method is known for finding the admittance assuming an aperture electric field of the form of the dominant mode, extension to a more general case is straightforward. Hence, in the future, only equation (2.17) will be considered.

3. APERTURE ANTENNA CHARACTERISTICS

This chapter will deal with the derivation of some important relations for the calculation of the aperture admittance and the radiation pattern of the radiating structures that are being considered. The results will apply to any aperture shape and to most media of physical interest where the antenna is radiating. The method used is new, yet quite simple, in that it bypasses the usual boundary-value approach.

A. Plane Wave Synthesis of Aperture Distributions

It is well known that an arbitrary time-harmonic field can be constructed by a superposition of plane waves, all of the same frequency, each with its appropriate amplitude, and traveling in all possible directions, real as well as complex (21).

The region $z < 0$ of Fig. 2.1 in which the waveguide is located will now be considered as a semi-infinite region of free space where plane waves propagating in every possible direction are incident on and reflected from the half-space $z > 0$, in such a way that on the plane $z = 0$ a specified electric field configuration is obtained. This field will be the assumed aperture electric field over the aperture area, and will vanish everywhere else. Looking at the problem in this way, the evaluation of the aperture admittance and the radiation pattern will be greatly simplified, particularly if the structure is radiating into an inhomogeneous medium. It will be shown that the double Fourier transform $\underline{\hat{E}}_o(u,v)$ of the aperture electric field $\underline{E}_o(x,y)$ is simply related to the amplitudes of the plane waves in the region $z < 0$. The calculation of the reflection and transmission coefficients for each plane wave will enable the computation of the aperture admittance and the radiation pattern.

A plane electromagnetic wave of frequency ω, traveling in free space can be represented by,

$$\begin{aligned} \underline{E}(\underline{r}) &= \hat{\underline{E}}\, e^{ik\underline{n}\cdot\underline{r}} \\ \underline{H}(\underline{r}) &= \eta\; \underline{n} \times \underline{E}\,(\underline{r}) \end{aligned} \tag{3.1}$$

where $k = \omega\sqrt{\varepsilon_o\mu_o}$, $\eta = \sqrt{\dfrac{\varepsilon_o}{\mu_o}}$, $\hat{\underline{E}}$ is the complex wave amplitude, $\underline{n}$ is a unit vector in the propagation direction, $\underline{n} \cdot \underline{E} = 0$, and an $e^{-i\omega t}$ time dependence is understood.

We now suppose that such a plane wave traveling in the region $z < 0$ is incident on the boundary at $z = 0$, and use the subscript i to denote the incident fields and the unit vector in the propagation direction of the incident wave.

The plane containing the vector in the direction of propagation $(\underline{n}_i)$ and the normal to the boundary is called the plane of incidence. It is well known that any plane electromagnetic wave can be resolved into two components, one with the electric vector polarized perpendicular to the plane of incidence, and another for which the electric vector lies in this plane. Let us consider a spherical system of coordinates, where the incident wave propagates in a radial direction (Fig. 3.1). Then from (3.1) the incident fields can be written as

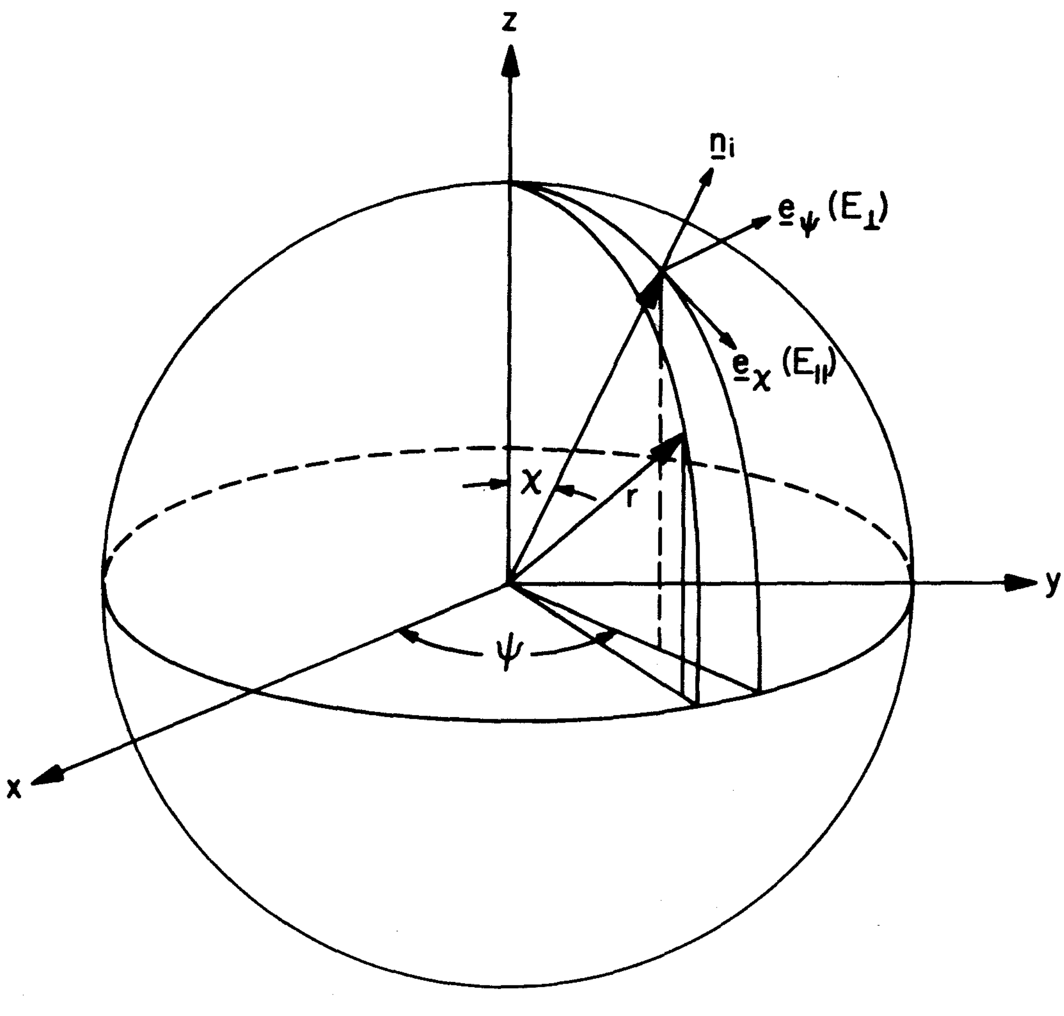


Fig. 3.1 A spherical system of coordinates for a plane electromagnetic wave.

$$\underline{E}_i(\underline{r}) = (\hat{E}_\perp \underline{e}_\psi + \hat{E}_\parallel \underline{e}_\chi)\, e^{ik\underline{n}_i \cdot \underline{r}}$$

$$\underline{H}_i(\underline{r}) = \eta(\hat{E}_\parallel \underline{e}_\psi - \hat{E}_\perp \underline{e}_\chi)\, e^{ik\underline{n}_i \cdot \underline{r}} \tag{3.2}$$

where $\underline{n}_i$, $\underline{e}_\chi$ and $\underline{e}_\psi$ are unit vectors in the radial, polar and azimuthal direction (Fig. 3.1). Clearly, the azimuthal component of the incident electric field is perpendicular to the plane of incidence, hence it is denoted as $\hat{E}_\perp$, while the polar component lies in this plane and is denoted as $\hat{E}_\parallel$. Often the perpendicular polarized component is referred to as transverse electric (TE), while the parallel polarized component is called transverse magnetic (TM).

The phase of the incident plane wave can be written as

$$\underline{n}_i \cdot \underline{r} = ux + vy + wz \tag{3.3}$$

where the direction cosines u, v, w of the vector $\underline{n}_i$ are expressed in terms of the polar and azimuthal angles χ and ψ as

$$u = \sin\chi \cos\psi$$

$$v = \sin\chi \sin\psi \tag{3.4}$$

$$w = \cos\chi$$

The direction cosines obviously satisfy the relation $u^2 + v^2 + w^2 = 1$.

Considering the geometry of the problem it will be more convenient to express the fields (3.2) in a circular cylindrical coordinate system with z as the longitudinal axis. Accordingly we have

$$\underline{E}_i(\underline{r}) = (\hat{E}_\perp \underline{e}_\psi + \hat{E}_\parallel \cos\chi\, \underline{e}_p - \hat{E}_\parallel \sin\chi\, \underline{e}_z)\, e^{ik(ux+vy+wz)}$$

$$\underline{H}_i(r) = \eta(\hat{E}_\parallel \underline{e}_\psi - \hat{E}_\perp \cos\chi\, \underline{e}_p + \hat{E}_\perp \sin\chi\, \underline{e}_z)\, e^{ik(ux+vy+wz)} \tag{3.5}$$

where $\underline{e}_p$ is a unit vector in the radial direction on the plane transverse to the z-axis, and we will let

$$p = \sin\chi = \sqrt{u^2 + v^2} \tag{3.6}$$

The unit vector in the z direction is now denoted as $\underline{e}_z$.

Part of this incident wave will be reflected back from the boundary at $z = 0$, while part will be transmitted into the region $z > 0$. The fields of the reflected wave have also the form of (3.1), but now the subscript r will be used to denote them as well as the unit vector in the propagation direction of the reflected wave. It follows from the laws of reflection at a plane boundary that this unit vector can be written as

$$\begin{aligned}\underline{n}_r &= \sin\chi\, \underline{e}_p - \cos\chi\, \underline{e}_z \\ &= u\, \underline{e}_x + v\, \underline{e}_y - w\, \underline{e}_z\end{aligned} \tag{3.7}$$

and hence the phase of the reflected wave is

$$\underline{n}_r \cdot \underline{r} = ux + vy - wz \tag{3.8}$$

Remembering that the perpendicular and parallel polarized components of the wave have different reflection coefficients, and denoting them by $\Gamma_\perp$ and $\Gamma_\parallel$ respectively, the reflected fields $\underline{E}_r(\underline{r})$ and $\underline{H}_r(\underline{r})$ can be written from (3.1) and (3.7) as

$$\begin{aligned} \underline{E}_r(\underline{r}) &= (\hat{E}_\perp \Gamma_\perp \underline{e}_\psi + \hat{E}_\parallel \Gamma_\parallel \cos\chi\, \underline{e}_p + \hat{E}_\parallel \Gamma_\parallel \sin\chi\, \underline{e}_z) e^{ik(ux+vy-wz)} \\ \underline{H}_r(\underline{r}) &= \eta(-\hat{E}_\parallel \Gamma_\parallel \underline{e}_\psi + \hat{E}_\perp \Gamma_\perp \cos\chi\, \underline{e}_p + \hat{E}_\perp \Gamma_\perp \sin\chi\, \underline{e}_z) e^{ik(ux+vy-wz)} \end{aligned} \tag{3.9}$$

The sum of (3.5) and (3.9) represents the total fields in the region $z < 0$, arising from a single plane wave, propagating in a given direction, which is incident on and reflected from the boundary at $z = 0$. The propagation direction of the incident wave changes as one varies the direction cosines u and v (or equivalently, the angles χ and ψ). Associating with the waves propagating in the cone $u + du$, $v + dv$ two complex amplitudes, one for perpendicular polarization $\hat{E}_\perp(u,v)k^2 dudv$, and another for parallel polarization $\hat{E}_\parallel(u,v)k^2 dudv$, and integrating (3.5) and (3.9) over all possible directions of propagation, we can represent the most general fields in the semi-infinite region $z < 0$, bounded by the surface at $z = 0$ (21). Thus,

$$\begin{aligned} \underline{E}(\underline{r}) = \iint_{-\infty}^{\infty} \Big\{ & \hat{E}_\perp (e^{ikwz} + \Gamma_\perp e^{-ikwz}) \underline{e}_\psi + \hat{E}_\parallel w\, (e^{ikwz} + \Gamma_\parallel e^{-ikwz}) \underline{e}_p \\ & - \hat{E}_\parallel \sqrt{1 - w^2}\, (e^{ikwz} - \Gamma_\parallel e^{-ikwz}) \underline{e}_z \Big\} e^{ik(ux + vy)} k^2 dudv \end{aligned} \tag{3.10a}$$

$$\underline{H}(\underline{r}) = \eta \iint_{-\infty}^{\infty} \Big\{ \hat{E}_{\|} (e^{ikwz} - \Gamma_{\|} e^{-ikwz}) \underline{e}_{\psi} - \hat{E}_{\perp} w (e^{ikwz} - \Gamma_{\perp} e^{-ikwz}) \underline{e}_{p}$$

$$+ \hat{E}_{\perp} \sqrt{1 - w^2} (e^{ikwz} + \Gamma_{\perp} e^{-ikwz}) \underline{e}_{z} \Big\} e^{ik(ux + vy)} k^2 dudv \tag{3.10b}$$

where $$w = \sqrt{1 - u^2 - v^2} \tag{3.11}$$

and $\hat{E}_{\perp}$, $\hat{E}_{\|}$, $\Gamma_{\perp}$, $\Gamma_{\|}$, $\underline{e}_{\psi}$ and $\underline{e}_{p}$ are all functions of u and v. The fact that the limits of integration are from $-\infty$ to $+\infty$ implies that complex angles for the propagation direction are also included. This is necessary in order to represent arbitrary fields in the region $z < 0$.

The amplitude functions $\hat{E}_{\perp}(u,v)$ and $\hat{E}_{\|}(u,v)$ can now be expressed in terms of the prescribed electric or magnetic field on the plane $z = 0$. If the electric field distribution is specified on the plane $z = 0$, then, the continuity of the tangential electric field requires that this must equal the transverse electric field at $z = 0$ obtained from (3.10a). Thus

$$\underline{E}_{o}(x,y) = \iint_{-\infty}^{\infty} \Big\{ \hat{E}_{\perp}(u,v)(1 + \Gamma_{\perp}(u,v)) \underline{e}_{\psi} + \hat{E}_{\|}(u,v) w (1 + \Gamma_{\|}(u,v)) \underline{e}_{p} \Big\}$$

$$e^{ik(ux + vy)} k^2 dudv \tag{3.12}$$

The expression within the brackets under the integral sign in Eq. (3.12) can be recognized as the two-dimensional Fourier transform of the specified electric field distribution $\underline{E}_o(x,y)$. Then

$$\hat{\underline{E}}_o(u,v) = \left[\frac{1}{2\pi}\right]^2 \iint_{-\infty}^{\infty} \underline{E}_o(x,y)\; e^{-ik(ux + vy)} dxdy$$
$$= \hat{E}_\perp(u,v)(1 + \Gamma_\perp(u,v))\underline{e}_\psi + \hat{E}_\parallel(u,v)w(1+ \Gamma_\parallel(u,v))\underline{e}_p \qquad (3.13)$$

It immediately follows from (3.13) that

$$\hat{E}_\perp(u,v) = \frac{\hat{\underline{E}}_o(u,v)\cdot\underline{e}_\psi}{1 + \Gamma_\perp(u,v)} = \frac{\hat{E}_{o\psi}(u,v)}{1 + \Gamma_\perp(u,v)} \qquad (3.14a)$$

and $$\hat{E}_\parallel(u,v) = \frac{\hat{\underline{E}}_o(u,v)\cdot\underline{e}_p}{w(1 + \Gamma_\parallel(u,v))} = \frac{\hat{E}_{op}(u,v)}{w(1 + \Gamma_\parallel(u,v)} \qquad (3.14b)$$

In terms of the specified electric field $\underline{E}_o(x,y)$ on the plane $z = 0$, the fields in the region $z < 0$ are now given by Eqs. (3.10), (3.13) and (3.14). Of course, the reflection coefficients $\Gamma_\perp(u,v)$ and $\Gamma_\parallel(u,v)$ for plane waves traveling in the (u,v) direction, and incident on the boundary at $z = 0$, must be known. These plane wave reflection coefficients depend also on the constitutive parameters of the medium in the region $z > 0$, which does not have to be homogeneous in the z direction.

Finally, the magnetic field on the plane $z = 0$, $\underline{H}_o(x,y)$, and its Fourier transform, $\hat{\underline{H}}_o(u,v)$ are given by,

$$\underline{H}_o(x,y) = \eta \iint_{-\infty}^{\infty} \left\{ \hat{E}_\parallel(u,v)(1 - \Gamma_\parallel(u,v))\underline{e}_\psi - \hat{E}_\perp(u,v)w(1 - \Gamma_\perp(u,v))\underline{e}_p \right\} e^{ik(ux + vy)} k^2 dudv \qquad (3.15)$$

$$\hat{\underline{H}}_o(u,v) = \left[\frac{1}{2\pi}\right]^2 \iint_{-\infty}^{\infty} \underline{H}_o(x,y)\ e^{-ik(ux + vy)}\ dxdy$$

$$= \eta \left(\hat{E}_{\|}(u,v)(1 - \Gamma_{\|}(u,v))\underline{e}_{\psi} - \hat{E}_{\perp}(u,v)w(1 - \Gamma_{\perp}(u,v))\underline{e}_p \right) \tag{3.16}$$

It can readily be seen that the magnetic field on the plane $z = 0$ is a function of the specified electric field on that plane.

B. Aperture Admittance

The results of the preceding paragraphs will now be used to obtain a general expression for the admittance of flush-mounted aperture antennas. For dominant mode aperture electric field approximation the admittance can be rewritten from (2.17) as

$$Y = \frac{1}{V_o^{\,2}} \iint_A \underline{E}_o^* (x,y) \times \underline{H}_o(x,y) \cdot \underline{e}_z\ dxdy \tag{3.17}$$

The asterisk denotes complex conjugates. Obviously this expression is the same as (2.17) since a real aperture field distribution is always assumed. Applying Parseval's theorem to (3.17) the admittance becomes

$$Y = \left[\frac{2\pi}{V_o}\right]^2 \iint_{-\infty}^{\infty} \hat{\underline{E}}_o^*(u,v) \times \hat{\underline{H}}_o(u,v) \cdot \underline{e}_z\ k^2 dudv \tag{3.18}$$

Using (3.13) and (3.16), and then (3.14)

$$\hat{\underline{E}}_o^* \times \hat{\underline{H}}_o \cdot \underline{e}_z = \eta \left\{ |\hat{E}_\perp|^2 \, w \, (1 + \Gamma_\perp^*)(1 - \Gamma_\perp) + |\hat{E}_\parallel|^2 \, w^* \, (1 + \Gamma_\parallel^*)(1 - \Gamma_\parallel) \right\}$$

$$= \eta \left\{ |\hat{E}_{o\psi}|^2 \, w \, \frac{1 - \Gamma_\perp}{1 + \Gamma_\perp} + |\hat{E}_{op}|^2 \, \frac{1}{w} \, \frac{1 - \Gamma_\parallel}{1 + \Gamma_\parallel} \right\} \tag{3.19}$$

The ratio of the tangential components of the magnetic and electric fields of the plane waves at the boundary surface $z = 0$ may be called the input admittance of the region $z > 0$, and is given, for each direction of polarization, in terms of the reflection coefficients by (22)

$$Y_{in_\perp}(u,v) = (\eta w) \, \frac{1 - \Gamma_\perp(u,v)}{1 + \Gamma_\perp(u,v)} \tag{3.20a}$$

and

$$Y_{in_\parallel}(u,v) = (\eta / w) \, \frac{1 - \Gamma_\parallel(u,v)}{1 + \Gamma_\parallel(u,v)} \tag{3.20b}$$

Relations (3.18), (3.19) and (3.20) can now be used to obtain a general expression for the admittance of a flush-mounted, waveguide-fed aperture antenna with the dominant mode aperture electric field approximation. The result is

$$Y = \frac{(2\pi k)^2}{V_o^{\,2}} \iint_{-\infty}^{\infty} \left\{ |\hat{E}_{o\psi}(u,v)|^2 \, Y_{in_\perp}(u,v) + |\hat{E}_{op}(u,v)|^2 Y_{in_\parallel}(u,v) \quad dudv \right\} \tag{3.21a}$$

or equivalently,

$$Y = \frac{(2\pi k)^2}{V_o^2} \int_o^{2\pi}\int_o^{\infty} \left\{ |\hat{E}_{o\psi}(p,\psi)|^2 \, Y_{in_\perp}(p) + |\hat{E}_{op}(p,\psi)|^2 \, Y_{in_\parallel}(p) \; pdpd\psi \right\} \tag{3.21b}$$

where it has been made clear that the input admittances depend only on p, that is to say, on the plane wave incidence angles only.

Sometimes it is more convenient to express the polar components of the transform of the aperture field, as they appear in the preceding expressions, in terms of their rectangle components, making use of the relations

$$\hat{E}_{o\psi}(u,v) = - \frac{v}{\sqrt{u^2 + v^2}} \hat{E}_{ox}(u,v) + \frac{u}{\sqrt{u^2 + v^2}} \hat{E}_{oy}(u,v) \tag{3.22a}$$

$$\hat{E}_{op}(u,v) = + \frac{u}{\sqrt{u^2 + v^2}} \hat{E}_{ox}(u,v) + \frac{v}{\sqrt{u^2 + v^2}} \hat{E}_{oy}(u,v) \tag{3.22b}$$

The expressions (3.21) are valid for antennas with any aperture shape, radiating into homogeneous as well as inhomogeneous media whose constitutive parameters vary only in the z-direction. In the light of expression (3.21) the aperture admittance can heuristically be interpreted as the "sum", with proper amplitudes, of all the input admittances of the plane waves that make up the assumed aperture field. In essence, the problem is now reduced to calculating the plane wave input admittances (3.20), which, in the case of an antenna radiating into a homogeneous medium, can readily be obtained. In case the permittivity of the medium is a function of the coordinate z, then the plane wave input admittances will be obtained, in the next chapter, by a special method.

If the region $z > 0$ consists of a dielectric slab of relative permittivity ε_1 and thickness d, for $0 < z < d$, and of a semi-infinite region of relative permittivity ε_2 for $z > d$, then the input admittance for either direction of polarization is given by the well-known expression from transmission-line theory (22),

$$Y_{in} = Y_1 \frac{Y_2 - i\, Y_1 \tan kdw_1}{Y_1 - i\, Y_2 \tan kdw_1} \tag{3.23}$$

where Y_1 and Y_2, the characteristic admittances of media of relative permittivity ε_1 and ε_2 are given, for waves of each polarization direction, by (22)

$$Y_{1\perp} = \eta w_1 \ , \qquad Y_{1\parallel} = \eta(\varepsilon_1/w_1) \tag{3.24}$$

$$Y_{2\perp} = \eta w_2 \ , \qquad Y_{2\parallel} = \eta(\varepsilon_2/w_2) \tag{3.25}$$

and

$$w_1 = \sqrt{\varepsilon_1 - u^2 - v^2} \ , \qquad w_2 = \sqrt{\varepsilon_2 - u^2 - v^2} \tag{3.26}$$

Making use of equations (3.24) and (3.25), equation (3.23) can be re-written for each direction of polarization as

$$Y_{in\,\perp} = \eta w_1 \frac{w_2 - iw_1 \tan kdw_1}{w_1 - iw_2 \tan kdw_1} \tag{3.27a}$$

and
$$Y_{in\parallel} = \eta \frac{\varepsilon_1}{w_1} \frac{\varepsilon_2 w_1 - i\varepsilon_1 w_2 \tan kdw_1}{\varepsilon_1 w_2 - i\varepsilon_2 w_1 \tan kdw_1} \tag{3.27b}$$

Usually, the region $z > d$ is free space, in which case we take $\varepsilon_2 = 1$ and $w_2 = w$ in equations (3.27).

If the region $z > 0$ consists only of a homogeneous dielectric medium of relative permittivity ε_2, then the input admittance is simply the characteristic admittance of the medium and is given by (3.25).

Since only outgoing waves may exist in the semi-infinite region outside, care must be taken in choosing the proper branch of w_2 in (3.27). In the dielectric slab, where standing waves are supported, the choice of a branch of w_1 is arbitrary. For the same reason, the proper branch of w_2 must be chosen in (3.25) in case the region $z > 0$ is a semi-infinite dielectric medium. The branches of w_1 and w_2 will always be selected such that,

$$\text{Re } w_1, \ \text{Im } w_1, \quad \text{Re } w_2, \quad \text{Im } w_2 > 0 \tag{3.28}$$

Expressions (3.21) will now be applied to find the aperture admittance of antennas fed by waveguides of various practical cross-sections and radiating either into semi-infinite homogeneous media or through homogeneous dielectric slabs into free space.

First, let us consider a slot of infinite extent in the y-direction, and of width a, fed by a parallel-plate waveguide (Fig. 3.2a). In this case the field quantities are independent of the y coordinate and the appropriate form of expression (3.21a) becomes

$$Y = \frac{2\pi k}{V_o^{\,2}} \int_{-\infty}^{\infty} \left\{ |\hat{E}_{oy}(u)|^2 \, Y_{in_\perp}(u) + |\hat{E}_{ox}(u)|^2 \, Y_{in_\parallel}(u) \right\} du \tag{3.29}$$

The dominant mode (TEM) electric field of a parallel-plate waveguide has the form,

$$\begin{aligned} \underline{E}_o(x) &= V_o \, \underline{e}_o(x) = \frac{V_o}{\sqrt{a}} \, \underline{e}_x \, , && \text{for} \quad |x| \leq a/2 \\ &= 0 && \text{for} \quad |x| > a/2 \end{aligned} \tag{3.30}$$

and its Fourier transform is,

$$\begin{aligned} \hat{\underline{E}}_o(u) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \underline{E}_o(x) e^{-ikux} \, dx \\ &= \frac{V_o \sqrt{\alpha}}{2\pi \sqrt{k}} \, \frac{\sin(\alpha u/2)}{(\alpha u/2)} \, \underline{e}_x \end{aligned} \tag{3.31}$$

where $\qquad \alpha = ka$

Substitution of (3.31) into (3.29) gives the aperture admittance of the infinite slot.

$$Y = \frac{\alpha}{\pi}\int_{o}^{\infty} \frac{\sin^2(\alpha u/2)}{(\alpha u/2)^2}\, Y_{in_{\parallel}}(u)\, du \tag{3.32}$$

Next, we consider a rectangular waveguide with smaller cross-sectional dimension a and larger dimension b (Fig. 3.2b). The dominant mode (TE_{10}) electric field can then be written as

$$\begin{aligned} \underline{E}_o(x,y) = V_o\, \underline{e}_o(x,y) &= V_o\sqrt{\frac{2}{ab}}\cos\frac{\pi y}{b}\,\underline{e}_x, \quad \text{for } |x| \le a/2,\ |y| \le b/2 \\ &= 0 \qquad\qquad \text{otherwise} \end{aligned} \tag{3.33}$$

The Fourier transform of this aperture field becomes

$$\begin{aligned} \hat{\underline{E}}_o(u,v) &= \left[\frac{1}{2\pi}\right]^2 \iint_{-\infty}^{\infty} \underline{E}_o(x,y)\, e^{-ik(ux+vy)}\, dx\, dy \\ &= \frac{V_o}{2\pi k}\sqrt{\frac{\alpha\beta}{8}}\, \frac{\sin(\alpha u/2)}{(\alpha u/2)}\, \frac{\cos(\beta v/2)}{(\pi/2)^2 - (\beta v/2)^2}\, \underline{e}_x \end{aligned} \tag{3.34}$$

where $\alpha = ka\ , \qquad \beta = kb$ (3.35)

Substitution of (3.34) into expressions (3.21) give the admittance of rectangular waveguide-fed apertures

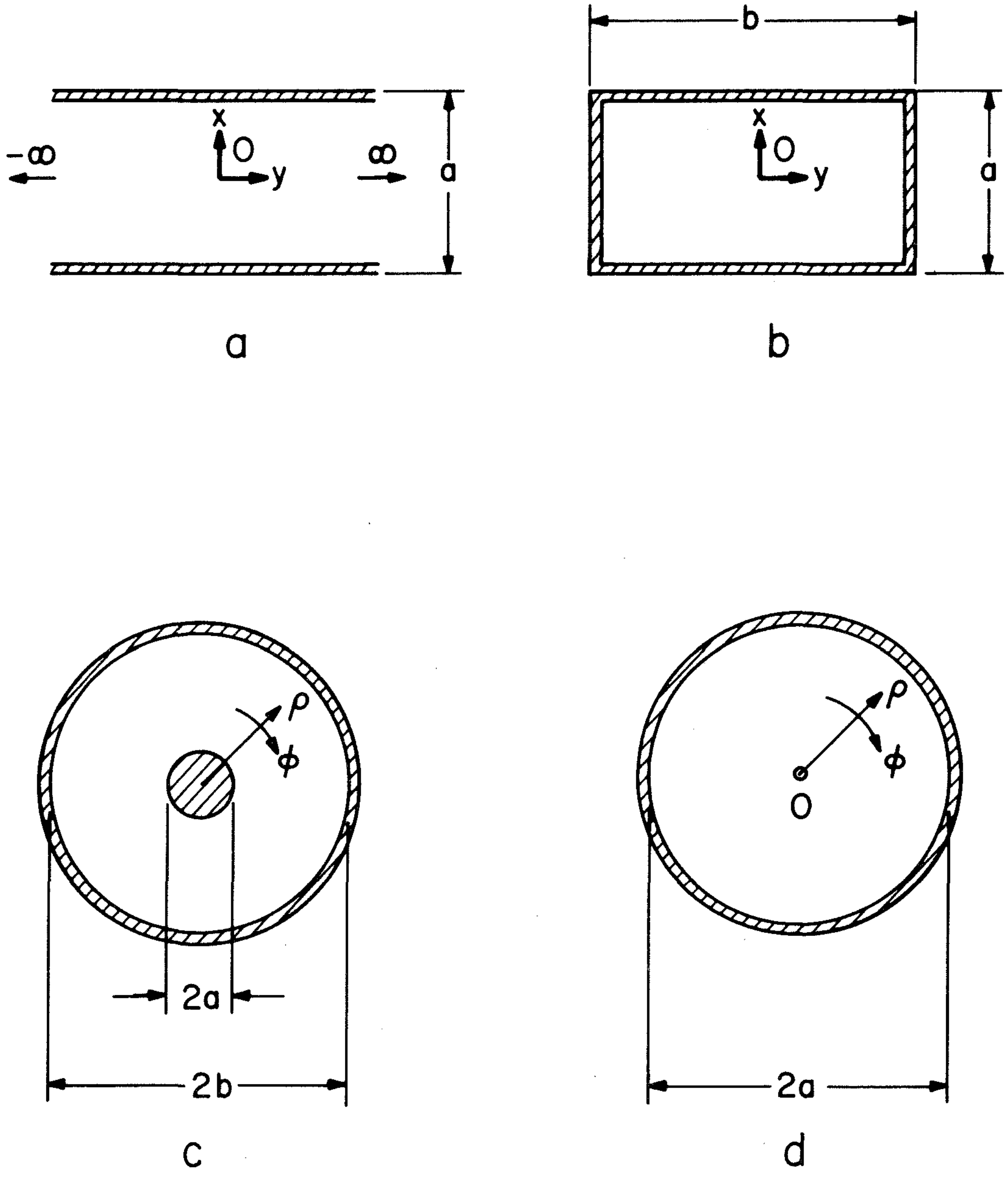


Fig. 3.2 Common waveguide cross-sections

(a) Parallel plate, (b) Rectangular,

(c) Coaxial, (d) Circular

$$Y = \frac{\alpha\beta}{8}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{\sin(\alpha u/2)}{(\alpha u/2)}\ \frac{\cos(\beta v/2)}{(\pi/2)^2 - (\beta v/2)^2}\right]^2 \left\{Y_{in_\perp}(u,v)\frac{v^2}{u^2+v^2} + Y_{in_\parallel}(u,v)\frac{u^2}{u^2+v^2}\right\}dudv \tag{3.36a}$$

or equivalently,

$$Y = \frac{\alpha\beta}{8}\int_{0}^{2\pi}\int_{0}^{\infty}\left[\frac{\sin\left(\frac{\alpha p}{2}\cos\psi\right)}{\left(\frac{\alpha p}{2}\cos\psi\right)}\ \frac{\cos\left(\frac{\beta p}{2}\sin\psi\right)}{\left(\frac{\pi}{2}\right)^2 - \left(\frac{\beta p}{2}\sin\psi\right)^2}\right]^2 \left\{Y_{in_\perp}(p)\cos^2\psi + Y_{in_\parallel}(p)\sin^2\psi\right\}pdpd\psi \tag{3.36b}$$

For a coaxial waveguide of inner conductor radius a, and outer conductor radius b (Fig. 3.2c), the dominant mode (TEM) electric field at the aperture is given by

$$\underline{E}_o(\rho,\phi) = V_o\,\underline{e}_o(\rho) = \frac{V_o}{\sqrt{2\pi\,\ell n(b/a)}}\,\frac{1}{\rho}\,\underline{e}_\rho, \quad \text{for} \quad a < \rho < b$$

$$= 0 \qquad\qquad \text{otherwise} \tag{3.37}$$

The transform of this field is readily found to be

$$\hat{\underline{E}}_o(\rho,\psi) = \left[\frac{1}{2\pi}\right]^2 \int_0^{2\pi}\int_0^{\infty} \underline{E}_o(\rho,\phi)\, e^{-ikp\rho\cos(\phi-\psi)}\,\rho\, d\rho\, d\phi$$

$$= \frac{i}{2\pi}\,\frac{V_o}{\sqrt{2\pi\,\ell n(\beta/\alpha)}}\,\frac{J_o(\beta p) - J_o(\alpha p)}{kp}\,\underline{e}_p \tag{3.38}$$

Hence, the admittance of an annular aperture fed by a coaxial waveguide becomes from (3.21b) and (3.38),

$$Y = \frac{1}{\ell n\ (\beta/\alpha)} \int_0^\infty \frac{\left[J_o(\beta p) - J_o(\alpha p)\right]^2}{p} Y_{in_{\|}}(p)\ dp \tag{3.39}$$

Finally we consider a circular waveguide of radius a (Fig. 3.2d) whose dominant mode (TE_{11}) electric field is given by

$$\begin{aligned} \underline{E}_o(\rho,\phi) &= V_o\ \underline{e}_o(\rho,\phi) \\ &= V_o \frac{A_{11}}{\rho} \left\{ J_1\left(\frac{x'_{11}\rho}{a}\right) \sin\phi\ \underline{e}_\rho + \frac{x'_{11}\rho}{a} J'_1\left(\frac{x'_{11}\rho}{a}\right) \cos\phi\ \underline{e}_\phi \right\}, \quad \text{for } 0 < \rho < a \\ &= 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } \rho > a \end{aligned} \tag{3.40}$$

where $$A_{11} = \sqrt{\frac{2}{\pi}}\ \frac{1}{\sqrt{(x'_{11})^2 - 1}} \frac{1}{J_1(x'_{11})}$$

and x'_{11} is the first root of $J'_1(x) = 0$

The transform of this field must then be calculated and the final result turns out to be

$$\hat{\underline{E}}_o(\rho,\psi) = \left[\frac{1}{2\pi}\right]^2 \int_0^{2\pi}\int_0^\infty \underline{E}_o(\rho,\phi) e^{-ikp\rho\cos(\phi-\psi)} \rho\, d\rho\ d\phi$$

$$= \frac{V_o}{2\pi} A_{11} J_1(x'_{11}) \left[\frac{J_1(\alpha p)}{kp} \sin\psi \, \underline{e}_p + \frac{x'_{11}(x'_{11}/\alpha) J'_1(\alpha p)}{k\left[(x'_{11}/\alpha)^2 - p^2\right]} \cos\psi \, \underline{e}_\psi \right] \tag{3.41}$$

The aperture admittance of a circular waveguide is found by substituting (3.41) into (3.21b), and is given by

$$Y = \frac{2}{(x'_{11})^2 - 1} \int_0^\infty \left\{ \left[\frac{J_1(\alpha p)}{p} \right]^2 Y_{in_\parallel}(p) + \left[\frac{x'_{11}(x'_{11}/\alpha) J'_1(\alpha p)}{(x'_{11}/\alpha)^2 - p^2} \right]^2 Y_{in_\perp}(p) \right\} p\,dp \tag{3.42}$$

Y_{in} in Eq. (3.32), (3.36), (3.39) and (3.42) are given either by Eqs. (3.25) or (3.27) depending on whether the antennas radiate directly or through homogeneous dielectric slabs into semi-infinite homogeneous media.

Most of the results that have just been obtained had been the subject of various investigations discussed in the Introduction. It has been shown here that all of these results follow quite simply from expressions (3.21).

The integrals in (3.32), (3.36), (3.39) and (3.42) have the form

$$\int_0^\infty f_\parallel(p)\, Y_{in_\parallel}(p)\, dp \quad \text{and} \quad \int_0^\infty f_\perp(p)\, Y_{in_\perp}(p)\, dp \tag{3.43}$$

where the integrations are carried along the real axis of the complex p-plane. The functions $f_\parallel(p)$ and $f_\perp(p)$ have no singularities, but in the case of lossless dielectric slabs singularities appear, for real

p, in $Y_{in_\parallel}(p)$ and $Y_{in_\perp}(p)$. Hence care must be taken in calculating the admittance of an aperture radiating through a lossless dielectric slab into a homogeneous half-space, which, in the present discussion, we take as free space. Accordingly, we let $\varepsilon_1 = \varepsilon$ in the dielectric slab and write $\varepsilon_2 = 1$, $w_2 = w$ in the region of free space. The singularities are poles of order one due to the zeros of the denominators of Y_{in}, in addition to a branch cut due to $w = \sqrt{1 - p^2}$. (No branch cut is needed for $w_1 = \sqrt{\varepsilon - p^2}$ since the Y_{in} are even functions of this variable.) The location of the poles is determined by the zeros of the denominators of Y_{in}, which are given, from equations (3.27) by,

$$D_\parallel(p) = \varepsilon w - iw_1 \tan (kd\, w_1) \tag{3.44a}$$

$$D_\perp(p) = w_1 - iw \tan (kd\, w_1) \tag{3.44b}$$

First, let us consider a dielectric slab with $\varepsilon > 1$. The poles occur, for real p, only in the range $1 < p^2 < \varepsilon$, and in this range (3.44) may be written as

$$-iD_\parallel(p) = \varepsilon\sqrt{p^2 - 1} - \sqrt{\varepsilon - p^2} \tan (\delta\sqrt{\varepsilon - p^2}) \tag{3.45a}$$

$$D_\perp(p) = \sqrt{\varepsilon - p^2} + \sqrt{p^2 - 1} \tan (\delta\sqrt{\varepsilon - p^2}) \tag{3.45b}$$

where $$\delta = kd \tag{3.46}$$

The roots of (3.45) determine the eigenvalues for surface wave propagation along plane dielectric slabs (23). Since the dielectric slab covering the waveguide aperture is located on a ground plane, the only surface wave modes that can exist are the even TM modes, for which $D_{\parallel}(p) = 0$, and the odd TE modes, for which $D_{\perp}(p) = 0$. In the calculation of the integrals (3.43) it will be necessary to find the residue due to each pole. The residue due to a pole at $p = p_n$ determined by $D_{\parallel}(p_n) = 0$ is

$$\mathrm{Res}(p_n) = -i \frac{f_{\parallel}(p_n)\ \varepsilon}{\delta p_n \left[1 + \left(\dfrac{\varepsilon - 1}{p_n^2 - 1} \right) \dfrac{\sin\left(2\delta\sqrt{\varepsilon - p_n^2}\right)}{\left(2\delta\sqrt{\varepsilon - p_n^2}\right)} \right]} \tag{3.47a}$$

and the residue due to a pole at $p = p_n$ determined by $D_{\perp}(p_n) = 0$ is

$$\mathrm{Res}(p_n) = -i \frac{f_{\perp}(p_n)\ (\varepsilon - p_n^2)}{\delta p_n \left[1 - \left(\dfrac{\varepsilon - 1}{p_n^2 - 1} \right) \dfrac{\sin\left(2\delta\sqrt{\varepsilon - p_n^2}\right)}{\left(2\delta\sqrt{\varepsilon - p_n^2}\right)} \right]} \tag{3.47b}$$

The onset of each surface wave mode occurs at a thickness given by

$$\delta = \frac{n\pi}{2\sqrt{\varepsilon - 1}} \qquad n = 0, 1, 2, 3, \ldots \tag{3.48}$$

where the even integers refer to the even modes, and the odd integers refer to the odd modes.

For a plasma slab the relative permittivity ε varies between unity and large negative values. In this case poles may occur, for real p, only in $Y_{in_{\parallel}}(p)$ and only for $\varepsilon < 0$ and $p > 1$. In this range (3.44a) may be written as

$$iD_{\parallel}(p) = |\varepsilon|\sqrt{p^2 - 1} - \sqrt{|\varepsilon| + p^2}\ \tanh\ (\delta\sqrt{|\varepsilon| + p^2}) \qquad (3.49)$$

For $\varepsilon < -1$ and for any plasma slab thickness $D_{\parallel}(p) = 0$ has at least one real root corresponding to a surface wave. In addition, in the range $0 > \varepsilon > -1.0363$ sufficiently thin plasma slabs support two more surface waves (24), one of which is a backward wave, i.e. a wave whose phase and group velocities along the interface are in opposite directions. This fact has been ignored in previous calculations of the admittance of apertures covered by homogeneous thin plasma slabs. The residue due to a pole at $p = p_i$ is found to be

$$\mathrm{Res}(p_i) = i\,\frac{f_{\parallel}(p_i)\,|\varepsilon|}{\delta p_i\left[1 - \left(\dfrac{|\varepsilon| + 1}{p_i^2 - 1}\right)\dfrac{\sinh(2\delta\sqrt{|\varepsilon| + p_i^2})}{(2\delta\sqrt{|\varepsilon| + p_i^2})}\right]} \qquad (3.50)$$

Whenever poles of the integrand lie on the real axis the path of integration in (3.43) must be deformed around them by semi-circular excursions in the complex p-plane. The choice of the path about each pole can be determined by considering an ideally lossless medium as the limiting case of a lossy medium with losses reduced to infinitesimal amounts. For a dielectric slab with $\varepsilon > 1$ the path of integration is shown in Fig. 3.3a, and for a plasma slab with $\varepsilon < 0$ in Fig. 3.3b. Except for a backward wave pole, marked b, the integration path passes below the poles.

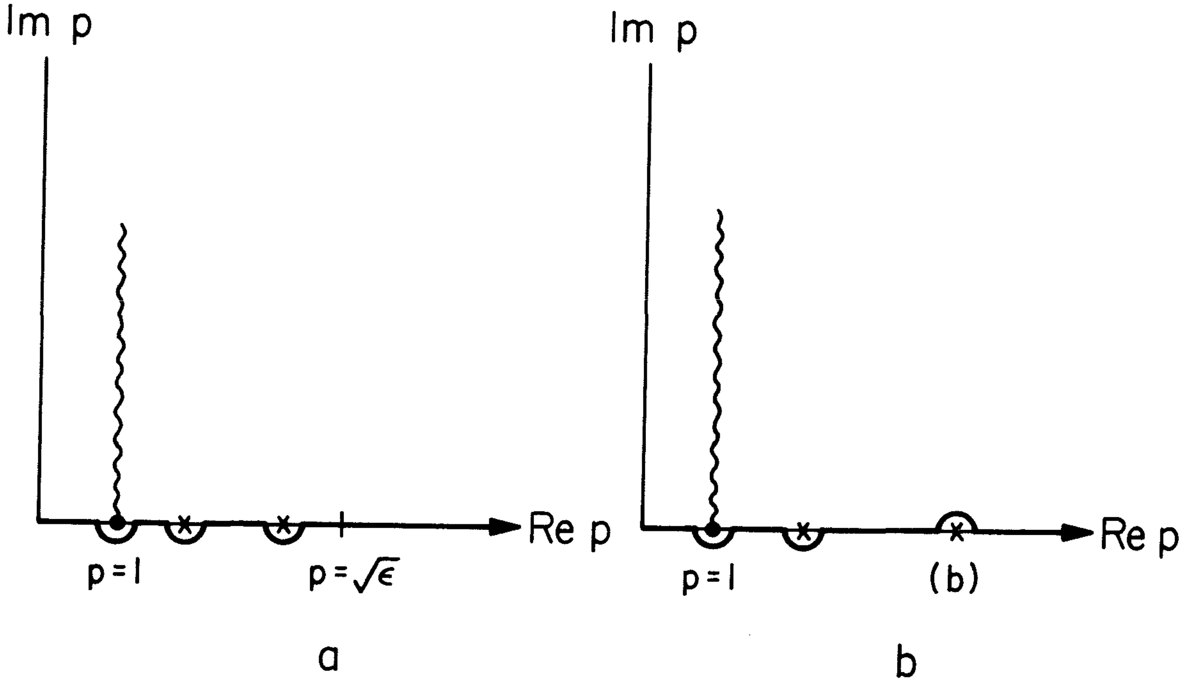


Fig. 3.3 Integration path for a lossless dielectric

(a) $\varepsilon > 1$, (b) $\varepsilon < 0$

The contribution of the surface wave poles to the admittance integrals is πi times the residue at the poles. ($-\pi i$ times the residue at the backward wave pole.) This contribution, which represents the amount of power confined within the slab, is real and may be called the surface wave conductance. Between poles the admittance integrals are evaluated numerically, and the result is added to the pole contributions.

In the case of lossy dielectric slabs no poles are located on the real p-axis and the numerical integrations are straightforward.

C. Radiation Pattern

In the first section of this chapter, the fields in the region $z < 0$ were represented as a superposition of plane waves incident on and reflected from the plane $z = 0$, in such a way that, on that plane the prescribed electric field was obtained. In a similar way, the fields transmitted through a dielectric slab located at $0 < z < d$ into a semi-infinite region of permittivity ε_2 at $z > d$, can be written as,

$$\underline{E}_t(\underline{r}) = \iint_{-\infty}^{\infty} \left\{ \hat{E}_\perp T_\perp \underline{e}_\psi + \hat{E}_\parallel T_\parallel \frac{w_2}{\sqrt{\varepsilon_2}} \underline{e}_p - \hat{E}_\parallel T_\parallel \frac{p}{\sqrt{\varepsilon_2}} \underline{e}_z \right\} e^{ikw_2(z-d)} e^{ik(ux+vy)} k^2 du dv \tag{3.51}$$

$$\underline{H}_t(\underline{r}) = \eta \sqrt{\varepsilon_2} \iint_{-\infty}^{\infty} \left\{ \hat{E}_\parallel T_\parallel \underline{e}_\psi - \hat{E}_\perp T_\perp \frac{w_2}{\sqrt{\varepsilon_2}} \underline{e}_p + \hat{E}_\perp T_\perp \frac{p}{\sqrt{\varepsilon_2}} \underline{e}_z \right\} e^{ikw_2(z-d)} e^{ik(ux+vy)} k^2 du dv$$

where the relations between $\hat{E}_\perp$ and $\hat{E}_\parallel$ and the transform of the prescribed electric field on the plane $z = 0$ are given by (3.14). $T_\perp$ and $T_\parallel$ are the slab transmission coefficients with respect to the electric field for perpendicular and parallel polarizations, respectively, and all these quantities as well as $\underline{e}_\psi$ and $\underline{e}_p$ are functions of u and v. Using relations (3.14) in equation (3.51) the transmitted fields can be written directly in terms of the transform of the prescribed aperture field:

$$\underline{E}_t(\underline{r}) = \iint_{-\infty}^{\infty} \left\{ \hat{E}_{o\psi} \frac{T_\perp}{1+\Gamma_\perp} \underline{e}_\psi + \hat{E}_{op} \frac{T_\parallel}{w(1+\Gamma_\parallel)} \left[\frac{w_2}{\sqrt{\varepsilon_2}} \underline{e}_p - \frac{p}{\sqrt{\varepsilon_2}} \underline{e}_z \right] \right\} e^{ikw_2(z-d)} e^{ik(ux+vy)} k^2 dudv$$

(3.52)

$$\underline{H}_t(\underline{r}) = \eta\sqrt{\varepsilon_2} \iint_{-\infty}^{\infty} \left\{ \hat{E}_{op} \frac{T_\parallel}{w(1+\Gamma_\parallel)} \underline{e}_\psi - \hat{E}_{o\psi} \frac{T_\perp}{1+\Gamma_\perp} \left[\frac{w_2}{\sqrt{\varepsilon_2}} \underline{e}_p - \frac{p}{\sqrt{\varepsilon_2}} \underline{e}_z \right] \right\} e^{ikw_2(z-d)} e^{ik(ux+by)} k^2 dudv$$

We can identify $T_\perp/[1+\Gamma_\perp]$ and $T_\parallel/[w(1+\Gamma_\parallel)]$ as the ratio of the total electric field amplitude transmitted at $z = d$ to the transverse electric field at $z = 0$, for perpendicular and parallel polarized plane waves, respectively.

If the region $0 < z < d$ consists of a homogeneous dielectric slab of permittivity ε_1, and we have a semi-infinite region of permittivity ε_2 for $z > d$, then $T_\perp/[1+\Gamma_\perp]$ and $T_\parallel/[w(1+\Gamma_\parallel)]$ are given by (22)

$$\frac{T_\perp}{1+\Gamma_\perp} = \frac{w_1}{w_1 \cos \delta w_1 - i\, w_2 \sin \delta w_1} \tag{3.53a}$$

$$\frac{T_\parallel}{w(1+\Gamma_\parallel)} = \frac{\sqrt{\varepsilon_2}\ \varepsilon_1}{\varepsilon_1 w_2 \cos \delta w_1 - i\, \varepsilon_2 w_1 \sin \delta w_1} \tag{3.53b}$$

In case the whole region $z > 0$ consists of a homogeneous dielectric medium of relative permittivity ε_2, then letting $\varepsilon_1 = \varepsilon_2$ in (3.53) we obtain,

$$\frac{T_\perp}{1 + \Gamma_\perp} = e^{i\delta w_2} \tag{3.54a}$$

and

$$\frac{T_\parallel}{w(1 + \Gamma_\parallel)} = \frac{\sqrt{\varepsilon_2}}{w_2} e^{i\delta w_2} \tag{3.54b}$$

In the event the permittivity of the medium $z > 0$ varies as a function of the coordinate z, then $T_\perp/[1 + \Gamma_\perp]$ and $T_\parallel/[w(1 + \Gamma_\parallel)]$ will be obtained by a method described in the next chapter.

Since the semi-infinite region where the antenna radiates is usually free space, we will henceforth take $\varepsilon_2 = 1$, $w_2 = w$ in equations (3.53) and (3.54). Likewise the transmitted fields in the free space region can be written from (3.52) as

$$\underline{E}_t(\underline{r}) = \iint_{-\infty}^{\infty} \left\{ \hat{E}_{o\psi} \frac{T_\perp}{1+\Gamma_\perp} \underline{e}_\psi + \hat{E}_{op} \frac{T_\parallel}{w(1+\Gamma_\parallel)} \underline{e}_\chi \right\} e^{ikw(z-d)} e^{ik(ux+vy)} k^2 du dv$$

$$\underline{H}_t(\underline{r}) = \eta \iint_{-\infty}^{\infty} \left\{ \hat{E}_{op} \frac{T_\parallel}{w(1+\Gamma_\parallel)} \underline{e}_\psi - \hat{E}_{o\psi} \frac{T_\perp}{1+\Gamma_\perp} \underline{e}_\chi \right\} e^{ikw(z-d)} e^{ik(ux+vy)} k^2 du dv \tag{3.55}$$

To find the radiation fields we need to calculate (3.55) for large kr. A typical field component is of the form

$$g(\underline{r}) = \iint_{-\infty}^{\infty} \hat{g}(u,v)\, e^{ikrf(u,v)}\, k^2 du dv \tag{3.56}$$

where, using spherical coordinates for the space variables, $f(u,v)$ becomes

$$f(u,v) = \cos\theta\sqrt{1 - u^2 - v^2} + \sin\theta\,(\cos\phi\, u + \sin\phi\, v) \tag{3.57}$$

The double integral (3.56) can be readily evaluated for large kr by applying twice the method of stationary phase. The point where $f_u(u,v) = f_v(u,v) = 0$, so that the phase is "stationary", is given by,

$$\begin{aligned} u_o &= \sin\theta\cos\phi \\ v_o &= \sin\theta\sin\phi \end{aligned} \tag{3.58}$$

and the result of the integration of (3.55) to first order in $1/kr$ turns out to be (25,26)

$$g(\underline{r}) = -\,2i\pi\,\hat{g}(u_o, v_o)\,k^2\cos\theta\,\frac{e^{ikr}}{kr} \tag{3.59}$$

The stationary phase integration of (3.56) is easy to interpret. We see that out of all the plane waves propagating in directions determined by u and v, and making up the fields at the observation point in the region $z > d$, only the wave that travels in a radial direction toward that point contributes significantly to the radiation field.

In the case of the waveguide aperture radiating through a lossless dielectric slab, singularities may appear on the path of integration, in the integrand of (3.56). Then, an evaluation of the integral by the method of steepest descent (15,27) reveals that these singularities, which correspond to surface waves, contribute to the radiation field only for $\theta = \pi/2$, and that for $\theta \neq \pi/2$ the result of the stationary phase analysis is still valid.

The far-zone fields follow from (3.55) and (3.59). Henceforth $\mathcal{E}(\theta,\phi)$ will be written for $\mathcal{E}(u_o = \sin\theta\cos\phi,\ v_o = \sin\theta\sin\phi)$ to clearly denote the angular dependence of the far-zone fields. Thus we have

$$E_\phi(\underline{r}) = -\,2i\pi\,\hat{E}_{o\psi}(\theta,\phi)\,\frac{T_\perp(\theta)}{1+\Gamma_\perp(\theta)}\,e^{-i\delta\cos\theta}\,k\cos\theta\,\frac{e^{ikr}}{r}$$

$$E_\theta(\underline{r}) = -\,2i\pi\,\hat{E}_{op}(\theta,\phi)\,\frac{T_\parallel(\theta)}{w(\theta)(1+\Gamma_\parallel(\theta))}\,e^{-i\delta\cos\theta}\,k\cos\theta\,\frac{e^{ikr}}{r} \qquad (3.60)$$

$$H_\phi(\underline{r}) = \eta\,E_\theta(\underline{r})$$

$$H_\theta(\underline{r}) = -\eta\,E_\phi(\underline{r}) \qquad (3.61)$$

For easy reference, the expressions for $T_\perp/[1+\Gamma_\perp]$ and $T_\parallel/[w(1+\Gamma_\parallel)]$ can be written down as a function of θ. From (3.53) we have, letting $\varepsilon_1 = \varepsilon$,

$$\frac{T_\perp(\theta)}{1+\Gamma_\perp(\theta)} = \frac{\sqrt{\varepsilon-\sin^2\theta}}{\sqrt{\varepsilon-\sin^2\theta}\,\cos(\delta\sqrt{\varepsilon-\sin^2\theta}) - i\cos\theta\,\sin(\delta\sqrt{\varepsilon-\sin^2\theta})} \qquad (3.62a)$$

and

$$\frac{T_{\parallel}(\theta)}{w(\theta)(1+\Gamma_{\parallel}(\theta))} = \frac{\varepsilon}{\varepsilon\cos\theta\ \cos(\delta\sqrt{\varepsilon-\sin^2\theta})-i\sqrt{\varepsilon-\sin^2\theta}\ \sin(\delta\sqrt{\varepsilon-\sin^2\theta})} \qquad (3.62b)$$

The radiation pattern, which is proportional to the angular dependent part of the far-zone Poynting's vector, can be written as

$$F(\theta,\phi)=k^2\cos^2\theta\left\{\left|\hat{E}_{o\psi}(\theta,\phi)\right|^2\left|\frac{T_{\perp}(\theta)}{1+\Gamma_{\perp}(\theta)}\right|^2 + \left|\hat{E}_{op}(\theta,\phi)\right|^2\left|\frac{T_{\parallel}(\theta)}{w(\theta)(1+\Gamma_{\parallel}(\theta))}\right|^2\right\} \qquad (3.63)$$

The far-zone fields and the radiation pattern of waveguide-fed apertures radiating through a dielectric slab into free space are given by (3.60), (3.61) and (3.63) in connection with (3.62). If the apertures were radiating directly into free space, i.e. if no dielectric slab were present, then the far-zone fields and the radiation pattern would still be given by (3.60), (3.61) and (3.63), but with $T/(1+\Gamma)=e^{i\delta w}$ as is readily evident from (3.54). Thus we see that the dielectric slab has the effect of multiplying the components of the far-zone fields with no dielectric slab present by the factors $T_{\perp}/(1+\Gamma_{\perp})$ or $T_{\parallel}/(1+\Gamma_{\parallel})$ (given by (3.62) in case of a homogeneous slab) and $e^{-i\delta w}$ which depend on θ and the parameters of the dielectric medium, provided that the same aperture field distribution is assumed with or without the dielectric slab.

To find the far-zone fields and the radiation pattern associated with particular configurations the appropriate aperture field transforms must be evaluated and substituted in (3.60) and (3.63). By way of example, we will find the radiation pattern of a circular aperture fed by a circular waveguide. The transform of the aperture field is given by (3.41), and hence, apart from ignorable constants, we have

$$\hat{E}_{o\psi}(\theta,\phi) = \frac{x'_{11}\,(x'_{11}/\alpha)\; J'_1\,(\alpha\sin\theta)}{k[(x'_{11}/\alpha)^2 - \sin^2\theta]}\,\cos\phi$$

$$\hat{E}_{op}(\theta,\phi) = \frac{J_1(\alpha\sin\theta)}{k\,\sin\,\theta}\,\sin\,\phi \tag{3.64}$$

The radiation patterns for each of the two principal planes, the xz-plane ($\phi = 0,\ \theta$ variable) and the yz-plane ($\phi = \pi/2,\ \theta$ variable) are given respectively by

$$F(\theta,0) = k^2\,|\hat{E}_{o\psi}(\theta,0)|^2 \left|\frac{T_\perp(\theta)}{1+\Gamma_\perp(\theta)}\right|^2 \cos^2\theta$$

and

$$F(\theta,\pi/2) = k^2|\hat{E}_{op}(\theta,\pi/2)|^2 \left|\frac{T_\parallel(\theta)}{1+\Gamma_\parallel(\theta)}\right|^2 \tag{3.65}$$

where $\hat{E}_{o\psi}(\theta,0)$ and $\hat{E}_{op}(\theta,\pi/2)$ can be obtained from (3.64), and where $T/(1+\Gamma)$ equals $e^{i\delta w}$ if the aperture radiates directly into free space, and is given by equations (3.62) if it radiates through a homogeneous dielectric slab into free space. It can further be noted that $F(\theta,0)$ and $F(\theta,\pi/2)$ are proportional to the angular dependent parts of $|E_\phi(r,\theta,0)|^2$ and $|E_\theta(r,\theta,\pi/2)|^2$ respectively.

4. REFLECTION AND TRANSMISSION PROPERTIES OF INHOMOGENEOUS DIELECTRIC SLABS

When a waveguide-fed aperture antenna is covered by an inhomogeneous dielectric slab whose permittivity varies in a direction perpendicular to the slab faces (the z-direction), then the expressions developed in the previous chapter are adequate to calculate the aperture admittance of the antenna as well as the radiation pattern, provided the reflection and transmission coefficients of the inhomogeneous dielectric slab are known. Accordingly, the present chapter will be devoted to the calculation of these coefficients, and related quantities, for inhomogeneous dielectric slabs.

If the properties of the dielectric slab vary only in the direction normal to its plane faces, then the reflection and transmission coefficients will again depend only on the angle of incidence χ and will be independent of the azimuthal variation of the incidence direction. Thus with no loss of generality the plane of incidence can be chosen as the xz plane and the fields taken to be independent of the coordinate y (Fig. 4.1). We then consider a dielectric slab for $0 < z < d$, whose relative permittivity is given by $\varepsilon(z)$. We assume the region $z < 0$ to be free space, and the region $z > d$ to consist of a homogeneous dielectric with relative permittivity ε_2. The permeability is taken to be equal to that of free space everywhere. We further suppose that a plane wave traveling in the region $z < 0$ is incident on the inhomo-

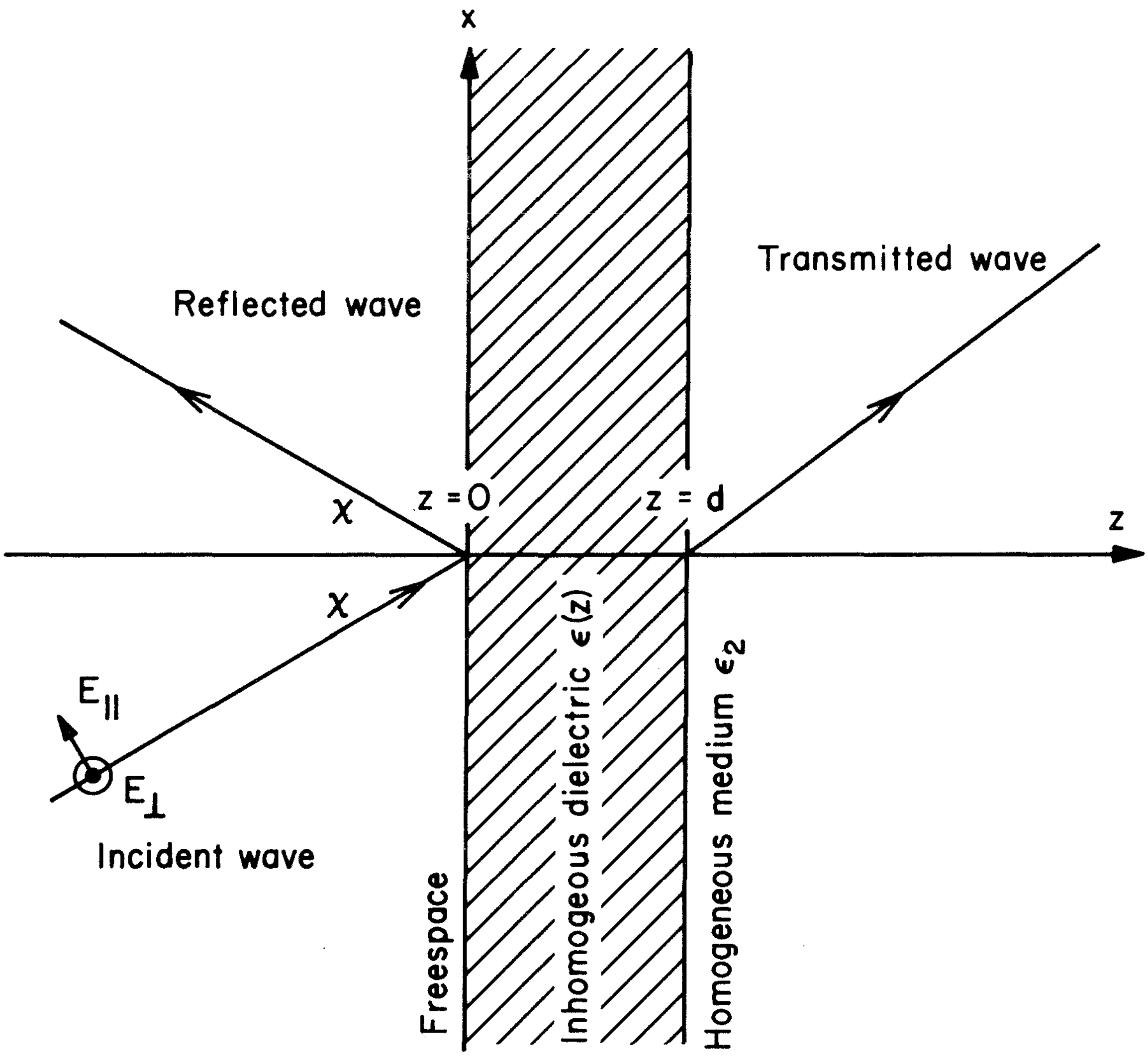


Fig. 4.1 Geometry of the inhomogeneous dielectric slab.

geneous dielectric slab with an angle of incidence given by χ, and is reflected by it. There will be a right traveling (transmitted) and a left traveling (reflected) wave in the inhomogeneous dielectric slab, as well as a transmitted plane wave in the region $z > d$. Our aim is to find two quantities related to the reflection and transmission coefficients, which are of interest in this report, without solving for the fields themselves.

The differential equations satisfied by the reflection and transmission coefficients in an inhomogeneous medium can be derived by the method of invariant imbedding (28,29,30). We will choose instead a purely mathematical way (31) which gets at the desired equations in a clear and straightforward manner, and has the added advantage of obtaining directly equations for the two quantities of primary interest, the input admittance and the ratio of the total electric field amplitude at the right face of the slab to the transverse field at the left face. The two cases of polarization must be treated separately. Accordingly, first the case of the electric field polarized perpendicular to the plane of incidence will be treated, and then the case of the electric field polarized in the plane of incidence will be discussed.

A. Perpendicular Polarization

In this case, keeping in mind that $\Gamma_{\perp}$ and $T_{\perp}$ are respectively the reflection and transmission coefficients of the slab, $w = \cos\chi$, $p = \sin\chi$, $w_2 = \sqrt{\varepsilon_2 - p^2}$, and letting $A(x) = e^{ikpx}$, the fields in the homogeneous regions can be written down in the following fashion:

For $z < 0$,

$$\begin{aligned} E_y(x,z) &= \quad (e^{ikwz} + \Gamma_\perp e^{-ikwz})\, A(x), \\ H_x(x,z) &= -\eta\, w(e^{ikwz} - \Gamma_\perp e^{-ikwz})\, A(x), \\ H_z(x,z) &= \eta\, p(e^{ikwz} + \Gamma_\perp e^{-ikwz})\, A(x). \end{aligned} \tag{4.1}$$

For $z > d$,

$$\begin{aligned} E_y(x,z) &= T_\perp \quad e^{ikw_2(z-d)}\, A(x), \\ H_x(x,z) &= -T_\perp \eta\, w_2 e^{ikw_2(z-d)}\, A(x), \\ H_z(x,z) &= T_\perp \eta\, p\, e^{ikw_2(z-d)}\, A(x). \end{aligned} \tag{4.2}$$

The fields in the inhomogeneous dielectric slab can now be defined in a form similar to (4.1). Interpreting $P_\perp(z)$ and $R_\perp(z)$ as the amplitudes of the transmitted and reflected waves in this region, and letting $w(z) = \sqrt{\varepsilon(z) - p^2}$, we can write for $0 < z < d$,

$$\begin{aligned} E_y(x,z) &= \quad (P_\perp(z) + R_\perp(z))\, A(x), \\ H_x(x,z) &= -\eta\, w(z)(P_\perp(z) - R_\perp(z))\, A(x), \\ H_z(x,z) &= \eta\, p \quad (P_\perp(z) + R_\perp(z))\, A(x). \end{aligned} \tag{4.3}$$

Note that E_y and H_z are related through Maxwell's equation $ikH_z = \eta\, \partial E_y/\partial x$. Substituting equations (4.3) into Maxwell's equations

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = -ik\eta\, \varepsilon(z) E_y \tag{4.4}$$

and

$$\eta \frac{\partial E_y}{\partial z} = -ikH_x \tag{4.5}$$

we find,

$$\frac{d}{dz}\,[w(z)\,(P_\perp - R_\perp)] - ikw^2(z)\,(P_\perp + R_\perp) = 0 \tag{4.6}$$

and

$$\frac{d}{dz}\,(P_\perp + R_\perp) - ikw(z)\,(P_\perp - R_\perp) = 0 \tag{4.7}$$

Eliminating first $dR_\perp/dz$ and then $dP_\perp/dz$ between equations (4.6) and (4.7) it is found that $P_\perp(z)$ and $R_\perp(z)$ satisfy the following pair of coupled equations:

$$\frac{dP_\perp}{dz} - ikw(z)\,P_\perp + \frac{1}{2w(z)}\,\frac{dw(z)}{dz}\,(P_\perp - R_\perp) = 0 \tag{4.8}$$

$$\frac{dR_\perp}{dz} + ikw(z)\,R_\perp - \frac{1}{2w(z)}\,\frac{dw(z)}{dz}\,(P_\perp - R_\perp) = 0 \tag{4.9}$$

Furthermore, it follows from the continuity of the tangential fields at $z = 0$ and $z = d$, that

$$1 + \Gamma_\perp = P_\perp(0) + R_\perp(0) \tag{4.10a}$$

$$w(1 - \Gamma_\perp) = w(0)\,(P_\perp(0) - R_\perp(0)) \tag{4.10b}$$

and

$$P_\perp(d) + R_\perp(d) = T_\perp \tag{4.11a}$$

$$w(d)\,(P_\perp(d) - R_\perp(d)) = w_2 T_\perp \tag{4.11b}$$

Since the amplitude of the incident wave was taken as unity, $P_\perp(z)$ is also the transmission coefficient at a point z, while

$R_{\perp}(z)/P_{\perp}(z) = \Gamma_{\perp}(z)$ is the reflection coefficient at the point z. The equation that the reflection coefficient $\Gamma_{\perp}(z)$ satisfies can be obtained from (4.8) and (4.9), and turns out to be a Riccati equation

$$\frac{d\Gamma_{\perp}}{dz} = \frac{1}{2w(z)} \frac{dw(z)}{dz} (1 - \Gamma_{\perp}^2) - i2kw(z)\Gamma_{\perp} \tag{4.12}$$

As a boundary condition we have

$$\Gamma_{\perp}(d) = \frac{w(d) - w_2}{w(d) + w_2} \tag{4.13}$$

which directly follows from (4.11). Also the relation between $\Gamma_{\perp}(0)$, and the reflection coefficient of the inhomogeneous dielectric slab $\Gamma_{\perp}$, is from (4.10),

$$\Gamma_{\perp} = \frac{[w - w(o)] + [w + w(0)]\ \Gamma_{\perp}(0)}{[w + w(0)] + [w - w(0)]\ \Gamma_{\perp}(0)} \tag{4.14}$$

An equation for the input admittance normalized to the admittance of free space,

$$y_{in\perp} = -\frac{H_x}{\eta E_y} = w(z)\ \frac{P_{\perp} - R_{\perp}}{R_{\perp} + R_{\perp}} \tag{4.15}$$

is also easily found from (4.6) and (4.7) to be

$$\frac{i}{k}\ \frac{dy_{in\perp}}{dz} = y_{in\perp}{}^2 - w^2(z) \tag{4.16}$$

The boundary condition is $y_{in\perp}(d) = w_2$, and the input admittance of the slab is given by $y_{in\perp}(0) = w(1-\Gamma_\perp/1+\Gamma_\perp)$.

As already mentioned the transmission coefficient satisfies equation (4.8):

$$\frac{dP_\perp}{dz} = h_\perp(z)\ P_\perp(z) \tag{4.17}$$

where $$h_\perp(z) = ikw(z) - \frac{1}{2w(z)}\frac{dw(z)}{dz}\ (1 - \Gamma_\perp(z)) \tag{4.18}$$

The solution of (4.17) is trivial once $\Gamma_\perp(z)$ is known from equation (4.12),

$$P_\perp(z) = P_\perp(0)\ e^{\int_0^z h_\perp(z)\ dz} \tag{4.19}$$

The boundary condition at $z = 0$ becomes from (4.10a) or (4.10b).

$$P_\perp(0) = \frac{2w}{[w + w(0)] + [w - w(0)]\Gamma_\perp(0)} \tag{4.20}$$

Finally, the transmission coefficient of the entire slab is given from (4.11a) or (4.11b) by

$$T_\perp = \frac{2w(d)}{w(d) + w_2}\ P_\perp(d) \tag{4.21}$$

or combining (4.19) and (4.21) by

$$T_{\perp} = \frac{2w(d)}{w(d) + w_2} \quad P_{\perp}(0) \; e^{\int_o^d h_{\perp}(z)dz} \tag{4.22}$$

$P(z)$ is the amplitude of the right-traveling wave at the point z when a wave of unit amplitude is incident on the left face of the dielectric slab. It will be convenient to define yet another function related to the transmission coefficient. Let $T(z)$ be the amplitude of the wave transmitted through the right face of the dielectric slab when a wave of unit amplitude is incident at the point z. Then, by definition,

$$P(z)\; T(z) = T \tag{4.23}$$

The equation satisfied by $T_{\perp}(z)$ is from (4.17) and (4.23),

$$-\frac{dT_{\perp}}{dz} = h_{\perp}(z)\; T_{\perp}(z) \tag{4.24}$$

and the boundary condition at $z = d$ becomes from (4.11) and (4.23)

$$T_{\perp}(d) = \frac{2w(d)}{w(d) + w_2} \tag{4.25}$$

Hence,

$$T_{\perp}(z) = T_{\perp}(d)\; e^{\int_z^d h_{\perp}(z)\, dz} \tag{4.26}$$

The transmission coefficient of the entire slab is given from (4.10) and (4.23) by

$$T_\perp = \frac{2w}{[w + w(0)] + [w - w(0)]\Gamma_\perp(0)}\, T_\perp(0) \tag{4.27}$$

which checks, as expected with (4.22).

The ratio of the total electric field amplitude at the right face of the slab to that at the point z inside the slab is given by $T_\perp(z)/[1 + \Gamma_\perp(z)]$. For radiation pattern calculations it is of interest to find the equation that this quantity satisfies in the inhomogeneous slab. First noting that

$$\frac{T_\perp(z)}{1 + \Gamma_\perp(z)} = \frac{T_\perp}{P_\perp(z) + R_\perp(z)} \tag{4.28}$$

we have from (4.28) and (4.7)

$$\frac{d}{dz}\left(\frac{T_\perp(z)}{1 + \Gamma_\perp(z)}\right) + ik\, y_{in\perp}(z)\left(\frac{T_\perp(z)}{1 + \Gamma_\perp(z)}\right) = 0 \tag{4.29}$$

and the appropriate boundary condition is from (4.11a) and (4.23)

$$\frac{T_\perp(d)}{1 + \Gamma_\perp(d)} = 1 \tag{4.30}$$

The the solution for (4.29) becomes

$$\frac{T_\perp(z)}{1 + \Gamma_\perp(z)} = e^{ik \int_z^d y_{in\perp}(z)dz} \tag{4.31}$$

Since, from (4.10a) and (4.23)

$$\frac{T_{\perp}(0)}{1 + \Gamma_{\perp}(0)} = \frac{T_{\perp}}{1 + \Gamma_{\perp}} , \tag{4.32}$$

we finally obtain

$$\frac{T_{\perp}}{1 + \Gamma_{\perp}} = e^{ik\int_0^d y_{in\perp}(z)dz} \tag{4.33}$$

As a partial check, we may apply equations (4.16) and (4.33), with which we are most interested, to the case of a homogeneous slab with relative permittivity ε_1 for $0 < z < d$, and a homogeneous semi-infinite region of relative permittivity ε_2 for $z > d$. In this case $w(z) = \sqrt{\varepsilon_1 - p^2} = w_1$ for $0 < z < d$, and equation (4.16) can be directly integrated, with the boundary condition $y(d) = w_2$. Writing y for $y_{in\perp}$, we have

$$i \int_{y(d)}^{y(z)} \frac{dy}{y^2 - w_1^2} = k \int_d^z dz ,$$

which, upon integration, becomes

$$\tan^{-1} \frac{y(d)}{iw_1} - \tan^{-1} \frac{y(z)}{iw_1} = k\ (d-z)w_1$$

and solving this latter equation for $y(z)$ we have

$$y(z) = w_1 \frac{y(d) - iw_1 \tan k(d-z)w_1}{w_1 - iy(d) \tan k(d-z)w_1}$$

which, setting $y(d) = w_2$, checks for $y(0)$ with (3.27a).

To calculate (4.33) we rewrite $y(z)$ as

$$y(z) = w_1 \frac{y(d) \cos k(d-z)w_1 - iw_1 \sin k(d-z)w_1}{w_1 \cos k(d-z)w_1 - iy(d) \sin k(d-z)w_1}$$

Letting

$$u = w_1 \cos k(d-z)w_1 - iy(d) \sin k(d-z)w_1$$

we have

$$\frac{du}{dz} = ikw_1 \, [y(d) \cos k(d-z)w_1 - iw_1 \sin k(d-z)w_1]$$

then

$$ik\int_0^d y(z)\, dz = \int_{u(0)}^{u(d)} \frac{du}{u} = \ln \frac{w_1}{w_1 \cos kdw_1 - iy(d) \sin kdw_1}$$

Hence
$$\frac{T_\perp}{1 + \Gamma_\perp} = \frac{w_1}{w_1 \cos kdw_1 - iw_2 \sin kdw_1}$$

which checks with (3.53a).

B. Parallel Polarization

In the case of the electric field polarized in the plane of incidence, the fields in the homogeneous regions can be written down as follows:

For $z < 0$,

$$E_x(x,z) = w\,(e^{ikwz} + \Gamma_{\parallel} e^{-ikwz})\,A(x)\ ,$$
$$E_z(x,z) = -p(e^{ikwz} - \Gamma_{\parallel} e^{-ikwz})\,A(x)\ , \tag{4.34}$$
$$H_y(x,z) = \eta(e^{ikwz} - \Gamma_{\parallel} e^{-ikwz})\,A(x)\ .$$

For $z > d$,

$$E_x(x,z) = \frac{w_2}{\sqrt{\varepsilon_2}}\,T_{\parallel}\,e^{ikw_2(z-d)}\,A(x)\ ,$$

$$E_z(x,z) = -\frac{p}{\sqrt{\varepsilon_2}}\,T_{\parallel}\,e^{ikw_2(z-d)}\,A(x)\ , \tag{4.35}$$

$$H_y(x,z) = \eta\sqrt{\varepsilon_2}\,T_{\parallel}\,e^{ikw_2(z-d)}\,A(x)\ .$$

The meaning of the symbols used above and in the following paragraph should be clear from our treatment of the previous case.

We then define the fields in the inhomogeneous dielectric slab, $0 < z < d$, as

$$E_x(x,z) = \frac{w(z)}{\sqrt{\varepsilon(z)}}\,(P_{\parallel}(z) + R_{\parallel}(z))\,A(x)\ ,$$
$$E_z(x,z) = -\frac{p}{\sqrt{\varepsilon(z)}}\,(P_{\parallel}(z) - R_{\parallel}(z))\,A(x)\ , \tag{4.36}$$
$$H_y(x,z) = \eta\sqrt{\varepsilon(z)}\,(P_{\parallel}(z) - R_{\parallel}(z))\,A(x)\ .$$

Note that H_y and E_z are related through Maxwell's equation $-ik\eta\varepsilon(z)E_z = \partial H_y/\partial x$. Substituting equations (4.36) into Maxwell's equations

$$\eta\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) = ik\ H_y \tag{4.37}$$

and

$$\frac{\partial H_y}{\partial z} = ik\eta\ \varepsilon(z)E_x \tag{4.38}$$

we find,

$$\frac{d}{dz}\left[\frac{w(z)}{\sqrt{\varepsilon(z)}}\ (P_{\parallel} + P_{\parallel})\right] - ik\ \frac{w^2(z)}{\sqrt{\varepsilon(z)}}\ (P_{\parallel} - R_{\parallel}) = 0 \tag{4.39}$$

and

$$\frac{d}{dz}\left[\sqrt{\varepsilon(z)}\,(P_{\parallel} - R_{\parallel})\right] - ikw(z)\sqrt{\varepsilon(z)}\,(P_{\parallel} + R_{\parallel}) = 0, \tag{4.40}$$

from which it follows that $P_{\parallel}(z)$ and $R_{\parallel}(z)$ satisfy the following pair of coupled equations

$$\frac{dP_{\parallel}}{dz} + ikw(z)P_{\parallel} + \frac{1}{2w(z)}\ \frac{dw(z)}{dz}\ (P_{\parallel} + R_{\parallel}) - \frac{1}{2\varepsilon(z)}\ \frac{d\varepsilon(z)}{dz}\ R_{\parallel} = 0 \tag{4.41}$$

$$\frac{dR_{\parallel}}{dz} + ikw(z)R_{\parallel} + \frac{1}{2w(z)}\ \frac{dw(z)}{dz}\ (P_{\parallel} + R_{\parallel}) - \frac{1}{2\varepsilon(z)}\ \frac{d\varepsilon(z)}{dz}\ P_{\parallel} = 0 \tag{4.42}$$

From the continuity of the tangential fields at $z = 0$ and $z = d$ we have the boundary conditions

$$w(1 + \Gamma_{\parallel}) = \frac{w(0)}{\sqrt{\varepsilon(0)}}\ (P_{\parallel}(0) + R_{\parallel}(0)) \tag{4.43a}$$

$$1 - \Gamma_{\parallel} = \sqrt{\varepsilon(0)}\ (P_{\parallel}(0) - R_{\parallel}(0)) \tag{4.43b}$$

and

$$\frac{w(d)}{\sqrt{\varepsilon(d)}}\ (P_{\parallel}(d) + R_{\parallel}(d)) = \frac{w_2}{\sqrt{\varepsilon_2}}\ T_{\parallel} \tag{4.44a}$$

$$\sqrt{\varepsilon(d)}\quad (P_{\parallel}(d) - R_{\parallel}(d)) = \sqrt{\varepsilon_2}\ T_{\parallel} \tag{4.44b}$$

The reflection coefficient $\Gamma_{\parallel}(z) = R_{\parallel}(z)/P_{\parallel}(z)$ satisfies a Riccati equation, which can easily be obtained from (4.41) and (4.42):

$$\frac{d\Gamma_{\parallel}}{dz} = \frac{\varepsilon(z)}{2w(z)}\frac{d}{dz}\left(\frac{w(z)}{\varepsilon(z)}\right)(\Gamma_{\parallel}^2 - 1) - i2kw(z)\Gamma_{\parallel} \tag{4.45}$$

subject to the boundary condition,

$$\Gamma_{\parallel}(d) = \frac{\dfrac{w_2}{\varepsilon_2} - \dfrac{w(d)}{\varepsilon(d)}}{\dfrac{w_2}{\varepsilon_2} + \dfrac{w(d)}{\varepsilon(d)}} \tag{4.46}$$

which directly follows from (4.44). The reflection coefficient of the inhomogeneous dielectric slab $\Gamma_{\parallel}$, follows from the relation between $\Gamma_{\parallel}(0)$ and $\Gamma_{\parallel}$ in equation (4.43), and is

$$\Gamma_{\parallel} = \frac{\left[\dfrac{w(0)}{\varepsilon(0)} - w\right] + \left[\dfrac{w(0)}{\varepsilon(0)} + w\right]\Gamma_{\parallel}(0)}{\left[\dfrac{w(0)}{\varepsilon(0)} + w\right] + \left[\dfrac{w(0)}{\varepsilon(0)} - w\right]\Gamma_{\parallel}(0)} \tag{4.47}$$

The equation for the normalized input admittance,

$$y_{in\parallel} = \frac{H_y}{\eta\ E_x} = \frac{\varepsilon(z)}{w(z)} \frac{P_\parallel - R_\parallel}{P_\parallel + R_\parallel} \tag{4.48}$$

can easily be obtained from (4.39) and (4.40). It is

$$\frac{i}{k} \frac{dy_{in\parallel}}{dz} = \frac{w^2(z)}{\varepsilon(z)} y_{in\parallel}^2 - \varepsilon(z) \tag{4.49}$$

The boundary condition is $y_\parallel(d) = \varepsilon_2/w_2$, and the input admittance of the slab is given by $y_{in}(0) = 1/w\ (1-\Gamma_\parallel/1+\Gamma_\parallel)$.

Since the incident electric field has unit amplitude the transmission coefficient is given by the same equation as (4.41),

$$\frac{dP_\parallel}{dz} = h_\parallel(z)\ P_\parallel(z) \tag{4.50}$$

where

$$h_\parallel(z) = ikw(z) - \frac{1}{2w(z)} \frac{dw(z)}{dz} (1 + \Gamma_\parallel(z)) + \frac{1}{2\varepsilon(z)} \frac{d\varepsilon(z)}{dz} \Gamma_\parallel(z) \tag{4.51}$$

With the knowledge of $\Gamma_\parallel(z)$ from equation (4.45) $P_\parallel(z)$ can readily be found. The boundary condition at $z = 0$ is from (4.43a) or (4.43b),

$$P_\parallel(0) = \frac{1}{\sqrt{\varepsilon(0)}} \frac{2w}{\left[\frac{w(0)}{\varepsilon(0)} + w\right] + \left[\frac{w(0)}{\varepsilon(0)} - w\right] \Gamma_\parallel(0)} \tag{4.52}$$

and

$$P_{\parallel}(z) = P_{\parallel}(0)\ e^{\int_0^z h_{\parallel}(z)\,dz} \tag{4.53}$$

The transmission coefficient of the entire slab is given from (4.44a) or (4.44b) by

$$T_{\parallel} = \frac{\sqrt{\varepsilon(d)}}{\sqrt{\varepsilon_2}} \frac{2\frac{w(d)}{\varepsilon(d)}}{\frac{w_2}{\varepsilon_2} + \frac{w(d)}{\varepsilon(d)}} P_{\parallel}(d) \tag{4.54}$$

As in the previous case, it is convenient to introduce the function $T(z)$ defined by equation (4.23). The equation satisfied by $T_{\parallel}(z)$ is from (4.50) and (4.23)

$$-\frac{dT_{\parallel}}{dz} = h_{\parallel}(z)\ T_{\parallel}(z) \tag{4.55}$$

and the boundary condition at $z = d$ is from (4.44b) and (4.23)

$$T_{\parallel}(d) = \frac{\sqrt{\varepsilon(d)}}{\sqrt{\varepsilon_2}} \frac{2\frac{w(d)}{\varepsilon(d)}}{\frac{w_2}{\varepsilon_2} + \frac{w(d)}{\varepsilon(d)}} \tag{4.56}$$

Hence

$$T_{\parallel}(z) = T_{\parallel}(d)\ e^{\int_z^d h_{\parallel}(z)\,dz} \tag{4.57}$$

The transmission coefficient of the entire slab is then given from

(4.43) and (4.23) by

$$T_{\parallel} = \frac{1}{\sqrt{\varepsilon(0)}} \frac{2w}{\left[\frac{w(0)}{\varepsilon(0)} + w\right] + \left[\frac{w(0)}{\varepsilon(0)} - w\right] \Gamma_{\parallel}(0)} T_{\parallel}(0) \tag{4.58}$$

The ratio of the total electric field amplitude at the right face of the slab to the transverse field at the point z inside the slab is given by $\sqrt{\varepsilon(z)}/w(z)\ [T_{\parallel}(z)/1 + \Gamma_{\parallel}(z)]$. The equation satisfied by this quantity can be written directly from (4.39) if we first note that

$$\frac{T_{\parallel}(z)}{1 + \Gamma_{\parallel}(z)} = \frac{T_{\parallel}}{P_{\parallel}(z) + R_{\parallel}(z)} \tag{4.59}$$

Then we have

$$\frac{d}{dz}\left(\frac{\sqrt{\varepsilon(z)}}{w(z)} \frac{T_{\parallel}(z)}{1 + \Gamma_{\parallel}(z)}\right) + ik \frac{w^2(z)}{\varepsilon(z)} y_{in\parallel}(z) \left(\frac{\sqrt{\varepsilon(z)}}{w(z)} \frac{T_{\parallel}(z)}{1 + \Gamma_{\parallel}(z)}\right) = 0 \tag{4.60}$$

The appropriate boundary condition is from (4.44a)

$$\frac{\sqrt{\varepsilon(d)}}{w(d)} \frac{T_{\parallel}(d)}{1 + \Gamma_{\parallel}(d)} = \frac{\sqrt{\varepsilon_2}}{w_2} \tag{4.61}$$

The solution of (4.60) then becomes

$$\frac{\sqrt{\varepsilon(z)}}{w(z)} \frac{T_{\parallel}(z)}{1 + \Gamma_{\parallel}(z)} = \frac{\sqrt{\varepsilon_2}}{w_2}\ e^{ik\int_z^d \frac{w^2(z)}{\varepsilon(z)} y_{in\parallel}(z)\, dz} \tag{4.62}$$

From (4.43a) and (4.23) we have

$$\frac{\sqrt{\varepsilon(0)}}{w(0)} \frac{T_{\parallel}(0)}{1 + \Gamma_{\parallel}(0)} = \frac{T_{\parallel}}{w(1 + \Gamma_{\parallel})} , \tag{4.63}$$

hence we finally obtain

$$\frac{T_{\parallel}}{w(1 + \Gamma_{\parallel})} = \frac{\sqrt{\varepsilon_2}}{w_2} e^{ik\int_0^d \frac{w^2(z)}{\varepsilon(z)} y_{in\parallel}(z)\, dz} \tag{4.64}$$

Again, as in the previous case, we will apply equations (4.49) and (4.64) to the case of a homogeneous dielectric slab of relative permittivity ε_1 and thickness d, adjacent to a semi-infinite region of relative permittivity ε_2. We can directly integrate equation (4.49) with $w(z) = w_1$ for $0 < z < d$, and boundary condition $y(d) = \varepsilon_2/w_2$. Writing now y for $y_{in\parallel}$, we have

$$i \frac{\varepsilon_1}{w_1^2} \int_{y(d)}^{y(z)} \frac{dy}{y^2 - \frac{\varepsilon_1^2}{w_1^2}} = k \int_d^z dz$$

which, upon integration, becomes

$$\tan^{-1} \frac{y(d)}{i(\varepsilon_1/w_1)} - \tan^{-1} \frac{y(z)}{i(\varepsilon_1/w_1)} = k(d-z)w_1$$

and, upon solving this latter equation for $y(z)$, we obtain,

$$y(z) = \frac{\varepsilon_1}{w_1} \frac{y(d) - i\,(\varepsilon_1/w_1)\tan k(d-z)w_1}{(\varepsilon_1/w_1) - i\,y(d)\tan k(d-z)w_1}$$

which, setting $y(d) = \varepsilon_2/w_2$, checks for $y(0)$ with (3.27b).

To calculate (4.64) we rewrite $y(z)$ as

$$y(z) = \frac{\varepsilon_1}{w_1} \frac{y(d)\cos k(d-z)w_1 - i(\varepsilon_1/w_1)\sin k(d-z)w_1}{(\varepsilon_1/w_1)\cos k(d-z)w_1 - i\,y(d)\sin k(d-z)w_1}$$

Letting

$$u = (\varepsilon_1/w_1)\cos k(d-z)w_1 - iy(d)\sin k(d-z)w_1$$

we have

$$\frac{du}{dz} = ikw_1[y(d)\cos k(d-z)w_1 - i(\varepsilon_1 w_1)\sin k(d-z)w_1] ,$$

then

$$ik\int_0^d \frac{w_1^{\;2}}{\varepsilon_1}\, y(z)dz = \int_{u(0)}^{u(d)} \frac{du}{u} = \ln\frac{u(d)}{u(0)}$$

$$= \ln \frac{\varepsilon_1/w_1}{(\varepsilon_1/w_1)\cos kdw_1 - i\,y(d)\sin kdw_1}$$

Hence $$\frac{T_{\parallel}}{w(1+\Gamma_{\parallel})} = \sqrt{\varepsilon_2}\,\frac{\varepsilon_1}{\varepsilon_1 w_2\cos kdw_1 - i\varepsilon_2 w_1\sin kdw_1}$$

which checks with (3.53b).

Direct integration of equations (4.16) and (4.49) is possible only for a few special $\varepsilon(z)$. In general, numerical methods must be used, which, in the present case, are fortunately quite simple.

A few remarks should now be made concerning some of the results obtained in this chapter. First, if the permittivity $\varepsilon(z)$ is discontinuous at one or more points in the inhomogeneous medium care must be exercised in the calculation of the reflection and transmission coefficients. Separate expressions must be written for the fields in each section of the medium where the permittivity is continuous, and the expressions for the tangential fields must be matched at the points of discontinuity of $\varepsilon(z)$, in exactly the same way as was done at the boundaries of the inhomogeneous medium.

However, $y_{in_\perp}(z)$, $y_{in_\parallel}(z)$, $[T_\perp(z)/1 + \Gamma_\perp(z)]$ and also $[\sqrt{\varepsilon(z)}/w(z)]\ [T_\parallel(z)/1 + \Gamma_\parallel(z)]$ are all continuous functions of z even if $\varepsilon(z)$ is not, since the former two are the ratio of the tangential magnetic to the tangential electric field at the point z in the medium, and the latter two are proportional to the reciprocal of the tangential electric field at the point z. Thus the calculation of these quantities presents an additional advantage over the calculation of the reflection and transmission coefficients.

If the inhomogeneous dielectric medium is a plasma there may be one or more points where $\varepsilon(z) = 0$. (Then the plasma frequency equals the frequency of the electromagnetic waves). In such a case the solution of equation (4.49) may require some ingenuity. Suppose that $\varepsilon(z_o) = 0$, and, assuming that the first derivative of $\varepsilon(z)$ does not vanish at z_o, $\varepsilon(z)$ can be written near z_o as

$$\varepsilon(z) = \left.\frac{d\varepsilon}{dz}\right|_{z=z_o} (z - z_o) = a(z - z_o) \qquad (4.65)$$

In a small interval of z near z_o, $z_o - \Delta z < z < z_o + \Delta z$, the differential equation (4.49) can be written as (writing y for $y_{in\|}$),

$$\frac{dy}{dz} = \frac{ikp^2}{a(z-z_o)} y^2 \qquad \text{or} \qquad \frac{dy}{y^2} = \frac{ikp^2}{a} \frac{dz}{(z-z_o)} \tag{4.66}$$

which can be directly integrated on this interval to give

$$\frac{1}{y(z)} = \frac{1}{y(z_o + \Delta z)} - \frac{ikp^2}{a} \ln \frac{|z-z_o|}{\Delta z} + \begin{cases} 0 & \text{if } z_o < z < z_o + \Delta z \\ \dfrac{\pi kp^2}{|a|} & \text{if } z_o - \Delta z < z < z_o \end{cases} \tag{4.67}$$

Thus, the solution around the point z_o can be found from (4.67), provided that the solution at $z_o + \Delta z$ is previously calculated. In particular,

$$\frac{1}{y(z_o - \Delta z)} = \frac{1}{y(z_o + \Delta z)} + \frac{\pi kp^2}{|a|} \tag{4.68}$$

The effect of the point z_o on the admittance is seen to be the addition, "in series", of a real admittance. Furthermore, it can easily be checked that at the point z_o y vanishes, while dy/dz becomes infinite.

Finally, it should be mentioned that if only the differential equations for the input admittances were of interest, they could be obtained in an even more direct and simple manner by making use of the fact that

$$\frac{dy_{in\perp}}{dz} = \frac{d}{dz}\left(\frac{H_x}{\eta E_y}\right) \quad \text{and} \quad \frac{dy_{in\parallel}}{dz} = \frac{d}{dz}\left(\frac{H_y}{\eta E_x}\right)$$

and making direct use of Maxwell's equations. The formulation presented in this chapter is preferred simply because it enables us to derive the equations for the other related quantities of interest as well.

5. RESULTS AND CONCLUSIONS

The main concern of the present report has been the presentation of a method of analyzing waveguide-fed aperture antennas of arbitrary cross-section and radiating into inhomogeneous media. However, it has been deemed useful that the method should be illustrated with examples which in themselves have some practical interest. Accordingly, the admittance and the radiation pattern of a circular aperture antenna fed by a circular waveguide and radiating into an inhomogeneous plasma slab has been calculated for a few interesting inhomogeneity profiles. Besides being of practical value, the circular aperture antenna presents a computational advantage over the rectangular, since the admittance expression contains a single rather than a double integral.

An inhomogeneous plasma slab of thickness d has been considered with the relative permittivity given by

$$\varepsilon(z) = 1 - \frac{(\omega_p/\omega)^2 f(z)}{1 + (\nu/\omega)^2} + i \frac{(\nu/\omega)(\omega_p/\omega)^2 f(z)}{1 + (\nu/\omega)^2} \tag{5.1}$$

where ω_p^2, the square of the peak plasma frequency, is proportional to the peak electron density in the plasma slab, and the electron density normalized to its peak value is given by the function $f(z)$, while the losses in the plasma are taken care of by an empirical collision frequency ν.

In the present report the plasma slab is chosen to have an inhomogeneous boundary layer for $0 < z < d_1$ and is homogeneous for $d_1 < z < d$. Three different electron density profiles have been chosen for the boundary layer region $0 < z < d_1$. These are,

a) a convex parabolic profile,

$$f(z) = z\,(2d_1 - z)/d_1^{\,2} \ ,$$

b) a linear profile,

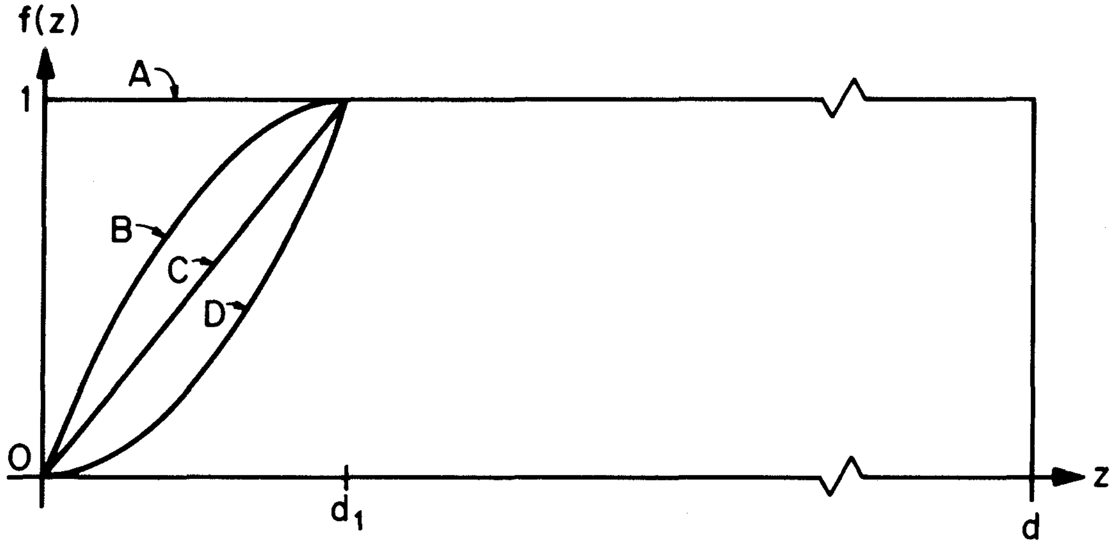
$$f(z) = z/d_1 \ ,$$

c) a concave parabolic profile,

$$f(z) = z^2/d_1^{\,2},$$

while $f(z) = 1$ for $d_1 < z < d$ for all three cases (Fig. 5.1). The homogeneous plasma slab for which $f(z) = 1$ for $0 < z < d$, has also been considered.

First, the calculations on the admittance of the circular aperture antenna are presented. For the inhomogeneous slabs, the input admittances $Y_{in_\perp}(p)$ and $Y_{in_\parallel}(p)$ have been calculated by numerically solving the differential equations (4.16) and (4.49) as a function of p, while for the homogeneous case they are given by expressions (3.27). These results have been used in the numerical integration of expression (3.42). It is easy to see that, at least for the homogeneous plasma slab, the integrand of (3.42) is of the order of p^{-3} for large p, and hence it is possible to soon terminate the integration without appreciable error. The integrand has been



A homogeneous profile

B convex parabolic profile

C linear profile

D concave parabolic profile

Fig. 5.1 Electron density profiles

checked for possible surface wave poles and such poles have been accounted for although their effect is negligible in the range of parameters under consideration.

The dominant mode of the circular waveguide is the TE_{11} mode. The aperture radius a has been chosen such that $ka = \alpha = 3\pi/4$, which is just below the cutoff of the next propagating mode (TM_{01}). The slab thickness has always been chosen to be $kd = \delta = 2\pi$.

The aperture admittance, Y, is normalized to the characteristic admittance of the waveguide dominant mode, $Y_o = \eta\sqrt{1 - (x_{11}'/\alpha)^2}$. We have computed the real as well as the negative of the imaginary part of $(Y/Y_o) = g - ib$, called the normalized conductance, g, and the normalized susceptance, b, respectively. g and b are plotted against $(\omega_p/\omega)^2$ in steps of 0.1 for $0.1 < (\omega_p/\omega)^2 < 1$ (underdense plasma), and in steps of 1.0 for $1 < (\omega_p/\omega)^2 < 10$ (overdense plasma). (Figs. 5-2 to 5-5). There is a discontinuity in the scale of the horizontal axis at $(\omega_p/\omega)^2 = 1$. The point $(\omega_p/\omega)^2 = 0$ corresponds always to the aperture radiating into free space. The value of (Y/Y_o) at that point (not shown on the figures) is found to be $1.156 + i\ 0.043$, which checks with previous calculations of this quantity (13).

Figures 5-2 and 5-3 compare the aperture admittance obtained from each of the three inhomogeneity profiles and from the homogeneous slab, when the plasma is lossy with $\nu/\omega = 0.4$. The thickness of the boundary layer is given by $\delta_1 = kd_1 = (1/20)\delta$ in Fig. 5-2 and by $\delta_1 = (1/10)\delta$ in Fig. 5-3. It is seen that the susceptance of a plasma with an inhomogeneous boundary layer is substantially decreased over the homogeneous slab. The least change occurs in the concave parabolic

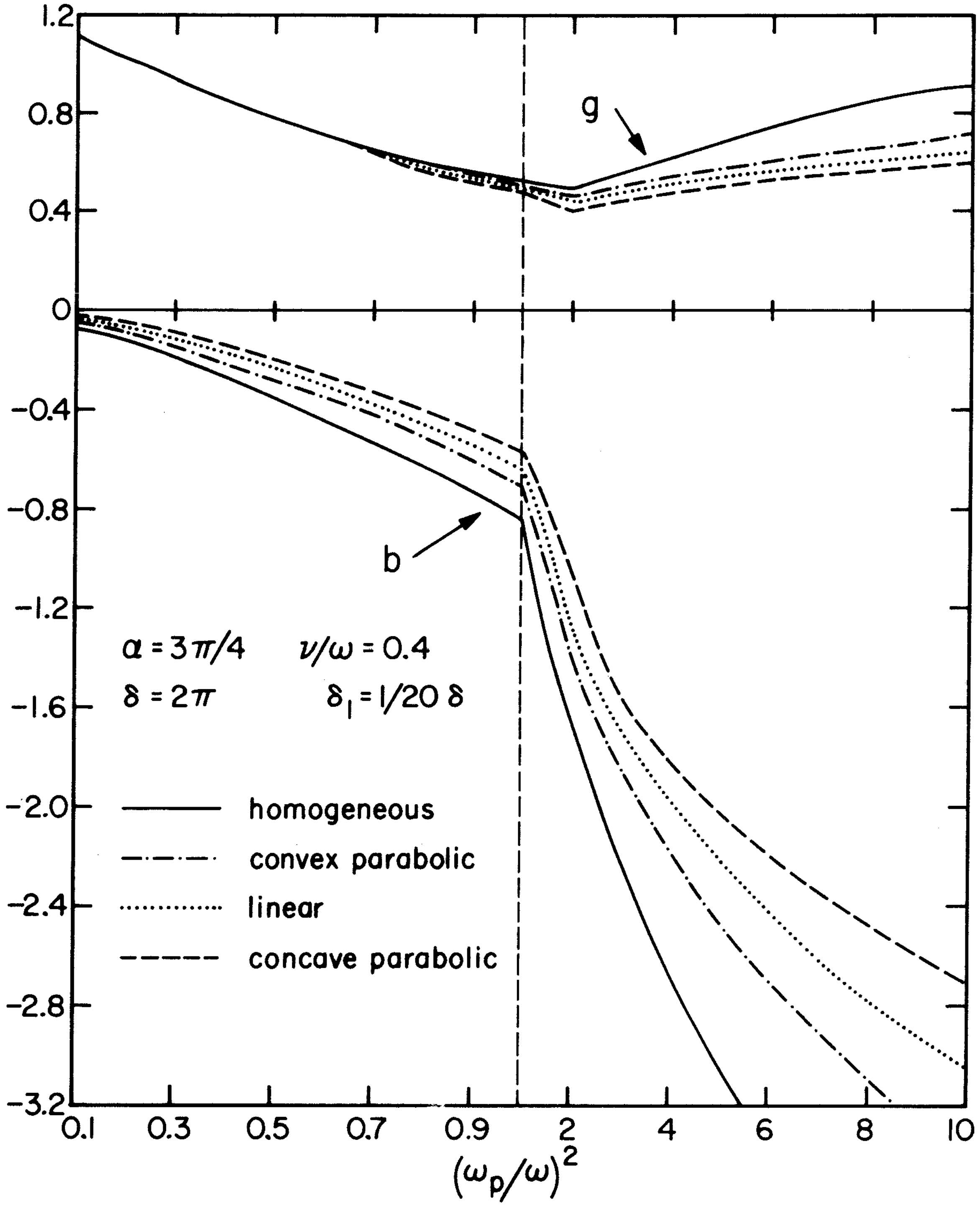


Fig. 5.2 Admittance of a circular aperture antenna coated with an inhomogeneous plasma slab ($\nu/\omega = 0.4$, $\delta_1 = 1/20\ \delta$)

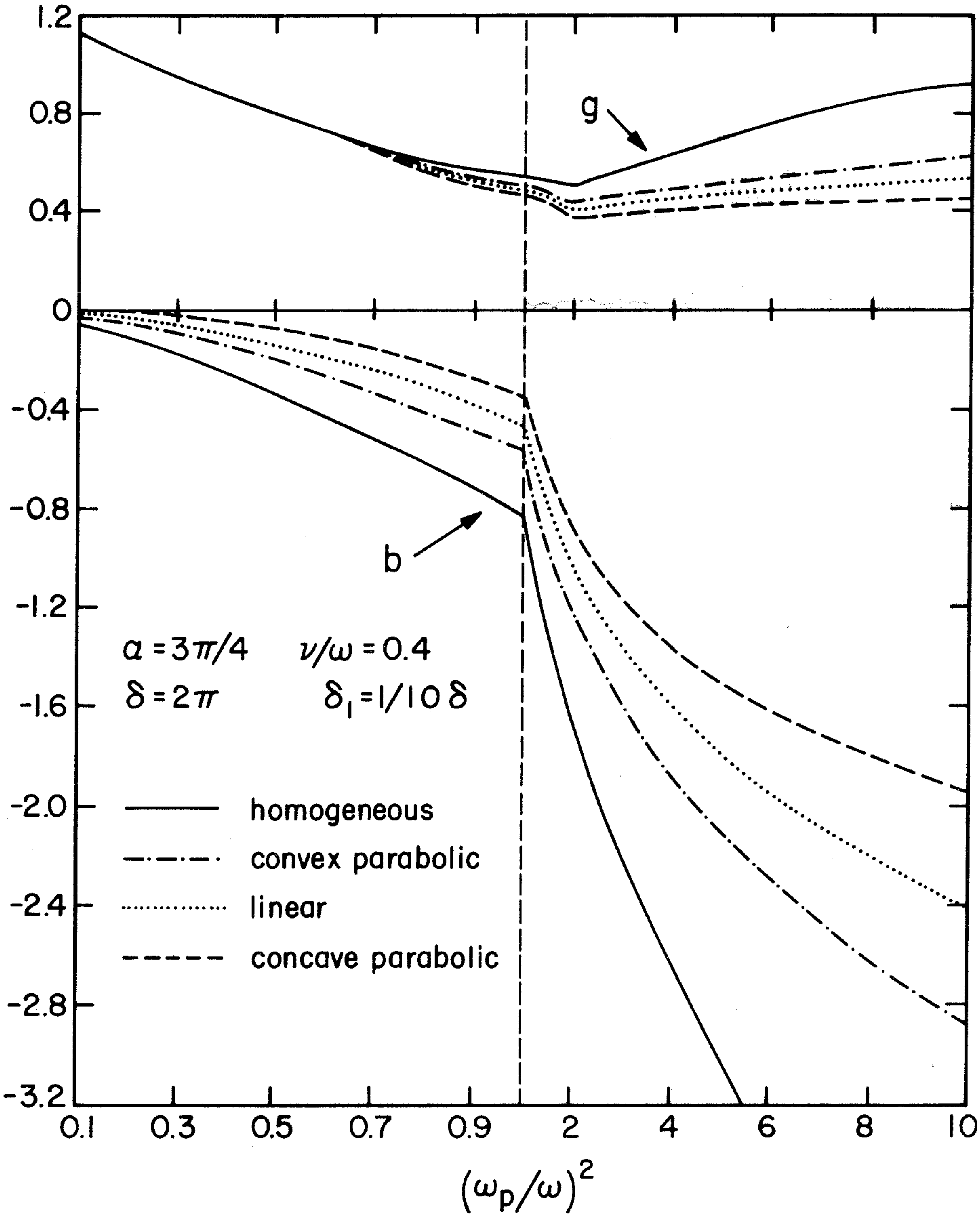


Fig. 5.3 Admittance of a circular aperture antenna coated with an inhomogeneous plasma slab (ν/ω = 0.4, δ_1 = 1/10 δ)

case where the "air gap" is the least and the greatest in the convex parabolic case where the "air gap" is the largest. The change in the conductance is appreciable only for $(\omega_p/\omega)^2 > 0.7$ and it seems to be quite insensitive to the inhomogeneity profile. Comparing Fig. 5-2 with Fig. 5-3 it is seen that increasing the boundary layer thickness to $\delta_1 = (1/10)\delta$ makes the decrease of the susceptance in the presence of "air gap" even more pronounced.

Figures 5-4 and 5-5 apply for an almost lossless plasma with $\nu/\omega = 0.025$. In Fig. 5-4 $\delta_1 = (1/20)\delta$, while in Fig. 5-5 $\delta_1 = (1/10)\delta$. The general behaviour of the curves is similar to those for $\nu/\omega = 0.4$. In particular, the susceptance is seen to be relatively independent of the collision frequency. The conductance decreases with decrease in collision frequency. However, it does not approach zero as the losses become vanishingly small in an inhomogeneous overdense plasma. This is due to the fact that at the point where the permittivity vanishes a real susceptance is added "in series" to Y, as was pointed out at the end of chapter 4.

Finally, the normalized aperture admittance Y/Y_o, and the reflection coefficient, Γ, of the dominant mode electric field are related by

$$(Y/Y_o) = \frac{1 - \Gamma}{1 + \Gamma} \tag{5.2a}$$

and

$$\Gamma = \frac{1 - (Y/Y_o)}{1 + (Y/Y_o)} \tag{5.2b}$$

Hence our knowledge of (Y/Y_o) yields readily information about Γ. In Figure 5-6 $|\Gamma|$, which is a measure of the power reflected back

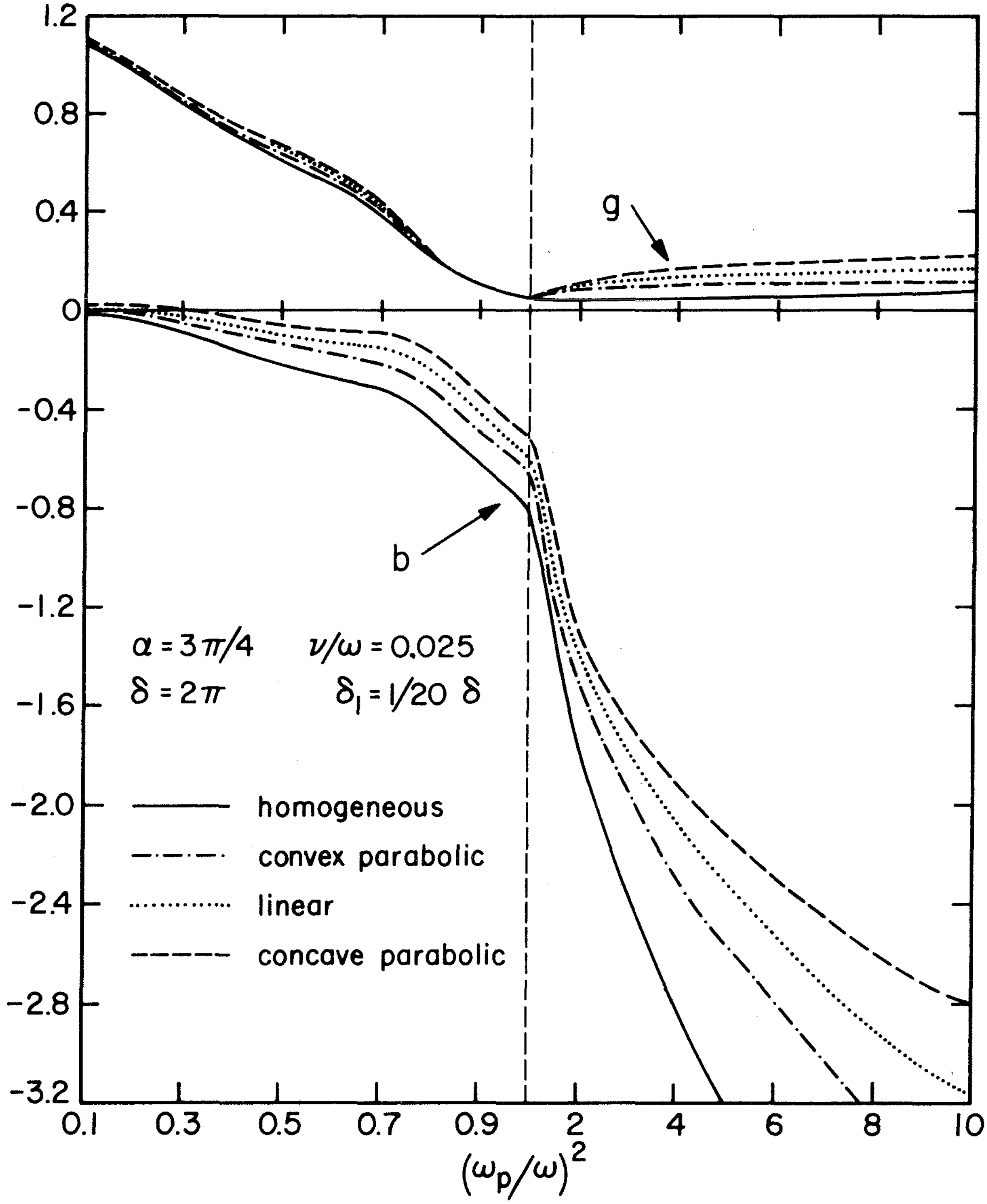


Fig. 5.4 Admittance of a circular aperture antenna coated with an inhomogeneous plasma slab (ν/ω = 0.025, δ_1 = 1/20 δ)

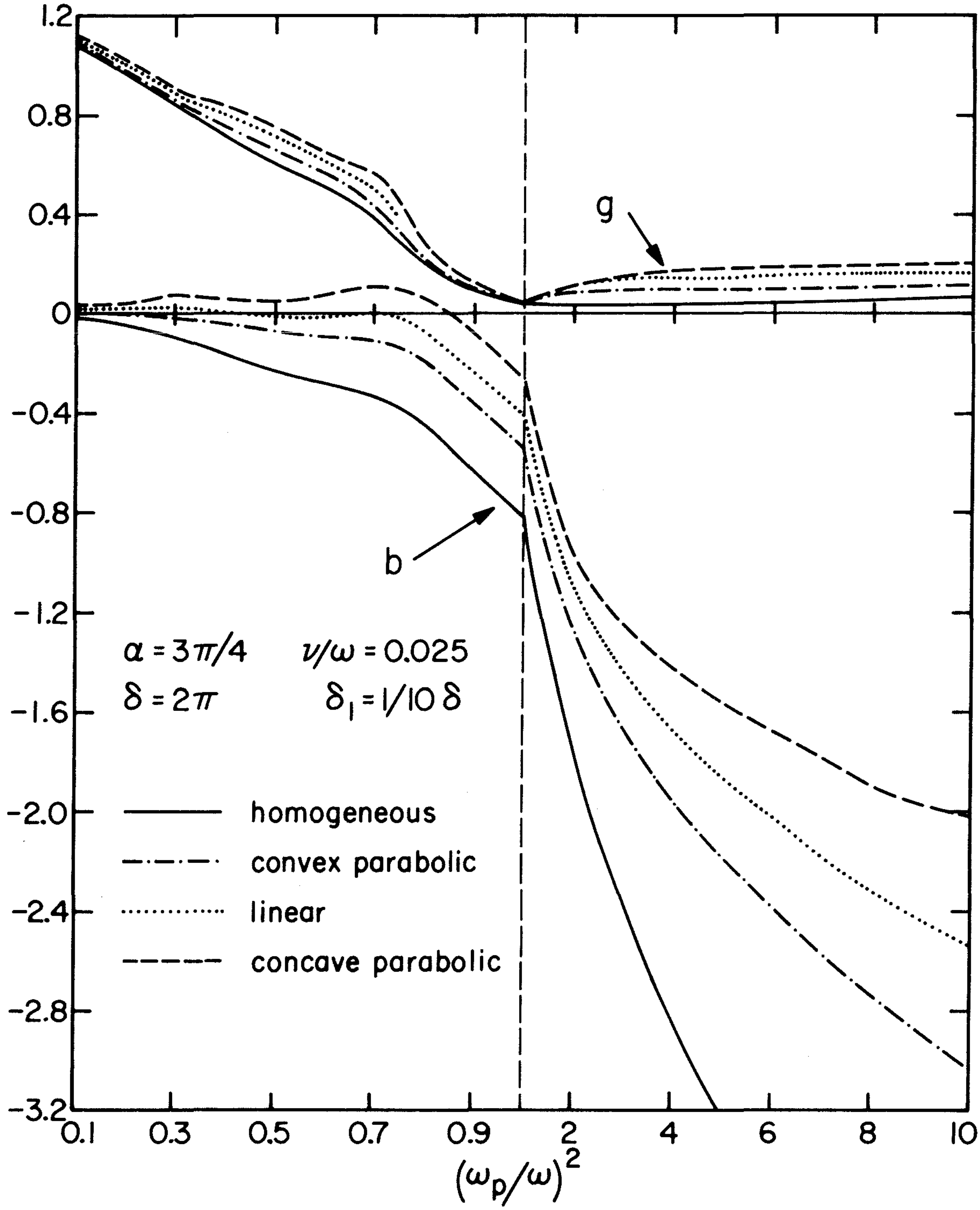


Fig. 5.5 Admittance of a circular aperture antenna coated with an inhomogeneous plasma slab ($\nu/\omega = 0.025$, $\delta_1 = 1/10\ \delta$)

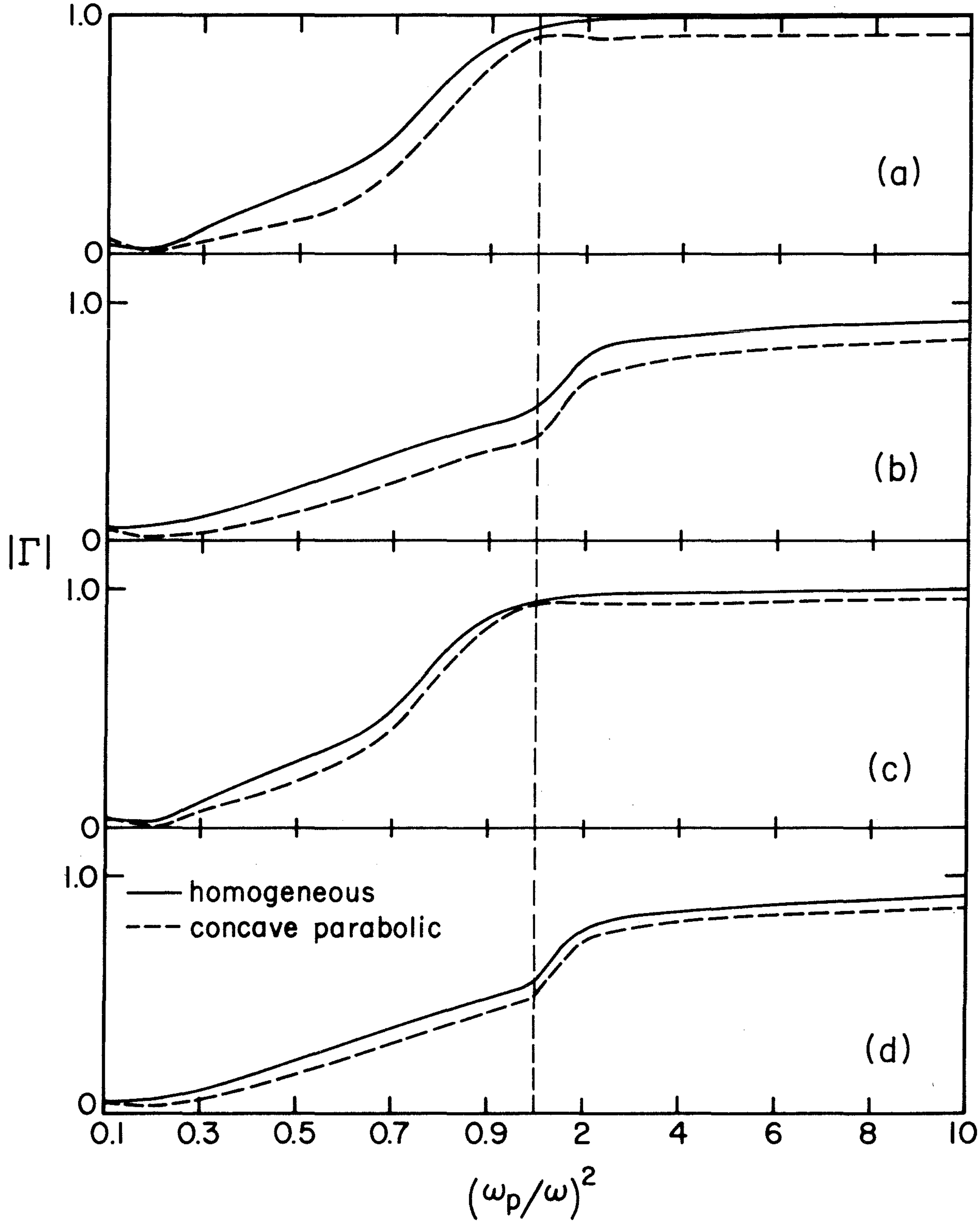


Fig. 5.6 Reflection coefficient of a plasma coated antenna (a) $\nu/\omega = 0.025$, $\delta_1 = 1/10\ \delta$, (b) $\nu/\omega = 0.4$, $\delta_1 = 1/10\ \delta$, (c) $\nu/\omega = 0.025$, $\delta_1 = 1/20\ \delta$, (d) $\nu/\omega = 0.4$, $\delta_1 = 1/20\ \delta$.

into the waveguide, is plotted against $(\omega_p/\omega)^2$ for each case of (ν/ω) and δ_1. Only the results for the homogeneous slab and the inhomogeneous plasma with a concave parabolic profile have been shown. For the two other profiles, the results fall between the ones shown. It can be concluded that the existance of an "air gap" as well as an increase in the collision frequency results in a decrease in $|\Gamma|$. It can also be noted that for an overdense plasma with low losses $|\Gamma|$ is insensitive to changes in $(\omega_p/\omega)^2$.

Next, the radiation pattern of the circular aperture antenna is discussed. Only the principal planes (the xz-plane, and the yz-plane) have been considered, and the ratio of the power radiated at $\theta = 0$ to that radiated in any direction in these planes is calculated in decibels. Thus for the xz-plane ($\phi = 0$, θ variable) we have plotted

$$10 \log_{10} \frac{F(0,0)}{F(\theta,0)} = 20 \log_{10} \left| \frac{E_\phi \ (r,0,0)}{E_\phi \ (r,\theta,0)} \right| ,$$

while for the yz-plane ($\phi = \pi/2$, θ variable)

$$10 \log_{10} \frac{F(0,\pi/2)}{F(\theta,\pi/2)} = 20 \log_{10} \left| \frac{E_\theta(r,0,\pi/2)}{E_\theta(r,\theta,\pi/2)} \right|$$

is plotted against θ. $F(\theta,0)$ and $F(\theta,\pi/2)$ are given by (3.65). Again, we have always taken $\alpha = 3\pi/4$ and $\delta = 2\pi$.

Fig. 5.7 shows the radiation patterns of a circular aperture antenna radiating into free space. The yz-plane radiation pattern shows that as θ approaches $\pi/2$ the electric field in that plane does

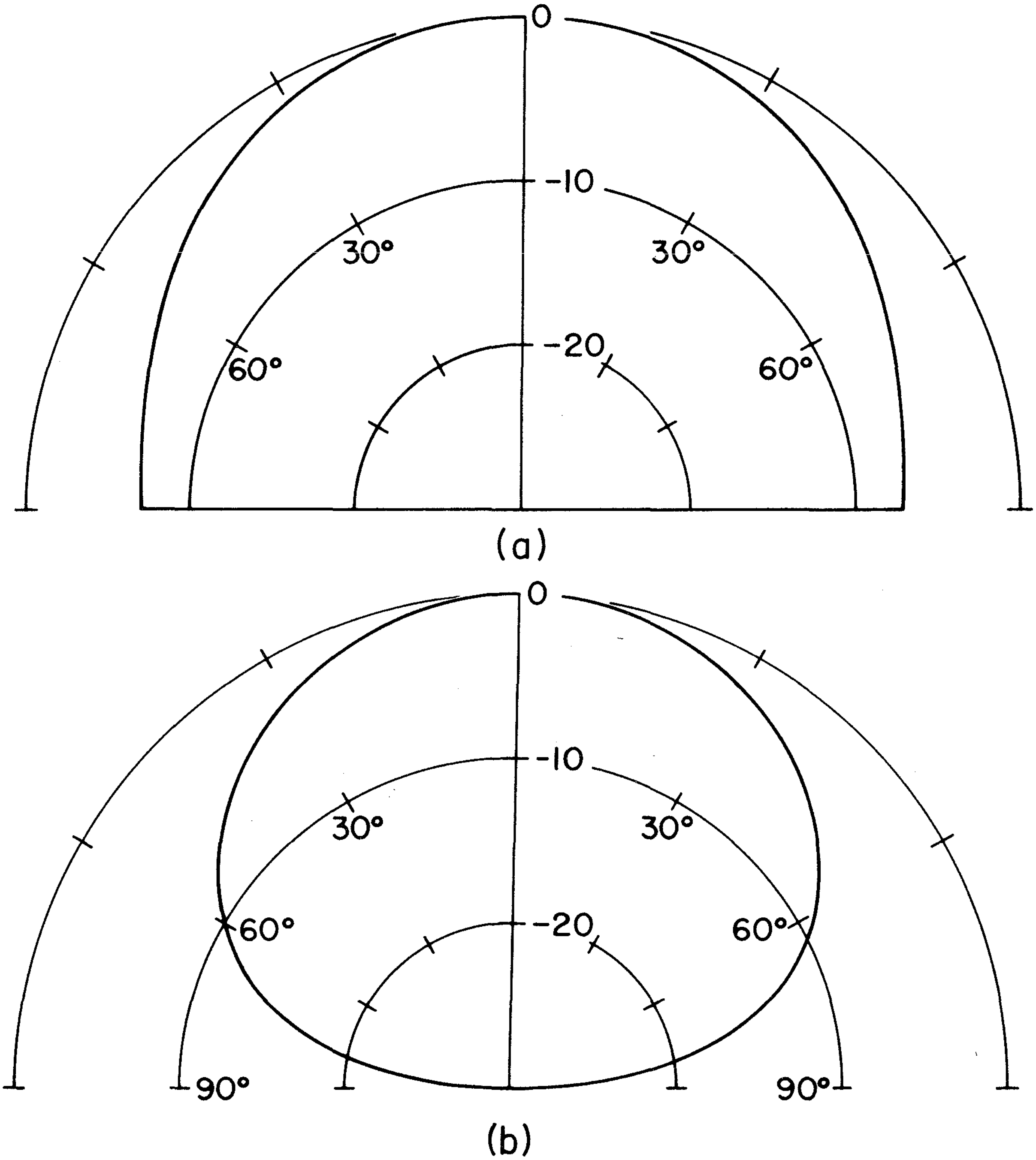


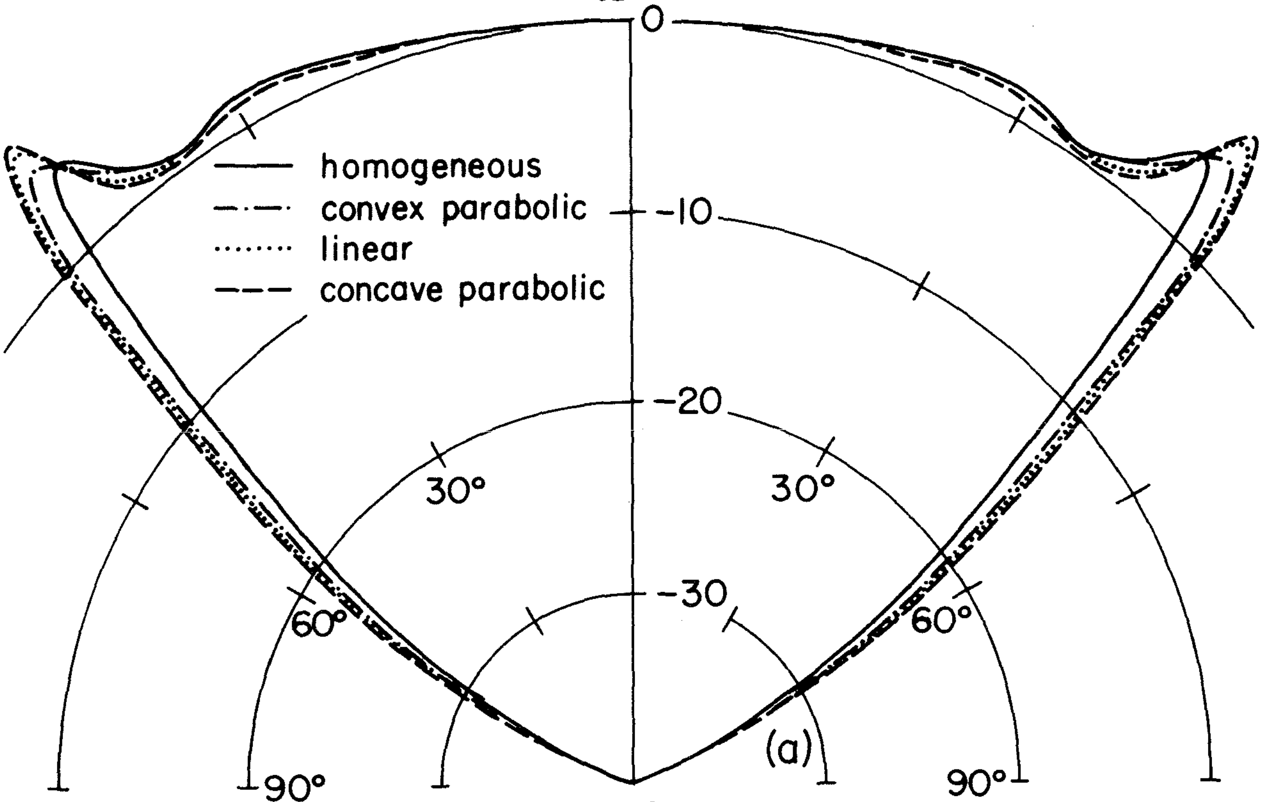
Fig. 5.7 **Radiation pattern** of a circular aperture antenna radiating into free space
(a) in yz-plane (b) in xz-plane.

not vanish, as it does in the xz-plane. This is to be expected since it represents the normal field on the conducting ground plane. However, it is found that even a thin dielectric layer over the aperture drastically reduces the yz-plane radiation near $\theta = \pi/2$ making it approach zero.

In Fig. 5-8 the antenna radiates into a lossless plasma, while in Fig. 5.9 the plasma is lossy with $\nu/\omega = 0.4$. In both cases we have chosen $(\omega_p/\omega)^2 = 1/2$. The yz-plane radiation changes slightly according to the inhomogeneity, while the xz-plane radiation is insensitive of the shape of the electron density profile. When the antenna radiates into a lossless plasma the radiation patterns have a wedge-like shape with a maximum near 45° and a sharp decrease of radiation at greater angles. This fact can be explained simply by remembering Snell's law. Since the plasma has a real positive permittivity smaller than that of free space, there exists a maximum permissible angle for plane waves refracted in the free space region. For $(\omega_p/\omega)^2 = 1/2$, this angle is 45°. When the plasma becomes lossy this fact is no longer true, the peaks disappear and the curves become smoother with the maximum at $\theta = 0$.

For $(\omega_p/\omega)^2 > 1$ the radiation in all directions is very weak, since the waves in the plasma are exponentially damped. In this case the shape of the radiation patterns would be smooth with no peaks, quite similar to Fig. 5.9.

The various advantages of the present method of analyzing aperture antennas have been discussed at length in the course of the report. One main advantage, as regards the numerical computation of the results, should be mentioned here. The time for obtaining numerical solutions of linear second order differential equations has been



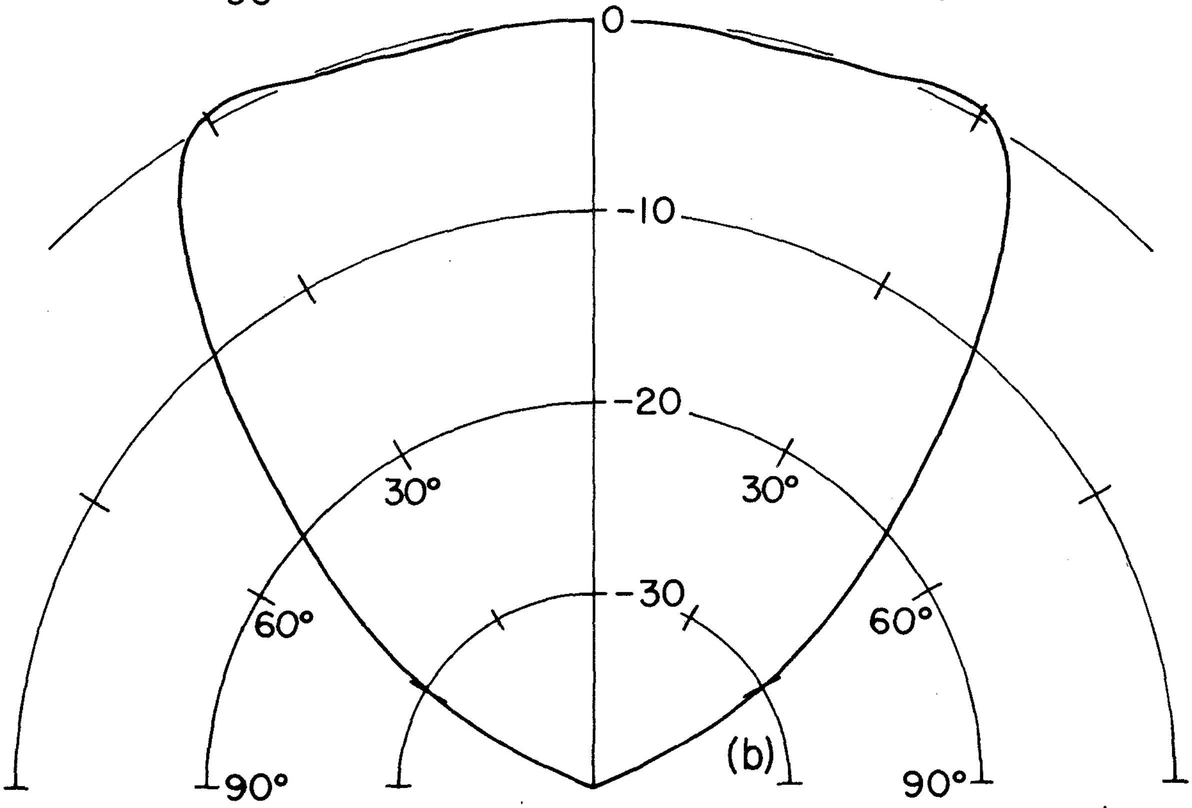


Fig. 5.8 Radiation pattern of a circular aperture antenna coated with a lossless inhomogeneous plasma slab ($\nu/\omega = 0$) $(\omega_p/\omega)^2 = 1/2$ (a) in yz-plane, (b) in xz-plane.

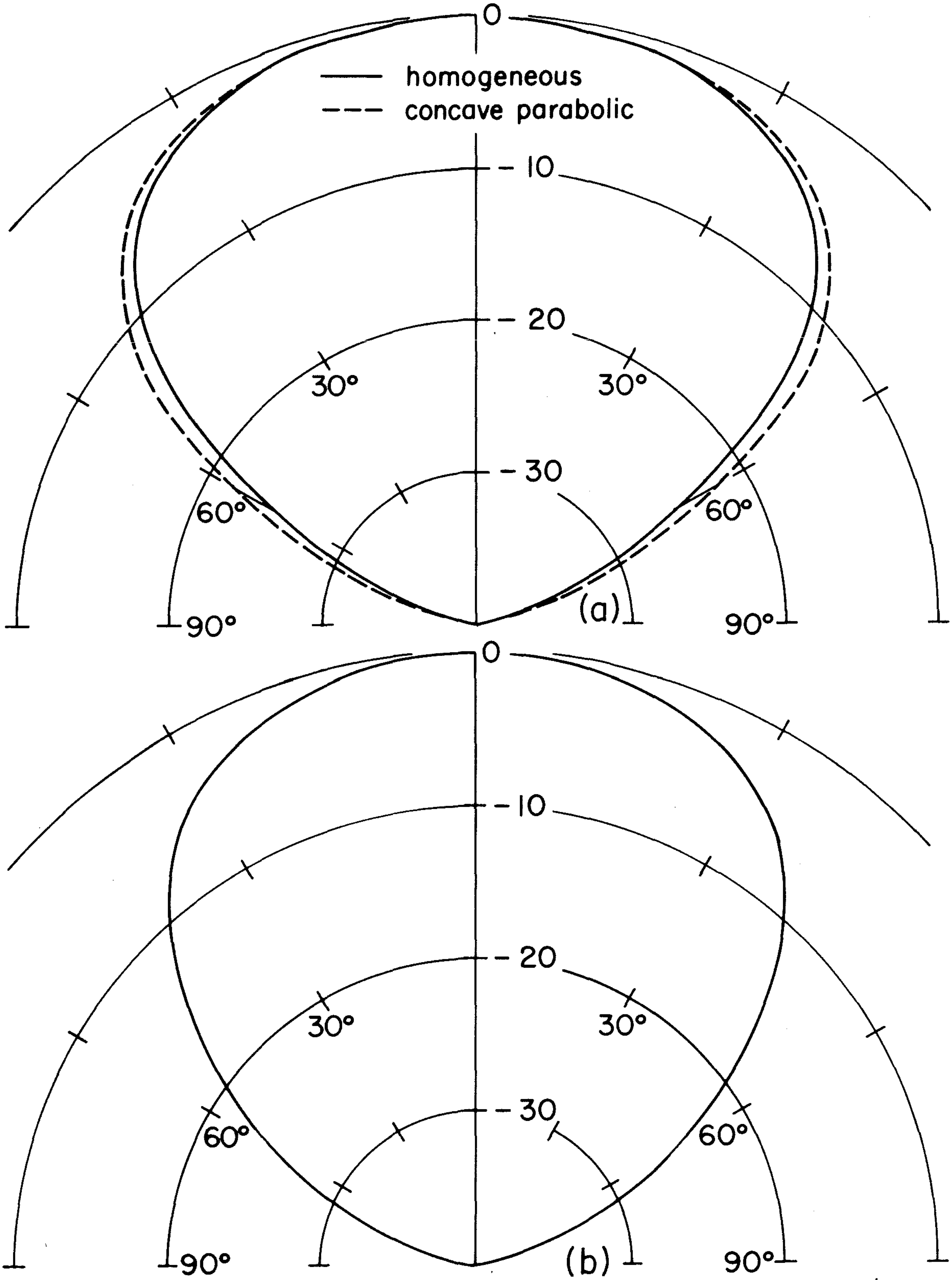


Fig. 5.9 Radiation pattern of a circular aperture antenna coated with a lossy plasma slab ($\nu/\omega = 0.4$) $(\omega_p/\omega)^2 = 1/2$ (a) in yz-plane, (b) in xz-plane.

eliminated in the present formulation. In this report we deal with non-linear first order equations which require a far less time for solution by a computer, than linear second order equations.

We will conclude with a discussion of the possible extensions of the method presented. First, the method could be extended to other than planar geometries. A treatment of cylindrical geometry, for example, would be particularly useful for the discussion of plasma covered cylindrical antennas, and would be quite straightforward to carry out. Second, the method could be used to apply to media other than the ones discussed in this report, such as turbulent, moving and anisotropic media. A medium whose contitutive parameters are functions of all three space coordinates would be harder to treat, and an extension of this method to such media may or may not be possible.

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