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MULTIPLE SOLUTIONS OF SINGULAR PERTURBATION PROBLEMS*

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Abstract. Under certain conditions on $g(x, u)$ we establish the existence and asymptotic behavior for small $\varepsilon > 0$ of *multiple* asymptotic solutions of the nonlinear boundary value problem

$$\varepsilon u'' + u' - g(x, u) = 0, \qquad 0 < x < 1,$$

$$u'(0) - au(0) = A \geqq 0, \qquad a > 0,$$

$$u'(1) + bu(1) = B > 0, \qquad b > 0.$$

Formal techniques of singular perturbation theory clearly reveal the mechanism which controls the appearance of multiple solutions. Their existence is then established rigorously by iteration schemes and the so-called "shooting method" for ordinary differential equations.

1. Introduction. We shall establish the existence and asymptotic behavior for small $\varepsilon > 0$ of multiple asymptotic solutions of the nonlinear boundary value problem

$$\varepsilon u'' + u' - g(x, u) = 0, \qquad 0 < x < 1, \tag{1.1}$$

$$u'(0) - au(0) = A \geqq 0, \qquad a > 0, \tag{1.2}$$

$$u'(1) + bu(1) = B > 0, \qquad b > 0. \tag{1.3}$$

In general, a function $u(x, \varepsilon)$ is said to be an asymptotic solution to order $O(\varepsilon^n)$ if the function satisfies the differential equation and boundary conditions to order $O(\varepsilon^n)$ as $\varepsilon \to 0$. More precisely, for this paper, we adopt the following definition.

DEFINITION. A function $u(x, \varepsilon)$ is an *asymptotic solution* of the boundary value problem (1.1)–(1.3) if $u(x, \varepsilon)$ satisfies (1.1), (1.2) and $u'(1, \varepsilon) + bu(1, \varepsilon) = B + O(\varepsilon)$ as $\varepsilon \to 0$.

Problems of this type occur in chemical reactor theory, and it has been found recently [1]–[3] that multiple *stable* steady states can occur in certain adiabatic tubular reactors. By considering the relevant physics in the various parts of the reactor, or equivalently by applying the formal techniques [4] of singular perturbation theory, the mechanism by which the multiple solutions occur is clearly revealed. We do this briefly in § 2, and this will provide us with useful insight regarding the properties of the equation and its solutions. The rest of the paper is devoted to rigorously establishing the existence and asymptotic behavior for small $\varepsilon > 0$ of the multiple asymptotic solutions of the nonlinear two-point boundary value problem (1.1)–(1.3).

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Our entire analysis is based on the so-called "shooting method" for ordinary differential equations. Accordingly, in § 3 we study the initial value problem

$$\varepsilon u'' + u' - g(x, u) = 0, \qquad x > 0, \tag{1.4}$$

$$u(0) = h \geqq 0, \tag{1.5}$$

$$u'(0) = A + ah. \tag{1.6}$$

Note that the initial conditions (1.5), (1.6) imply that the boundary condition (1.2) is satisfied. Specific properties of $g(x, u)$ are stated, and we then prove that for all $\varepsilon > 0$ the initial value problem (1.4)–(1.6) possesses a unique solution $u(x, \varepsilon, h)$ and that $u(x, \varepsilon, h)$ and $u'(x, \varepsilon, h)$ depend continuously on h for $h \geqq 0$, and $u(x, \varepsilon, h)$ depends continuously on ε for sufficiently small $\varepsilon > 0$.

In § 4 we show that the boundary value problem (1.1)–(1.3) possesses many distinct (and we state precisely how many) asymptotic solutions. This is accomplished by demonstrating that there exist many distinct values of h such that for each of these values of h the solution $u(x, \varepsilon, h)$ of the initial value problem (1.4)–(1.6) also satisfies $u'(1, \varepsilon, h) + bu(1, \varepsilon, h) = B + O(\varepsilon)$ for sufficiently small $\varepsilon > 0$. Furthermore, we show that on the subinterval $0 < \delta \leqq x \leqq 1$ each asymptotic solution $u(x, \varepsilon)$ possesses the property that $u(x, \varepsilon) - v(x) = O(\varepsilon)$ and $u'(x, \varepsilon) - v'(x) = O(\varepsilon)$ for sufficiently small $\varepsilon > 0$, where $v(x)$ is the solution of an appropriate reduced problem (that is, the problem $v' - g(x, v) = 0$ subject to an appropriate boundary condition).

Our analysis and specific results are confined to the problem (1.1)–(1.3) for simplicity. However, our proofs and results can be extended to problems more general than (1.1). For example, it is relatively easy to extend our proofs to the case where we allow u' in (1.1) to have a positive nonlinear coefficient $f(x, u)$. Furthermore, with somewhat more work the results of the present paper taken together with those of [5] allow us to obtain quite similar results for equations of the form $\varepsilon u'' + f(x, u, u')u' - g(x, u) = 0$ for classes of f and g which occur in problems in fluid and gas dynamics.

2. Formal methods and multiple solutions. The reason for the existence of multiple solutions is clearly revealed by an application of the formal matching techniques of singular perturbation theory [4]. For $0 < \varepsilon \ll 1$ we find that there is a boundary layer of thickness $O(\varepsilon)$ near $x = 0$. Away from this boundary layer the first term of the asymptotic expansion (the outer expansion) is given by

$$u' - g(x, u) = 0, \qquad 0 < x \leqq 1, \tag{2.1}$$

$$u'(1) + bu(1) = B. \tag{2.2}$$

Evaluating (2.1) at $x = 1$, we find that (2.1) and (2.2) together imply that

$$g(1, u(1)) = B - bu(1). \tag{2.3}$$

Clearly, the solutions of (2.3) provide the proper initial conditions for (2.1). Figure 1 illustrates a case where there are four roots $\alpha_i, i = 1, \cdots, 4$, of (2.3) for some nonlinearity $g = g(u)$ which is sketched.

Our formalism suggests that there are as many solutions for small $\varepsilon > 0$ as there are roots of (2.3) (later, we shall have to modify this slightly), and the first

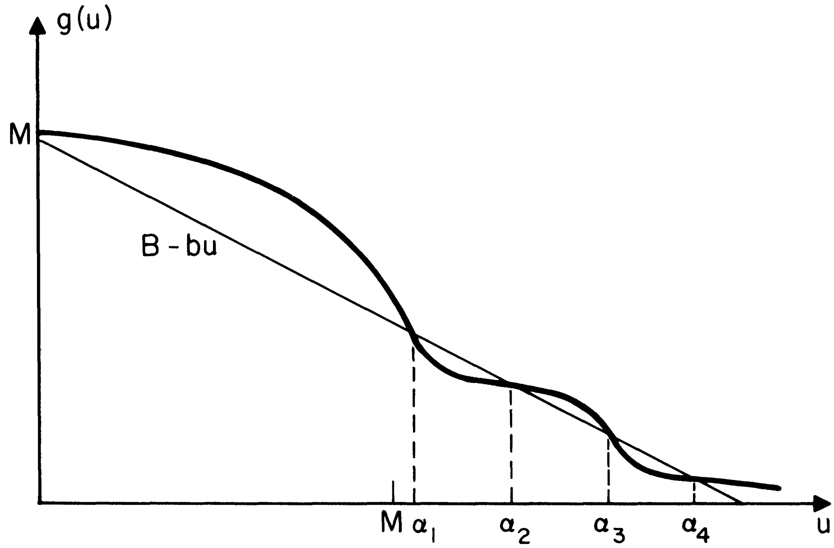

FIG. 1.

term in the outer expansion of each solution is given by

$$v' - g(x, v) = 0, \tag{2.4}$$

$$v(1) = \alpha_i. \tag{2.5}$$

In the boundary layer we introduce a new length $\tilde{x} = x/\varepsilon$ and let $u(x) \equiv u(\varepsilon\tilde{x}) \equiv w(\tilde{x})$. Then, the first term of the expansion (the inner expansion) near $x = 0$ is given by

$$w'' + w' = 0, \tag{2.6}$$

$$w'(0) - aw(0) = A, \tag{2.7}$$

$$w(\infty) = v(0). \tag{2.8}$$

The boundary condition (2.8) expresses the proper condition for matching the inner and outer solutions.

This procedure can be continued to generate succeeding terms in an asymptotic expansion, and from this procedure we could, in fact, construct an expansion which is uniformly valid on the interval $0 \leqq x \leqq 1$. Alternatively, we could employ a "two-timing" formalism to obtain the same answer. We shall not pursue this further, however, because the mechanism controlling the appearance of multiple solutions when ε is small is already clear. Quite simply, multiplicity is governed by the roots, $\alpha = \alpha_i$, of the equation

$$g(1, \alpha) = B - b\alpha. \tag{2.9}$$

Each root α_i of (2.9) gives rise to an appropriate "reduced problem" (2.4), (2.5), and as we shall see, each solution $v_i(x)$ of (2.4), (2.5) can be an asymptotic solution of (1.1)–(1.3) on any subinterval $0 < \delta \leqq x \leqq 1$ for sufficiently small $\varepsilon > 0$. (We shall also see that sufficiently small values of α_i may not generate an asymptotic solution.) We shall now proceed to give a rigorous investigation of the existence and multiplicity of asymptotic solutions of (1.1)–(1.3).

3. The shooting method. For all the work in §§ 3 and 4 the conditions imposed on g will be:

H.1 $g(x, u)$ is continuously differentiable in the region

$$R = \{(x, u) | 0 \leqq x \leqq 1, u \geqq 0\}.$$

H.2: $g(x, u) \geqq 0$ on R.

H.3: $0 \leqq u_1 \leqq u_2$ implies that $g(x, u_1) \geqq g(x, u_2)$.

H.4: $g(x, u)$ satisfies a Lipschitz condition in R; that is, there exists a constant k such that for all $(x, u) \in R$,

$$|g(x, u) - g(x, v)| \leqq k|u - v|.$$

H.5: The equation $g(1, \alpha) = B - b\alpha$ possesses N roots α_i, $i = 1, \cdots, N$, such that $0 < \alpha_1 \leqq \alpha_2 \leqq \cdots \leqq \alpha_N$.

Conditions H.1 to H.4 imply that as a function of u for $u \geqq 0$ the nonlinearity $g(x, u)$ is a reasonably smooth, positive, Lipschitz continuous, nonincreasing function. Condition H.5 simply guarantees that there exists at least one root of $g(1, \alpha) = B - b\alpha$, and from the formalism of § 2 we suspect that for small positive $\varepsilon > 0$ a solution of (1.1)–(1.3) will not exist if a root α_1 does not exist. Note that the conditions H.1 to H.3 imply that $g(x, u)$ is uniformly bounded above on R. Thus, $g(x, u) \leqq M < \infty$ on R, and since g is positive and monotone nonincreasing in u, we can take

$$M = \max_{0 \leqq x \leqq 1} [g(x, 0)].$$

For the rest of this paper M shall have this meaning. We wish to point out that these conditions are satisfied in many rate functions in chemical kinetics.

Write the differential equation (1.4) as $\varepsilon u'' + u' = g(x, u)$, and consider it as a first order equation in u' with initial condition $u'(0) = A + ah$. Then,

$$u'(x) = (A + ah)\, e^{-x/\varepsilon} + \frac{1}{\varepsilon}\int_0^x e^{-(x-t)/\varepsilon} g(t, u(t))\, dt. \tag{3.1}$$

Clearly, $u'(x) \geqq 0$ on $0 \leqq x \leqq 1$ if $u(x)$ exists on $0 \leqq x \leqq 1$. Integrating (3.1) and using the condition that $u(0) = h$, and performing an integration by parts, we obtain

$$u(x) = h + \varepsilon(A + ah)(1 - e^{-x/\varepsilon}) + \int_0^x [1 - e^{-(x-t)/\varepsilon}] g(t, u(t))\, dt. \tag{3.2}$$

For later convenience we shall write (3.1) and (3.2) respectively as

$$u'(x) = S[u], \qquad u(x) = T[u], \tag{3.3}$$

where the operators S and T are defined as

$$S[u] \equiv (A + ah)\, e^{-x/\varepsilon} + \frac{1}{\varepsilon}\int_0^x e^{-(x-t)/\varepsilon} g(t, u(t))\, dt, \tag{3.4}$$

$$T[u] = h + \varepsilon(A + ah)(1 - e^{-x/\varepsilon}) + \int_0^x [1 - e^{-(x-t)/\varepsilon}] g(t, u(t))\, dt. \tag{3.5}$$

The conditions H.2 and H.3 imply the following lemma which is basic for all of our results.

LEMMA 3.1. *Let y_1 and y_2 be continuously differentiable nonnegative functions of x defined on $0 \leqq x \leqq 1$. If $y_1(x) \leqq y_2(x)$, then $S[y_1] \geqq S[y_2]$ and $T[y_1] \geqq T[y_2]$.*

Define the sequences $\{u_n(x)\}$ and $\{u_n'(x)\}$ by

$$u_0(x) \equiv h, \qquad u_{n+1}(x) = T[u_n], \qquad n = 0, 1, 2, \cdots, \tag{3.6}$$

$$u_0'(x) \equiv 0, \qquad u_{n+1}'(x) = S[u_n], \qquad n = 0, 1, 2, \cdots. \tag{3.7}$$

Clearly, $u_1(x) \geqq u_0(x) \equiv h$, $u_2(x) \geqq u_0(x) \equiv h$, $u_1'(x) \geqq u_0'(x) \equiv 0$ and $u_2'(x) \geqq u_0'(x) \equiv 0$. These facts and Lemma 3.1 immediately imply the next lemma.

LEMMA 3.2.

$$u_0 \leqq u_1, \quad u_1 \geqq u_2, \quad u_2 \leqq u_3, \quad u_3 \geqq u_4, \quad \cdots,$$
$$u_0 \leqq u_2, \quad u_1 \geqq u_3, \quad u_2 \leqq u_4, \quad u_3 \geqq u_5, \quad \cdots,$$
$$u_0' \leqq u_1', \quad u_1' \geqq u_2', \quad u_2' \leqq u_3', \quad u_3' \geqq u_4', \quad \cdots,$$
$$u_0' \leqq u_2', \quad u_1' \geqq u_3', \quad u_2' \leqq u_4', \quad u_3' \geqq u_5', \quad \cdots;$$

that is, for any positive integers k and l,

$$u_0 \leqq u_2 \leqq u_4 \leqq \cdots \leqq u_{2l} \leqq \cdots \leqq u_{2k+1} \leqq \cdots \leqq u_5 \leqq u_3 \leqq u_1, \tag{3.8}$$

and

$$u_0' \leqq u_2' \leqq u_4' \leqq \cdots \leqq u_{2l}' \leqq \cdots \leqq u_{2k+1}' \leqq \cdots \leqq u_5' \leqq u_3' \leqq u_1'. \tag{3.9}$$

That the alternating pincer movement (for fixed h) converges to the unique solution of (1.4)–(1.6) is the content of the following theorem.

THEOREM 3.3. *Let $g(x, u)$ satisfy* H.1 *to* H.4. *Then, for any $h \geqq 0$ the sequences $\{u_n(x)\}$ and $\{u_n'(x)\}$ defined by* (3.6) *and* (3.7) *converge respectively to the unique solution $u(x)$ of the initial value problem* (1.4)–(1.6) *and to its derivative $u'(x)$ on the interval* $0 \leqq x \leqq 1$.

Proof. First, we prove that for all $n \geqq 1$ we have

$$|u_n - u_{n-1}| \leqq \frac{(A + ah)k^n x^n}{n!} + \frac{M^n k^n x^n}{n!}, \tag{3.10}$$

$$|u_n' - u_{n-1}'| \leqq \frac{(A + ah)k^n x^{n-1}}{(n-1)!} + \frac{M^n k^n x^n}{(n-1)!}. \tag{3.11}$$

Here k is the Lipschitz constant of condition H.4, and M is the uniform upper bound on $g(x, u)$.

We now proceed by induction. Using the fact that $\varepsilon(1 - e^{-x/\varepsilon}) \leqq x$, we obtain

$$\begin{aligned} |u_1 - u_0| &= \varepsilon(A + ah)(1 - e^{-x/\varepsilon}) + \int_0^x [1 - e^{-(x-t)/\varepsilon}]g(t, u(t))\,dt \\ &\leqq (A + ah)x + Mx \leqq (A + ah)kx + Mkx. \end{aligned} \tag{3.12}$$

We have used the fact that $x^2/2 \leqq x$ on $0 \leqq x \leqq 1$, and we see also that we must take $k \geqq 1$. Similarly,

$$
\begin{aligned}
|u_1' - u_0'| &= (A + ah)\, e^{-x/\varepsilon} + \frac{1}{\varepsilon}\int_0^x e^{-(x-t)/\varepsilon} g(t, u(t))\, dt \\
&\leqq (A + ah) + \frac{M}{\varepsilon}\int_0^x e^{-(x-t)/\varepsilon}\, dt \qquad (3.13)\\
&\leqq (A + ah) + Mk.
\end{aligned}
$$

Hence, (3.10) and (3.11) are valid for $n = 1$. Now, assume that (3.10) and (3.11) are valid for all integers up to and including a given integer n. We must prove that they are valid for $n + 1$. Using H.4 and the induction hypotheses (3.10) and (3.11), we obtain

$$
\begin{aligned}
|u_{n+1} - u_n| &\leqq \int_0^x [1 - e^{-(x-t)/\varepsilon}]|g(t, u_n(t)) - g(t, u_{n-1}(t))|\, dt \\
&\leqq k \int_0^x |u_n(t) - u_{n-1}(t)|\, dt \\
&\leqq k \int_0^x \left[\frac{(A + ah)k^n t^n}{n!} + \frac{M^n k^n t^n}{n!}\right] dt \\
&= \frac{(A + ah)k^{n+1}x^{n+1}}{(n + 1)!} + \frac{M^{n+1}k^{n+1}x^{n+1}}{(n + 1)!}.
\end{aligned}
$$

To obtain the last line of the inequality we have used the facts that we can take $M \geqq 1$ and $x^{n+2}/(n + 2) \leqq 1$ on $0 \leqq x \leqq 1$. Similarly,

$$
|u_{n+1}' - u_n'| \leqq \frac{(A + ah)k^{n+1}x^n}{n!} + \frac{M^{n+1}k^{n+1}x^n}{n!}.
$$

Therefore, we have verified that (3.10) and (3.11) hold for all $n \geqq 1$. Now, write $u_n(x)$ as

$$
u_n(x) = u_0(x) + \sum_{j=1}^{n} [u_j(x) - u_{j-1}(x)] \qquad (3.14)
$$

with a similar formula for $u_n'(x)$. The estimates (3.10) and (3.11) immediately imply that in the limit as $n \to \infty$ the series in (3.14) converges absolutely and uniformly on the interval $0 \leqq x \leqq 1$. Consequently, the limit functions $u(x) = \lim_{n\to\infty} [u_n(x)]$ and $u'(x) = \lim_{n\to\infty} [u_n'(x)]$ exist and are continuous (since each $u_n(x)$ and $u_n'(x)$ is continuous), and it then follows in the usual manner that $u(x)$ is a solution of (1.4)–(1.6) on $0 \leqq x \leqq 1$ with derivative $u'(x)$.

We shall now prove the uniqueness of the solution $u(x)$. Suppose that $\tilde{u}(x)$ is another solution. Then, $\tilde{u}(x) \geqq u_0(x) \equiv h$, and hence,

$$
T[\tilde{u}] = \tilde{u} \leqq u_1 = T[u_0].
$$

In the same way we show that $u_{2n} \leqq \tilde{u} \leqq u_{2n+1}$. As we have just showed, the sequence $\{u_n(x)\}$ converges (i.e., the pincer closes). Then,

$$\tilde{u}(x) = \lim_{n \to \infty} [u_n(x)] = u(x).$$

This completes the proof.

We wish to note here for future use that the solution $u(x, \varepsilon, h)$ of the initial value problem (1.4)–(1.6), and its derivative, depend continuously on h for all $h \geqq 0$. This follows from the uniform convergence of the $\{u_n(x)\}$ which are clearly continuously differentiable in ε and h.

The preceding analysis was suggested by the classical paper of Hermann Weyl [6] who obtained a similar alternating process for the Blasius problem of fluid dynamics.

4. Multiple solutions and their asymptotic expansions. We shall now show that under the conditions H.1 to H.5 every root α_i of (2.9) can give rise to an asymptotic solution $u_i(x, \varepsilon)$ of the boundary value problem (1.1)–(1.3). Furthermore, we shall prove that corresponding to any α_i the asymptotic solution $u_i(x, \varepsilon)$ possesses the property that $u_i(x, \varepsilon) - v_i(x) = O(\varepsilon)$ and $u_i'(x, \varepsilon) - v_i'(x) = O(\varepsilon)$ as $\varepsilon \to 0$ uniformly on any subinterval $0 < \delta \leqq x \leqq 1$, where $v_i(x)$ is the solution of the reduced problem (2.4), (2.5).

In order to prove the existence of multiple asymptotic solutions of the boundary value problem (1.1)–(1.3) for sufficiently small $\varepsilon > 0$ we shall need the following lemmas.

LEMMA 4.1. *If* $|u'(x, \varepsilon, h)| < C$ *for sufficiently small* $\varepsilon > 0$, *where* C *is independent of* ε, *then for any* $x \in (0, 1]$ *we have*

$$\int_0^x \frac{1}{\varepsilon} e^{-(x-t)/\varepsilon} g(t, u(t, \varepsilon, h))\, dt - g(x, u(x, \varepsilon, h)) = O(\varepsilon) \tag{4.1}$$

for sufficiently small $\varepsilon > 0$, *where* $u(x, \varepsilon, h)$, *for fixed* $h \geqq 0$, *is the unique solution of the initial value problem* (1.4)–(1.6).

Proof. First, note that

$$\int_0^x \frac{1}{\varepsilon} e^{-(x-t)/\varepsilon}\, dt = 1 - e^{-x/\varepsilon}.$$

Then,

$$\left| \int_0^x \frac{1}{\varepsilon} e^{-(x-t)/\varepsilon} g(t, u(t, \varepsilon, h))\, dt - g(x, u(x, \varepsilon, h)) \right|$$

$$= \left| \frac{1}{(1 - e^{-x/\varepsilon})} \int_0^x \frac{1}{\varepsilon} e^{-(x-t)/\varepsilon} [(1 - e^{-x/\varepsilon}) g(t, u(t, \varepsilon, h)) - g(x, u(x, \varepsilon, h))]\, dt \right|$$

$$\leqq \frac{1}{(1 - e^{-x/\varepsilon})} \int_0^x \frac{1}{\varepsilon} e^{-(x-t)/\varepsilon} |g(t, u(t, \varepsilon, h)) - g(x, u(x, \varepsilon, h))|\, dt$$

$$+ \frac{e^{-x/\varepsilon}}{(1 - e^{-x/\varepsilon})} \int_0^x \frac{1}{\varepsilon} e^{-(x-t)/\varepsilon} g(t, u(t, \varepsilon, h))\, dt$$

$$\leqq \frac{\max |dg/dt|}{(1 - e^{-x/\varepsilon})} \int_0^x \frac{1}{\varepsilon} e^{-(x-t)/\varepsilon} (x - t)\, dt + e^{-x/\varepsilon} \max |g|$$

$$= \frac{\max |dg/dt|}{(1 - e^{-x/\varepsilon})} [e^{-x/\varepsilon}(-x - \varepsilon) + \varepsilon] + e^{-x/\varepsilon} \max |g|.$$

Here we have used the mean value theorem and the facts that g and $dg/dt = g_t + g_u u'$ are bounded. The lemma now follows.

As an immediate consequence of applying Lemma 4.1 to (3.1) we obtain the following lemma.

LEMMA 4.2. *If* $|u'(x, \varepsilon, h)| < C$ *for sufficiently small* $\varepsilon > 0$, *where* C *is independent of* ε, *then for all* $h \geqq 0$ *the solution* $u(x, \varepsilon, h)$ *of the initial value problem* (1.4)–(1.6) *satisfies*

$$u'(x, \varepsilon, h) - g(x, u(x, \varepsilon, h)) = O(\varepsilon) \tag{4.2}$$

for sufficiently small $\varepsilon > 0$ *on any subinterval* $0 < \delta \leqq x \leqq 1$.

Now, define J as the number of roots α_i of $g(1, \alpha) = B - b\alpha$ which exceed the quantity $M + O(\varepsilon)$ for sufficiently small $\varepsilon > 0$. For example, $J = 4$ for the situation illustrated in Fig. 1, and $J = 3$ for the situation illustrated in Fig. 2.

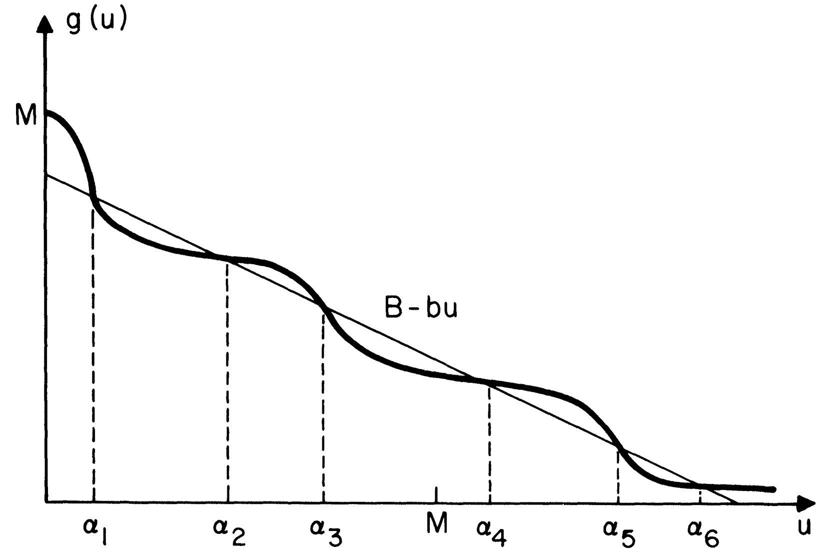

FIG. 2

We shall now prove that there exist J asymptotic solutions of the boundary value problem (1.1)–(1.3) and that on any subinterval $0 < \delta \leqq x \leqq 1$ each asymptotic solution and its derivative is asymptotic to the solution and its derivative of the reduced problem (2.4), (2.5). (Here we are assuming that $J \geqq 1$. Later we shall discuss the situation where roots α_i of $g(1, \alpha) = B - b\alpha$ exist but where $J = 0$.)

THEOREM 4.3. *Let* $g(x, u)$ *satisfy* H.1 *to* H.5. *Let* $v_i(x)$, $i = N - J + 1, \cdots, N$, *denote the solution on* $0 \leqq x \leqq 1$ *of the reduced problem*

$$v' - g(x, v) = 0, \tag{4.3}$$

$$v(1) = \alpha_i, \tag{4.4}$$

where α_i, $i = N - J + 1, \cdots, N$, *are the* J *roots of* $g(1, \alpha) = B - b\alpha$ *which exceed the quantity* $M + O(\varepsilon)$ *for sufficiently small* $\varepsilon > 0$. *Then, for all sufficiently small* $\varepsilon > 0$ *there exist* J *asymptotic solutions* $u_i(x, \varepsilon)$, $i = N - J + 1, \cdots, N$, *of* (1.1)–

(1.3) *such that* $u_i(x, \varepsilon) - v_i(x) = O(\varepsilon)$ *and* $u_i'(x, \varepsilon) - v_i'(x) = O(\varepsilon)$ *as* $\varepsilon \to 0$ *uniformly on any subinterval* $0 < \delta \leqq x \leqq 1$.

Proof. First, we prove that there exist J asymptotic solutions. Now consider the solution $u(x, \varepsilon, h)$ of the initial value problem (1.4)–(1.6). Equation (3.2) implies that for any $\varepsilon > 0$ we can choose h so large that $u(1, \varepsilon, h)$ is arbitrarily large. Furthermore, $g(x, u) \leqq M$ on R and (3.2) imply that

$$u(1, \varepsilon, h) < h + \varepsilon(A + ah)\, e^{-1/\varepsilon} + M(1 + \varepsilon + \varepsilon\, e^{-1/\varepsilon}).$$

Thus, for sufficiently small $\varepsilon > 0, u(1, \varepsilon, h) = M + h + O(\varepsilon)$. Hence, for sufficiently small $\varepsilon > 0$, $u(1, \varepsilon, h)$ varies continuously from $M + O(\varepsilon)$ to infinity as h varies from 0 to infinity. Therefore, $u(1, \varepsilon, h)$ takes on the values α_i, $i = N - J + 1, \cdots, N$, as h varies. Now, let $h_i = h_i(\varepsilon)$ denote the value of h for which $u(1, \varepsilon, h)$ takes on the value α_i; that is, $u(1, \varepsilon, h_i(\varepsilon)) = \alpha_i$ for sufficiently small $\varepsilon > 0$. Then,

$$g(1, u(1, \varepsilon, h_i(\varepsilon))) = B - bu(1, \varepsilon, h_i(\varepsilon)) \tag{4.5}$$

for sufficiently small $\varepsilon > 0$. If we can show that $|u'(1, \varepsilon, h_i(\varepsilon))| < C$ for sufficiently small $\varepsilon > 0$ where C is independent of ε, then Lemma 4.2 and (4.5) imply that $u'(1, \varepsilon, h_i) + bu(1, \varepsilon, h_i) = B + O(\varepsilon)$ for sufficiently small $\varepsilon > 0$. Therefore, if $|u'(1, \varepsilon, h_i(\varepsilon))| < C$ for sufficiently small $\varepsilon > 0$, then for each root $\alpha_i, i = N - J + 1, \cdots, N$, of $g(1, \alpha) = B - b\alpha$ there exists an $h_i = h_i(\varepsilon)$ such that corresponding to that value of h_i there exists an asymptotic solution $u_i(x, \varepsilon)$ of the boundary value (1.1)–(1.3). Therefore, if $|u'(1, \varepsilon, h_i(\varepsilon))| < C$ for sufficiently small $\varepsilon > 0$, then there exist J asymptotic solutions. To show that $|u'(1, \varepsilon, h_i(\varepsilon))| < C$ for sufficiently small $\varepsilon > 0$ note that (3.2) implies that

$$\begin{aligned} u(1, \varepsilon, h_i(\varepsilon)) = \alpha_i &= h_i(\varepsilon) + \varepsilon(A + ah_i(\varepsilon))(1 - e^{-1/\varepsilon}) \\ &\quad + \int_0^1 [1 - e^{-(1-t)/\varepsilon}] g(t, u(t, \varepsilon, h_i(\varepsilon)))\, dt. \end{aligned} \tag{4.6}$$

Since all terms on the right of (4.6) are positive, then $h_i(\varepsilon) < \alpha_i$. That is, $h_i(\varepsilon) = O(1)$ as $\varepsilon \to 0$. From this together with $|g(x, u)| < M$ we conclude from (3.1) that $u'(1, \varepsilon, h_i(\varepsilon))$ is bounded independent of ε for sufficiently small $\varepsilon > 0$, and therefore there exist J asymptotic solutions. Note that in a similar way it follows that $|u'(x, \varepsilon, h_i(\varepsilon))| < c$ for all $x \in (0, 1]$.

Lemma 4.2 and the preceding paragraph imply that each asymptotic solution $u_i(x, \varepsilon)$ satisfies

$$u_i' - g(x, u_i) = O(\varepsilon),$$

$$u_i(1, \varepsilon) = \alpha_i,$$

for sufficiently small $\varepsilon > 0$ on $0 < \delta \leqq x \leqq 1$. Let $v_i(x)$ denote the solution of (4.3), (4.4) on $0 < \delta \leqq x \leqq 1$. Then, standard theorems on ordinary differential equations (for example, Theorem 5 of W. Hurewicz [7, p. 9]) immediately imply that $u_i(x, \varepsilon) - v_i(x) = O(\varepsilon)$ and $u_i'(x, \varepsilon) - v_i'(x) = O(\varepsilon)$ as $\varepsilon \to 0$ uniformly as $\varepsilon \to 0$ on $0 < \delta \leqq x \leqq 1$. This completes the proof.

It is clear that there exist functions $g(x, u)$ satisfying H.1 to H.5 such that roots α_i exist but $J = 0$. Our Theorem 4.3 does not apply here, and we can draw no conclusions as to whether or not the solutions of (4.3), (4.4) approximate solutions of (1.1)–(1.3) on $0 < \delta \leqq x \leqq 1$ for sufficiently small $\varepsilon > 0$. The formal matching techniques of singular perturbation theory [4] indicate that a root α_1 may exist but that the corresponding solution of (4.3), (4.4) is *not* an approximate solution of (1.1)–(1.3) on $0 < \delta \leqq x \leqq 1$ for sufficiently small $\varepsilon > 0$. There are other situations for which such solutions may not exist. For example, let $y = u'$ and write (1.1) as

$$\frac{dy}{du} = \frac{g(x, u) - y}{\varepsilon y}. \tag{4.7}$$

Figure 3 represents a sketch of the phase-plane trajectories corresponding to (4.7) for small $\varepsilon > 0$ for the same function g used in Fig. 1. A necessary condition for the existence of a solution of (1.1)–(1.3) is that a trajectory intersect both the

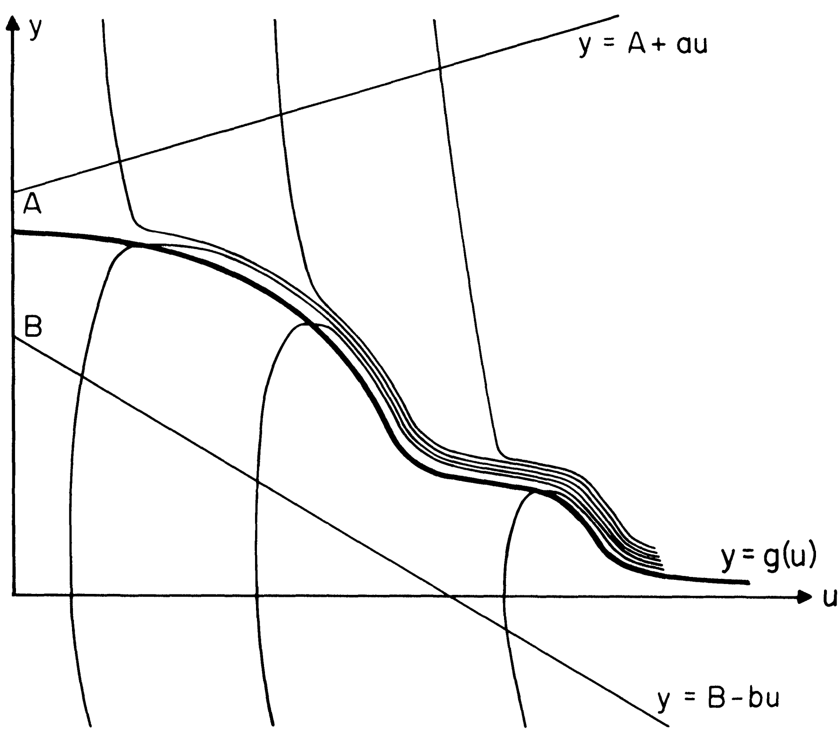

FIG. 3

lines $y = A + au$ and $y = B - bu$. The situation depicted in Fig. 3 represents a case in which no such trajectory exists. It seems reasonable to expect that under such conditions a solution of (1.1)–(1.3) will not exist. Thus, for example, if $B - bu < g(x, u)$ for all u and $A > M = \max_{0 \leqq x \leqq 1} [g(x, 0)]$, we expect that (1.1)–(1.3) has no solution for sufficiently small $\varepsilon > 0$.

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