Note on space-times that admit constant electromagnetic fields*

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All space–times that admit a covariantly constant, test, electromagnetic field are constructed. All solutions to the Einstein–Maxwell equations with constant electromagnetic field are given.

1. INTRODUCTION

Honig, Schücking, and Vishveshwara recently gave an elegant discussion[1] of the motion of a charged test particle in any space-time $(M, \mathbf{g})$ with a covariantly constant (hence source-free) electromagnetic field **F**. But they did not address the question: Which $(M, \mathbf{g})$ admit such an **F**, even a test field **F**? This note answers this question. The argument will be familiar to mathematicians, but physicists may find it novel. For simplicity, we will confine the work to some local neighborhood in $(M, \mathbf{g})$.

2. HOLONOMY OF SPACE-TIME

Choose any point $p \in M$ in space-time. Fix, once and for all, a simply connected neighborhood U of p. Parallelly carry an orthonormal tetrad O_p around any closed path in U beginning and ending at p, yielding some new tetrad O'_p: $O'_p = \Lambda O_p$, where Λ is some homogeneous Lorentz transformation depending only on the choice of path. The set of all such Λ at p for all possible paths forms Cartan's (local) *holonomy group*[2,3,4] $H(p)$. $H(p)$ is a subgroup of the homogeneous Lorentz group L; $H(p)$ is in fact independent of $p \in U$. Roughly, the higher the symmetry of $(M, \mathbf{g})$, the *smaller* $H(p)$ is.

Let **T** be any geometric-object field on $(M, \mathbf{g})$ for which covariant differentiation ∇ is defined. Can there exist in U a covariantly constant **T**, $\nabla \mathbf{T} = 0$? If so, **T** can be uniquely constructed by giving its value $\mathbf{T}(p)$ at *one* point $p \in U$, and then carrying **T** parallelly all over U along paths in U. This construction must be path-independent; equivalently, $\mathbf{T}(p)$ must be left invariant at p when it is carried parallelly around any closed path beginning and ending at p. That is, $\mathbf{T}(p)$ must be invariant under the action of $H(p)$:

Lemma (see e.g., Schouten[2]*):* $(M, \mathbf{g})$ (locally) admits a covariantly constant, test, field **T** iff there exists $\mathbf{T}(p)$ at any point $p \in M$, invariant under the holonomy group $H(p)$.

3. CONSTANT, TEST, ELECTROMAGNETIC FIELD

The differential problem, "solve $\nabla \mathbf{F} = 0$," is thus reduced to an algebraic problem (Cartan's favorite trick!): Choose a point p; find an electromagnetic field tensor $\mathbf{F}(p) \neq 0$ such that its invariance group $G[\mathbf{F}(p)]$,

$$G[\mathbf{F}(p)] \equiv \left\{ \Lambda \in L \mid \Lambda \mathbf{F}(p) = \mathbf{F}(p) \right\},$$

contains the holonomy group: $H(p) \subseteq G[\mathbf{F}(p)] \subseteq L$.

Choose a favored observer at p with tetrad $O_p = (\mathbf{e}_{\hat{t}}, \mathbf{e}_{\hat{x}}, \mathbf{e}_{\hat{y}}, \mathbf{e}_{\hat{z}})$ so that the tetrad components $F_{\mu\nu}$ of $\mathbf{F}(p)$ reduce to one of two canonical forms[5]: "$\mathbf{F}(p)$ null" or "$\mathbf{F}(p)$ nonnull."

(A) $\mathbf{F}(p)$ null: $F_{\hat{t}\hat{x}} = -F_{\hat{x}\hat{t}} = F_{\hat{x}\hat{z}} = -F_{\hat{z}\hat{x}} = A = \text{const}$, other components vanish. $G[\mathbf{F}(p)]$ is generated by the two null rotations[6] which leave invariant the null direction defined bv $\mathbf{k} = \mathbf{e}_{\hat{t}} + \mathbf{e}_{\hat{z}}$. Define the spinor[7] $o^A(p)$ by $o^A(p)\bar{o}^{\dot{A}}(p) = k^{A\dot{A}}$; then $G[\mathbf{o}^A(p)] = G[\mathbf{F}(p)]$. Therefore (by lemma) $(M, \mathbf{g})$ must admit a covariantly constant spinor field $\mathbf{o}^A$ (equivalently, a covariantly constant, complex, null bivector field $\mathbf{F} + i^*\mathbf{F}$). By a result of Ehlers and Kundt,[8] it is necessary and sufficient that in some local coordinates

$$\mathbf{g} = 2K(u, x, y)du^2 + 2du\,dv + dx^2 + dy^2, \tag{1a}$$

and

$$\mathbf{F} = 2^{1/2} A\, du \wedge dx. \tag{1b}$$

(B) $\mathbf{F}(p)$ nonnull: $F_{\hat{t}\hat{z}} = -F_{\hat{z}\hat{t}} = A \cos\theta = \text{const.}$, $F_{\hat{x}\hat{y}} = -F_{\hat{y}\hat{x}} = A \sin\theta = \text{const.}$, other components vanish. $G[\mathbf{F}(p)]$ is the direct product of the one-parameter group of boosts in the (tz) plane with the commuting one-parameter group of rotations in the (xy) plane. It is a fundamental result[3] that if the tangent space T_p is reducible under $H(p)$, then $(M, \mathbf{g})$ is correspondingly reducible into the direct product of (pseudo-) Riemannian manifolds of lower dimension. Here, T_p reduces to the (tz) and (xy) planes under $H(p)$, so $(M, \mathbf{g}) = (M_+, \mathbf{g}_+) \otimes (M, g_-)$, where $(M_+, \mathbf{g}_+)$ is a Lorentzian 2-manifold and $(M_-, \mathbf{g}_-)$ is a Riemannian 2-manifold. The vector fields $\mathbf{e}_{\hat{t}}$ and $\mathbf{e}_{\hat{z}}$ lie entirely in $(M_+, \mathbf{g}_+)$; $\mathbf{e}_{\hat{x}}$ and $\mathbf{e}_{\hat{y}}$ lie entirely in $(M_-, \mathbf{g}_-)$. In some coordinates,

$$\mathbf{g} = \mathbf{g}_+ + \mathbf{g}_-, \tag{2a}$$

where

$$\mathbf{g}_+ = g_{+ab}(t, z)dx^a dx^b, \qquad x^a \equiv (t, z), \tag{2b}$$

$$\mathbf{g}_- = g_{-ij}(x, y)dx^i dx^j, \qquad x^i \equiv (x, y), \tag{2c}$$

and

$$\mathbf{F} = A \cos\theta(-g_+)^{1/2} dt \wedge dz + A \sin\theta(g_-)^{1/2} dx \wedge dy. \tag{2d}$$

Equations (1) and (2) give all space-times $(M, \mathbf{g})$, and all test fields **F**, that solve $\nabla \mathbf{F} = 0$.

4. ELECTROVAC SOLUTIONS

Now impose the Einstein-Maxwell equations for a nontest **F**, to find all solutions with covariantly constant **F**. The resulting electrovac space-times are well known.

(A) **F** null: The Einstein-Maxwell equations for Eqs. (1) read

$$(\partial_x^2 + \partial_y^2)K(u, x, y) = -4A^2. \tag{3}$$

The general solution K is a linear superposition, $K = K_{\text{em}} + K_{\text{grav}}$, where K_{em} is the particular solution,

$$K_{\text{em}} = -A^2(x^2 + y^2),$$

and K_{grav} is any homogeneous solution,

$$(\partial_x^2 + \partial_y^2)K_{\rm grav}(u,x,y) = 0.$$

$K_{\rm em}$ represents the transverse, "monopole," always focusing gravitational disturbance due to **F**; this disturbance is homogeneous for local observers (despite the dependence of $K_{\rm em}$ on x and y). $K_{\rm grav}$ represents an arbitrary "plane-fronted gravitational wave with parallel rays"[8] ("pp wave"). So the electrovac space-time given by Eqs. (1) and (3) describes an arbitrary, gravitational pp wave traversing a region of constant, null **F**, such that the wave direction is everywhere parallel to the Poynting vector.

(B) F nonnull: The Einstein-Maxwell questions for Eqs. (2) imply that (M_+, g_+) is a two-dimensional anti-deSitter space-time[9] of radius A^{-1}, and that (M_-, g_-) is a 2-sphere of radius A^{-1}. This solution is the "Bertotti-Robinson magnetic universe"; for discussions see Bertotti,[10] Robinson,[11] and Lindquist,[12] and Exercise 32.1 of Misner, Thorne, and Wheeler.[13]

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