Anisotropic KPZ growth in $2+1$ dimensions: fluctuations and covariance structure

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Abstract

In [5] we studied an interacting particle system which can be also interpreted as a stochastic growth model. This model belongs to the anisotropic KPZ class in $2+1$ dimensions. In this paper we present the results that are relevant from the perspective of stochastic growth models, in particular: (a) the surface fluctuations are asymptotically Gaussian on a $\sqrt{\ln t}$ scale and (b) the correlation structure of the surface is asymptotically given by the massless field.

1 Introduction

We consider growth models of an interface described, at least on a coarse-grained level, by a height function $x \mapsto h(x,t)$, where $t \in \mathbb{R}$ is the time and $x \in \mathbb{R}^d$ (or $\mathbb{Z}^d$) is the position on a d-dimensional substrate. When the growth is *local*, then on a macroscopic level the speed of growth v is a function of the slope only

$$v = v(\nabla h). \tag{1}$$

The class of models we consider are *stochastic* with a *smoothing mechanism* leading to a non-random limit shape,

$$h_{\mathrm{ma}}(\xi) := \lim_{t\to\infty} \frac{h(\xi t, t)}{t}, \tag{2}$$

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and in the Kardar-Parisi-Zhang (KPZ) class, related to the KPZ equation

$$\frac{\partial h}{\partial t} = \nu \Delta h + \frac{\lambda}{2}(\nabla h)^2 + \eta, \tag{3}$$

with η a local space-time noise. (3) is a mesoscopic equation, in which the smoothing mechanism is modeled by the $\nu\Delta h$ term, which is physically related with the surface tension. η is the stochastic part, while the non-linear term $\frac{\lambda}{2}(\nabla h)^2$ is coming from Taylor series of $v(\nabla h)$ around $\nabla h = 0$. Indeed, (3) assumes that one sets the system of coordinates such that ∇h is small. It is then possible to make the choice so that the 0th and 1st Taylor coefficients vanish. In [26] it was argued that for the long time behavior, higher order terms become irrelevant.

Before describing the two-dimensional case, let us spend one paragraph on the $d = 1$ situation. By non-rigorous arguments, the scaling exponent in $d = 1$ were correctly derived to be $1/3$ for the fluctuations and $2/3$ for the spatial correlations, see [3, 4, 20]. This means that the height function h rescaled as

$$h_t^{\rm resc}(u) := \frac{h(\xi t + u t^{2/3}, t) - t h_{\rm ma}\left((\xi t + u t^{2/3})/t\right)}{t^{1/3}}, \tag{4}$$

in the long time limit will converge to a well-defined stochastic process. The analysis of simplified models in the KPZ class gives the following results for noise-free initial conditions (results expected to hold for the whole class by universality). If $h_{\rm ma}$ is straight (i.e. second derivative is zero) around ξ, then the limit process is the Airy$_1$ process [7, 8], if $h_{\rm ma}$ is curved, the process is the Airy$_2$ process [23, 24, 36, 38]. Further results as the transition between these processes [10] or correlations at different times [6, 9, 22] with also some unexpected results [15] are also available. See [12, 16, 17, 34] for reviews and [1, 2, 18, 25, 32, 35, 37] for related works.

The main reason of this paper is to present results on the anisotropic class in $d = 2$. Going back to the KPZ equation, one realizes that in two dimensions one should write

$$\frac{\partial h}{\partial t} = \nu \Delta h + Q(\nabla h) + \eta, \tag{5}$$

where Q is a quadratic form. It turns out that there are two classes, depending on the signature of the quadratic form Q.

(a) Isotropic KPZ: $\mathrm{sign}(Q) = (+,+)$ or $(-,-)$, i.e., the Hessian of $v = v(\nabla h)$ has two eigenvalues of the same sign. In this case, the fluctuations grow as t^{β} for some $\beta > 0$, but renormalization arguments used in the $d = 1$

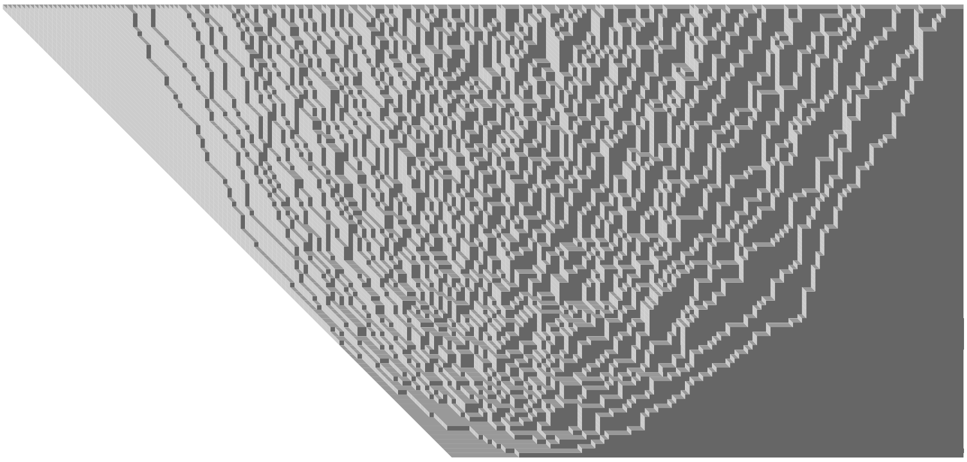

Figure 1: A configuration of the model analyzed with $N = 100$ particles at time $t = 25$. In [14] there is a Java animation of the model.

case fail to give an answer. Numerical studies on discrete models belonging to the KPZ class [19,40] as well as numerical solution of the equation itself [31], indicate that $\beta = 0.240 \pm 0.001$ (ruling out the value 1/4). For this case, essentially no analytic results are available.

(b) Anisotropic KPZ (=AKPZ): $\mathrm{sign}(Q) = (+, -)$. In this situation, the behavior is quite different and the fluctuations increase much slower, only as $\sqrt{\ln t}$. The first time the AKPZ was considered was by Wolf in [41], who was interested in growth on vicinal surfaces. They are surfaces with a small tilt with respect to a high-symmetry plane of the crystalline structure of the solid, and they are intrinsically asymmetric. By detailed one-loop computations, Wolf deduced that the non-linearity should be irrelevant for the roughness, which should grow in time as $\ln(t)$, i.e.,

$$\mathrm{Var}(h(\xi t, t)) \sim \ln(t), \qquad \text{as } t \to \infty, \tag{6}$$

exactly as in the $\lambda = 0$ case (Edwards-Wilkinson model [13]). On the numerical side, this drastic difference have been tested positively soon after [21] on the model used by Wolf directly, but also on a deposition model [30]. On the analytic side, Prähofer and Spohn [33] considered a microscopic model in the AKPZ class and reproduced Wolf's prediction, namely (6). Moreover, they computed the local correlations obtaining

$$\lim_{t\to\infty} \mathrm{Var}(h(x,t) - h(x',t)) \sim \ln|x - x'|, \tag{7}$$

for large $|x - x'| \to \infty$, but not growing with t.

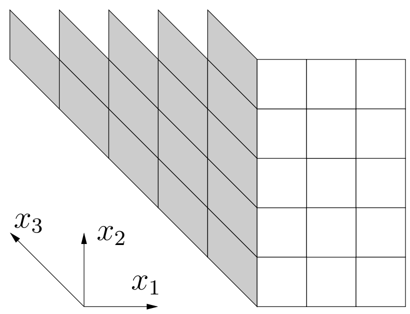

Figure 2: Initial condition. The corner of the crystal is at position $(0, 0, 0)$.

In our recent work [5] we consider a growth model in the AKPZ class, for which we determine the detailed correlation structure on a macroscopic scale. Below we explain the model and present the results relevant under the perspective of growth models (see Figure 1 for an illustration of the surface we analyzed).

2 The model

We consider a simplified model of anisotropic evaporation of a crystal. As initial condition we have a perfect corner on the positive octant, see Figure 2. The dynamics is as follows. The evaporation occurs by detaching of columns along the x_2-direction. Each cube at (x_1, x_2, x_3) such that both its x_3-facet and x_1-facet are visible, evaporates with rate one, and by doing so it takes away also all the cubes in the x_2-column above it: $\{(x_1, x_2', x_3) \text{ s.t. } x_2' > x_2\}$. This dynamics is intrinsically asymmetric, since the x_2 and x_3 directions play different roles.

There are several possible choices of coordinate which can be used to describe the interface as in Figure 1. The one we have chosen in [5] is the following. Consider the two-dimensional projection as in Figure 1, whose points can be described by $(x_1 - y, y = x_3 + x_2)$. For convenience, we take $x = x_1$ as the first coordinate and y as the second one, while the height function will be given by x_3:

$$(x, y, h_t(x, y)) = (x_1, x_2 + x_3, x_3). \tag{8}$$

The surface grows with a non-zero speed. Thus we rescale our coordinates as $x_1 = \xi t$, $x_2 = \zeta t$, $x_3 = h_t(\xi t, \zeta t)$ so that $x = \xi t$, $y = \zeta t - h_t(\xi t, \zeta t)$. Moreover,

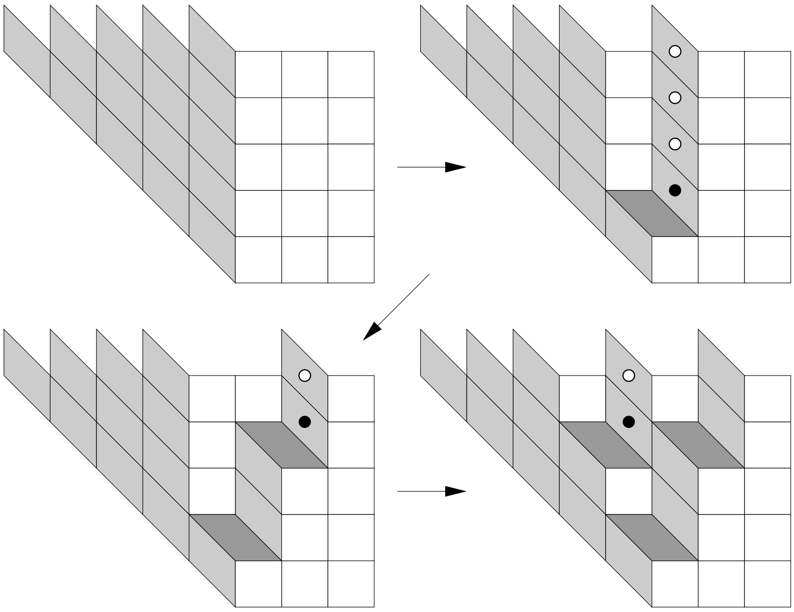

Figure 3: An example of the first three moves. The moved cubes are the ones with the black circles, and also the ones with white circles which are moved because they are in a column above the black ones.

in the long time limit, our surface has a *deterministic limit shape*

$$\boldsymbol{h}(\xi,\zeta) := \lim_{t\to\infty} \frac{h_t(\xi t,\zeta t)}{t}. \tag{9}$$

The limit shape consists of facets joined by a curved region in the coordinate range

$$(1-\sqrt{\zeta})^2 < \xi < (1+\sqrt{\zeta})^2. \tag{10}$$

Explicitly, for (ξ,ζ) in the curved region,

$$\boldsymbol{h}(\xi,\zeta) = \frac{1}{\pi}\int_{\xi}^{(1+\sqrt{\zeta})^2} \arccos\left(\frac{1+\zeta-\xi'}{2\sqrt{\zeta}}\right) \mathrm{d}\xi'. \tag{11}$$

In the curved region, in the long time limit, one locally sees a flat interface characterized by the two slopes

$$\boldsymbol{h}_\xi := \frac{\partial \boldsymbol{h}}{\partial \xi}, \quad \boldsymbol{h}_\zeta := \frac{\partial \boldsymbol{h}}{\partial \zeta}. \tag{12}$$

The two slopes determine uniquely a stationary measure, and since our *growth is short range*, the speed of growth v will be a function of $\boldsymbol{h}_\xi$ and $\boldsymbol{h}_\zeta$ only.

Indeed, we get

$$v = v(\boldsymbol{h}_\xi, \boldsymbol{h}_\zeta) = -\frac{1}{\pi}\frac{\sin(\pi \boldsymbol{h}_\xi)\sin(\pi \boldsymbol{h}_\zeta)}{\sin(\pi(\boldsymbol{h}_\xi + \boldsymbol{h}_\zeta))}. \tag{13}$$

From this expression, we can compute the determinant of the Hessian of v with the result

$$\begin{vmatrix} \partial_{\boldsymbol{h}_\xi}\partial_{\boldsymbol{h}_\xi} v & \partial_{\boldsymbol{h}_\xi}\partial_{\boldsymbol{h}_\zeta} v \\ \partial_{\boldsymbol{h}_\zeta}\partial_{\boldsymbol{h}_\xi} v & \partial_{\boldsymbol{h}_\zeta}\partial_{\boldsymbol{h}_\zeta} v \end{vmatrix} = -4\pi^2 \frac{\sin(\pi \boldsymbol{h}_\xi)^2 \sin(\pi \boldsymbol{h}_\zeta)^2}{\sin(\pi(\boldsymbol{h}_\xi + \boldsymbol{h}_\zeta))^4} < 0 \tag{14}$$

for $\boldsymbol{h}_\xi, \boldsymbol{h}_\zeta, \boldsymbol{h}_\xi + \boldsymbol{h}_\zeta \in (0,1)$ (which is equivalent to being in the curved region).

Thus, our stochastic growth model has a smoothing mechanism determining deterministic limit shape, the growth rule is local, and the Hessian has eigenvalues of opposite signs. Therefore, our model belongs to the anisotropic KPZ class (AKPZ) in $2+1$ dimensions.

3 Results

Logarithmic fluctuations

The first result concerns Wolf's prediction of the logarithmic growth of the fluctuations. In Theorem 1.2 of [5] we prove the following result.

Theorem 1. *Consider any macroscopic point in the curved region, i.e.,* $(x, y) = (\xi t, \zeta t)$ *with* (ξ, ζ) *satisfying (10). Then, the random variable*

$$\widetilde{h}_t(\xi, \zeta) := \frac{h_t(x, y) - \mathbb{E}(h_t(x, y))}{\sqrt{\ln t}} \tag{15}$$

converges in the $t \to \infty$ *limit to a normal random variable with mean zero and variance* $1/(2\pi^2)$*.*

Complex structure and local measure

To present the other results, we need to define the following mapping. For any (ξ, ζ) satisfying (10), define the map Ω to the upper (complex) half-plane $\mathbb{H} := \{z \in \mathbb{C} \text{ s.t. } \mathrm{Im}(z) > 0\}$ as follows:

$$|\Omega| = \sqrt{\zeta}, \quad |1 - \Omega| = \sqrt{\xi}. \tag{16}$$

Geometrically, the point Ω is the intersection in $\mathbb{H}$ of the circles[1] $B(0, \sqrt{\zeta})$ and $B(1, \sqrt{\xi})$ (see Figure 4). Also, denote the angles of the triangle with

[1]The notation $B(x, r)$ denotes the circle centered at x of radius r.

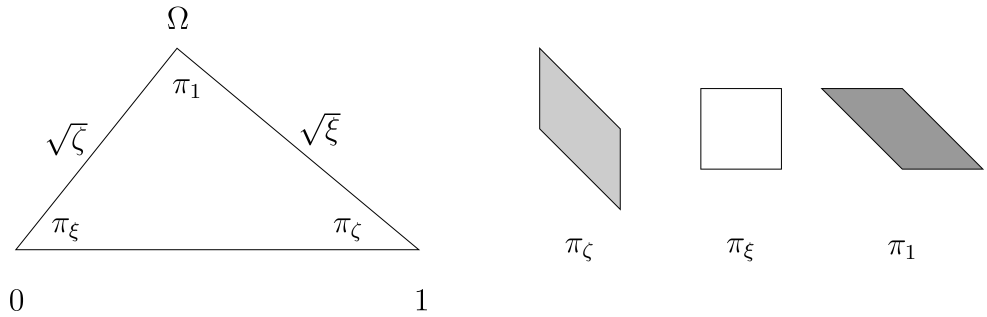

Figure 4: Map Ω and types of facets with associated angles.

vertices $0, 1, \Omega$ by

$$\pi_\zeta = -\arg(1-\Omega), \quad \pi_\xi = \arg(\Omega), \quad \pi_1 = \pi - \pi_\zeta - \pi_\xi. \tag{17}$$

Now we discuss the local measure in the curved part of the surface. If one focuses around macroscopic point (ξ, ζ), asymptotically one sees a translation invariant measure parameterized by the gradient of the surface. The surface can be also viewed as a random tiling of the plane by three types of lozenges indicated in Figure 4. Then, the measure one sees locally is the unique translation invariant Gibbs measure on lozenge tilings with prescribed proportions of lozenges $(\pi_\zeta/\pi, \pi_\xi/\pi, \pi_1/\pi)$ (see [11, 27, 29] and references therein for discussions of these Gibbs measures). From the tiling point of view, it is easy to understand the connection between the angles π_ξ, π_ζ and the asymptotic slopes $\boldsymbol{h}_\xi, \boldsymbol{h}_\zeta$:

$$\boldsymbol{h}_\xi = -\pi_\zeta/\pi, \quad \boldsymbol{h}_\zeta = 1 - \pi_\xi/\pi. \tag{18}$$

Indeed, along the ξ-direction, the height changes whenever there are lozenges associated to π_ζ, while along the ζ direction it changes whenever there are lozenges not associated with π_ξ (varying ζ with fixed ξ corresponds to moving along the $(1, -1)$ direction in our figures). Finally, the growth velocity takes a very simple form in terms of Ω:

$$v = \mathrm{Im}(\Omega)/\pi. \tag{19}$$

Interestingly enough, although we study a stochastic growth model (out of equilibrium), the limit shape is related to a variational problem appearing for uniformly distributed lozenge tilings of a given domain. The map Ω satisfies the complex Burgers equation

$$(1-\Omega)\frac{\partial \Omega}{\partial \xi} = \Omega \frac{\partial \Omega}{\partial \zeta}, \tag{20}$$

which has been identified as the Euler-Lagrange equation for the surface free energy for fixed domains, see [28] and references therein. The limit shape gives a solution of (20) and it is encoded in the equation

$$\xi\Omega + \zeta(1-\Omega) = \Omega(1-\Omega), \quad \Omega \in \mathbb{H},\, \xi,\zeta \geq 0. \tag{21}$$

Indeed, for any $\Omega \in \mathbb{H}$, there is a unique couple $(\xi,\zeta) \in \mathbb{R}_+^2$ satisfying (21), and then the limit shape $\boldsymbol{h}(\xi,\zeta)$ can be simply expressed as the integral of the slope as in (11).

Macroscopic correlations

Finally, let us discuss the covariance structure of different macroscopic points in the curved part of the surface. At first sight, in view of (15) one might think that we have to compute $\mathbb{E}(\prod_{i=1}^m \widetilde{h}_t(\xi_i,\zeta_i))$ for distinct (ξ_i,ζ_i) satisfying (10). However, this turns out to converge to zero in the $t \to \infty$ limit.

Our result is that, after mapping by Ω, the height function converges to the massless field on $\mathbb{H}$, which we denote GFF. The value of GFF at a point cannot be defined. However, one can think of expectations of products of values of GFF at different points as being finite and equal to

$$\begin{aligned} &\mathbb{E}[\mathrm{GFF}(z_1)\cdots\mathrm{GFF}(z_m)] \\ &\quad= \begin{cases} 0 & \text{if } m \text{ is odd,} \\ \sum\limits_{\text{pairings } \sigma} \mathcal{G}(z_{\sigma(1)}, z_{\sigma(2)})\cdots\mathcal{G}(z_{\sigma(m-1)}, z_{\sigma(m)}) & \text{if } m \text{ is even,} \end{cases} \end{aligned} \tag{22}$$

where

$$\mathbb{E}[\mathrm{GFF}(z)\,\mathrm{GFF}(w)] = \mathcal{G}(z,w) := -\frac{1}{2\pi}\ln\left|\frac{z-w}{z-\bar{w}}\right|, \tag{23}$$

is the Green function of the Laplace operator on $\mathbb{H}$ with Dirichlet boundary conditions. In (22) one recognizes Wick formula for Gaussian processes. For further details on the massless field (also called Gaussian Free Field), see e.g. [39].

Our result on the correlations on a *macroscopic* scale is as follows (Theorem 1.3 in [5]).

Theorem 2. *For any $m = 1,2,\ldots$, consider any m distinct pairs $\{(\xi_i,\zeta_i), i = 1,\ldots,m\}$ which satisfy (10). Denote*

$$H_t(\xi,\zeta) := \sqrt{\pi}\,[h_t(\xi t,\zeta t) - \mathbb{E}(h_t(\xi t,\zeta t))] \tag{24}$$

and $\Omega_i := \Omega(\xi_i,\zeta_i)$. Then,

$$\lim_{t\to\infty} \mathbb{E}(H_t(\xi_1,\zeta_1)\cdots H_t(\xi_m,\zeta_m)) = \mathbb{E}[\mathrm{GFF}(\Omega_1)\cdots\mathrm{GFF}(\Omega_m)] \tag{25}$$

At first sight it might look strange that in Theorem 1 we have $\sqrt{\ln t}$ scaling, while in Theorem 2 it is not present. If we consider H_t as random variable, Theorem 1 tells us that, in the $t \to \infty$ limit, H_t does not converge to a well-defined random variable (the $\sqrt{\ln t}$ normalization is missing). On the other hand, Theorem 2 tells us that the macroscopic correlations of H_t are non-trivial. The reason is that asymptotically the interface H_t will take values in the space of distributions, not ordinary functions.

Our Theorem 1.3 in [5] is actually more general. Indeed, we obtain the limiting correlations also for the height function at different times, provided that the space-time events at which we consider the height function satisfy a "space-like condition". When we consider heights at different times, we cannot use t anymore as a large parameter. We use L instead, set $t = \tau L$, and denote by $(\nu L, \eta L, \tau L)$ a space-time point ($\nu = \tau\xi$, $\eta = \tau\zeta$). Then, the condition (10) that a point is in the curved region becomes

$$(\sqrt{\tau} - \sqrt{\eta})^2 < \nu < (\sqrt{\tau} + \sqrt{\eta})^2. \tag{26}$$

Then, the mapping Ω is now from a triple (ν, η, τ) to $\mathbb{H}$ given by the condition

$$|\Omega| = \sqrt{\eta/\tau}, \quad |1 - \Omega| = \sqrt{\nu/\tau}. \tag{27}$$

Then the full result of our Theorem 1.3 in [5] is the following.

Theorem 3. *For any $m = 1, 2, \ldots$, consider any m distinct triples $\{(\nu_i, \eta_i, \tau_i), i = 1, \ldots, m\}$, which satisfy (26) and the space-like condition*

$$\tau_1 \leq \tau_2 \leq \ldots \leq \tau_m, \quad \eta_1 \geq \eta_2 \geq \ldots \geq \eta_m. \tag{28}$$

Denote

$$H_L(\nu, \eta, \tau) := \sqrt{\pi}\,[h_{\tau L}(\nu L, \eta L) - \mathbb{E}(h_{\tau L}(\nu L, \eta L))] \tag{29}$$

and $\Omega_i := \Omega(\nu_i, \eta_i, \tau_i)$. Then,

$$\lim_{L\to\infty} \mathbb{E}(H_L(\nu_1, \eta_1, \tau_1) \cdots H_L(\nu_m, \eta_m, \tau_m)) = \mathbb{E}[\mathrm{GFF}(\Omega_1) \cdots \mathrm{GFF}(\Omega_m)]. \tag{30}$$

There is an interesting fact to notice. Consider for simplicity the following two pairs of space-time points

$$\{(\nu_1, \eta_1, 1), (\nu_2, \eta_2, 1) \text{ with } \eta_2 < \eta_1\} \tag{31}$$

and

$$\{(\nu_1, \eta_1, 1), (\nu', \eta', \tau) \text{ s.t. } \Omega(\nu_2, \eta_2, 1) = \Omega(\nu', \eta', \tau), \tau > 1\}. \tag{32}$$

Then, Theorem 3 tell us that the correlations of H_L at these two pairs are asymptotically the same. Also, remark that space-time points have the same image by Ω on $\mathbb{H}$ if and only if they have the same macroscopic slopes defined in (12), see (18).

This a priori unexpected phenomenon, can be explained as follows. Along the space-time curves with constant Ω, the height decorrelates on a much slower time scale than L. The same fact has been already observed in KPZ growth on a $d = 1$ substrate [15] (there the correct scale is L instead of $L^{2/3}$ for spatial correlations).

As a consequence of this interpretation, we conjecture that Theorem 3 holds *without the space-like condition* (28), provided that $\Omega_1, \ldots, \Omega_m$ remain pairwise distinct.

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